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Non-uniqueness of positive ground states of non-linear Schrödinger equations

Juan Dávila, Manuel del Pino and Ignacio Guerra

Abstract

Existence of a positive, decaying radial solution to the problem

$$\Delta u - u + u^p + \lambda u^q = 0 \quad \text{in } \mathbb{R}^N,$$

when $\lambda > 0$ and $1 < q < p < (N+2)/(N-2)$ has been known for a long time. For $\lambda = 0$, it is well known that this solution is unique. While uniqueness conditions for rather general non-linearities have been found, the issue has remained elusive for this problem. We prove that uniqueness is in general not true. We find that if $N = 3$, $1 < q < 3$, λ is fixed sufficiently large, and $p < 5$ is taken sufficiently close to 5, then there are at least three positive decaying radial solutions.

1. *Introduction*

We consider the non-linear Schrödinger equation

$$i\psi_t = \Delta\psi + |\psi|^{p-1}\psi + |\psi|^{q-1}\psi \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $N \geqslant 3$ and the powers p and q are superlinear and subcritical, namely

$$1 < q < p < \frac{N+2}{N-2}.$$

This equation is a natural non-scaling invariant extension of the extensively studied *defocusing* equation

$$i\psi_t = \Delta\psi + |\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

A complete theory on the basic issues of well-posedness, asymptotic behaviour and blow-up for (1.1) was developed by Tao, Visan and Zhang [**29**].

In this paper, we are interested in standing-wave solutions of problem (1.1), namely finite-energy solutions of the form

$$\psi(x,t) = e^{-i\beta t} Q(x).$$

Assuming that $\beta = \alpha^2$ with $\alpha > 0$ and renormalizing through the scaling

$$Q(x) = \alpha^{2/(p-1)} v(\alpha x),$$

we obtain the following equation for v:

$$\Delta v - v + |v|^{p-1}v + \lambda |v|^{q-1}v = 0 \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where

$$\lambda = \alpha^{2(q-p)/(p-1)}.$$

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In this paper, we are interested in positive decaying solutions of equation (1.2) (sometimes called ground states), namely solutions of the problem

$$\begin{aligned} &\Delta v - v + v^p + \lambda v^q = 0, \quad v > 0 \quad \text{in } \mathbb{R}^N, \\ &v(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{aligned} \tag{1.3}$$

where $\lambda > 0$ and $1 < q \leqslant p < (N+2)/(N-2)$.

In the case of a pure power non-linearity, namely the problem

$$\begin{aligned} &\Delta v - v + v^p = 0, \quad v > 0 \quad \text{in } \mathbb{R}^N, \\ &v(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{aligned} \tag{1.4}$$

existence of a radially symmetric solution was first established by Strauss [**28**] for $1 < p < (N+2)/(N-2)$. When $p \geqslant (N+2)/(N-2)$ no solution exists, as it follows from Pohozaev's identity [**25**]. Solutions of (1.4) (and also those of (1.3)) are necessarily radially symmetric up to translations owing to the classical Gidas, Ni and Nirenberg result [**13**]. In [**15**], Kwong established uniqueness of the radially symmetric solution of (1.4).

Berestycki and Lions [**6**] found that existence of radial solutions also holds for the more general problem

$$\begin{aligned} &\Delta v - v + f(v) = 0, \quad v > 0 \quad \text{in } \mathbb{R}^N, \\ &v(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{aligned} \tag{1.5}$$

where f is of class C^1 and there exist $p \in (1, (N+2)/(N-2))$ and $t_0 > 0$ such that

$$f(0) = f'(0) = 0, \quad \frac{t_0^2}{2} < \int_0^{t_0} f(t)\, dt, \quad |f(t)| \leqslant C(1+t^p) \quad \text{for all } t > 0.$$

These conditions obviously hold for the sum of subcritical powers (1.3); see also [**1**, **4**, **5**, **7**, **10**–**12**, **24**] for related existence results.

On the other hand, *uniqueness* of radial solutions of (1.5) is known only under more restrictive assumptions; see, for instance, [**8**, **15**–**18**, **21**, **22**, **27**] and also [**3**, **9**] for uniqueness in balls.

The most general extension of Kwong's result is due to Serrin and Tang [**27**]: a radial solution of (1.5) is unique if there exists a $b > 0$ such that $(f(v) - v)(v - b) > 0$ for $v \neq b$ and the quotient $(f'(v)v - v)/(f(v) - v)$ is a non-increasing function of $v \in (b, \infty)$.

However, $f(v) = v^p + \lambda v^q$ *does not* satisfy the latter condition for large v, unless $p = q$. Thus, uniqueness of radial solutions of problem (1.3), the most natural extension of the single power case (1.4), has remained conspicuously open.

The purpose of this paper is to establish the rather striking fact that Kwong's uniqueness result is in general *not true* for problem (1.3) when $p \neq q$. In fact, we establish that in dimension $N = 3$ and suitable ranges for the parameters p, q and λ, problem (1.3) has *at least three solutions*.

Thus we consider in what follows the problem

$$\begin{cases} \Delta v - v + v^p + \lambda v^q = 0, \quad v > 0 \quad \text{in } \mathbb{R}^3, \\ v(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{cases} \tag{1.6}$$

where $\lambda > 0$, and $1 < q < p < 5$.

1.1. *Main result*

Our main result reads as follows.

THEOREM 1.1. *Let $1 < q < 3$. Then for each λ sufficiently large, there exists a number $1 < p_0 < 5$ so that for all $p_0 < p < 5$ problem* (1.6) *has at least three solutions.*

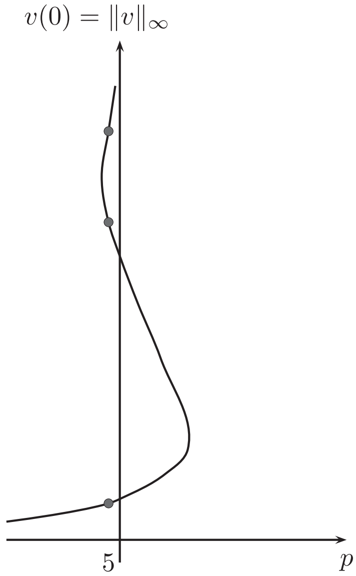

FIGURE 1. *Bifurcation diagram in p for (1.6) for λ sufficiently large and fixed.*

It is illustrative to depict the above result in terms of bifurcation diagrams. The picture in Figure 1, obtained from numerical simulations, represents the branch of positive solutions for a fixed, large number λ, when we let p act as a parameter of the problem and q is fixed. The branch in p crosses the critical exponent $p = 5$ then it turns backwards crossing again $p = 5$ and finally it turns right developing an asymptote at $p = 5$. We distinguish in this picture for a given p slightly below 5, a *large* solution and a *small* solution. Those parts of the branch will be described in detail. The third solution can be found by a degree-theoretical argument.

The restriction $1 < q < 3$ is essential in our proof. Moreover, if $q > 3$, then the branch appears numerically monotone. This seems also the case in dimensions $N \geqslant 4$. Establishing uniqueness in those scenarios (except for λ small, which is easy by perturbations) appears as a challenging problem.

1.2. *The small solution*

The lower part of the branch represents a small solution of size of order $O(\lambda^{-1/(q-1)})$. The change of variables $v(x) = \lambda^{-1/(q-1)} w(x)$ takes problem (1.6) into the form

$$\begin{cases} \Delta w - w + \tau w^p + w^q = 0, \quad w > 0 \text{ in } \mathbb{R}^3, \\ w(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \end{cases}$$

where $\tau = \lambda^{-(p-1)/(q-1)}$.

In Lemma 5.1, we find a solution for any λ large as regular perturbation of the unique solution for $\tau = 0$.

1.3. *The large solution*

The upper part of the branch diverges in size by *bubbling*. Let us write $p = 5 - \varepsilon$, where we regard ε as a small positive parameter. It turns out that the following scaling is convenient:

$$v(x) = \varepsilon^{-2/(4-\varepsilon)} u(x/\varepsilon),$$

so that problem (1.6) becomes

$$\begin{cases} \Delta u + u^{5-\varepsilon} + \lambda \varepsilon^{\alpha} u^{q} - \varepsilon^{2} u = 0, \quad u > 0 \text{ in } \mathbb{R}^3, \\ u(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty, \end{cases} \tag{1.7}$$

where

$$\alpha := \frac{5-q}{2} - \frac{\varepsilon(q-1)}{2(4-\varepsilon)}. \tag{1.8}$$

As $\varepsilon \to 0$, problem (1.7) approaches formally to

$$\Delta u + u^5 = 0, \quad u > 0 \text{ in } \mathbb{R}^3,$$

whose unique radial solutions are given by the functions

$$w_\mu(y) = 3^{1/4} \left(\frac{\mu}{\mu^2 + |y|^2} \right)^{1/2}.$$

As we will see, there is a solution of (1.7) which comes as a perturbation of w_μ for the choice $\mu = \pi/32$. In terms of the original problem (1.6), the following result holds.

THEOREM 1.2. *Let $1 < q < 3$, $\lambda \geqslant 0$ be given, and write $p = 5 - \varepsilon$. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of (1.6) of the form*

$$u_\varepsilon(x) = 3^{1/4} \left(\frac{1}{1 + (32/\pi)^2 \varepsilon^{-2} |x|^2} \right)^{1/2} \varepsilon^{-1/2} \sqrt{32/\pi}(1 + o(1)), \tag{1.9}$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$.

1.4. *The central solution and λ large*

As a by-product of the proofs, we will see that the large and the small solution are both non-degenerate, and that their Morse indices are both equal to 1. This information yields their local degrees in a suitable space, and the existence of a third solution will follow from a global degree argument. The number λ in Theorem 1.1 has to be fixed prior to letting p approach 5. Indeed, as we find in Lemma 5.3, when we fix p, if λ is too large, then there is only one solution. The set of positive solutions when we fix $p = 5 - \varepsilon$ and consider λ as its parameter can be depicted by the diagram in Figure 2, obtained by numerical simulations. Computing how large λ can be taken in Theorem 1.1, depending on ε, corresponds intuitively to locating the upper turning point P_ε in Figure 2. For λ near that point, we see two large solutions which can be explicitly described for $2 \leqslant q < 3$ as follows.

THEOREM 1.3. *Assume $2 \leqslant q < 3$. There exists a number λ_0 such that for each $0 < \bar{\lambda} < \lambda_0$ and the number*

$$\lambda = \begin{cases} \bar{\lambda} \varepsilon^{-(3-q)/2} & \text{if } 2 < q < 3, \\ \bar{\lambda} \varepsilon^{-1/2} |\log \varepsilon|^{-1} & \text{if } q = 2, \end{cases}$$

in problem (1.6), there exist two positive numbers d_- and d_+ such that for all ε there are two solutions $v_\pm$ of the form

$$v_\pm(x) = 3^{1/4} \left(\frac{1}{1 + d_\pm^4 \varepsilon^{-2} |x|^2} \right)^{1/2} \varepsilon^{-1/2} d_\pm (1 + o(1)), \tag{1.10}$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$.

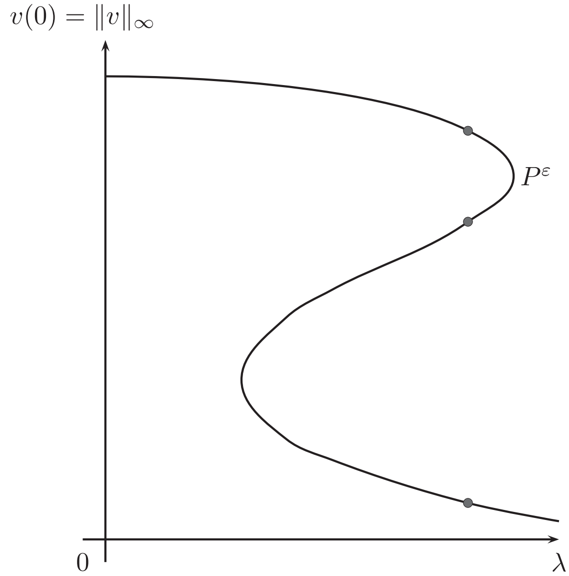

FIGURE 2. *Bifurcation diagram in* λ *for* (1.6) *in* $p = 5 - \varepsilon$, $\varepsilon > 0$ *small and fixed.*

In the case $1 < q < 2$, it is also possible to find these two solutions but the proof is different and will be addressed in future work. The numbers λ_0 and $d_\pm$ can be explicitly computed as follows. Let us consider the function

$$f(\mu) = b_1 \mu^{-(3-q)/2} - b_1 \frac{\pi}{32} \mu^{-(5-q)/2} \quad \text{where } b_1 = \frac{4}{5-q} \frac{q+1}{3^{(q-1)/4}} \frac{\Gamma((1/2)(q+1))}{\pi^{1/2}\Gamma((q-2)/2)},$$

whose maximum value is computed as

$$\lambda_0 := \max_{\mu>0} f(\mu) = f(\mu_0) = b_1 \left(\frac{\pi}{32}\right)^{-(3-q)/2} \left(\frac{5-q}{3-q}\right)^{-(5-q)/2} \frac{2}{3-q}, \quad \mu_0 = \frac{5-q}{3-q}\frac{\pi}{32}. \quad (1.11)$$

Thus, given $0 < \bar{\lambda} < \lambda_0$, the equation $\bar{\lambda} = f(\mu)$ has exactly two solutions

$$\frac{\pi}{32} < \mu^-(\bar{\lambda}) < \mu_0 < \mu^+(\bar{\lambda}). \quad (1.12)$$

As we will see, the numbers $d_\pm$ in (1.10) are simply given by

$$d_\pm = \mu^\pm(\bar{\lambda})^{-1/2}.$$

The solutions of (1.6) in the (λ, v) space can be identified with a set in the (λ, m)-plane, where $m = v(0) = \|v\|_\infty$ as in Figure 2. Our result can be portrayed as representing approximately the upper turning point as

$$P^\varepsilon \sim (\lambda_0 \varepsilon^{-(3-q)/2}, 3^{1/4}(\mu_0)^{-1/2}\varepsilon^{-1/2}),$$

while the set is itself near this point approximated by the graph

$$\lambda = \varepsilon^{-(3-q)/2} f(3^{1/2}(\varepsilon m^2)^{-1}) \quad \text{for } m \sim \varepsilon^{-1/2}.$$

The proofs are based on a Lyapunov–Schmidt reduction method along the lines of that used in [**23**, **24**]. We first prove Theorem 1.2 in Section 4 after some preliminaries in Section 2, a computation of the energy in Subsection 2.2, and a study of the linearized operator in Section 3. Theorem 1.1 is proved in Section 5. In Section 6, we carry out the proof of Theorem 1.3.

2. *First approximation of the large solution*

We assume that $1 < q < 3$ and $\lambda \geqslant 0$ are given. As we have discussed, to prove Theorem 1.2 it is convenient to express problem (1.6) in its equivalent form

$$\begin{cases} \Delta u + u^{5-\varepsilon} + \lambda \varepsilon^{\alpha} u^{q} - \varepsilon^2 u = 0, \quad u > 0 \text{ in } \mathbb{R}^3, \\ u(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty, \end{cases} \tag{2.1}$$

with $\alpha = (5-q)/2 - \varepsilon(q-1)/2(4-\varepsilon)$, via the change of variables $v(x) = \varepsilon^{-2/(4-\varepsilon)} u(x/\varepsilon)$. Thus, letting

$$w(y) = 3^{1/4} \frac{1}{(1+|y|^2)^{1/2}}, \quad w_\mu(y) = \mu^{-1/2} w(y/\mu),$$

the idea is to look of a solution of (2.1) which is a perturbation of w_μ for a suitable choice of μ. It turns out that a more convenient first approximation than w_μ is its *projection* U_μ defined as the unique solution of the problem

$$\begin{cases} \Delta U_\mu - \varepsilon^2 U_\mu = -w_\mu^5 \quad \text{in } \mathbb{R}^3, \\ U_\mu(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty. \end{cases} \tag{2.2}$$

Let us write

$$f_\varepsilon(u) = u^{5-\varepsilon} + \lambda \varepsilon^{\alpha} u^{q}.$$

Searching for a solution u of (2.1) of the form $u = U_\mu + \phi$ yields the following equation for ϕ:

$$\begin{cases} L_\varepsilon \phi + N(\phi) + R = 0 \quad \text{in } \mathbb{R}^3, \\ \phi(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow +\infty, \end{cases} \tag{2.3}$$

where

$$\begin{aligned} L_\varepsilon \phi &= \Delta\phi + f_\varepsilon'(U_\mu)\phi - \varepsilon^2 \phi, \quad N(\phi) = f_\varepsilon(U_\mu + \phi) - f_\varepsilon(U_\mu) - f_\varepsilon'(U_\mu)\phi, \\ R &= \Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu. \end{aligned} \tag{2.4}$$

We will use a Lyapunov–Schmidt reduction scheme to solve problem (2.3). To this end, it is important to get some basic estimates for U_μ.

2.1. *Basic estimates for U_μ*

First, by the maximum principle we readily find

$$0 < U_\mu \leqslant w_\mu \quad \text{in } \mathbb{R}^3.$$

Define the positive function $\pi_\mu := w_\mu - U_\mu$. Then

$$\Delta \pi_\mu - \varepsilon^2 \pi_\mu = -\varepsilon^2 w_\mu \quad \text{in } \mathbb{R}^3.$$

The following estimates hold.

LEMMA 2.1. *Assume that $\delta \leqslant \mu \leqslant \delta^{-1}$ for some $\delta > 0$. For any $0 < \sigma < 1$, we have the expansion*

$$\mu^{1/2} \pi_\mu(y) = 4\pi 3^{1/4} \varepsilon \mu H(\varepsilon y) - \varepsilon^2 \mu^2 D_0(y/\mu) + \varepsilon^{3-\sigma} \theta_\varepsilon(y),$$

where $H(x) = (1 - e^{-|x|})/4\pi|x|$,

$$D_0(y) = \tfrac{1}{2}[|y|^{-1} \log(|y| + \sqrt{|y|^2+1}) + \sqrt{|y|^2+1} - |y|], \tag{2.5}$$

and $|\theta_\varepsilon(y)| \leqslant C(1+\varepsilon|y|)^{-1+\sigma}$ for all $y \in \mathbb{R}^3$.

Proof. Let us define the Green's function $G := G(x)$ by

$$-\Delta G(x) + G(x) = \delta_0(x), \quad G(x) \longrightarrow 0 \text{ as } |x| \longrightarrow \infty. \tag{2.6}$$

Take $H(x) = 1/4\pi|x| - G(x)$, so

$$\Delta H(x) - H(x) = -\frac{1}{4\pi|x|}, \quad H(x) - \frac{1}{4\pi|x|} \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty.$$

Note that $H(x) = (1 - e^{-|x|})/4\pi|x|$ is the explicit solution of the problem. Let D_0 be the unique continuous solution of the problem

$$\Delta D_0 = D_1(y) := 3^{1/4}\left[\frac{1}{(1+|y|^2)^{1/2}} - \frac{1}{|y|}\right]$$

with $D_0(y) \to 0$ as $|y| \to \infty$. Since $D_1 < 0$, we have $D_0 > 0$, in fact D_0 is given by (2.5). Define

$$S(y) = \mu^{1/2}\pi_\mu(y) - 4\pi 3^{1/4}\varepsilon\mu H(\varepsilon y) + \varepsilon^2\mu^2 D_0(y/\mu).$$

Clearly, S satisfies

$$-\Delta S + \varepsilon^2 S = \varepsilon^4\mu^2 D_0(y/\mu) > 0 \quad \text{in } \mathbb{R}^3.$$

By the maximum principle $S > 0$ in $\mathbb{R}^3$. Taking $\bar{S}(x) = S(x/\varepsilon)$,

$$-\Delta\bar{S} + \bar{S} = \varepsilon^2\mu^2 D_0(x/(\varepsilon\mu)) \quad \text{in } \mathbb{R}^3.$$

Since $D_0(y) \sim |y|^{-1}\log(|y|)$ as $|y| \to \infty$, we have $D_0(x/(\varepsilon\mu)) \leqslant C(\varepsilon/|x|)^{1-\sigma}$ for any $0 < \sigma < 1$. Then $\bar{S}(x) \leqslant \varepsilon^2(\varepsilon/(1+|x|))^{1-\sigma}$ for all $x \in \mathbb{R}^3$. □

LEMMA 2.2. *We have*

$$w_\mu(y) - U_\mu(y) \leqslant C\frac{\varepsilon}{1+\varepsilon|y|} \quad \textit{for all } y \in \mathbb{R}^3, \tag{2.7}$$

$$U_\mu(y) \leqslant C\varepsilon^{-4}|y|^{-5} \quad \textit{for } |y| \geqslant 1/\varepsilon. \tag{2.8}$$

Proof. Let $P(x) = w_\mu(x/\varepsilon) - U_\mu(x/\varepsilon)$. The P satisfies

$$-\Delta P + P = w_\mu(x/\varepsilon) \quad \text{in } \mathbb{R}^3.$$

Since $w_\mu(x/\varepsilon) \leqslant C\varepsilon/|x|$, using $v(x) = \varepsilon/|x|$ as a barrier in a set $|x| \geqslant R/\varepsilon$ with $R > 0$ a large constant, we get $P(x) \leqslant C\varepsilon/(1+|x|)$ for all $|x| \geqslant R/\varepsilon$. On the other hand, $P(x) \leqslant \varepsilon$ near the origin and we deduce (2.7).

To prove (2.8), we use as barrier the function $v(y) = \varepsilon^{-4}|y|^{-5}$. It satisfies $\Delta v - \varepsilon^2 v \leqslant -\varepsilon^{-2}|y|^{-5}$ for $|y| \geqslant R/\varepsilon$ with $R > 0$ a large constant. Since $v(y) = R\varepsilon$ for $|y| = R/\varepsilon$ and $U_\mu(y) \leqslant w_\mu(y) \leqslant C/|y|$ for all $|y| \geqslant 0$, we get $Av(y) \geqslant U_\mu(y)$ for $|y| = R/\varepsilon$, for some constant $A > 0$. By the maximum principle, $U_\mu(y) \leqslant Av(y)$ for all $|y| \geqslant R/\varepsilon$. □

We will also need the functions

$$Z_\mu = \frac{\partial w_\mu}{\partial\mu} \tag{2.9}$$

and

$$\tilde{Z}_\mu = \frac{\partial U_\mu}{\partial\mu}, \tag{2.10}$$

which satisfies

$$\begin{cases} \Delta\tilde{Z}_\mu - \varepsilon^2\tilde{Z}_\mu = -5w_\mu^4 Z_\mu & \text{in } \mathbb{R}^3, \\ \tilde{Z}_\mu(y) \longrightarrow 0 & \text{as } |y| \longrightarrow \infty. \end{cases}$$

As in the proof of (2.7), we can also show the following lemma.

LEMMA 2.3.

$$|\tilde{Z}_\mu(y) - Z_\mu(y)| \leqslant \frac{C\varepsilon}{1+\varepsilon|y|} \quad \text{for all } |y| \geqslant 0, \tag{2.11}$$

$$|\tilde{Z}_\mu(y)| \leqslant \frac{C}{1+|y|} \quad \text{for all } |y| \geqslant 0, \tag{2.12}$$

$$|\tilde{Z}_\mu(y)| \leqslant C\varepsilon^{-4}|y|^{-5} \quad \text{for all } |y| \geqslant 1/\varepsilon. \tag{2.13}$$

2.2. *Energy expansion for U_μ*

Solutions of problem (2.1) are critical points of the energy functional

$$E(u) = E_p(u) + E_\lambda(u),$$

where $p = 5 - \varepsilon$,

$$E_p(u) = \frac{1}{2}\int_{\mathbb{R}^3} |Du|^2\, dy + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3} |u|^2\, dy - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}\, dy$$

and

$$E_\lambda(u) = -\lambda\varepsilon^\alpha \frac{1}{q+1}\int_{\mathbb{R}^3} |u|^{q+1}\, dy.$$

LEMMA 2.4. *Assume $1 < q < 3$, $\lambda > 0$ and $\delta > 0$ be fixed. Then there exist positive constants a_0, a_1, a_2, a_3 for such that $\delta < \mu < \delta^{-1}$*

$$E(U_\mu) = a_0 + \varepsilon\Psi(\mu) - a_2\varepsilon\log\varepsilon - a_3\varepsilon + \varepsilon\Theta_\varepsilon(\mu),$$

where

$$\Psi(\mu) = a_1\mu - a_2\log\mu,$$

and $\Theta_\varepsilon(\mu) \to 0$ as $\varepsilon \to 0$ in the C^1 norm in the interval $\delta \leqslant \mu \leqslant \delta^{-1}$.

Since a_1 and a_2 are positive, the critical point of Ψ is $\mu = a_2/a_1$ and $d = \mu^{-1/2}$.

Proof. For $u = U_\mu$, we have

$$E_5(U_\mu) = -\frac{1}{6}\int_{\mathbb{R}^3} |U_\mu|^6\, dy + \frac{1}{2}\int_{\mathbb{R}^3} w_\mu^5 U_\mu\, dy,$$

writing $U_\mu = w_\mu - \pi_\mu$, we have

$$E_5(U_\mu) = \frac{1}{3}\int_{\mathbb{R}^3} w_\mu^6(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^3} w_\mu^5\pi_\mu\, dy + \mathcal{R},$$

where

$$\mathcal{R} = -\frac{1}{6}\int_{\mathbb{R}^3} [|w_\mu - \pi_\mu|^6 - w_\mu^6 + 6w_\mu^5\pi_\mu]\, dy.$$

Using Lemma 2.1, we have

$$\frac{1}{3}\int_{\mathbb{R}^3} w_\mu^6(y)\, dy - \frac{1}{2}\int_{\mathbb{R}^3} w_\mu^5\pi_\mu\, dy = a_0 + a_1\varepsilon\mu,$$

where

$$a_0 = \frac{1}{3}\int_{\mathbb{R}^3} w^6(y)\, dy = \frac{1}{4}\sqrt{3}\pi^2, \quad a_1 = \frac{1}{2}\int_{\mathbb{R}^3} w(y)^5 3^{1/4}\, dy = 2\pi\sqrt{3}.$$

Now using (2.7), we have

$$\mathcal{R} = -5 \int_{\mathbb{R}^3} \int_0^1 (w_\mu - t\pi_\mu)^4 \pi_\mu^2 (1-t)\, dt\, dy = O(\varepsilon^2).$$

So we have the following energy expansion

$$E_5(U_\mu) = a_0 + a_1 \varepsilon\mu + O(\varepsilon^2).$$

On the other hand,

$$E_p(U_\mu) - E_5(U_\mu) = (p-5)[a_2 \log(\mu) + a_3] + o(p-5),$$

where

$$a_2 = \frac{1}{12} \int_{\mathbb{R}^3} w(y)^6\, dy = \frac{\sqrt{3}\pi^2}{16}, \quad a_3 = \frac{1}{36} \int_{\mathbb{R}^3} w(y)^6 [6 \log(w(y)) - 1]\, dy.$$

For $2 < q < 3$, we have

$$E_\lambda(U_\mu) = -\lambda a_4 (\varepsilon\mu)^{(5-q)/2} + O(\varepsilon^2).$$

where

$$a_4 = \frac{1}{q+1} \int_{\mathbb{R}^3} w^{q+1}(y)\, dy = \frac{3^{(q+1)/4}}{(q+1)} \frac{\pi^{3/2}\Gamma((q-2)/2)}{\Gamma(1/2(q+1))},$$

and the energy has the form

$$E(U_\mu) = a_0 + a_1\varepsilon\mu - \lambda a_4 (\varepsilon\mu)^{(5-q)/2} + (p-5)[a_2 \log(\mu) + a_3]. \tag{2.14}$$

For $q = 2$, we have the following estimate

$$E(U_\mu) = a_0 + a_1\mu + \lambda a_4 \log(\varepsilon\mu)(\varepsilon\mu)^{3/2} + (p-5)[a_2 \log(\mu) + a_3], \tag{2.15}$$

where $a_4 = 4\pi/3^{1/4}$. In fact, we have

$$\int_{\mathbb{R}^3} U_\mu(y)^3\, dy = \int_{B_0(1/\varepsilon)} U_\mu(y)^3\, dy + \int_{\mathbb{R}^3 \setminus B_0(1/\varepsilon)} U_\mu(y)^3\, dy \tag{2.16}$$

$$= -4\pi 3^{3/4} \mu^{3/2} \log(\varepsilon\mu) + O(1), \tag{2.17}$$

since $\int_0^a r^2/(1+r^2)^{3/2} = \log(a + \sqrt{1+a^2}) - a/\sqrt{1+a^2}$. For $1 < q < 2$, we have

$$\int_{\mathbb{R}^3} (w_\mu - \pi_\mu)^{q+1} = \mu^{(q+1)/2} \varepsilon^{q-2} (3^{1/4} 4\pi)^{q+1} \int_{\mathbb{R}^3} G^{q+1} + o(1),$$

and the energy has the form

$$E(U_\mu) = a_0 + a_1\varepsilon\mu - \lambda a_4 (\varepsilon\mu)^{(q+1)/2} + (p-5)[a_2 \log(\mu) + a_3], \tag{2.18}$$

where $a_4 = (3^{1/4} 4\pi)^{q+1} \int_{\mathbb{R}^3} G^{q+1}(x)/(q+1)\, dx$.

Combining (2.14), (2.15) and (2.18), and taking $p = 5 - \varepsilon$, we obtain the result. □

3. *Solvability for the linearized operator around* U_μ

In this section, we analyse the linear equation

$$\begin{cases} \Delta\phi + pU_\mu^{p-1}\phi + \lambda\varepsilon^\alpha q U_\mu^{q-1}\phi - \varepsilon^2\phi = h + c_1 Z_\mu w_\mu^4 & \text{in } \mathbb{R}^3, \\ \phi(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow +\infty, \\ \int_{\mathbb{R}^3} \phi Z_\mu w_\mu^4 = 0, \end{cases} \tag{3.1}$$

where U_μ is the function introduced in (2.2), $p = 5 - \varepsilon$, $1 < q < 5$ and $\alpha = (5-q)/2 - (5-p)(q-1)/2(p-1)$.

Let us define the following norms, for function $\phi, h : \mathbb{R}^3 \to \mathbb{R}$:

$$\|\phi\|_* = \sup_{|y|\leqslant 1/\varepsilon} (1+|y|^2)^{(\theta-2)/2}|\phi(y)| + \sup_{|y|\geqslant 1/\varepsilon} \varepsilon^2|y|^\theta|\phi(y)| \tag{3.2}$$

and

$$\|h\|_{**} = \sup_{|y|\geqslant 0} (1+|y|^2)^{\theta/2}|h(y)| \tag{3.3}$$

with θ in the range $2<\theta<3$ (so that $r^{2-\theta}$ is superharmonic).

The objective of this section is to prove the following result.

Lemma 3.1. *Let $0<\delta<1$ and $\bar\lambda\geqslant 0$ be fixed. Then there exists $\varepsilon_0=\varepsilon_0(\delta,\bar\lambda)>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$, $0\leqslant\lambda\leqslant\bar\lambda$, $\delta\leqslant\mu\leqslant\delta^{-1}$, and for any radial h with $\|h\|_{**}<\infty$ there exists a unique radial ϕ with $\|\phi\|_*<+\infty$ and $c_1\in\mathbb{R}$ solution of (3.1), moreover there exists $C>0$ such that*

$$\|\phi\|_* \leqslant C\|h\|_{**}, \quad |c_1|\leqslant C\|h\|_{**}. \tag{3.4}$$

We first prove an a priori estimate for solutions of a simpler problem:

$$\begin{cases} \Delta\phi + pU_\mu^{p-1}\phi - \varepsilon^2\phi = h, \\ \phi(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty, \\ \int_{\mathbb{R}^3} Z_\mu w_\mu^4\phi = 0, \quad \int_{\mathbb{R}^3} \frac{\partial w_\mu}{\partial x_i} w_\mu^4\phi = 0, \quad i=1,2,3 \end{cases} \tag{3.5}$$

with $|p-5|=\varepsilon$. In order for it to be useful in a later situation, we do not assume here h,ϕ to be radial.

Lemma 3.2. *Assume that $\delta\leqslant\mu\leqslant\delta^{-1}$ where $0<\delta<1$ is fixed. There is C such that if $\varepsilon>0$ is sufficiently small, for any h,ϕ solution of (3.5) we have*

$$\|\phi\|_* \leqslant C\|h\|_{**}. \tag{3.6}$$

Proof. By contradiction, suppose that there exist ϕ_n, h_n, μ_n, ε_n, $|p_n-5|=\varepsilon_n$ such that

$$\|\phi_n\|_*=1, \quad \|h_n\|_{**}\longrightarrow 0, \quad \mu_n\in[\delta,\delta^{-1}], \quad \varepsilon_n\longrightarrow 0,$$

and such that ϕ_n, h_n solve (3.5).

We claim that $\phi_n\to 0$ uniformly on compact sets of $\mathbb{R}^3$. Indeed, assume otherwise. Then up to a subsequence $\mu_n\to\mu>0$ and $\phi_n\to\phi$ uniformly on compact subsets of $\mathbb{R}^3$, where $\phi\not\equiv 0$ and satisfies

$$\Delta\phi + 5w_\mu^4\phi = 0 \quad \text{in } \mathbb{R}^3.$$

We also know that $\|\phi\|_*\leqslant 1$ which implies that ϕ is bounded. Since w_μ is non-degenerate, it is well known, see [**26**], that $\phi = c_0Z_\mu + \sum_{i=1}^3 c_i(\partial w_\mu/\partial x_i)$ for some $c_0,\dots,c_3\in\mathbb{R}$. But taking the limit in the orthogonality condition in (3.5), we obtain

$$\int_{\mathbb{R}^3} Z_\mu w_\mu^4\phi = 0 \quad \int_{\mathbb{R}^3} \frac{\partial w_\mu}{\partial x_i} w_\mu^4\phi = 0,$$

so $\phi=0$, which is a contradiction.

This proves that $\phi_n\to 0$ uniformly on compact sets of $\mathbb{R}^3$. We will obtain now an estimate for $\|\phi_n\|_*$ using suitable barriers. Let $0<\sigma<1$ with $\sigma<\theta$, $\delta>0$ and $r_0>0$ to be fixed later on. Define

$$\bar\phi(x) = r^{2-\theta} + \delta r^{-\sigma}, \quad r=|x|.$$

Then

$$\begin{aligned}(\Delta + p_n w^{p_n-1} - \varepsilon_n^2)\bar\phi &= (2-\theta)(3-\theta)r^{-\theta} + p_n w^{p_n-1} r^{2-\theta} - \varepsilon_n^2 r^{2-\theta} \\ &\quad + \delta[-\sigma(1-\sigma)r^{-\sigma-2} + p_n w^{p_n-1} r^{-\sigma} - \varepsilon_n^2 r^{-\sigma}] \\ &= (2-\theta)(3-\theta)r^{-\theta} + O(r^{-4+O(\varepsilon_n)})r^{2-\theta} - \varepsilon_n^2 r^{2-\theta} \\ &\quad + \delta[-\sigma(1-\sigma)r^{-\sigma-2} + +O(r^{-4+O(\varepsilon_n)})r^{-\sigma} - \varepsilon_n^2 r^{-\sigma}] \\ &\leqslant -C_\theta r^{-\theta} \quad \text{for } r \geqslant r_0,\end{aligned}$$

where $C_\theta > 0$ depends only on θ, if we chose $r_0 > 0$ large depending on θ and σ. Define

$$v_n(x) = \left(\sup_{|y|=r_0} |\phi_n(y)| r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde h_n\|_{**} + \frac{1}{n}\right)\bar\phi(x) - \phi_n(x),$$

which satisfies

$$(\Delta + p_n w^{p_n-1} - \varepsilon_n^2)v_n \leqslant 0 \quad \text{for } |x| \geqslant r_0$$

and

$$v_n(x) \geqslant 0 \quad \text{for } |x| = r_0.$$

Since $|\phi_n(x)| \leqslant \varepsilon_n^{-2}|x|^{-\theta}$ for $|x| \geqslant 1/\varepsilon_n$, we can find $r_n \geqslant 1/\varepsilon_n$ such that for $|x| \geqslant r_n$ we have

$$v_n(x) \geqslant 0 \quad \text{for } |x| \geqslant r_n.$$

By the maximum principle, we deduce that

$$v_n(x) \geqslant 0 \quad \text{for } |x| \geqslant r_0,$$

which means

$$\phi_n(x) \leqslant \left(\sup_{|y|=r_0} |\phi_n(y)| r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde h_n\|_{**} + \frac{1}{n}\right)(|x|^{2-\theta} + \delta|x|^{-\sigma}) \quad \text{for } |x| \geqslant r_0.$$

By a similar argument,

$$|\phi_n(x)| \leqslant \left(\sup_{|y|=r_0} |\phi_n(y)| r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde h_n\|_{**} + \frac{1}{n}\right)(|x|^{2-\theta} + \delta|x|^{-\sigma}) \quad \text{for } |x| \geqslant r_0.$$

Letting $\delta \to 0$, we obtain

$$|\phi_n(x)| \leqslant \left(\sup_{|y|=r_0} |\phi_n(y)| r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde h_n\|_{**} + \frac{1}{n}\right)|x|^{2-\theta} \quad \text{for } |x| \geqslant r_0. \tag{3.7}$$

Let

$$\bar\phi(x) = r^{-\theta} + \delta r^{-\sigma}, \quad r = |x|,$$

with σ as before. Then

$$\begin{aligned}(\Delta + p_n w^{p_n-1} - \varepsilon_n^2)\bar\phi &= -\theta(1-\theta)r^{-\theta-2} + p_n w^{p_n-1} r^{-\theta} - \varepsilon_n^2 r^{-\theta} \\ &\quad + \delta[-\sigma(1-\sigma)r^{-\sigma-2} + p_n w^{p_n-1} r^{-\sigma} - \varepsilon_n^2 r^{-\sigma}] \\ &\leqslant -\frac{\varepsilon_n^2}{2} r^{-\theta} \quad \text{for } r \geqslant \frac{M}{\varepsilon_n},\end{aligned}$$

where $M > 0$ is a constant that depends only on θ. So

$$(\Delta + p_n w^{p_n-1} - \varepsilon_n^2)\left(\frac{2}{\varepsilon_n^2}\|\tilde h_n\|_{**}\bar\phi - \phi_n\right) \leqslant 0 \quad \text{for } r \geqslant \frac{M}{\varepsilon_n}.$$

Since

$$\bar\phi\left(\frac{M}{\varepsilon_n}\right) \geqslant M^{-\theta}\varepsilon_n^{\theta},$$

and by (3.7)

$$|\phi_n(x)| \leqslant \left(|\phi_n(r_0)|r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde{h}_n\|_{**} + \frac{1}{n}\right) M^{2-\theta}\varepsilon_n^{\theta-2} \quad \text{for } |x| = \frac{M}{\varepsilon_n},$$

we have

$$|\phi_n(x)| \leqslant \left(|\phi_n(r_0)|r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde{h}_n\|_{**} + \frac{1}{n}\right) M^2\varepsilon_n^{-2}\bar{\phi}\left(\frac{M}{\varepsilon_n}\right) \quad \text{for } |x| = \frac{M}{\varepsilon_n}.$$

We also have

$$|\phi_n(x)| \leqslant \left(\sup_{|y|=r_0} |\phi_n(y)|r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde{h}_n\|_{**} + \frac{1}{n}\right) M^2\varepsilon_n^{-2}\bar{\phi}(x)$$

for $|x|$ sufficiently large. By the maximum principle,

$$|\phi_n(x)| \leqslant \left(\left(\sup_{|y|=r_0} |\phi_n(y)|r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde{h}_n\|_{**} + \frac{1}{n}\right) M^2\varepsilon_n^{-2} + \frac{2}{\varepsilon_n^2}\|\tilde{h}_n\|_{**}\right)\bar{\phi}(x)$$

for all $|x| \geqslant M/\varepsilon_n$. Letting $\delta \to 0$, we obtain

$$|\phi_n(r)| \leqslant \left(\left(\sup_{|y|=r_0} |\phi_n(y)|r_0^{\theta-2} + \frac{1}{C_\theta}\|\tilde{h}_n\|_{**} + \frac{1}{n}\right) M^2\varepsilon_n^{-2} + \frac{2}{\varepsilon_n^2}\|\tilde{h}_n\|_{**}\right)|x|^{-\theta}$$

for all $|x| \geqslant M/\varepsilon_n$. This and (3.7) imply that $\|\phi_n\|_* \to 0$ as $n \to \infty$, which is a contradiction, and establishes (3.6). □

We derive now an a priori estimate for the solutions of:

$$\begin{cases} \Delta\phi + pU_\mu^{p-1}\phi + \lambda\varepsilon^\alpha qU_\mu^{q-1}\phi - \varepsilon^2\phi = h, \\ \phi(y) \to 0 \quad \text{as } |y| \longrightarrow \infty, \\ \int_{\mathbb{R}^3} Z_\mu w_\mu^4\phi = 0, \quad \int_{\mathbb{R}^3} \frac{\partial w_\mu}{\partial x_i} w_\mu^4\phi = 0, \quad i = 1,2,3, \end{cases} \tag{3.8}$$

with $|p-5| = \varepsilon$ and $1 < q < 5$. Again this is done without assuming ϕ, h to be radial.

Lemma 3.3. *Assume that $\delta \leqslant \mu \leqslant \delta^{-1}$ where $0 < \delta < 1$ is fixed. There is C such that if $\varepsilon > 0$ is sufficiently small, for any h, ϕ solution of (3.8) we have*

$$\|\phi\|_* \leqslant C\|h\|_{**}. \tag{3.9}$$

Proof. We claim that

$$\|U_\mu^{q-1}\phi\|_{**} \leqslant C\varepsilon^{q-3}\|\phi\|_*. \tag{3.10}$$

Since $U_\mu \leqslant w_\mu$, it is sufficient to prove

$$\|w_\mu^{q-1}\phi\|_{**} \leqslant C\varepsilon^{q-3}\|\phi\|_*.$$

We have that

$$\begin{aligned} \sup_{|y|\leqslant 1/\varepsilon} (1+|y|^2)^{\theta/2} w_\mu^{q-1}|\phi(y)| &\leqslant C \sup_{|y|\leqslant 1/\varepsilon} (1+|y|^2)^{\theta/2}|y|^{-(q-1)}|\phi(y)| \\ &\leqslant C\|\phi\|_* \sup_{|y|\leqslant 1/\varepsilon} |y|^{3-q}. \end{aligned}$$

Therefore,

$$\sup_{|y|\leqslant 1/\varepsilon} |y|^\theta w_\mu^{q-1}(y)|\phi(y)| \leqslant C\|\phi\|_*\varepsilon^{q-3}.$$

Now we analyse the case $|y| \geqslant 1/\varepsilon$:

$$\begin{aligned}\sup_{|y|\geqslant 1/\varepsilon} |y|^{\theta} w_\mu^{q-1}(y)|\phi(y)| &\leqslant C \sup_{|y|\geqslant 1/\varepsilon} |y|^{\theta}|y|^{-(q-1)}|\phi(y)| \\ &\leqslant C \sup_{|y|\geqslant 1/\varepsilon} |y|^{\theta}|y|^{-(q-1)}|y|^{-\theta}\varepsilon^{-2}\|\phi\|_* \\ &= C\varepsilon^{-2}\|\phi\|_* \sup_{|y|\geqslant 1/\varepsilon} |y|^{-(q-1)} \\ &\leqslant C\|\phi\|_*\varepsilon^{q-3},\end{aligned}$$

since $q > 1$. This proves (3.10).

Then, using estimate (3.6), we deduce that

$$\|\phi\|_* \leqslant C\|h\|_{**} + C\varepsilon^{\alpha+q-3}\|\phi\|_*.$$

Since $\alpha = (5-q)/2 + O(\varepsilon)$, we see that $\alpha + q - 3 > 0$, which proves the desired estimate. □

Proof of Lemma 3.1. We first prove the estimate (3.4). Assume that h, ϕ are radial and ϕ satisfies (3.1). Then Lemma 3.3 shows that $\|\phi\|_*$ is finite. Let $\eta \in C_0^\infty(B_{2R}(0))$ be such that $\eta \equiv 1$ in $B_R(0)$, $|\nabla\eta| \leqslant CR^{-1}$, $|\Delta\eta| \leqslant CR^{-2}$. Multiplying (3.1) by $Z_\mu\eta$ and then letting $R \to \infty$, we get

$$c_1 \int_{\mathbb{R}^3} Z_\mu^2 w_\mu^4 = \int_{\mathbb{R}^3} (pU_\mu^{p-1} - 5w_\mu^4)\phi Z_\mu + \lambda\varepsilon^\alpha q \int_{\mathbb{R}^3} U_\mu^{q-1}\phi Z_\mu - \varepsilon^2 \int_{\mathbb{R}^3} \phi Z_\mu - \int_{\mathbb{R}^3} hZ_\mu.$$

To verify this, we need to estimate

$$\int_{B_R(0)} |\phi\Delta\eta| Z_\mu \leqslant \frac{C}{R^2}\|\phi\|_* \int_R^{2R} \frac{1}{\varepsilon^2 r^{\theta+1}} r^2\, dr \leqslant C\varepsilon^{-2}R^{-1-\theta}\|\phi\|_*$$

and

$$\int_{B_R(0)} |\phi||\nabla\eta||\nabla Z_\mu| \leqslant C\varepsilon^{-2}R^{-1-\theta}\|\phi\|_*,$$

and they converge to 0 as $R \to \infty$. We also have

$$\varepsilon^2 \int_{\mathbb{R}^3} |\phi Z_\mu| \leqslant C\|\phi\|_*\varepsilon^{\theta-2} \tag{3.11}$$

and

$$\left|\int_{\mathbb{R}^3} (pU_\mu^{p-1} - 5w_\mu^4)\phi Z_\mu\right| \leqslant C\varepsilon\|\phi\|_*,$$

using (2.7). Similarly,

$$\lambda\varepsilon^\alpha \int_{\mathbb{R}^3} U_\mu^{q-1}|Z_\mu\phi| \leqslant C\varepsilon^{\theta-2}, \tag{3.12}$$

and $|\int_{\mathbb{R}^3} hZ_\mu| \leqslant C\|h\|_{**}$. The inequalities (3.11) and (3.12) show that

$$|c_1| \leqslant o(1)\|\phi\|_* + C\|h\|_{**},$$

where $o(1) \to 0$ as $\varepsilon \to 0$. This together with (3.9) yields (3.4).

To prove existence of a solution of (3.1), consider the Hilbert space

$$H = \left\{\phi \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} Z_\mu w_\mu^4 \phi = 0\right\}$$

with inner product $\langle \phi_1, \phi_2\rangle = \int_{\mathbb{R}^3} \nabla\phi_1\nabla\phi_2 + \varepsilon^2\int_{\mathbb{R}^3}\phi_1\phi_2$. For $h : \mathbb{R}^3 \to \mathbb{R}$, with $\|h\|_{**} < +\infty$, the variational problem of finding $\phi \in H$ such that

$$\langle \phi, \psi\rangle = \int_{\Omega_\lambda} (pU_\mu^{p-1}\phi + \lambda\varepsilon^\alpha U_\mu^{q-1} + h)\phi \quad \text{for all } \phi \in H$$

is a weak formulation of (3.1). Using the Riesz representation theorem, this variational problem is equivalent to solve

$$\phi + K(\phi) = \tilde{h}, \tag{3.13}$$

where $\tilde{h} \in H$ and $K : H \to H$ is a compact operator. Any solution ϕ of (3.13) is a weak solution of (3.1) and by standard regularity theory $\phi \in C(\mathbb{R}^3)$. Moreover, we can prove that this solution has finite $\| \, \|_*$ norm using barriers, and hence estimate (3.4) holds. When $\tilde{h} = 0$, then this argument shows that $\phi = 0$. By the Fredholm alternative, there is a solution $\phi \in H$ of (3.13) giving a solution of (3.1). □

4. *Proof of Theorem* 1.2 *and non-degeneracy of the solution*

For the proof, we will solve the problem in two steps: first we use the linear theory devised in the previous section to solve a projected version of the problem, and then we will find the right value of μ in such a way that we actually have a solution to the full problem. We have the validity of the following result.

Proposition 4.1. *For $\varepsilon > 0$ sufficiently small, there is a unique ϕ_μ and c solution of*

$$\begin{cases} L_\varepsilon \phi + N(\phi) + R = cZ_\mu w_\mu^4 \quad \text{in } \mathbb{R}^3, \quad \phi(x) \longrightarrow 0 \text{ as } |x| \longrightarrow +\infty, \\ \int_{\mathbb{R}^3} \phi \tilde{Z} = 0, \end{cases} \tag{4.1}$$

and such that $\|\phi_\mu\| \leqslant C\varepsilon$, $|c| \leqslant C\varepsilon$.

For the proof, we start by estimating R, which was defined in (2.4).

Lemma 4.2. *Assume $1 < q < 5$. Suppose that $\delta \leqslant \mu \leqslant \delta^{-1}$ where $0 < \delta < 1$ is fixed and that $\tilde{\lambda} \geqslant 0$ is a constant. Then, choosing $2 < \theta < 3$ appropriately in the norms (3.2), (3.3), there exists $\varepsilon_0 = \varepsilon_0(\tilde{\lambda}) > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$, $0 \leqslant \lambda \leqslant \tilde{\lambda}$, we have*

$$\|R\|_{**} \leqslant C\varepsilon, \tag{4.2}$$

$$\|\partial_\mu R\|_{**} \leqslant C\varepsilon. \tag{4.3}$$

Proof. We compute $R = U_\mu^{5-\varepsilon} - w_\mu^5 + \lambda \varepsilon^\alpha U_\mu^q$. We claim that

$$\|U_\mu^{5-\varepsilon} - w_\mu^{5-\varepsilon}\|_{**} \leqslant C\varepsilon. \tag{4.4}$$

Indeed, using (2.7) we get

$$\begin{aligned} \sup_{|y| \leqslant 1/\varepsilon} (1+|y|^2)^{\theta/2} |U_\mu^{5-\varepsilon} - w_\mu^{5-\varepsilon}| &\leqslant C \sup_{|y| \leqslant 1/\varepsilon} (1+|y|^2)^{\theta/2} w_\mu^{5-\varepsilon-1} |U_\mu - w_\mu| \\ &\leqslant C\varepsilon \sup_{|y| \leqslant 1/\varepsilon} \frac{(1+r)^{\theta-4+O(\varepsilon)}}{1+r\varepsilon} \leqslant C\varepsilon, \end{aligned}$$

since we work with $2 < \theta < 3$. Also

$$\sup_{r \geqslant 1/\varepsilon} (U_\mu^{5-\varepsilon} + w_\mu^{5-\varepsilon}) \leqslant C\varepsilon^{4-\theta+O(\varepsilon)} \leqslant C\varepsilon^{4-\theta} \leqslant C\varepsilon,$$

and we obtain (4.4).

By direct calculation,

$$\|w^{5-\varepsilon} - w^5\|_{**} \leqslant C\varepsilon.$$

To estimate the term $\lambda\varepsilon^{\alpha}U_{\mu}^{q}$, we use the inequality $U_{\mu} \leqslant w_{\mu}$ to get

$$\lambda\varepsilon^{\alpha} \sup_{0\leqslant|y|\leqslant 1/\varepsilon} (1+|y|^2)^{\theta/2}U_{\mu}^{q} \leqslant \begin{cases} C\lambda\varepsilon^{\alpha} & \text{if } \theta < q, \\ C\lambda\varepsilon^{\alpha+q-\theta} & \text{if } \theta \geqslant q. \end{cases}$$

Using (2.8), we find

$$\lambda\varepsilon^{\alpha} \sup_{|y|\geqslant 1/\varepsilon} (1+|y|^2)^{\theta/2}U_{\mu}^{q} \leqslant C\lambda\varepsilon^{\alpha-4q} \sup_{|y|\geqslant 1/\varepsilon} (1+|y|)^{\theta/2}|y|^{-5q} \leqslant C\lambda\varepsilon^{\alpha+q-\theta}.$$

Note that $\alpha + q = q/2 + 5/2 + O(\varepsilon) > 3$. Therefore, fixing θ in the range

$$2 < \theta < \frac{3+q}{2}, \tag{4.5}$$

we get estimate (4.2).

Regarding the derivative of R, we have

$$\partial_{\mu}R = (5-\varepsilon)U_{\mu}^{5-\varepsilon-1}\tilde{Z}_{\mu} - 5w_{\mu}^{4}Z_{\mu} + \lambda\varepsilon^{\alpha}qU_{\mu}^{q-1}\tilde{Z}_{\mu}.$$

Owing to (2.11) and (2.12), the proof of estimate (4.3) for $\|\partial_{\mu}R\|_{**}$ is similar to that of $\|R\|_{**}$. □

4.1. *Proof of Proposition* 4.1

Let T be the linear operator that to h with $\|h\|_{**} < +\infty$ associates the unique solution ϕ of (3.1) with $\|\phi\|_{*} < +\infty$, constructed in Lemma 3.1. Then problem (4.1) can be written as the fixed point problem

$$\phi = -T(N(\phi) + R),$$

which we can solve by the fixed point mapping principle. For this, let E be the Banach space of continuous radial functions $\phi : \mathbb{R}^3 \to \mathbb{R}$ with $\|\phi\|_{*} < \infty$, endowed with this norm. Let $\bar{B}_{\rho} \subset E$ be the closed ball in E centred at zero with radius $\rho > 0$, where ρ will be chosen later on.

Owing to (3.4),

$$\|T(N(\phi) + R)\|_{*} \leqslant C(\|N(\phi)\|_{**} + \|R\|_{**}).$$

We estimate $\|N(\phi_1) - N(\phi_2)\|_{**}$ for $\|\phi_1\|_{*}, \|\phi_2\|_{*} \leqslant \rho$, by writing

$$N(\phi_1) - N(\phi_2) = \int_0^1 N'(\phi_2 + t(\phi_1 - \phi_2))\,dt(\phi_1 - \phi_2).$$

We see that

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leqslant K\|\phi_1 - \phi_2\|_{*},$$

where

$$K_{\rho} = \sup_{\|\phi\|_{*}\leqslant\rho} \left[\sup_{r\leqslant 1/\varepsilon} r^2|f_{\varepsilon}'(U_{\mu}+\phi) - f_{\varepsilon}'(U_{\mu})| + \sup_{r\geqslant 1/\varepsilon} \varepsilon^{-2}|f_{\varepsilon}'(U_{\mu}+\phi) - f_{\varepsilon}'(U_{\mu})| \right]. \tag{4.6}$$

We compute

$$\sup_{r\leqslant 1/\varepsilon} r^2|(U_{\mu}+\phi)^{p-1} - U_{\mu}^{p-1}| \leqslant C(\|\phi\|_{*} + \varepsilon^{\min((\theta-2)(p-1)-2,0)}\|\phi\|_{*}^{p-1}) \tag{4.7}$$

and

$$\sup_{r\geqslant 1/\varepsilon} \varepsilon^{-2}|(U_{\mu}+\phi)^{p-1} - U_{\mu}^{p-1}| \leqslant C(\varepsilon^{\theta+p-6}\|\phi\|_{*} + \varepsilon^{(\theta-2)(p-1)-2}\|\phi\|_{*}^{p-1}). \tag{4.8}$$

If $2 \leqslant q < 3$, then we obtain

$$K_{\rho} \leqslant C\lambda\varepsilon^{\alpha}(\varepsilon^{\theta+q-6} + \varepsilon^{(\theta-2)(q-1)-2})\rho,$$

and if $1 < q < 2$, then we get

$$K_\rho \leqslant C\lambda\varepsilon^{\alpha+(\theta-2)(q-1)-2}\rho^{q-1}.$$

Take $\rho = A\varepsilon$ for some A to be fixed. Then for $\|\phi_1\|_*, \|\phi_2\|_* \leqslant A\varepsilon$,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leqslant C\varepsilon^a\|\phi_1 - \phi_2\|_*,$$

where $a > 0$ (for any $2 < \theta < 3$). This and the estimate for R in (4.2) (valid for $\theta > 2$ in the range (4.5)) show that taking A large enough, $-T(N(\phi) + R)$ is a contraction from $\bar{B}_{A\varepsilon}$ to itself, and therefore it has a unique fixed point in this set.

PROPOSITION 4.3. *The solution ϕ_μ, $c(\mu)$ constructed in Proposition 4.1 is C^1 with respect to μ and satisfies*

$$\|\partial_\mu\phi_\mu\|_* + |c'(\mu)| \leqslant C\varepsilon. \tag{4.9}$$

Proof. The differentiability of ϕ_μ, $c(\mu)$ with respect to μ follows from the differentiability of R, the operator T defined by Lemma 3.1 and the contraction mapping principle, by a standard argument. We will prove next estimate (4.9). Differentiating (4.1) with respect to μ, we find for $v = \partial_\mu\phi_\mu$

$$\begin{aligned}\Delta v + f_\varepsilon'(U_\mu)v - \varepsilon^2 v + (f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))(\tilde{Z}_\mu + v) + \frac{\partial R}{\partial\mu}\\ = c'Z_\mu w_\mu^4 + c\frac{\partial(Z_\mu w_\mu^4)}{\partial\mu},\end{aligned}$$

in $\mathbb{R}^3$, where $\tilde{Z}_\mu$ is given by (2.10). Let $\tilde{v} = v - aZ_\mu w_\mu^4$, where $a \in \mathbb{R}$ is chosen so that $\int_{\mathbb{R}^3}\tilde{v}Z_\mu w_\mu^4 = 0$. Differentiating the orthogonality condition in (4.1), we see that $a = O(\varepsilon)$. The function $\tilde{v}$ satisfies

$$\begin{aligned}\Delta\tilde{v} + f_\varepsilon'(U_\mu)\tilde{v} - \varepsilon^2\tilde{v} + \alpha[\Delta(Z_\mu w_\mu^4) + f_\varepsilon'(U_\mu)Z_\mu w_\mu^4 - \varepsilon^2 Z_\mu w_\mu^4]\\ + (f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))(\tilde{Z}_\mu + \tilde{v} + aZ_\mu w_\mu^4) + \frac{\partial R}{\partial\mu} = c'Z_\mu w_\mu^4 + c\frac{\partial(Z_\mu w_\mu^4)}{\partial\mu}\end{aligned}$$

in $\mathbb{R}^3$. Therefore, applying Lemma 3.1 we obtain

$$\|\tilde{v}\|_* + |c'| \leqslant C\varepsilon + C\left\|(f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))(\tilde{Z}_\mu + \tilde{v} + aZ_\mu w_\mu^4) + \frac{\partial R}{\partial\mu}\right\|_{**}, \tag{4.10}$$

where we have used that $a = O(\varepsilon)$ and $c = O(\varepsilon)$. Using the function K_ρ introduced in (4.6), we can estimate

$$\|(f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))\tilde{v}\|_{**} \leqslant C\varepsilon^b\|\tilde{v}\|_* \tag{4.11}$$

for some $b > 0$. Similarly, since $a = O(\varepsilon)$ and $\|Z_\mu w_\mu^4\|_* \leqslant C$ and using the estimates for K_ρ, we find

$$\|(f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))aZ_\mu w_\mu^4\|_{**} \leqslant C\varepsilon. \tag{4.12}$$

Next we claim that

$$\|(f_\varepsilon'(U_\mu + \phi) - f_\varepsilon'(U_\mu))\tilde{Z}_\mu\|_{**} \leqslant C\varepsilon. \tag{4.13}$$

The computations for the term u^p in f_ε are similar as before, using K_ρ, the estimates (4.7), (4.8) and $\|\tilde{Z}_\mu\|_* \leqslant C$. Regarding the term $\lambda\varepsilon^\alpha u^q$, in the case $2 \leqslant q < 3$ we compute

$$\begin{aligned}\varepsilon^\alpha \sup_{r\leqslant 1/\varepsilon} r^\theta|((U_\mu + \phi)^{q-1} - U_\mu^{q-1})\tilde{Z}_\mu| &\leqslant \varepsilon^\alpha \sup_{r\leqslant 1/\varepsilon} r^\theta(|U_\mu|^{q-2} + |\phi|^{q-2})|\phi||\tilde{Z}_\mu|\\ &\leqslant C\varepsilon^{\alpha+q-2},\end{aligned}$$

and note that $\alpha + q - 2 > 1$. Also, using (2.13)

$$\varepsilon^\alpha \sup_{r \geqslant 1/\varepsilon} r^\theta (|U_\mu|^{q-2} + |\phi|^{q-2})|\phi||\tilde{Z}_\mu| \leqslant C\varepsilon^{\alpha+q-2}.$$

In the case $1 < q < 2$, a similar calculation shows that

$$\varepsilon^\alpha \sup_{r \geqslant 0} r^\theta |\phi|^{q-1}|\tilde{Z}_\mu| \leqslant C\varepsilon^{\alpha+(\theta-1)(q-2)}.$$

We note that in the case $1 < q < 2$, choosing θ in the interval (4.5) implies $2 < \theta < 2 + (q-1)/2(2-q)$ which gives $\alpha + (\theta-1)(q-2) > 1$ for $\varepsilon > 0$ small. Therefore, we obtain (4.13).

Using the bounds (4.10)–(4.13) and the estimate for $\|\partial_\mu R\|_{**}$ in (4.3), we deduce

$$\|\tilde{v}\|_* + |c'| \leqslant C\varepsilon^b \|\tilde{v}\|_* + C\varepsilon.$$

Thus, for $\varepsilon > 0$ small we deduce the validity of (4.9). □

4.2. *Variational reduction and the proof of the theorem*

Next we adjust μ such that $c = 0$. We consider the energy functional

$$E(u) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - F_\varepsilon(u),$$

where $F_\varepsilon(u) = \int_0^u f_\varepsilon(s)\, ds$, and define

$$\tilde{E}(\mu) = E(U_\mu + \phi_\mu).$$

Lemma 4.4. *We have the expansion*

$$\tilde{E}(\mu) = E(U_\mu) + o(\varepsilon),$$

as $\varepsilon \to 0$ *where this error is in* C^1 *norm for* μ *in an interval of the form* $[\delta, \delta^{-1}]$.

The proof of this estimate is similar to the one of del Pino, Dolbeault and Musso [**23**, Lemma 4].

Proof of Theorem 1.2. Testing equation (4.1) against $\tilde{Z}_\mu$, we obtain

$$\int_{\mathbb{R}^3} \phi L_\varepsilon \tilde{Z}_\mu + \int_{\mathbb{R}^3} N(\phi)\tilde{Z}_\mu + \int_{\mathbb{R}^3} R\tilde{Z}_\mu = c\int_{\mathbb{R}^3} \tilde{Z}_\mu \tilde{Z}_\mu w_\mu^4.$$

A calculation shows that the equation $c = 0$ is equivalent to

$$\int_{\mathbb{R}^3} R\tilde{Z}_\mu + o(\varepsilon) = 0, \tag{4.14}$$

as $\varepsilon \to 0$ where $o(\varepsilon)$ depends continuously on μ for μ in (δ, δ^{-1}). We observe that

$$\int_{\mathbb{R}^3} R\tilde{Z}_\mu = \tilde{E}'(\mu).$$

By Lemma 2.4,

$$\tilde{E}(U_\mu) = c_\varepsilon + \varepsilon\Psi(\mu) + o(\varepsilon),$$

where

$$\Psi(\mu) = a_1\mu - a_2 \log \mu$$

with $a_1, a_2 > 0$ and $o(\varepsilon)$ is uniform in C^1 for μ in $[\delta, \delta^{-1}]$. The function Ψ has a unique critical point $\mu^* > 0$, which is moreover non-degenerate. Then, owing to Lemma 4.4, equation (4.14) can be rewritten in the form

$$\varepsilon(\Psi'(\mu) + o(1)) = 0,$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$ in $[\delta, \delta^{-1}]$. Since μ^* is a non-degenerate critical point of Ψ, it follows that for $\varepsilon > 0$ small there is a unique solution μ of (4.14) close to μ^*. The construction is concluded. □

4.3. *Non-degeneracy and Morse index*

We will prove that the solution just built is non-degenerate in the sense that the linearized operator only contains trivial solutions, and in addition we will compute its Morse index as a critical point of the associated energy.

We recall the notation $f_\varepsilon(u) = u^{5-\varepsilon} + \lambda\varepsilon^\alpha u^q$. Let μ_ε be the unique number close to μ^* such that $\tilde{E}'(\mu_\varepsilon) = 0$. Let u_ε be the solution constructed before for $\varepsilon > 0$ small, having the form $u_\varepsilon = U_{\mu_\varepsilon} + \phi_{\mu_\varepsilon}$. We shall denote in the following:

$$U_\mu = U_{\mu_\varepsilon}, \quad \phi = \phi_{\mu_\varepsilon} \quad \text{and} \quad w_\mu = w_{\mu_\varepsilon}.$$

We need to show that if ψ is a bounded solution of

$$\Delta\psi + f'_\varepsilon(u_\varepsilon)\psi - \varepsilon^2\psi = 0 \quad \text{in } \mathbb{R}^3,$$

then ψ is a linear combination of the functions $\partial u_\varepsilon/\partial x_i$, $i = 1, 2, 3$. We note that for convenient $c_1, c_2, c_3 \in \mathbb{R}$ the function $\tilde{\psi} = \psi - \sum_{i=1}^3 c_i(\partial u_\varepsilon/\partial x_i)$ satisfies

$$\int_{\mathbb{R}^3} \frac{\partial w_\mu}{\partial x_j}\tilde{\psi} w_\mu^4 = 0, \quad j = 1, 2, 3. \tag{4.15}$$

Indeed, this system is equivalent to

$$\int_{\mathbb{R}^3} \psi \frac{\partial w_\mu}{\partial x_j} w_\mu^4 = \sum_{i=1}^3 c_i \int_{\mathbb{R}^3} \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial w_\mu}{\partial x_j} w_\mu^4,$$

which is diagonal with the diagonal elements bounded away from 0. Replacing ψ with $\tilde{\psi}$, we may assume that ψ satisfies (4.15) and it is sufficient to prove that $\psi = 0$.

Let us write $\psi = \psi^\perp - \alpha_1 \tilde{Z}_\mu$ where α_1 is such that

$$\int_{\mathbb{R}^3} \psi^\perp Z_\mu w_\mu^4 = 0, \tag{4.16}$$

and where Z_μ, $\tilde{Z}_\mu$ are defined in (2.9) and (2.10), respectively. Then $\psi^\perp$ satisfies

$$\Delta\psi^\perp + f'_\varepsilon(U_\mu + \phi)\psi^\perp - \varepsilon^2\psi^\perp - \alpha_1(f'_\varepsilon(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu) = 0 \quad \text{in } \mathbb{R}^3.$$

Multiplying this equation by $\tilde{Z}_\mu$ and integrating, we obtain

$$\alpha_1 \int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu)\tilde{Z}_\mu = \int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu)\psi^\perp. \tag{4.17}$$

We want to estimate the integral

$$I = \int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu)\tilde{Z}_\mu. \tag{4.18}$$

Let us define the energy of the ansatz as

$$J(\mu) = E(U_\mu) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla U_\mu|^2 - F_\varepsilon(U_\mu) + \frac{\varepsilon^2}{2}U_\mu^2,$$

and let us compute

$$J'(\mu) = -\int_{\mathbb{R}^3} (\Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu)\tilde{Z}_\mu,$$

$$J''(\mu) = -\int_{\mathbb{R}^3} (\Delta \tilde{Z}_\mu + f'_\varepsilon(U_\mu)\tilde{Z}_\mu - \varepsilon^2 \tilde{Z}_\mu)\tilde{Z}_\mu - \int_{\mathbb{R}^3} (\Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu)\frac{\partial \tilde{Z}_\mu}{\partial \mu}.$$

Differentiating (2.2) with respect to μ yields

$$\begin{cases} \Delta \tilde{Z}_\mu - \varepsilon^2 \tilde{Z}_\mu = -5w_\mu^4 Z_\mu \quad \text{in } \mathbb{R}^3, \\ \tilde{Z}_\mu(y) \to 0 \quad \text{as } |y| \longrightarrow \infty, \end{cases} \tag{4.19}$$

so

$$\begin{aligned} I &= \int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu + \phi)\tilde{Z}_\mu + \Delta\tilde{Z}_\mu - \varepsilon^2 \tilde{Z}_\mu)\tilde{Z}_\mu \\ &= -J''(\mu) + \int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu + \phi) - f'_\varepsilon(U_\mu))\tilde{Z}_\mu^2 - \int_{\mathbb{R}^3} (\Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu)\frac{\partial \tilde{Z}_\mu}{\partial \mu} \\ &= -J''(\mu) + \int_{\mathbb{R}^3} 20 w_\mu^3 \phi Z_\mu^2 + \int_{\mathbb{R}^3} [(f'_\varepsilon(U_\mu + \phi) - f'_\varepsilon(U_\mu))\tilde{Z}_\mu^2 - 20 w_\mu^3 \phi Z_\mu^2] \\ &\quad - \int_{\mathbb{R}^3} (\Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu)\frac{\partial \tilde{Z}_\mu}{\partial \mu}. \end{aligned}$$

But differentiating (4.19) with respect to μ gives

$$\Delta \frac{\partial \tilde{Z}_\mu}{\partial \mu} - \varepsilon^2 \frac{\partial \tilde{Z}_\mu}{\partial \mu} + 20 w_\mu^3 Z_\mu^2 + 5 w_\mu^4 \frac{\partial Z_\mu}{\partial \mu} = 0.$$

Multiplying this equation by ϕ_μ, integrating and evaluating at $\mu = \mu_\varepsilon$, so that $c = 0$ in equation (4.1), we find

$$\int_{\mathbb{R}^3} (f'_\varepsilon(U_\mu)\phi + N(\phi) + R)\frac{\partial \tilde{Z}_\mu}{\partial \mu} = \int_{\mathbb{R}^3} \left(20 w_\mu^3 Z_\mu^2 + 5 w_\mu^4 \frac{\partial Z_\mu}{\partial \mu}\right)\phi.$$

We solve from here $\int_{\mathbb{R}^3} 20 w_\mu^3 Z_\mu^2 \phi$ and replace it in the formula for I, recalling that $R = \Delta U_\mu + f_\varepsilon(U_\mu) - \varepsilon^2 U_\mu$:

$$\begin{aligned} I = -J''(\mu_\varepsilon) &+ \int_{\mathbb{R}^3} \left(f'_\varepsilon(U_\mu)\frac{\partial \tilde{Z}_\mu}{\partial \mu} - 5 w_\mu^4 \frac{\partial Z_\mu}{\partial \mu}\right)\phi + \int_{\mathbb{R}^3} N(\phi)\frac{\partial \tilde{Z}_\mu}{\partial \mu} \\ &+ \int_{\mathbb{R}^3} [(f'_\varepsilon(U_\mu + \phi) - f'_\varepsilon(U_\mu))\tilde{Z}_\mu^2 - 20 w_\mu^3 \phi Z_\mu^2]. \end{aligned}$$

We need to show that all terms in RHS of the above expression, except $F''(\mu)$, are $o(\varepsilon)$ as $\varepsilon \to 0$. We start estimating

$$A := \int_{\mathbb{R}^3} [(f'_\varepsilon(U_\mu + \phi) - f'_\varepsilon(U_\mu))\tilde{Z}_\mu^2 - 20 w_\mu^3 \phi Z_\mu^2] = A_1 + A_2 + A_3,$$

where

$$\begin{aligned} A_1 &= \int_{\mathbb{R}^3} p((U_\mu + \phi)^{p-1} - U_\mu^{p-1} - (p-1)U_\mu^{p-2}\phi)\tilde{Z}_\mu^2, \\ A_2 &= \int_{\mathbb{R}^3} p(p-1)U_\mu^{p-2}\phi \tilde{Z}_\mu^2 - 20 w_\mu^3 \phi Z_\mu^2, \\ A_3 &= \lambda \varepsilon^\alpha \int_{\mathbb{R}^3} (q(U_\mu + \phi)^{q-1} - q U_\mu^{q-1})\tilde{Z}_\mu^2. \end{aligned}$$

Let us estimate A_3. In the case $1 < q < 2$, we estimate $|(U_\mu + \phi)^{q-1} - U_\mu^{q-1}| \leqslant C|\phi|^{q-1}$. Using that $|\phi(r)| \leqslant \varepsilon^{-2} r^{-\theta} \|\phi\|_* \leqslant C\varepsilon^{-1} r^{-\theta}$ for $r \geqslant 1/\varepsilon$ and (2.13), we estimate

$$\varepsilon^\alpha \int_{r \geqslant 1/\varepsilon} |(U_\mu + \phi)^{q-1} - U_\mu^{q-1}| \tilde{Z}_\mu^2 \leqslant C\varepsilon^\alpha \int_{r=1/\varepsilon}^\infty (\varepsilon^{-1} r^{-\theta})^{q-1} (\varepsilon^{-4} r^{-5})^2 r^2 \, dr$$
$$\leqslant C\varepsilon^{\alpha + (\theta-1)(q-1)-1}.$$

Since α has the form (1.8) and $q > 1$, we see that $\alpha + (\theta - 1)(q - 1) - 1 > 1$ for $\varepsilon > 0$ small. Also, since $|\phi(r)| \leqslant C\varepsilon(1 + r)^{2-\theta}$ and $|\tilde{Z}_\mu(r)| \leqslant (1 + r)^{-1}$ for $r \leqslant 1/\varepsilon$,

$$C\varepsilon^\alpha \int_{r \leqslant 1/\varepsilon} |(U_\mu + \phi)^{q-1} - U_\mu^{q-1}| \tilde{Z}_\mu^2 \leqslant \varepsilon^\alpha \int_0^\infty (\varepsilon(1 + r)^{2-\theta})^{q-1} (1 + r)^{-2} r^2 \, dr.$$

Therefore, $A_3 = o(\varepsilon)$ as $\varepsilon \to 0$. Similarly, it is possible to verify that $A_1 = o(\varepsilon)$, $A_2 = o(\varepsilon)$ as $\varepsilon \to 0$.

It follows that

$$I = -J''(\mu_\varepsilon) + o(\varepsilon), \quad \text{as } \varepsilon \to 0. \tag{4.20}$$

We estimate the right-hand side of (4.17)

$$\int_{\mathbb{R}^3} |(f_\varepsilon'(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu)\psi^\perp| \leqslant C\varepsilon \|\psi^\perp\|_*. \tag{4.21}$$

We observe that $\psi^\perp$ satisfies (4.16) and (4.15) because $\int_{\mathbb{R}^3} \tilde{Z}_\mu (\partial w_\mu / \partial x_j) = 0$. Therefore, we may apply Lemma 3.3 and obtain

$$\|\psi^\perp\|_* \leqslant C\|\alpha_1(f_\varepsilon'(U_\mu + \phi)\tilde{Z}_\mu - 5w_\mu^4 Z_\mu)\|_{**} \leqslant C\varepsilon|\alpha_1|. \tag{4.22}$$

Combining (4.17) and (4.20)–(4.22), we find

$$|\alpha_1(-J''(\mu_\varepsilon) + o(\varepsilon))| \leqslant C\varepsilon^2 |\alpha_1|.$$

Since $J''(\mu_\varepsilon) = \Psi''(\mu_\varepsilon)\varepsilon + o(\varepsilon)$ as $\varepsilon \to 0$, and $\Psi''(\mu_\varepsilon) \neq 0$, we deduce from this that $\alpha_1 = 0$. This implies that $\psi = \psi^\perp$ and from (4.22) we obtain that $\psi = 0$, which is the desired non-degeneracy of the solution u_ε.

We comment here on the claim that u_ε has Morse index equal 1. By Morse index, we mean the largest integer k such that there is a subspace $N \subset C_0^\infty(\mathbb{R}^3)$ of dimension k on which the quadratic form

$$Q(\varphi) = \int_{\mathbb{R}^3} |\nabla\varphi|^2 + \varepsilon^2\varphi^2 - pu_\varepsilon^{p-1}\varphi^2 - \lambda q\varepsilon^\alpha u_\varepsilon^{q-1}\varphi^2$$

is negative definite.

It is convenient to introduce the eigenvalue problem

$$\Delta\psi + f_\varepsilon'(u_\varepsilon)\psi - \varepsilon^2\psi + \nu w_\mu^4 \psi = 0 \quad \text{in } \mathbb{R}^3 \tag{4.23}$$

with $\psi \in H^1(\mathbb{R}^3)$. Owing to the weight w_μ^4, the embedding from $H^1(\mathbb{R}^3)$ to $L^2(w_\mu^4\, dx)$ is compact and the theory provides a sequence of eigenvalues $\nu_{j,\varepsilon} \to \infty$ as $j \to \infty$ with associated eigenfunctions $\psi_{j,\varepsilon} \in H^1(\mathbb{R}^3)$. These eigenvalues can be obtained variationally

$$\nu_{j,\varepsilon} = \inf\left\{ \frac{Q(\varphi)}{\int_{\mathbb{R}^3} w_\mu^4 \varphi^2} : \varphi \in C_0^\infty(\mathbb{R}^3),\ \langle\varphi, \psi_{i,\varepsilon}\rangle = 0,\ i = 1, \ldots, j-1 \right\},$$

where $\langle\varphi_1, \varphi_2\rangle = \int_{\mathbb{R}^3} \varphi_1\varphi_2 w_\mu^4$. Then the Morse index of u_ε is the same as the number of negative eigenvalues of (4.23).

The limit eigenvalue problem

$$\Delta\psi + 5w_\mu^4\psi + \nu w_\mu^4\psi = 0 \quad \text{in } \mathbb{R}^3$$

is known to have a negative eigenvalue $\nu_1 = -4$ with associated eigenfunction $\psi_1 = w_\mu$. The second eigenvalue is 0 with eigenfunctions given by Z_μ and $\partial w_\mu/\partial x_i$, $i = 1, 2, 3$.

The eigenvalue $\nu_{1,\varepsilon}$ is simple, and the eigenfunction is radial, has exponential decay and converges as $\varepsilon \to 0$ (after normalization) to a multiple of ψ_1. Also $\nu_{1,\varepsilon} \to \nu_1$ as $\varepsilon \to 0$.

Now suppose that ψ_ε is an eigenfunction with eigenvalue $\nu_\varepsilon < 0$, $\nu_\varepsilon \neq \nu_{1,\varepsilon}$. Let us consider first the case that ν_ε stays away from zero. Then one can prove that ψ_ε converges, after normalizing $\|\psi_\varepsilon\|_{L^2} = 1$, to an eigenfunction ψ associated to a negative eigenvalue $\nu < 0$. The case $\nu = \nu_1$ can be discarded because ψ is $L^2(w_\mu^4\,dx)$ orthogonal to ψ_1, since ψ_ε is $L^2(w_\mu^4\,dx)$ orthogonal to $\psi_{1,\varepsilon}$. The case $\nu_1 < \nu < 0$ can be discarded because the limit eigenvalue problem has only one negative eigenvalue.

In the case $\nu_\varepsilon \to 0$ as $\varepsilon \to 0$, we argue as follows. We define

$$\tilde\psi_\varepsilon = \psi_\varepsilon - \sum_{i=1}^{3} c_{i,\varepsilon}\frac{\partial u_\varepsilon}{\partial x_i}$$

with $c_{i,\varepsilon}$ chosen so that (4.15) holds for $\tilde\psi_\varepsilon$. Note that

$$|c_{i,\varepsilon}| \leqslant C\|\psi_\varepsilon\|_*. \tag{4.24}$$

We write $\tilde\psi_\varepsilon = \psi_\varepsilon^\perp - \alpha_1\tilde Z_\mu$ so that (4.16) holds for $\psi_\varepsilon^\perp$. Observe that $\psi_\varepsilon^\perp$ also satisfies (4.15). We compute

$$\begin{aligned}\Delta\psi_\varepsilon^\perp + f_\varepsilon'(u_\varepsilon)\psi_\varepsilon^\perp - \varepsilon^2\psi_\varepsilon^\perp + \nu_\varepsilon w_\mu^4\psi_\varepsilon^\perp + \nu_\varepsilon\sum_{i=1}^{3} c_{i,\varepsilon} w_\mu^4\frac{\partial u_\varepsilon}{\partial x_i}&\\ = \alpha_1(f_\varepsilon'(u_\varepsilon)\tilde Z_\mu - 5w_\mu^4 Z_\mu + \nu_\varepsilon w_\mu^4\tilde Z_\mu)&\end{aligned} \tag{4.25}$$

in $\mathbb{R}^3$. We multiply this equation by $\tilde Z_\mu$ and obtain,

$$\alpha_1\left(I + \nu_\varepsilon\int_{\mathbb{R}^3} w_\mu^4\tilde Z_\mu^2\right) = \int_{\mathbb{R}^3}(f_\varepsilon'(u_\varepsilon)\tilde Z_\mu - 5w_\mu^4 Z_\mu)\psi_\varepsilon^\perp + \nu_\varepsilon\int_{\mathbb{R}^3} w_\mu^4\psi_\varepsilon^\perp\tilde Z_\mu,$$

where I is the integral (4.18). Owing to (4.20), we find

$$\alpha_1\left(-J''(\mu_\varepsilon) + o(\varepsilon) + \nu_\varepsilon\int_{\mathbb{R}^3} w_\mu^4\tilde Z_\mu^2\right) \leqslant C(\varepsilon + |\nu_\varepsilon|)\|\psi_\varepsilon^\perp\|_*.$$

Using Lemma 3.3 and equation (4.25), we obtain

$$\|\psi_\varepsilon^\perp\|_* \leqslant C\|\alpha_1(f_\varepsilon'(u_\varepsilon)\tilde Z_\mu - 5w_\mu^4 Z_\mu + \nu_\varepsilon w_\mu^4\tilde Z_\mu)\|_{**} + C\left\|\sum_{i=1}^{3} c_{i,\varepsilon}\nu_\varepsilon w_\mu^4\frac{\partial u_\varepsilon}{\partial x_i}\right\|_{**}. \tag{4.26}$$

As in (4.22),

$$\|\alpha_1(f_\varepsilon'(u_\varepsilon)\tilde Z_\mu - 5w_\mu^4 Z_\mu + \nu_\varepsilon w_\mu^4\tilde Z_\mu)\|_{**} \leqslant C|\alpha_1|(\varepsilon + |\nu_\varepsilon|).$$

Therefore, (4.26) and (4.24) yield

$$\|\psi_\varepsilon^\perp\|_* \leqslant C|\alpha_1|(\varepsilon + |\nu_\varepsilon|) + C|\nu_\varepsilon|\|\psi_\varepsilon^\perp\|_*.$$

Then for $\varepsilon > 0$ small we obtain

$$\|\psi_\varepsilon^\perp\|_* \leqslant C|\alpha_1|(\varepsilon + |\nu_\varepsilon|). \tag{4.27}$$

Therefore,

$$\alpha_1\left(-J''(\mu_\varepsilon) + o(\varepsilon) + \nu_\varepsilon\int_{\mathbb{R}^3} w_\mu^4\tilde Z_\mu^2\right) \leqslant C(\varepsilon + |\nu_\varepsilon|)^2|\alpha_1|.$$

Since $J''(\mu_\varepsilon) = \Psi''(\mu_\varepsilon)\varepsilon + o(\varepsilon)$ as $\varepsilon \to 0$, $\Psi''(\mu_\varepsilon) > 0$, and $\nu_\varepsilon < 0$, $\nu_\varepsilon \to 0$ as $\varepsilon \to 0$, we can conclude that $\alpha_1 = 0$ for $\varepsilon > 0$ small. This implies that $\tilde{\psi}_\varepsilon = \psi_\varepsilon^\perp$ and then (4.27) implies that $\tilde{\psi}_\varepsilon = 0$, which gives that ψ_ε is a linear combination of the functions $\partial u_\varepsilon/\partial x_i$. But $\nu_\varepsilon < 0$ and this implies $c_{i,\varepsilon} = 0$, so $\psi_\varepsilon = 0$, which is a contradiction.

5. *Proof of Theorem* 1.1

The proof of Theorem 1.1 is based on Theorem 1.2 which provides a large solution and Lemma 5.1 that gives the existence of a small solution. A degree-theoretical argument will give the third solution.

We start with the existence of small solutions (for λ large). This solution can also be constructed by a shooting argument, using the results in [**11**].

LEMMA 5.1. *Fix $1 < q < 5$ and consider $p \in [p_1, p_2]$, where $1 < p_1 < p_2$ are fixed (here p_2 need not be subcritical). Then there exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$,* (1.3) *has a solution u_λ, which depends continuously on λ and satisfies $\|u\|_{L^\infty} \leqslant C\lambda^{-1/(q-1)}$.*

Proof. By the change of variables $u(x) = \lambda^{-1/(q-1)}v(x)$, problem (1.3) gets rewritten as

$$\begin{cases} \Delta v + \lambda^{-\gamma}v^p + v^q - v = 0, & v > 0 \text{ in } \mathbb{R}^3, \\ v(x) \longrightarrow 0 \quad \text{as } |x| \to +\infty, \end{cases} \tag{5.1}$$

where $\gamma = (p-1)/(q-1) > 0$. Let $v_0 \in H^1(\mathbb{R}^3)$ be the unique radially symmetric solution of

$$\Delta v + v^q - v = 0, \quad v > 0 \text{ in } \mathbb{R}^3. \tag{5.2}$$

We look then for a solution of (5.1) of the form $v = v_0 + \phi$. Then equation (5.1) becomes

$$L\phi + N_1(\phi) + N_2(\phi) = 0 \quad \text{in } \mathbb{R}^3,$$

where

$$L\phi = \Delta\phi + qv_0^{p-1}\phi - \phi,$$
$$N_1(\phi) = \lambda^{-\gamma}(v_0 + \phi)_+^p, \quad N_2(\phi) = (v_0 + \phi)_+^q - v_0^q - qv_0^{q-1}\phi.$$

Problem (5.1) can be solved by the contraction mapping theorem in the space E of radial continuous functions $\phi : \mathbb{R}^3 \to \mathbb{R}$, with the norm

$$\|\phi\|_\sigma = \sup_{x\in\mathbb{R}^3} e^{\sigma|x|}|\phi(x)|,$$

where $\sigma > 0$ is fixed and small. Using the non-degeneracy of v_0, see [**20**, Appendix C] and also [**15**, **16**], it can be shown that L is invertible from E to E. We look for a solution ϕ of

$$\phi = L^{-1}(N_1(\phi) + N_2(\phi)).$$

In fact, we have

$$\|N_1(\phi_1) - N_1(\phi_2)\|_\sigma \leqslant \lambda^{-\gamma}\|\phi_1 - \phi_2\|_\sigma, \quad \|N_1(0)\|_\sigma \leqslant C\lambda^{-\gamma}$$

and

$$\|N_2(\phi_1) - N_2(\phi_2)\|_\sigma \leqslant \lambda^{-\min\{q-1,1\}\gamma}\|\phi_1 - \phi_2\|_\sigma.$$

Then we find a unique solution $\phi \in E$ with $\|\phi\|_\sigma \leqslant A\lambda^{-\gamma}$ and A large. □

Next we compute the total degree of the solutions of (1.3). For this purpose, we introduce the operator

$$T(u) = G * (u_+^p + \lambda u_+^q)$$

for $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : u \text{ is radial}\}$, where G is the Green function defined in (2.6). Fixed points of T in $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ are automatically solutions of (1.3).

We can write $T = G * A(u)$, where $A(u) = u_+^p + \lambda u_+^q$. By the lemma of Strauss [**28**], $A : H^1_{\mathrm{rad}}(\mathbb{R}^3) \to L^6_{\mathrm{rad}}(\mathbb{R}^3)$ is completely continuous. Since G is C^∞ with exponential decay, $u \in L^6_{\mathrm{rad}}(\mathbb{R}^3) \mapsto G * u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ is a bounded linear operator, and we get that $T : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is completely continuous.

For $1 < q < p < 5$, there is an apriori bound for solutions of (1.3), that is, there is $R > 0$ such that for any solution u of (1.3) we have

$$\|u\|_{H^1(\mathbb{R}^3)} < R. \tag{5.3}$$

Indeed, using a blow-up argument and the non-existence result of Gidas and Spruck [**14**], there exists R such that for any solution u of (1.3) satisfies

$$\|u\|_{L^\infty(\mathbb{R}^3)} = u(0) \leqslant R.$$

Then a barrier argument gives

$$u(x) \leqslant Ce^{-c|x|} \quad \text{for all } x \in \mathbb{R}^3$$

for some $c > 0$ (see, for example, [**6**]). This implies the apriori estimate (5.3). Moreover, this estimate is uniform for bounded λ.

Then, for $R > 0$ large enough, the Leray–Schauder degree $\deg(I - T, B_R(0), 0)$ is well defined.

LEMMA 5.2. *For all* $\lambda \geqslant 0$, *if* $R > 0$ *is large, then* $\deg(I - T, B_R(0), 0) = 0$.

Proof. We introduce a family of operators $T_t : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ defined by

$$T_t(u) = G * ((tg + u_+)^p + u_+^q),$$

where $t \geqslant 0$ and $g(x) \geqslant 0$, is a radial C^∞ function with compact support such that $g = 1$ in the unit ball $B_1(0)$. The same argument that leads to the apriori estimate (5.3) shows that for any $L > 0$, there exists $R > 0$ such that for any $t \in [0, L]$ and any fixed point $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ of T_t we have

$$\|u\|_{H^1(\mathbb{R}^3)} < R.$$

Then by homotopy invariance of the degree,

$$\deg(I - T_0, B_R(0), 0) = \deg(I - T_L, B_R(0), 0).$$

We claim, that the above total degree is zero if L is large, which we can prove by showing that T_L as no fixed points. Suppose to the contrary that T_L has a fixed point $u \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$. Then u solves

$$\Delta u + (u + Lg(x))^p + \lambda u^q - u = 0 \quad \text{in } \mathbb{R}^3, \tag{5.4}$$

and decays to zero exponentially as $|x| \to +\infty$.

Let $\varphi_1 \in H^1(\mathbb{R}^3)$, $\varphi > 0$ be the principal eigenfunction of

$$-\Delta\varphi + \varphi = \mu\tilde{g}\varphi \quad \text{in } \mathbb{R}^3,$$

where $\tilde{g} \geqslant 0$ is a smooth non-trivial function with compact support in the unit ball. The existence of this principal eigenfunction associated to an eigenvalue $\mu > 0$ can be found in [**19**]. We normalize the eigenfunction φ so that $\varphi(0) = 1$, and note that it decays exponentially to zero. Multiplying (5.4) by φ and integrating in $\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} (u + Lg)^p \varphi + \lambda u^q \varphi = \mu \int_{\mathbb{R}^3} \tilde{g} u \varphi.$$

If we choose L large enough, then we have

$$(u + L)^p \geqslant \mu \|\tilde{g}\|_{L^\infty} u + 1 \quad \forall u \geqslant 0,$$

and therefore, $(u + Lg)^p + \lambda u^q \geqslant \mu \tilde{g} u + 1$ in $B_0(1)$. This yields

$$\int_{\mathbb{R}^3} \varphi \leqslant 0,$$

which is impossible, and we conclude that (5.4), has no solutions. □

Lemma 5.3. *Fix $1 < q < p < 5$. Then for all λ sufficiently large (depending on p, q), (1.6) has a unique radial solution.*

Proof. We proceed by contradiction. Suppose that for a sequence $\lambda_n \to +\infty$, there are two different radial solutions $v_{1,n}$, $v_{2,n}$ of (5.1). Using a blow-up argument, we can show that $v_{1,n}$, $v_{2,n}$ remain uniformly bounded in $\mathbb{R}^3$, and then that they converge uniformly on compact sets to the unique radially symmetric solution v_0 of (5.2).

Let

$$w_n = \frac{v_{1,n} - v_{2,n}}{\|v_{1,n} - v_{2,n}\|_{L^\infty(\mathbb{R}^3)}}.$$

Then w_n satisfies

$$\Delta w_n + \lambda_n^{-\gamma} A_n(x) w_n + B_n(x) w_n - w_n = 0 \quad \text{in } \mathbb{R}^3,$$

where

$$A_n = \frac{v_{1,n}^p - v_{2,n}^p}{v_{1,n} - v_{2,n}}, \quad B_n = \frac{v_{1,n}^q - v_{2,n}^q}{v_{1,n} - v_{2,n}}.$$

Using a barrier, we get $|w_n(x)| \leqslant Ce^{-\delta|x|}$ for some constants $C, \delta > 0$ and all large n. Therefore, there is some $x_n \in \mathbb{R}^3$ such that $|w_n(x_n)| = 1$, and x_n remains bounded. By elliptic regularity, up to subsequence $w_n \to w$ uniformly on compact sets, and w is bounded and satisfies

$$\Delta w + q v_0^{q-1} w - w = 0 \quad \text{in } \mathbb{R}^3.$$

By the non-degeneracy of v_0, deduce that $w \equiv 0$ (see [**2**, p. 47]). But also up to subsequence $x_n \to x_0$ and hence $|w(x_0)| = 1$, which yields a contradiction. □

Proof of Theorem 1.1. Let λ_0 be as in Lemma 5.2. The solution u_λ of (1.6) constructed in that lemma for $\lambda \geqslant \lambda_0$ is continuous with respect to λ, and is also isolated in the space E in that lemma. By elliptic regularity, it is isolated also in $H^1_{\text{rad}}(\mathbb{R}^3)$. Therefore, the local degree of T around u_λ is well defined. But for $\lambda > 0$ very large the total degree is zero, there is uniqueness of non-trivial solutions, and the zero solution has local degree 1. Therefore, the local degree of T around u_λ is -1 for all $\lambda \geqslant \lambda_0$.

By Theorem 1.2, for any $\bar{\lambda} > 0$ there exists $\bar{\varepsilon} > 0$ such that for $0 < \varepsilon \leqslant \bar{\varepsilon}$ and $0 \leqslant \lambda \leqslant \bar{\lambda}$ there exists a solution $U_{\lambda,\varepsilon}$ of (1.6) of the form (1.9). In particular,

$$U_{\varepsilon,\lambda}(0) = C\varepsilon^{-1/2}(1 + o(1)) \tag{5.5}$$

as $\varepsilon \to 0$, and this is uniform for $0 \leqslant \lambda \leqslant \bar{\lambda}$. Moreover, this solution is non-degenerate in the space of radial functions by Theorem 1.2.

Fix $\bar\lambda > \lambda_0$ and $\varepsilon > 0$ small, and let $\lambda_0 \leqslant \lambda \leqslant \bar\lambda$. We note that $U_{\varepsilon,\lambda} \neq u_\lambda$ because $\|u\|_{L^\infty} \leqslant C\lambda^{-1/(q-1)}$ and (5.5). Since

$$\deg(I - T_\lambda, B_R(0), 0) = 0$$

for $\lambda_0 \leqslant \lambda \leqslant \bar\lambda$ (R is fixed large), $U_{\varepsilon,\lambda}$ is non-degenerate and the local degrees of u_λ and 0 are -1,1, respectively, by degree theory we conclude that there exists a third solution of (1.6). □

6. *Three solutions*

In this section, we sketch the proof of Theorem 1.3 in the case $2 < q < 3$. The case $q = 2$ is analogous.

We look for solution of problem

$$(P_\varepsilon) \quad \begin{cases} \Delta u + u^{5-\varepsilon} + \bar\lambda\varepsilon^{-(3-q)/2}u^q - u = 0 & \text{in } \mathbb{R}^3, \\ u > 0 \quad \text{in } \mathbb{R}^3 & u \text{ in } H^1(\mathbb{R}^3). \end{cases}$$

By the rescaling in Section 2, we obtain

$$\begin{cases} \Delta u + u^{5-\varepsilon} + \bar\lambda\varepsilon^{\bar\alpha}u^q - \varepsilon^2 u = 0, & u > 0 \text{ in } \mathbb{R}^3, \\ u(y) \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty, \end{cases} \tag{6.1}$$

where

$$\bar\alpha = 1 - \frac{\varepsilon(q-1)}{2(4-\varepsilon)}.$$

To prove Theorem 1.3, we follow the proof of Theorem 1.2. For that, we need to study the solvability of the linear problem (3.8) with $\lambda = \bar\lambda$ and $\alpha = \bar\alpha$. This is done in Lemma 3.3, for (3.8). Rewriting the proof of Lemma 3.3, now using $\lambda = \bar\lambda$ and $\alpha = \bar\alpha$, we obtain the result. For problem (6.1), we can prove the error estimates (4.2) and (4.3) as in Lemma 4.2, using that $2 < q < 3$, and $\lambda = \bar\lambda$ and $\alpha = \bar\alpha$. Note that now $\bar\lambda < \lambda_0$ in Lemma 4.2. The expansion of the energy is different and is given in the next lemma.

Lemma 6.1. *Assume $2 < q < 3$, $\bar\lambda > 0$ and $\delta > 0$ be fixed. Then there exist positive constants a_0, a_1, a_2, a_3, a_4 for such that $\delta < \mu < \delta^{-1}$*

$$E(U_\mu) = a_0 + \varepsilon\Psi(\mu) - a_2\varepsilon\log\varepsilon - a_3\varepsilon + \varepsilon\Theta_\varepsilon(\mu),$$

where

$$\Psi(\mu) = a_1\mu - \bar\lambda\mu^{(5-q)/2}a_4 - a_2\log\mu,$$

and $\Theta_\varepsilon(\mu) \to 0$ as $\varepsilon \to 0$ in the C^1 norm in the interval $\delta \leqslant \mu \leqslant \delta^{-1}$.

Combining the solvability of the linear problem, the error estimates and the above lemma, we can conclude the proof of Theorem 1.3. Note that Ψ has two non-degenerate critical points for each $0 < \bar\lambda < \lambda_0$. In fact,

$$\Psi'(\mu) = a_1 - \bar\lambda\frac{5-q}{2}\mu^{(3-q)/2}a_4 - a_2\mu^{-1}$$

is negative for small and large μ, and has a unique critical point that is a maximum. In this maximum point $\mu_{\max}$, the function $\Psi'(\mu_{\max})$ is positive if and only if $0 < \bar\lambda < \lambda_0$, where λ_0 is given by (1.11). In this case, the equation $\Psi'(\mu) = 0$ has two positive solutions $\mu^\pm(\bar\lambda)$ satisfying (1.12). Note that μ^- is a local minimum and μ^+ is a local maximum of $\Psi(\mu)$. Following the argument in Section 4, the solution u_ε^- has Morse index 1 and u_ε^+ has Morse index 2.

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Juan Dávila and Manuel del Pino
Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS)
Universidad de Chile
Casilla 170 Correo 3
8370459 Santiago
Chile

jdavila@dim.uchile.cl
delpino@dim.uchile.cl

Ignacio Guerra
Departamento de Matemática y Ciencia de la Computacion
Universidad de Santiago
9170125 Santiago
Chile

ignacio.guerra@usach.cl