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Nonlinear magnetoelastostatics: Energy functionals and their second variations

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Abstract
Two variational principles for nonlinear magnetoelastostatics are studied, considering a magnetosensitive body completely surrounded by free space extending to infinity. The functionals depend on the deformation function as one of the independent variables, and on either the scalar magnetic potential or the magnetic vector potential as the independent magnetic variable. Alternative representations for the energy densities are given for free space, from which simple expressions for the first and second variations of the functionals are obtained.

Keywords
Magnetoelasticity, variational formulations, second variation, free space, energy function

1. Introduction

Recent years have seen an increasing interest in studying theoretically the behaviour of a class of smart materials called magnetosensitive (MS) elastomers; see, for example, the works by Dorfmann and Ogden [1, 2], Kankanala and Triantafyllidis [3] and Steigmann and co-workers [4, 5].[1] MS elastomers can exhibit large nonlinear elastic deformations when subjected to external magnetic fields. The presence of large deformations and the coupling between elastic deformations and the magnetic field imply that the task of solving boundary-value problems can be particularly difficult; therefore, there is a need to develop numerical methods for solution of boundary-value problems; see, for example, Bustamante et al. [10, 11] for some numerical solutions of the magnetoelastic problem and also Vu et al. [12] for a similar nonlinear electroelastic case. Variational formulations can be used not only to develop numerical methods such as the finite element method, but also to help in the study of other topics such as the appearance of instability phenomena; see, for example, for the elastic nonlinear boundary-value problem (no magnetic interactions) Sections 88 and 89 in Truesdell and Noll [13] and Ball and Marsden [14] and Section 20 in Šilhavý [15]. In Section 8 of Barham et al. [16] we see similar use of the second variation in the study of the behaviour of a magnetoelastic membrane.

In Bustamante et al. [17], several variational principles were reviewed and energy functionals that depend on the magnetic field and the magnetic induction were presented, considering not only the magnetoelastic body, but also the surrounding free space.[2]

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In the present work, we consider the energy functional presented in Bustamante et al. [17], and we rewrite some of the expressions for the energy functionals for the surrounding free space using the device of a ‘fictitous’ extension of the deformation field in the body to the surrounding space. With these modified forms for the functionals, expressions for the first and the second variations are obtained.

The structure of the paper is described now. In Section 2 we summarize the basic equations of nonlinear magnetoelasticity, introducing a Lagrangian form for the energy density for the vacuum space outside the body, from which simple expressions for the Maxwell stresses and magnetic variables can be obtained. In Section 3 new expressions for the energy functional are presented, which are used in Sections 4 and 5 in order to calculate the first and the second variations, considering first the scalar magnetic potential as the independent magnetic variable and thereafter the vector magnetic potential as the independent magnetic variable. In Section 6 some concluding remarks are provided.

2. Basic equations

2.1. Kinematics

We consider a deformable magnetically sensitive body that is initially in an unstressed configuration, which we denote by $\mathcal{B}_r$, with boundary $\partial\mathcal{B}_r$. Let $\mathbf{X}$ denote the position vector of a material point within the body in this configuration. As a result of applied mechanical loads and an applied magnetic field the body occupies a deformed configuration, denoted $\mathcal{B}$, with boundary $\partial\mathcal{B}$, within which the material point $\mathbf{X}$ occupies the position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, where the vector function $\boldsymbol{\chi}$, which is a one-to-one, orientation-preserving mapping with suitable regularity properties, describes the (quasi-static) deformation from $\mathcal{B}_r$ to $\mathcal{B}$ and is defined for $\mathbf{X} \in \mathcal{B}_r \cup \partial\mathcal{B}_r$.

The deformation gradient tensor $\mathbf{F}$ relative to $\mathcal{B}_r$ is defined by

$$\mathbf{F} = \operatorname{Grad}\boldsymbol{\chi}, \quad \mathbf{X} \in \mathcal{B}_r, \tag{1}$$

where Grad denotes the gradient operator with respect to $\mathbf{X}$. In Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, where X_α and x_i, $\alpha, i \in \{1, 2, 3\}$ are the components of $\mathbf{X}$ and $\mathbf{x}$ respectively. Greek subscripts are associated with $\mathcal{B}_r$ and Roman subscripts with $\mathcal{B}$. We also adopt the notation

$$J = \det\mathbf{F} \tag{2}$$

with the standard convention $J > 0$.

For detailed discussion of the relevant background in nonlinear elasticity and continuum mechanics we refer to, for example, Ogden [19] and Holzapfel [20].

2.2. The equations of magnetostatics: Eulerian form

We denote by $\mathbf{H} = \mathbf{H}(\mathbf{x})$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})$, respectively, the magnetic field and the magnetic induction vectors in the deformed configuration configuration $\mathcal{B}$. These vector fields satisfy the equations

$$\operatorname{curl}\mathbf{H} = \mathbf{0}, \quad \operatorname{div}\mathbf{B} = 0, \tag{3}$$

which are the appropriate specializations of Maxwell's equations in the absence of distributed currents and time dependence. They apply both within magnetizable and nonmagnetizable material, including free space. In vacuum or in nonmagnetizable material $\mathbf{B}$ and $\mathbf{H}$ are related simply through

$$\mathbf{B} = \mu_0\mathbf{H}, \tag{4}$$

where the constant μ_0 is the permeability of free space. In magnetizable material there is an additional vector, the magnetization $\mathbf{M}$, which is defined by

$$\mathbf{M} = \mu_0^{-1}\mathbf{B} - \mathbf{H}. \tag{5}$$

In the present paper we shall not make use of $\mathbf{M}$, but we remark that equation (5) gives an explicit expression for $\mathbf{M}$ in terms of either $\mathbf{H}$ or $\mathbf{B}$ as the independent variable when $\mathbf{B}$ (respectively $\mathbf{H}$) is given in terms of $\mathbf{H}$ (respectively $\mathbf{B}$) by a constitutive law. The magnetization itself is also often used as the independent magnetic variable, as, for example, in the work of Kankanala and Triantafyllidis [3].

We consider that the material is located in free space. Then, across the boundary $\partial\mathcal{B}$, the fields $\mathbf{H}$ and $\mathbf{B}$ have to satisfy certain continuity conditions. Let open square brackets signify the jump in the enclosed quantity in passing from the inside to the outside of the body. For example, $[\![\mathbf{H}]\!] = \mathbf{H}^{\mathrm{o}} - \mathbf{H}^{\mathrm{i}}$, where $^{\mathrm{o}}$ and $^{\mathrm{i}}$ designate 'outside' and 'inside', respectively. Then, in the absence of surface currents, the continuity conditions satisfied by the fields are

$$\mathbf{n} \times [\![\mathbf{H}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0 \quad \text{on } \partial\mathcal{B}, \tag{6}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$. Here we consider that the material in $\mathcal{B}$ has no internal surfaces of discontinuity.

2.3. The equations of magnetostatics: Lagrangian form

We now define Lagrangian counterparts of $\mathbf{H}$ and $\mathbf{B}$, defined in the reference configuration $\mathcal{B}_r$. These are denoted $\mathbf{H}_l = \mathbf{H}_l(\mathbf{X})$ and $\mathbf{B}_l = \mathbf{B}_l(\mathbf{X})$, respectively, and they are given by 'pull-back' operations on $\mathbf{H}$ and $\mathbf{B}$ via

$$\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}, \quad \mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \tag{7}$$

respectively, where $^{\mathrm{T}}$ denotes the transpose of a second-order tensor.

Assuming appropriate regularity of the deformation, the standard kinematical identities

$$\mathrm{Div}\,(J\mathbf{F}^{-1}\mathbf{B}) = J\mathrm{div}\,\mathbf{B}, \quad \mathbf{F}\mathrm{Curl}\,(\mathbf{F}^{\mathrm{T}}\mathbf{H}) = J\mathrm{curl}\,\mathbf{H} \tag{8}$$

ensure that the pair of equations in equation (3) is entirely equivalent to the pair

$$\mathrm{Curl}\,\mathbf{H}_l = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{B}_l = 0, \tag{9}$$

where Curl and Div are the divergence and curl operators with respect to $\mathcal{B}_r$.

The counterparts of the jump conditions, equation (6), are

$$\mathbf{N} \times [\![\mathbf{H}_l]\!] = \mathbf{0}, \quad \mathbf{N} \cdot [\![\mathbf{B}_l]\!] = 0 \quad \text{on } \partial\mathcal{B}_r, \tag{10}$$

where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_r$.

2.4. Equilibrium and stress

Following Dorfmann and Ogden [1,2] we work with the so-called 'total Cauchy stress tensor', denoted $\boldsymbol{\tau}$, which is symmetric and incorporates terms that may be considered, equivalently, as magnetic body forces rather than stresses. In terms of $\boldsymbol{\tau}$ the equilibrium equation has the form

$$\mathrm{div}\,\boldsymbol{\tau} + \rho\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}, \tag{11}$$

where $\mathbf{f}$ is the *mechanical* body force per unit mass and ρ is the mass density of the material in the configuration $\mathcal{B}$.

The appropriate version of the total stress tensor $\boldsymbol{\tau}$ in the Lagrangian context is the 'total nominal stress tensor', here denoted $\mathbf{T}$ and related to $\boldsymbol{\tau}$ by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau}. \tag{12}$$

The equilibrium equation (11) may then be written in the alternative (and equivalent) form

$$\mathrm{Div}\,\mathbf{T} + \rho_r\mathbf{f} = \mathbf{0} \quad \text{in } \mathcal{B}_r, \tag{13}$$

where $\rho_r = \rho J$ is the mass density of the material in $\mathcal{B}_r$.

Outside the material it is convenient to define the Maxwell stress tensor, denoted $\boldsymbol{\tau}_{\mathrm{m}}$, which, in view of the connection given in equation (4), may be written in several equivalent forms. In what follows we shall use the forms

$$\boldsymbol{\tau}_{\mathrm{m}} = \mu_0\left[\mathbf{H}\otimes\mathbf{H} - \tfrac{1}{2}(\mathbf{H}\cdot\mathbf{H})\mathbf{I}\right] = \mu_0^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right], \tag{14}$$

where $\mathbf{I}$ is the identity tensor.

The boundary condition to be satisfied by $\boldsymbol{\tau}$ on the part of the boundary where the traction is prescribed, say $\partial\mathcal{B}^\tau$, is

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_\mathrm{a} + \mathbf{t}_\mathrm{m} \quad \text{on } \partial\mathcal{B}^\tau, \tag{15}$$

where $\mathbf{t}_\mathrm{a}$ is the applied mechanical traction per unit area of $\partial\mathcal{B}$ and $\mathbf{t}_\mathrm{m}$ is the corresponding load due to the Maxwell stress and is defined by

$$\mathbf{t}_\mathrm{m} = \boldsymbol{\tau}_\mathrm{m}\mathbf{n} \quad \text{on } \partial\mathcal{B}. \tag{16}$$

The corresponding traction boundary condition for $\mathbf{T}$ is

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + \mathbf{t}_\mathrm{M} \quad \text{on } \partial\mathcal{B}_r^\tau, \tag{17}$$

where $\mathbf{t}_\mathrm{a}\mathrm{d}a = \mathbf{t}_\mathrm{A}\mathrm{d}A$, $\mathbf{t}_\mathrm{m}\mathrm{d}a = \mathbf{t}_\mathrm{M}\mathrm{d}A$, $\mathrm{d}A$ and $\mathrm{d}a$ are surface area elements on $\partial\mathcal{B}_r$ and $\partial\mathcal{B}$, respectively, and

$$\mathbf{t}_\mathrm{M} = \mathbf{T}_\mathrm{m}^\mathrm{T}\mathbf{N}, \quad \mathbf{T}_\mathrm{m} = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{m} \quad \text{on } \partial\mathcal{B}_r. \tag{18}$$

We may also consider that on part of the boundary, say $\partial\mathcal{B}_r^x$, where $\partial\mathcal{B}_r^x \cup \partial\mathcal{B}_r^\tau = \partial\mathcal{B}_r$, the position $\mathbf{x}$ is prescribed: thus,

$$\mathbf{x} \equiv \boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on } \partial\mathcal{B}_r^x, \tag{19}$$

where $\boldsymbol{\xi}$ is a given function.

2.5. Constitutive laws

For the magnetoelastic constitutive laws we adopt the formulation developed by Dorfmann and Ogden [1, 2]. In particular, we consider total 'energy' density functions for the material defined per unit reference volume, specifically $\Omega(\mathbf{F}, \mathbf{B}_l)$ and $\Omega^*(\mathbf{F}, \mathbf{H}_l)$, which are Legendre duals with respect to the variables $\mathbf{B}_l$ and $\mathbf{H}_l$, with the connection

$$\Omega(\mathbf{F}, \mathbf{B}_l) = \Omega^*(\mathbf{F}, \mathbf{H}_l) + \mathbf{H}_l \cdot \mathbf{B}_l. \tag{20}$$

Switching from one formulation to the other requires that there is a one-to-one relationship between $\mathbf{B}_l$ and $\mathbf{H}_l$, but otherwise each formulation stands separately without the assumption of such a relationship. With $\mathbf{B}_l$ as the independent magnetic variable, then, based on Ω, we have the constitutive relations

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \tag{21}$$

while if $\mathbf{H}_l$ is the independent magnetic variable, based on Ω^* we have

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial\Omega^*}{\partial\mathbf{H}_l}. \tag{22}$$

In the present paper we consider materials that are not subject to internal mechanical constraints, although extension of the theory to allow for constraints such as incompressibility is straightforward.

It is possible to define an energy function using a Lagrangian form of the magnetization, as in Kankanala and Triantafyllidis [3], but this does not yield such a compact form for the stress as the expressions above and we do not therefore adopt the magnetization as an independent magnetic variable in the present paper.

Let the region exterior to the material in the configuration $\mathcal{B}$ be denoted $\mathcal{B}'$, with boundary consisting of $\partial\mathcal{B}' = \partial\mathcal{B}$ and the boundary at infinity, denoted $\partial\mathcal{B}^\infty$. We distinguish between $\partial\mathcal{B}'$ and $\partial\mathcal{B}$ only by virtue of the fact that their outward normals have opposite signs. We also denote the exterior of $\mathcal{B}_r$ by $\mathcal{B}_r'$, with boundary $\partial\mathcal{B}_r' = \partial\mathcal{B}_r$. Within $\mathcal{B}'$ the magnetic energy per unit volume can be written in either of the forms

$$\tfrac{1}{2}\mathbf{H} \cdot \mathbf{B} = \tfrac{1}{2}\mu_0\mathbf{H} \cdot \mathbf{H} = \tfrac{1}{2}\mu_0^{-1}\mathbf{B} \cdot \mathbf{B}.$$

It is convenient temporarily to introduce an energy density function $\omega(\mathbf{B})$, analogous to $\Omega(\mathbf{F}, \mathbf{B}_l)$ but defined per unit volume in $\mathcal{B}'$, and its Legendre dual $\omega^*(\mathbf{H}) = \omega(\mathbf{B}) - \mathbf{H} \cdot \mathbf{B}$. Then

$$\omega(\mathbf{B}) = \tfrac{1}{2}\mu_0^{-1}\mathbf{B} \cdot \mathbf{B}, \quad \omega^*(\mathbf{H}) = -\tfrac{1}{2}\mu_0\mathbf{H} \cdot \mathbf{H},$$

and, analogously to equations $(21)_2$ and $(22)_2$,

$$\mathbf{H} = \frac{\partial \omega}{\partial \mathbf{B}}, \quad \mathbf{B} = -\frac{\partial \omega^*}{\partial \mathbf{H}}.$$

It is now of considerable advantage to introduce in the exterior of $\mathcal{B}_r$ a continuation of the deformation function $\boldsymbol{\chi}$ and the corresponding deformation gradient so that both are continuous across $\partial\mathcal{B}_r$. This is a purely mathematical device which has no physical significance but enables the deforming boundary $\partial\mathcal{B}$ to be accommodated by the fields exterior to the material. The end result is independent of the choice of this continuation. Accordingly, we introduce Lagrangian forms of the field variables in $\mathcal{B}_r'$, namely $\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}$ and $\mathbf{H}_l = \mathbf{F}^{\mathrm{T}}\mathbf{H}$, as in equation (7). Also, we extend the definition of $\mathbf{T}_{\mathrm{m}}$ from the boundary $\partial\mathcal{B}_r$, as indicated in equation $(18)_2$, so that $\mathbf{T}_{\mathrm{m}} = J\mathbf{F}^{-1}\boldsymbol{\tau}_{\mathrm{m}}$ in $\mathcal{B}_r'$.

Then $J\omega$ and $J\omega^*$ can be expressed in terms of the Lagrangian variables as

$$J\omega = \tfrac{1}{2}\mu_0^{-1}J^{-1}(\mathbf{F}\mathbf{B}_l)\cdot(\mathbf{F}\mathbf{B}_l), \quad J\omega^* = -\tfrac{1}{2}\mu_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l).$$

These are the energy densities per unit volume in $\mathcal{B}_r'$, and we introduce the functions[3]

$$\Omega_{\mathrm{e}}(\mathbf{F}, \mathbf{B}_l) = \tfrac{1}{2}\mu_0^{-1}J^{-1}(\mathbf{F}\mathbf{B}_l)\cdot(\mathbf{F}\mathbf{B}_l), \quad \Omega_{\mathrm{e}}^*(\mathbf{F}, \mathbf{H}_l) = -\tfrac{1}{2}\mu_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l) \tag{23}$$

to represent them, where the subscript e indicates 'external'. These are the analogues in $\mathcal{B}_r'$ of $\Omega(\mathbf{F}, \mathbf{B}_l)$ and $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ in $\mathcal{B}_r$ but, unlike the latter, have the specific forms of equation (23). With the definitions in equation (23), the formulas

$$\mathbf{T}_{\mathrm{m}} = \frac{\partial \Omega_{\mathrm{e}}}{\partial \mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial \Omega_{\mathrm{e}}}{\partial \mathbf{B}_l}, \quad \mathbf{T}_{\mathrm{m}} = \frac{\partial \Omega_{\mathrm{e}}^*}{\partial \mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial \Omega_{\mathrm{e}}^*}{\partial \mathbf{H}_l} \tag{24}$$

may be established. The proofs of these formulas are straightforward but nontrivial, as follows:

$$\begin{aligned}\frac{\partial \Omega_{\mathrm{e}}}{\partial \mathbf{F}} &= \mu_0^{-1}J^{-1}\mathbf{B}_l \otimes \mathbf{F}\mathbf{B}_l - \tfrac{1}{2}\mu_0^{-1}J^{-1}(\mathbf{F}\mathbf{B}_l)\cdot(\mathbf{F}\mathbf{B}_l)\mathbf{F}^{-1} \\ &= \mu_0^{-1}J\mathbf{F}^{-1}\left[\mathbf{B}\otimes\mathbf{B} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{B})\mathbf{I}\right] = J\mathbf{F}^{-1}\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{T}_{\mathrm{m}},\end{aligned}$$

in which the standard formula $\partial J/\partial \mathbf{F} = J\mathbf{F}^{-1}$ has been used:

$$\frac{\partial \Omega_{\mathrm{e}}}{\partial \mathbf{B}_l} = \mu_0^{-1}J^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{B}_l = \mathbf{F}^{\mathrm{T}}(\mu_0^{-1}\mathbf{B}) = \mathbf{F}^{\mathrm{T}}\mathbf{H} = \mathbf{H}_l;$$

$$\begin{aligned}\frac{\partial \Omega_{\mathrm{e}}^*}{\partial \mathbf{F}} &= \mu_0 J\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l \otimes \mathbf{F}^{-\mathrm{T}}\mathbf{H}_l - \tfrac{1}{2}\mu_0 J(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\cdot(\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l)\mathbf{F}^{-1} \\ &= \mu_0 J\mathbf{F}^{-1}\left[\mathbf{H}\otimes\mathbf{H} - \tfrac{1}{2}(\mathbf{H}\cdot\mathbf{H})\mathbf{I}\right] = J\mathbf{F}^{-1}\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{T}_{\mathrm{m}};\end{aligned}$$

this also makes use of the formula $\partial J/\partial \mathbf{F} = J\mathbf{F}^{-1}$, and additionally an expression for $\partial(\mathbf{F}^{-1})/\partial\mathbf{F}$, which in index notation may be written

$$\frac{\partial (\mathbf{F}^{-1})_{\beta j}}{\partial F_{i\alpha}} = -(\mathbf{F}^{-1})_{\alpha j}(\mathbf{F}^{-1})_{\beta i};$$

finally,

$$\frac{\partial \Omega_{\mathrm{e}}^*}{\partial \mathbf{H}_l} = -\mu_0 J\mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}\mathbf{H}_l = -J\mathbf{F}^{-1}(\mu_0\mathbf{H}) = -J\mathbf{F}^{-1}\mathbf{B} = -\mathbf{B}_l.$$

3. Energy functionals

In Bustamante et al. [17] variational principles based on the energy functions $\Omega(\mathbf{F}, \mathbf{B}_l)$ and $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ were constructed. In the former case the equation $\operatorname{Div} \mathbf{B}_l = 0$ was used to introduce a vector potential $\mathbf{A}_l(\mathbf{X})$ such that $\mathbf{B}_l = \operatorname{Curl} \mathbf{A}_l$, and in the latter case the equation $\operatorname{Curl} \mathbf{H}_l = \mathbf{0}$ was used to introduce a scalar potential, here denoted $\varphi_l(\mathbf{X})$, such that $\mathbf{H}_l = -\operatorname{Grad} \varphi_l$. Thus the magnetic arguments of the energy functionals are the potentials rather than the fields themselves. These potentials are Lagrangian in form. The corresponding Eulerian forms are denoted $\mathbf{A}(\mathbf{x})$ and $\varphi(\mathbf{x})$, with $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and $\mathbf{H} = -\operatorname{grad} \varphi$ and the connections

$$\mathbf{A}_l(\mathbf{X}) = \mathbf{F}^{\mathrm{T}}(\mathbf{X})\mathbf{A}(\boldsymbol{\chi}(\mathbf{X})), \quad \varphi_l(\mathbf{X}) = \varphi(\boldsymbol{\chi}(\mathbf{X})). \tag{25}$$

As in Bustamante et al. [17] we may take φ_l and $\mathbf{A}_l$ to be continuous across $\partial\mathcal{B}_r$. These ensure, respectively, that the boundary conditions in equations $(10)_1$ and $(10)_2$ are satisfied.

We write the energy functional based on $\Omega(\mathbf{F}, \mathbf{B}_l)$ as

$$\begin{aligned}\Pi\{\mathbf{A}_l, \mathbf{x}\} = &\int_{\mathcal{B}_r} \Omega(\mathbf{F}, \mathbf{B}_l)\,\mathrm{d}V + \int_{\mathcal{B}'_r} \Omega_{\mathrm{e}}(\mathbf{F}, \mathbf{B}_l)\,\mathrm{d}V \\ &- \int_{\partial\mathcal{B}_r^{\tau}} \mathbf{x}\cdot\mathbf{t}_{\mathrm{A}}\,\mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{x}\cdot\mathbf{f}\,\mathrm{d}V + \int_{\partial\mathcal{B}^{\infty}} (\mathbf{H}_a \times \mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a,\end{aligned} \tag{26}$$

and that based on $\Omega^*(\mathbf{F}, \mathbf{H}_l)$ as

$$\begin{aligned}\Pi^*\{\varphi_l, \mathbf{x}\} = &\int_{\mathcal{B}_r} \Omega^*(\mathbf{F}, \mathbf{H}_l)\,\mathrm{d}V + \int_{\mathcal{B}'_r} \Omega^*_{\mathrm{e}}(\mathbf{F}, \mathbf{H}_l)\,\mathrm{d}V \\ &- \int_{\partial\mathcal{B}_r^{\tau}} \mathbf{x}\cdot\mathbf{t}_{\mathrm{A}}\,\mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{x}\cdot\mathbf{f}\,\mathrm{d}V - \int_{\partial\mathcal{B}^{\infty}} \varphi\mathbf{B}_a\cdot\mathbf{n}\,\mathrm{d}a.\end{aligned} \tag{27}$$

The field $\mathbf{B}_{\mathrm{a}} = \mu_0 \mathbf{H}_{\mathrm{a}}$ is prescribed on the boundary $\partial\mathcal{B}^{\infty}$ and is independent of the deformation, i.e. it is of 'dead' type. Moreover, we assume that the boundary at infinity is fixed and there is therefore no need to distinguish between its Eulerian and Lagrangian forms.

We note that there are minor differences from Bustamante et al. [17]. In particular, we have used the notations Π and Π^* for the functionals instead of Π^* and Π, respectively. Second, we have used Ω_{e} and Ω^*_{e} instead of their explicit forms in equation (23). Third, we are considering that the mechanical traction is prescribed on only part of the boundary $\partial\mathcal{B}_r^{\tau}$ instead of the whole boundary, and that on the remaining part of the boundary $\partial\mathcal{B}_r^{x}$ the placement $\mathbf{x}$ is prescribed according to equation (19). As in Bustamante et al. [17] we consider the traction $\mathbf{t}_{\mathrm{A}}$ to correspond to a dead load. Finally, we are assuming for simplicity that the body force $\mathbf{f}$ (per unit mass) is constant, whereas, more generally, as in Bustamante et al. [17], it could be taken to be a conservative field. We also note that, in contrast to the other integrals, the integrals over the boundary at infinity are written in Eulerian form since we assume this boundary is fixed and, by use of Nanson's formula $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}A$, a vector identity and the connections, equation (7), we may establish the identities $(\mathbf{H}\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}a = (\mathbf{H}_l\times\mathbf{A}_l)\cdot\mathbf{N}\,\mathrm{d}A$ and $\varphi\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}a = \varphi_l\mathbf{B}_l\cdot\mathbf{N}\,\mathrm{d}A$.

We next consider the first and second variations of the functionals Π and Π^*, beginning with Π^* since the analysis required is slightly less involved than for Π.

4. First and second variations of the functional Π^*

4.1. The first variation

For consistency with the approach to the second variation we repeat the derivation of the first variation but in a somewhat different form from that given in Bustamante et al. [17], partly with different notation. In considering the first variations in the functions φ_l (or φ) and $\boldsymbol{\chi}$, we use the following notation. The variation of $\boldsymbol{\chi}$ is denoted $\delta\mathbf{x}$, the notation δ_φ signifies a variation in which the function φ (and hence φ_l) is varied (at fixed $\mathbf{x}$), while δ_x signifies a variation due to variation in the function $\boldsymbol{\chi}$. In particular, it is important to distinguish $\delta_\varphi\varphi_l$ and $\delta_x\varphi_l$: the first is the variation in φ_l at fixed $\mathbf{x}$; the second is the variation in φ_l induced by the variation in $\mathbf{x}$, bearing in mind the functional dependence of φ, and hence of φ_l, on $\mathbf{x}$. Note that $\delta_x\varphi_l = \mathrm{grad}\varphi \cdot \delta\mathbf{x}$ although we shall not use this form explicitly. For completeness, we note that $\delta_\varphi \mathbf{x} = \mathbf{0}$. We shall also denote by $\delta\varphi_l$ the total variation of φ_l, i.e. the sum $\delta_\varphi\varphi_l + \delta_x\varphi_l$ (and similarly $\delta\varphi = \delta_\varphi\varphi + \delta_x\varphi$). The variations are taken to have sufficient regularity for validity of the ensuing analysis. For compatibility with the boundary condition on $\partial\mathcal{B}_r^x$ and with the fact that the boundary at infinity is fixed we must have $\delta\mathbf{x} = \mathbf{0}$ on $\partial\mathcal{B}_r^x$ and $\partial\mathcal{B}^\infty$.

For the functions φ_l and $\boldsymbol{\chi}$ we first consider the variation of $\Pi^*\{\varphi_l, \mathbf{x}\}$ with respect to φ_l at fixed $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. From equation (27) this gives

$$\delta_\varphi \Pi^*\{\varphi_l, \mathbf{x}\} = -\int_{\mathcal{B}_r} \frac{\partial \Omega^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta_\varphi\varphi_l \,\mathrm{d}V - \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta_\varphi\varphi_l \,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \delta_\varphi\varphi \mathbf{B}_a \cdot \mathbf{n}\,\mathrm{d}a. \tag{28}$$

The corresponding variation with respect to $\mathbf{x}$ is obtained as

$$\begin{aligned}\delta_x \Pi^*\{\varphi_l, \mathbf{x}\} = &\int_{\mathcal{B}_r} \frac{\partial \Omega^*}{\partial \mathbf{F}} : \mathrm{Grad}\,\delta\mathbf{x}\,\mathrm{d}V - \int_{\mathcal{B}_r} \frac{\partial \Omega^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta_x\varphi_l \,\mathrm{d}V \\ &+ \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}^*}{\partial \mathbf{F}} : \mathrm{Grad}\,\delta\mathbf{x}\,\mathrm{d}V - \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta_x\varphi_l \,\mathrm{d}V \\ &- \int_{\partial\mathcal{B}_r^\tau} \mathbf{t}_\mathrm{A} \cdot \delta\mathbf{x}\,\mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{f} \cdot \delta\mathbf{x}\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \mathbf{B}_a \cdot \mathbf{n}\,\delta_x\varphi\,\mathrm{d}a.\end{aligned} \tag{29}$$

The total variation in Π^*, $\delta\Pi^* = \delta_\varphi\Pi^* + \delta_x\Pi^*$ can now be obtained by summing the expressions in equations (28) and (29). Thus,

$$\begin{aligned}\delta \Pi^*\{\varphi_l, \mathbf{x}\} = &\int_{\mathcal{B}_r} \frac{\partial \Omega^*}{\partial \mathbf{F}} : \mathrm{Grad}\,\delta\mathbf{x}\,\mathrm{d}V - \int_{\mathcal{B}_r} \frac{\partial \Omega^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta\varphi_l \,\mathrm{d}V \\ &+ \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}^*}{\partial \mathbf{F}} : \mathrm{Grad}\,\delta\mathbf{x}\,\mathrm{d}V - \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}^*}{\partial \mathbf{H}_l} \cdot \mathrm{Grad}\,\delta\varphi_l \,\mathrm{d}V \\ &- \int_{\partial\mathcal{B}_r^\tau} \mathbf{t}_\mathrm{A} \cdot \delta\mathbf{x}\,\mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{f} \cdot \delta\mathbf{x}\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty} \mathbf{B}_a \cdot \mathbf{n}\,\delta\varphi\,\mathrm{d}a.\end{aligned} \tag{30}$$

By introducing the notations $\mathbf{T}$ and $\mathbf{B}_l$ from equation (22) in $\mathcal{B}_r$ and $\mathbf{T}_\mathrm{m}$ and $\mathbf{B}_l$ from equation (24) in $\mathcal{B}_r'$ followed by use of the divergence theorem, the definition in equation (18) of the Maxwell traction $\mathbf{t}_\mathrm{M}$, the fact that $\delta\mathbf{x} = \mathbf{0}$ on $\partial\mathcal{B}_r^x$ and $\partial\mathcal{B}^\infty$, and recalling that $\partial\mathcal{B}_r'$ and $\partial\mathcal{B}_r$ have opposite outward normals, we may rearrange equation (30) in the form

$$\begin{aligned}\delta \Pi^*\{\varphi_l, \mathbf{x}\} = &-\int_{\mathcal{B}_r\cup\mathcal{B}_r'} \mathrm{Div}\,\mathbf{B}_l\,\delta\varphi_l\,\mathrm{d}V - \int_{\partial\mathcal{B}_r} [\![\mathbf{B}_l]\!] \cdot \mathbf{N}\,\delta\varphi_l\,\mathrm{d}A + \int_{\partial\mathcal{B}^\infty} (\mathbf{B} - \mathbf{B}_\mathrm{a}) \cdot \mathbf{n}\,\delta\varphi\,\mathrm{d}a \\ &- \int_{\mathcal{B}_r} (\mathrm{Div}\,\mathbf{T} + \rho_r\mathbf{f}) \cdot \delta\mathbf{x}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r^\tau} (\mathbf{T}^\mathrm{T}\mathbf{N} - \mathbf{t}_\mathrm{A} - \mathbf{t}_\mathrm{M}) \cdot \delta\mathbf{x}\,\mathrm{d}A \\ &- \int_{\mathcal{B}_r'} (\mathrm{Div}\,\mathbf{T}_\mathrm{m}) \cdot \delta\mathbf{x}\,\mathrm{d}V.\end{aligned} \tag{31}$$

The connections $\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}$ and $\mathbf{T}_\mathrm{m} = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{m}$ enable the two integrals over $\mathcal{B}_r'$ to be written together in Eulerian form as

$$-\int_{\mathcal{B}'} [\mathrm{div}\,\mathbf{B}\,\delta\varphi + (\mathrm{div}\,\boldsymbol{\tau}_\mathrm{m}) \cdot \delta\mathbf{x}]\,\mathrm{d}v, \tag{32}$$

which is independent of the specific choice of the continuation of the deformation into $\mathcal{B}_r'$.

By the fundamental theorem of the calculus of variations it follows that $\delta\Pi^* = 0$ for all variations $\delta\mathbf{x}$ and $\delta\varphi_l$ subject to the restrictions indicated if and only if

1. In $\mathcal{B}_r$,

$$\mathrm{Div}\,\mathbf{T} + \rho_r\mathbf{f} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{B}_l = 0, \tag{33}$$

where

$$\mathbf{T} = \frac{\partial\Omega^*}{\partial\mathbf{F}}, \quad \mathbf{B}_l = -\frac{\partial\Omega^*}{\partial\mathbf{H}_l}, \quad \mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}, \quad \mathbf{H}_l = -\mathrm{Grad}\varphi; \tag{34}$$

2. in $\mathcal{B}'$,

$$\mathrm{div}\,\mathbf{B} = 0, \quad \mathrm{div}\,\boldsymbol{\tau}_\mathrm{m} = \mathbf{0}, \tag{35}$$

where

$$\mathbf{B} = \mu_0\mathbf{H}, \quad \mathbf{H} = -\mathrm{grad}\varphi, \quad \boldsymbol{\tau}_\mathrm{m} = \mathbf{B}\otimes\mathbf{H} - \tfrac{1}{2}(\mathbf{B}\cdot\mathbf{H})\mathbf{I}; \tag{36}$$

3. on $\partial\mathcal{B}_r^\tau$,

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + \mathbf{t}_\mathrm{M}, \tag{37}$$

where $\mathbf{t}_\mathrm{A}$ is the applied mechanical traction per unit area and $\mathbf{t}_\mathrm{M} = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{m}$;

4. on $\partial\mathcal{B}_r$,

$$\mathbf{N}\cdot[\![\mathbf{B}_l]\!] = 0, \tag{38}$$

where $\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}$;

5. on $\partial\mathcal{B}^\infty$,

$$\mathbf{n}\cdot\mathbf{B} = \mathbf{n}\cdot\mathbf{B}_\mathrm{a}. \tag{39}$$

Note that $\mathrm{div}\,\boldsymbol{\tau}_\mathrm{m} = \mathbf{0}$ follows also from the equations $\mathrm{curl}\,\mathbf{H} = \mathbf{0}$, $\mathrm{div}\,\mathbf{B} = 0$, $\mathbf{B} = \mu_0\mathbf{H}$. Moreover, the equations applying in $\mathcal{B}'$ and on its boundaries no longer involve use of $\mathbf{F}$ within $\mathcal{B}'$. Only the value of $\mathbf{F}$ on $\partial\mathcal{B}_r$ (or $\partial\mathcal{B}$) is required.

4.2. The second variation

Let $\delta^2\mathbf{x}$ denote the second variation in $\mathbf{x}$, which, like $\delta\mathbf{x}$, is required to vanish on $\partial\mathcal{B}_r^\tau$ and $\partial\mathcal{B}^\infty$. We denote by $\delta^2\varphi_l$ the total second variation in φ_l, i.e. $\delta_\varphi\delta\varphi_l + \delta_x\delta\varphi_l$. In calculating the second variation $\delta^2\Pi^*$ the terms involving $\delta_\varphi\delta\varphi_l$ and $\delta_x\delta\varphi_l$ combine in a similar way to $\delta_\varphi\varphi_l$ and $\delta_x\varphi_l$ and we can therefore omit the counterpart of the step leading to equation (30). This yields

$$\begin{aligned}
&\delta^2\ \Pi^*\ \{\varphi_l, \mathbf{x}\} \\
&= \int_{\mathcal{B}_r}\left\{\left(\frac{\partial^2\Omega^*}{\partial\mathbf{F}\partial\mathbf{F}} : \delta\mathbf{F}\right) : \delta\mathbf{F} + 2\left(\frac{\partial^2\Omega^*}{\partial\mathbf{F}\partial\mathbf{H}_l} : \delta\mathbf{F}\right)\cdot\delta\mathbf{H}_l + \left(\frac{\partial^2\Omega^*}{\partial\mathbf{H}_l\partial\mathbf{H}_l}\cdot\delta\mathbf{H}_l\right)\cdot\delta\mathbf{H}_l\right\}\mathrm{d}V \\
&+ \int_{\mathcal{B}_r'}\left\{\left(\frac{\partial^2\Omega_\mathrm{e}^*}{\partial\mathbf{F}\partial\mathbf{F}} : \delta\mathbf{F}\right) : \delta\mathbf{F} + 2\left(\frac{\partial^2\Omega_\mathrm{e}^*}{\partial\mathbf{F}\partial\mathbf{H}_l} : \delta\mathbf{F}\right)\cdot\delta\mathbf{H}_l + \left(\frac{\partial^2\Omega_\mathrm{e}^*}{\partial\mathbf{H}_l\partial\mathbf{H}_l}\cdot\delta\mathbf{H}_l\right)\cdot\delta\mathbf{H}_l\right\}\mathrm{d}V \\
&+ \int_{\mathcal{B}_r}\mathbf{T} : \delta^2\mathbf{F}\,\mathrm{d}V + \int_{\mathcal{B}_r'}\mathbf{T}_\mathrm{m} : \delta^2\mathbf{F}\,\mathrm{d}V - \int_{\partial\mathcal{B}_r^\tau}\mathbf{t}_\mathrm{A}\cdot\delta^2\mathbf{x}\,\mathrm{d}A - \int_{\mathcal{B}_r}\rho_r\mathbf{f}\cdot\delta^2\mathbf{x}\,\mathrm{d}V \\
&- \int_{\mathcal{B}_r\cup\mathcal{B}'}\mathbf{B}_l\cdot\delta^2\mathbf{H}_l\,\mathrm{d}V - \int_{\partial\mathcal{B}^\infty}\varphi\mathbf{B}_a\cdot\mathbf{n}\,\delta^2\varphi\,\mathrm{d}a,
\end{aligned} \tag{40}$$

where, for compactness of expression, we have introduced the notations $\delta\mathbf{F} = \mathrm{Grad}\,\delta\mathbf{x}$, $\delta^2\mathbf{F} = \mathrm{Grad}\,\delta^2\mathbf{x}$ and written $\delta\mathbf{H}_l = -\mathrm{Grad}\,\delta\varphi_l$ and $\delta^2\mathbf{H}_l = -\mathrm{Grad}\,\delta^2\varphi_l$. By using essentially the same manipulations as for the first variation and the results of the first variation it is easy to show that the third and fourth lines in the above vanish separately. Thus, we have finally for the second variation

$$\begin{aligned}
&\delta^2\ \Pi^*\ \{\varphi_l, \mathbf{x}\}\\
&= \int_{\mathcal{B}_r} \left\{ \left(\frac{\partial^2 \Omega^*}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega^*}{\partial \mathbf{F} \partial \mathbf{H}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{H}_l + \left(\frac{\partial^2 \Omega^*}{\partial \mathbf{H}_l \partial \mathbf{H}_l} \cdot \delta \mathbf{H}_l \right) \cdot \delta \mathbf{H}_l \right\} \mathrm{d}V\\
&+ \int_{\mathcal{B}_r'} \left\{ \left(\frac{\partial^2 \Omega_\mathrm{e}^*}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega_\mathrm{e}^*}{\partial \mathbf{F} \partial \mathbf{H}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{H}_l + \left(\frac{\partial^2 \Omega_\mathrm{e}^*}{\partial \mathbf{H}_l \partial \mathbf{H}_l} \cdot \delta \mathbf{H}_l \right) \cdot \delta \mathbf{H}_l \right\} \mathrm{d}V.
\end{aligned} \tag{41}$$

The integral over $\mathcal{B}_r'$ can be expressed in Eulerian form using the expressions given in the Appendix, and we note, in particular, that the resulting expression is independent of the extension of the deformation into $\mathcal{B}_r'$. Moreover, the final term in this integral may be written as

$$-\mu_0 \int_{\mathcal{B}'} \delta \mathbf{H} \cdot \delta \mathbf{H}\, \mathrm{d}v, \tag{42}$$

which is negative definite. Overall, however, the integral over $\mathcal{B}'$, is in general indefinite. The two other terms in this integral are responsible for accommodating, within $\mathcal{B}'$, the effect of the deforming boundary.

5. First and second variations of the functional Π

5.1. The first variation

Let $\delta_A \mathbf{A}_l$ and $\delta_A \mathbf{A}$ denote the first variations in $\mathbf{A}_l$ and $\mathbf{A}$ at fixed $\mathbf{x}$, respectively, and $\delta_x \mathbf{A}_l$ and $\delta_x \mathbf{A}$ the corresponding variations induced by a variation $\delta \mathbf{x}$ in $\mathbf{x}$. The total variations are denoted $\delta \mathbf{A}_l$ and $\delta \mathbf{A}$. Note the connection $\mathbf{A}_l = \mathbf{F}^\mathrm{T} \mathbf{A}$.

For the functions $\mathbf{A}_l$ and $\boldsymbol{\chi}$ we first consider the variation of $\Pi\{\mathbf{A}_l, \mathbf{x}\}$ with respect to $\mathbf{A}_l$ at fixed $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. With the notation $\delta_A \mathbf{B}_l = \mathrm{Curl}\, \delta_A \mathbf{A}_l$, the variation of the functional (26) with respect to $\mathbf{A}_l$ at fixed $\mathbf{x}$ gives

$$\delta_A \Pi\{\mathbf{A}_l, \mathbf{x}\} = \int_{\mathcal{B}_r} \frac{\partial \Omega}{\partial \mathbf{B}_l} \cdot \delta_A \mathbf{B}_l\, \mathrm{d}V + \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}}{\partial \mathbf{B}_l} \cdot \delta_A \mathbf{B}_l\, \mathrm{d}V + \int_{\partial \mathcal{B}^\infty} (\mathbf{H}_a \times \delta_A \mathbf{A}) \cdot \mathbf{n}\, \mathrm{d}a. \tag{43}$$

Next we compute the first variation of Π with respect to $\mathbf{x}$. This yields

$$\begin{aligned}
\delta_x \Pi\{\mathbf{A}_l, \mathbf{x}\} &= \int_{\mathcal{B}_r} \frac{\partial \Omega}{\partial \mathbf{F}} : \delta \mathbf{F}\, \mathrm{d}V + \int_{\mathcal{B}_r} \frac{\partial \Omega}{\partial \mathbf{B}_l} \cdot \delta_x \mathbf{B}_l\, \mathrm{d}V,\\
&+ \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}}{\partial \mathbf{F}} : \delta \mathbf{F}\, \mathrm{d}V + \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}}{\partial \mathbf{B}_l} \cdot \delta_x \mathbf{B}_l\, \mathrm{d}V\\
&- \int_{\partial \mathcal{B}_r^\tau} \mathbf{t}_\mathrm{A} \cdot \delta \mathbf{x}\, \mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{f} \cdot \delta \mathbf{x}\, \mathrm{d}V + \int_{\partial \mathcal{B}^\infty} (\mathbf{H}_a \times \delta_x \mathbf{A}) \cdot \mathbf{n}\, \mathrm{d}a.
\end{aligned} \tag{44}$$

In arriving at the integral over $\partial \mathcal{B}^\infty$ we have made use of the connection $\delta_x \mathbf{A}_l = \mathbf{F}^\mathrm{T}[\delta_x \mathbf{A} + (\mathrm{grad}\, \delta \mathbf{x})^\mathrm{T} \mathbf{A}]$, Nanson's formula, the vector identity $\mathbf{F}^\mathrm{T} \mathbf{u} \times \mathbf{F}^\mathrm{T} \mathbf{v} = J \mathbf{F}^{-1}(\mathbf{u} \times \mathbf{v})$ for two vectors $\mathbf{u}$ and $\mathbf{v}$, and the fact that, since $\delta \mathbf{x} = \mathbf{0}$ on $\partial \mathcal{B}^\infty$, $(\mathbf{n} \times \mathrm{grad})\delta \mathbf{x} = \mathbf{0}$ on $\partial \mathcal{B}^\infty$.

The sum of the latter two expressions yields the total variation

$$\begin{aligned}
\delta \Pi\{\mathbf{A}_l, \mathbf{x}\} &= \int_{\mathcal{B}_r} \frac{\partial \Omega}{\partial \mathbf{F}} : \delta \mathbf{F}\, \mathrm{d}V + \int_{\mathcal{B}_r} \frac{\partial \Omega}{\partial \mathbf{B}_l} \cdot \delta \mathbf{B}_l\, \mathrm{d}V\\
&+ \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}}{\partial \mathbf{F}} : \delta \mathbf{F}\, \mathrm{d}V + \int_{\mathcal{B}_r'} \frac{\partial \Omega_\mathrm{e}}{\partial \mathbf{B}_r} \cdot \delta \mathbf{B}_l\, \mathrm{d}V\\
&- \int_{\partial \mathcal{B}_r^\tau} \mathbf{t}_\mathrm{A} \cdot \delta \mathbf{x}\, \mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{f} \cdot \delta \mathbf{x}\, \mathrm{d}V + \int_{\partial \mathcal{B}^\infty} (\mathbf{H}_a \times \delta \mathbf{A}) \cdot \mathbf{n}\, \mathrm{d}a.
\end{aligned} \tag{45}$$

By making use of the identity

$$\mathbf{H}_l \cdot \operatorname{Curl} \delta\mathbf{A}_l = \operatorname{Curl} \mathbf{H}_l \cdot \delta\mathbf{A}_l - \operatorname{Div}(\mathbf{H}_l \times \delta\mathbf{A}_l),$$

followed by the divergence theorem for each of $\mathcal{B}_r$ and $\mathcal{B}'_r$ we obtain

$$\begin{aligned} \delta\Pi\{\mathbf{A}_l, \mathbf{x}\} = & \int_{\mathcal{B}_r \cup \mathcal{B}'_r} \operatorname{Curl} \mathbf{H}_l \cdot \delta\mathbf{A}_l \, \mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{N} \times [\![\mathbf{H}_l]\!]) \cdot \delta\mathbf{A}_l \, \mathrm{d}A \\ & - \int_{\partial\mathcal{B}^\infty} [\mathbf{n} \times (\mathbf{H} - \mathbf{H}_\mathrm{a})] \cdot \delta\mathbf{A} \, \mathrm{d}A - \int_{\mathcal{B}_r} (\operatorname{Div} \mathbf{T} + \rho_r \mathbf{f}) \cdot \delta\mathbf{x} \, \mathrm{d}V \\ & + \int_{\partial\mathcal{B}_r} (\mathbf{T}^\mathrm{T}\mathbf{N} - \mathbf{t}_\mathrm{A} - \mathbf{t}_\mathrm{M}) \cdot \delta\mathbf{x} \, \mathrm{d}A + \int_{\mathcal{B}'_r} \operatorname{Div} \mathbf{T}_\mathrm{m} \cdot \delta\mathbf{x}. \end{aligned} \tag{46}$$

Again we have made use of the connection $\mathbf{F}^{-\mathrm{T}}\delta\mathbf{A}_l = \delta\mathbf{A} + (\operatorname{grad} \delta\mathbf{x})^\mathrm{T}\mathbf{A}$ and the fact that $(\mathbf{n} \times \operatorname{grad})\delta\mathbf{x} = \mathbf{0}$ on $\partial\mathcal{B}^\infty$. Similarly to the situation in respect of Π^* the two integrals over $\mathcal{B}'_r$ can be expressed in Eulerian form as

$$\int_{\mathcal{B}'} \{\operatorname{curl} \mathbf{H} \cdot [\delta\mathbf{A} + (\operatorname{grad} \delta\mathbf{x})^\mathrm{T}\mathbf{A}] + \operatorname{div} \boldsymbol{\tau}_\mathrm{m} \cdot \delta\mathbf{x}\} \, \mathrm{d}V, \tag{47}$$

which is independent of the continuation of the deformation into $\mathcal{B}'$.

By the fundamental theorem of the calculus of variations it follows that $\delta\Pi = 0$ for all variations $\delta\mathbf{x}$ and $\delta\mathbf{A}_l$ subject to the restrictions indicated if and only if

1. In $\mathcal{B}_r$,

$$\operatorname{Div} \mathbf{T} + \rho_r \mathbf{f} = \mathbf{0}, \quad \operatorname{Curl} \mathbf{H}_l = \mathbf{0}, \tag{48}$$

where

$$\mathbf{T} = \frac{\partial\Omega}{\partial\mathbf{F}}, \quad \mathbf{H}_l = \frac{\partial\Omega}{\partial\mathbf{B}_l}, \quad \mathbf{F} = \operatorname{Grad} \boldsymbol{\chi}, \quad \mathbf{B}_l = \operatorname{Curl} \mathbf{A}_l; \tag{49}$$

2. in $\mathcal{B}'$,

$$\operatorname{curl} \mathbf{H} = 0, \quad \operatorname{div} \boldsymbol{\tau}_\mathrm{m} = \mathbf{0}, \tag{50}$$

where

$$\mathbf{H} = \mu_0^{-1}\mathbf{B}, \quad \mathbf{B} = \operatorname{curl} \mathbf{A}, \quad \boldsymbol{\tau}_\mathrm{m} = \mathbf{B} \otimes \mathbf{H} - \tfrac{1}{2}(\mathbf{B} \cdot \mathbf{H})\mathbf{I}; \tag{51}$$

3. on $\partial\mathcal{B}_r^\tau$,

$$\mathbf{T}^\mathrm{T}\mathbf{N} = \mathbf{t}_\mathrm{A} + \mathbf{t}_\mathrm{M}, \tag{52}$$

where $\mathbf{t}_\mathrm{A}$ is the applied mechanical traction per unit area and $\mathbf{t}_\mathrm{M} = J\mathbf{F}^{-1}\boldsymbol{\tau}_\mathrm{m}$;

4. on $\partial\mathcal{B}_r$,

$$\mathbf{N} \times [\![\mathbf{H}_l]\!] = \mathbf{0}, \tag{53}$$

where $\mathbf{H}_l = \mathbf{F}^\mathrm{T}\mathbf{H}$;

5. on $\partial\mathcal{B}^\infty$,

$$\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}_\mathrm{a}. \tag{54}$$

As for the case of Π^*, $\operatorname{div} \boldsymbol{\tau}_\mathrm{m} = \mathbf{0}$ follows also from the equations $\operatorname{curl} \mathbf{H} = \mathbf{0}$, $\operatorname{div} \mathbf{B} = 0$, $\mathbf{B} = \mu_0\mathbf{H}$. Moreover, the equations applying in $\mathcal{B}'$ and on its boundaries no longer involve use of $\mathbf{F}$ within $\mathcal{B}'$. Only the value of $\mathbf{F}$ on $\partial\mathcal{B}_r$ (or $\partial\mathcal{B}$) is required to define $\mathbf{H}_l = \mathbf{F}^\mathrm{T}\mathbf{H}$ thereon.

5.2. The second variation

From equation (45) we calculate the second variation in the form

$$
\begin{aligned}
&\delta^2\ \Pi\ \{\mathbf{A}_l, \mathbf{x}\} \\
&= \int_{\mathcal{B}_r} \left[\left(\frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{B}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{B}_l + \left(\frac{\partial^2 \Omega}{\partial \mathbf{B}_l \partial \mathbf{B}_l} \cdot \delta \mathbf{B}_l \right) \cdot \delta \mathbf{B}_l \right] \mathrm{d}V \\
&+ \int_{\mathcal{B}'_r} \left[\left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{F} \partial \mathbf{B}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{B}_l + \left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{B}_l \partial \mathbf{B}_l} \cdot \delta \mathbf{B}_l \right) \cdot \delta \mathbf{B}_l \right] \mathrm{d}V \\
&+ \int_{\mathcal{B}_r} \mathbf{T} : \delta^2 \mathbf{F}\, \mathrm{d}V + \int_{\mathcal{B}'_r} \mathbf{T}_\mathrm{m} : \delta^2 \mathbf{F}\, \mathrm{d}V - \int_{\partial \mathcal{B}_r} \mathbf{t}_\mathrm{A} \cdot \delta^2 \mathbf{x}\, \mathrm{d}A - \int_{\mathcal{B}_r} \rho_r \mathbf{f} \cdot \delta^2 \mathbf{x}\, \mathrm{d}V \\
&+ \int_{\mathcal{B}_r \cup \mathcal{B}'_r} \mathbf{H}_l \cdot \delta^2 \mathbf{B}_l\, \mathrm{d}V + \int_{\mathcal{B}\infty} (\mathbf{H}_\mathrm{a} \times \delta^2 \mathbf{A}) \cdot \mathbf{n} \mathrm{d}a.
\end{aligned}
\tag{55}
$$

By similar calculations to those used for the first variation of Π but involving the second variations of $\mathbf{x}, \mathbf{A}_l$ and $\mathbf{A}$ the third and fourth lines in the above can be shown to vanish separately by using the first variation results, and we have finally

$$
\begin{aligned}
&\delta^2\ \Pi\ \{\mathbf{A}_l, \mathbf{x}\} \\
&= \int_{\mathcal{B}_r} \left[\left(\frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega}{\partial \mathbf{F} \partial \mathbf{B}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{B}_l + \left(\frac{\partial^2 \Omega}{\partial \mathbf{B}_l \partial \mathbf{B}_l} \cdot \delta \mathbf{B}_l \right) \cdot \delta \mathbf{B}_l \right] \mathrm{d}V \\
&+ \int_{\mathcal{B}'_r} \left[\left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{F} \partial \mathbf{F}} : \delta \mathbf{F} \right) : \delta \mathbf{F} + 2 \left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{F} \partial \mathbf{B}_l} : \delta \mathbf{F} \right) \cdot \delta \mathbf{B}_l + \left(\frac{\partial^2 \Omega_\mathrm{e}}{\partial \mathbf{B}_l \partial \mathbf{B}_l} \cdot \delta \mathbf{B}_l \right) \cdot \delta \mathbf{B}_l \right] \mathrm{d}V.
\end{aligned}
\tag{56}
$$

The integral over $\mathcal{B}'_r$ can be expressed in Eulerian form in a similar way to the second variation of Π^* using the results in the Appendix and is independent of the continuation of the deformation into $\mathcal{B}'_r$. In particular, the final term in the integral over $\mathcal{B}'_r$ can be written

$$
\mu_0^{-1} \int_{\mathcal{B}'} \delta \mathbf{B} \cdot \delta \mathbf{B}\, \mathrm{d}v, \tag{57}
$$

which is positive definite, although overall the integral over $\mathcal{B}'$ is in general indefinite.

6. Final remarks

In this work we have revisited the variational principles studied in Bustamante et al. [17], with particular reference to obtaining the second variations of the two functionals involved, one based on the magnetic scalar potential and the other on the magnetic vector potential. We have also presented an alternative approach to obtaining the first variations. This approach is based on using general, rather than specific, expressions for the energy densities in the space surrounding the material body, which are given in equation (23). These are Lagrangian expressions, denoted as Ω_e and Ω_e^*, which use an extension to the surrounding space of the deformation within the body. This fictitious device is particularly useful for obtaining simple formulas for the second variations of the functionals given in equations (41) and (56), but the end result is independent of the particular form of this extension.

In the classical theory of nonlinear elasticity (with no magnetic interactions), the sign of the second variation has been linked to different properties of the solutions of the boundary-value problem, and has been used to study, among other things, the appearance of instability phenomena and to establish restrictions on the energy densities such that existence of solutions would ensure (see, for example, Ball and Marsden [14] and Sections 88 and 89 of Truesdell and Noll [13]). In the magnetoelastostatic case, the expressions in equations (41) and (56) could be used for similar analyses with the study of restrictions on Ω and Ω^*, the deformation and the magnetic field vectors such that the second variation *in the body* is positive, bearing in mind the fact that the contribution to the second variation *outside the body* is in general indefinite. In Section 4 of Kankanala and Triantafyllidis [3] and Section 8 of Barham et al. [16]) similar analyses have been considered, although for formulations based on the use of the magnetization as the independent magnetic variable. Analysis of this kind will be illustrated in a separate paper.

Notes

1. The interested reader can find most of the theoretical background for nonlinear magneto- and electroelastic interactions, for example, in the monographs by Brown [6], by Hutter and van de Ven [7], the detailed presentation in the book by Maugin [8] and Chapter F and Section IV of Chapter G of the article by Truesdell and Toupin [9].
2. This last remark is important, considering that in magnetoelasticity, the Maxwell equations must be satisfied not only for the bodies, but also in the surrounding vacuum space (see, for example, Kovetz [18]).
3. Recently, Vu and Steinmann [21] have introduced a similar stored energy function (in Lagrangian form) for free space, in the context of the similar electroelastic problem (see, for example equation (95) in that work), but they were concerned with the development of weak formulations and did not used this stored energy function to calculate the second variation.

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Conflict of interest

None declared.

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Appendix

The second derivatives of Ω_{e} *and* Ω_{e}^*

The first derivatives of Ω_{e} and Ω_{e}^* are given following equation (24). Here we derive corresponding formulas for their second derivatives, working mainly in index notation First, it follows simply that

$$\frac{\partial^2 \Omega_{\mathrm{e}}}{\partial B_{l\alpha} \partial B_{l\beta}} = \mu_0^{-1} J^{-1} F_{k\alpha} F_{k\beta}, \quad \frac{\partial^2 \Omega_{\mathrm{e}}}{\partial \mathbf{B}_l \partial \mathbf{B}_l} = \mu_0^{-1} J^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}.$$

For the mixed derivative we obtain

$$\frac{\partial^2 \Omega_{\mathrm{e}}}{\partial F_{i\alpha} \partial B_{l\beta}} = \mu_0^{-1} J^{-1} [\delta_{\alpha\beta} F_{i\gamma} B_{l\gamma} + B_{l\alpha} F_{i\beta} - (\mathbf{F}^{-1})_{\alpha i} F_{k\beta} F_{k\gamma} B_{l\gamma}],$$

and for the second derivative with respect to the deformation gradient:

$$\frac{\partial^2 \Omega_{\mathrm{e}}}{\partial F_{i\alpha} \partial F_{j\beta}} = \mu_0^{-1} J^{-1} [\delta_{ij} B_{l\alpha} B_{l\beta} - B_{l\alpha} (\mathbf{F}\mathbf{B}_l)_i (\mathbf{F}^{-1})_{\beta j} - B_{l\beta} (\mathbf{F}\mathbf{B}_l)_j (\mathbf{F}^{-1})_{\alpha i}]$$
$$+ \frac{1}{2} \mu_0^{-1} J^{-1} (\mathbf{F}\mathbf{B}_l) \cdot (\mathbf{F}\mathbf{B}_l) [(\mathbf{F}^{-1})_{\alpha i} (\mathbf{F}^{-1})_{\beta j} + (\mathbf{F}^{-1})_{\alpha j} (\mathbf{F}^{-1})_{\beta i}]$$

The corresponding Eulerian forms of these expressions are obtained by pushing forward with $J\mathbf{F}^{-1}$, respectively $\mathbf{F}$, where a derivative with respect to $\mathbf{B}_l$, respectively $\mathbf{F}$, is involved. This yields

$$J^2 (\mathbf{F}^{-1})_{p\alpha} (\mathbf{F}^{-1})_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}}{\partial B_{l\alpha} \partial B_{l\beta}} = \mu_0^{-1} J \delta_{pq},$$

$$J F_{p\alpha} (\mathbf{F}^{-1})_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}}{\partial F_{i\alpha} \partial B_{l\beta}} = \mu_0^{-1} J (\delta_{pq} B_i + \delta_{iq} B_p - \delta_{ip} B_q),$$

$$F_{p\alpha} F_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}}{\partial F_{i\alpha} \partial F_{j\beta}} = \mu_0^{-1} J (\delta_{ij} B_p B_q - \delta_{jq} B_i B_p - \delta_{ip} B_j B_q) + \tfrac{1}{2} \mu_0^{-1} J (\mathbf{B} \cdot \mathbf{B}) (\delta_{ip} \delta_{jq} + \delta_{jp} \delta_{iq}).$$

The second derivatives of Ω_{e}^* are obtained similarly but we give them here only in Eulerian form. In the case of a derivative with respect to $\mathbf{H}_l$ the push forward is with $\mathbf{F}$:

$$F_{p\alpha} F_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}^*}{\partial H_{l\alpha} \partial H_{l\beta}} = -\mu_0 J \delta_{pq},$$

$$F_{p\alpha} F_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}^*}{\partial F_{i\alpha} \partial H_{l\beta}} = \mu_0 J (\delta_{pq} H_i + \delta_{iq} H_p - \delta_{ip} H_q),$$

$$F_{p\alpha} F_{q\beta} \frac{\partial^2 \Omega_{\mathrm{e}}^*}{\partial F_{i\alpha} \partial F_{j\beta}} = \mu_0 J (\delta_{ip} H_j H_q + \delta_{jq} H_i H_p - \delta_{iq} H_j H_p - \delta_{jp} H_i H_q - \delta_{pq} H_i H_j)$$
$$+ \tfrac{1}{2} \mu_0 J (\mathbf{H} \cdot \mathbf{H}) (\delta_{iq} \delta_{jp} - \delta_{ip} \delta_{jq}).$$