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On a new class of electro-elastic bodies. II. Boundary value problems

R. Bustamante[1] and K. R. Rajagopal[2]

[1]Departamento de Ingeniería Mecánica, Universidad de Chile Beaucheff 850, Santiago Centro, Santiago, Chile

[2]Department of Mechanical Engineering, University of Texas A&M, College Station, TX 77843-3123, USA

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Author for correspondence:
R. Bustamante
e-mail: rogbusta@ing.uchile.cl

In part I of this two-part paper, a new theoretical framework was presented to describe the response of electro-elastic bodies. The constitutive theory that was developed consists of two implicit constitutive relations: one that relates the stress, stretch and the electric field, and the other that relates the stress, the electric field and the electric displacement field. In part II, several boundary value problems are studied within the context of such a construct. The governing equations allow for nonlinear coupling between the electric and stress fields. We consider boundary value problems wherein both homogeneous and inhomogeneous deformations are considered, with the body subject to an electric field. First, the extension and the shear of an electro-elastic slab subject to an electric field are studied. This is followed by a study of the problem of a thin circular plate and a long cylindrical tube, both subject to an inhomogeneous deformation and an electric field. In all the boundary value problems considered, the relationships between the stress and the linearized strain are nonlinear, in addition to the nonlinear relation to the electric field. It is emphasized that the theories that are currently available are incapable of modelling such nonlinear relations.

1. Introduction

In part I of this two-part paper [1], we extended the implicit constitutive theory proposed by Rajagopal [2–4] for describing the response of elastic bodies that lead to models that are neither Cauchy elastic nor Green elastic (see Truesdell & Noll [5] for a definition of the same), to the electro-elastic response of materials. This generalization involves two sets of implicit constitutive

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relations, one between the stress, the Cauchy–Green tensor and the electric field and an implicit relation between the stress, the electric field and the electric displacement field. The theory developed was also restricted to a simplified form of Maxwell's equations. Unlike the studies of Rajagopal & Srinivasa [6,7] that consider implicit constitutive relations for elastic bodies within the context of a thermodynamic framework, our generalization to electro-elasticity was not within the context of a fully thermodynamic framework but was restricted to mechanical, electrical and magnetic effects. After developing implicit constitutive relations that are capable of describing large deformations, we obtained approximations wherein the displacement gradient and the electric displacement are assumed to be small; the constitutive relations, however, yet being nonlinear. As discussed in part I, such constitutive relations are capable of describing the nonlinear response that is observed in piezoelectric bodies, which the classical small displacement gradient theories are incapable of describing.

A special subclass of the fully implicit constitutive relations is one wherein explicit constitutive relations are prescribed for the Cauchy–Green stretch in terms of the stress, the electric field and the electric displacement vector. This constitutive relation can also be approximated by assuming that the displacement gradient and the electric displacement vector are appropriately small. This part of the paper is devoted to the study of boundary value problems corresponding to both homogeneous and inhomogeneous states of stress, within the context of such an approximation. We first consider the homogeneous state of stress of a slab, first when subject to traction, and then when subject to shear, the slab being in the presence of an electric field in both problems. This is followed by the analysis of a thin circular plate which is inflated in the radial direction, wherein the state of stress is inhomogeneous. Finally, we study the inhomogeneous inflation of a long cylindrical annulus. In all the boundary value problems that were studied, the strains remain very small, though the stresses and the electrical field are large, and, more importantly, the relationship between the stress and the linearized strain is nonlinear. We cannot emphasize enough the fact that unlike the present theory, the theories that are currently in place are incapable of describing a nonlinear relationship between the linearized strain and the stress.

2. Basic equations

(a) Kinematics and the equations of electrostatics

Let $\mathbf{X} \in \kappa_R(\mathcal{B})$ denote a particle belonging to a body $\mathcal{B}$ in the reference configuration $\kappa_R(\mathcal{B})$, and let $\mathbf{x} \in \kappa_t(\mathcal{B})$ denote the position of the same particle in the current configuration $\kappa_t(\mathcal{B})$, at time t. We shall assume that the mapping $\boldsymbol{\chi}$, which assigns the position $\mathbf{x}$ at time t, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is sufficiently smooth so as to make all the derivatives that are taken, meaningful. The displacement $\mathbf{u}$, the deformation gradient $\mathbf{F}$, the Cauchy–Green stretch tensors $\mathbf{b}$ and $\mathbf{c}$ and the linearized strain $\boldsymbol{\varepsilon}$ are defined through

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right]. \tag{2.1}$$

More details concerning kinematics can be found in references [8,9]. In this paper, we are interested in studying quasi-static problems of electro-elastic bodies.

We denote by $\mathbf{E}$ and $\mathbf{D}$, respectively, the electric field and the electric displacement in the current configuration. The fields $\mathbf{E}$ and $\mathbf{D}$ satisfy a simplified form of Maxwell's equations in the absence of magnetic interactions, distributed charges and time dependence, namely

$$\operatorname{curl} \mathbf{E} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0. \tag{2.2}$$

In vacuum $\mathcal{B}'$, the following relation is valid:

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \tag{2.3}$$

where ϵ_0 is the electric permittivity in vacuum.

The polarization field $\mathbf{P}$ for condensed matter is defined as

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}. \tag{2.4}$$

Across a surface of discontinuity in the body or the boundary $\partial\kappa_t(\mathcal{B})$, considering there is no distribution of electric surface charges, the fields $\mathbf{E}$ and $\mathbf{D}$ have to satisfy the continuity conditions

$$\mathbf{n} \times [\![\mathbf{E}]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\mathbf{D}]\!] = 0, \tag{2.5}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\kappa_t(\mathcal{B})$. The double brackets represent the jump across the surface of discontinuity, for example, $[\![\mathbf{D}]\!] = \mathbf{D}^{\mathrm{o}} - \mathbf{D}^{\mathrm{i}}$, where $\mathbf{D}^{\mathrm{o}}$ and $\mathbf{D}^{\mathrm{i}}$ would be the electric displacements on either side of the boundary, respectively (evaluated very close to the surface of discontinuity). More detail about the theory of electromagnetism can be found, for example, in Kovetz [10].

There are different ways in which the equilibrium equations can be used when dealing with electromagnetic interactions [11]. A simple formulation is based on the use of a 'total stress' tensor $\boldsymbol{\tau}$, which incorporates in its definition a term related with the electric body forces [12]. This total stress tensor is symmetric and in the current configuration the equilibrium equation is of the form (in the absence of time dependence)

$$\operatorname{div} \boldsymbol{\tau} + \rho \mathbf{f} = \mathbf{0}. \tag{2.6}$$

The continuity condition across a surface of discontinuity of the surface of the body $\partial\kappa_t(\mathcal{B})$ in the current configuration is of the form [12,13]

$$[\![\boldsymbol{\tau}]\!]\mathbf{n} = \mathbf{0}, \tag{2.7}$$

where if $\mathbf{t}_{\mathrm{a}}$ is the mechanical traction per unit area, then the above condition implies that

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \boldsymbol{\tau}_{\mathrm{m}}\mathbf{n}, \tag{2.8}$$

where $\boldsymbol{\tau}_{\mathrm{m}}$ is the Maxwell stress due to the electric field outside the material near the boundary of the body [11]

$$\boldsymbol{\tau}_{\mathrm{m}} = \mathbf{D}^{\mathrm{o}} \otimes \mathbf{E}^{\mathrm{o}} - \tfrac{1}{2}(\mathbf{D}^{\mathrm{o}} \cdot \mathbf{E}^{\mathrm{o}})\mathbf{I}. \tag{2.9}$$

(b) Some new constitutive relations for electro-elastic bodies

We [1] proposed the following implicit relations to describe the response of electro-elastic bodies:

$$\mathfrak{f}(\boldsymbol{\tau}, \mathbf{b}, \mathbf{E}) = \mathbf{0}, \quad \mathfrak{l}(\boldsymbol{\tau}, \mathbf{E}, \mathbf{D}) = \mathbf{0}, \tag{2.10}$$

where $\mathfrak{f}$ is a tensor implicit relation and $\mathfrak{l}$ is a vector implicit relation. In this paper, we work with the subclass of (2.10), which is a consequence of a linearization based on $\|\nabla\mathbf{u}\| \sim O(\delta)$, $\delta \ll 1$. We consider the special subclass that takes the form

$$\boldsymbol{\varepsilon} = \hat{\mathfrak{f}}(\boldsymbol{\tau}, \mathbf{E}), \quad \mathfrak{l}(\boldsymbol{\tau}, \mathbf{E}, \mathbf{D}) = \mathbf{0}. \tag{2.11}$$

For isotropic functions $\hat{\mathfrak{f}}$ and $\mathfrak{l}$, (2.11) leads to (see §3.3 of Bustamante & Rajagopal [1])

$$\boldsymbol{\varepsilon} = \hat{\alpha}_0\mathbf{I} + \hat{\alpha}_1\boldsymbol{\tau} + \hat{\alpha}_2\boldsymbol{\tau}^2 + \hat{\alpha}_3\mathbf{E}\otimes\mathbf{E} + \hat{\alpha}_4(\mathbf{E}\otimes\boldsymbol{\tau}\mathbf{E} + \boldsymbol{\tau}\mathbf{E}\otimes\mathbf{E}) + \hat{\alpha}_5(\mathbf{E}\otimes\boldsymbol{\tau}^2\mathbf{E} + \boldsymbol{\tau}^2\mathbf{E}\otimes\mathbf{E}) \tag{2.12}$$

and

$$\beta_0\mathbf{E} + \beta_1\boldsymbol{\tau}\mathbf{E} + \beta_2\boldsymbol{\tau}^2\mathbf{E} + \beta_3\mathbf{D} + \beta_4\boldsymbol{\tau}\mathbf{D} + \beta_5\boldsymbol{\tau}^2\mathbf{D} + \beta_6[\boldsymbol{\tau}(\mathbf{E}\times\mathbf{D}) + (\boldsymbol{\tau}\mathbf{D})\times\mathbf{E})] = \mathbf{0}, \tag{2.13}$$

where $\hat{\alpha}_i$, $i = 0, \ldots, 5$ are scalar functions that depends on the invariants (see Spencer [14])

$$I_1 = \operatorname{tr}\boldsymbol{\tau}, \quad I_2 = \operatorname{tr}\boldsymbol{\tau}^2, \quad I_3 = \operatorname{tr}\boldsymbol{\tau}^3, \quad I_4 = \mathbf{E}\cdot\mathbf{E}, \quad I_5 = \mathbf{E}\cdot(\boldsymbol{\tau}\mathbf{E}), \quad I_6 = \mathbf{E}\cdot(\boldsymbol{\tau}^2\mathbf{E}) \tag{2.14}$$

and β_j, $j = 0, 1, \ldots, 6$ are scalar functions that depend on the invariants (2.14) and the invariants

$$I_7 = \mathbf{D} \cdot \mathbf{D}, \quad I_8 = \mathbf{D} \cdot (\boldsymbol{\tau}\mathbf{D}), \quad I_9 = \mathbf{D} \cdot (\boldsymbol{\tau}^2\mathbf{D}), \quad I_{10} = (\mathbf{D} \cdot \mathbf{E})^2, \quad I_{11} = [\mathbf{D} \cdot (\boldsymbol{\tau}\mathbf{E})]^2. \tag{2.15}$$

We shall also consider the approximation to (2.10) under the assumptions $\|\nabla\mathbf{u}\| \sim O(\delta)$ and $\|\mathbf{D}\| \sim O(\delta)$ with $\delta \ll 1$; in fact, we shall consider the special subclass of electro-elastic bodies defined by

$$\boldsymbol{\varepsilon} = \hat{\mathfrak{f}}(\boldsymbol{\tau}, \mathbf{E}), \quad \mathbf{D} = \hat{\mathfrak{l}}(\boldsymbol{\tau}, \mathbf{E}). \tag{2.16}$$

For isotropic functions $\hat{\mathfrak{f}}$ and $\hat{\mathfrak{l}}$, $(2.16)_1$ reduces to (2.12), whereas $(2.16)_2$ leads to

$$\mathbf{D} = \hat{\beta}_0\mathbf{E} + \hat{\beta}_1\boldsymbol{\tau}\mathbf{E} + \hat{\beta}_2\boldsymbol{\tau}^2\mathbf{E}. \tag{2.17}$$

The functions $\hat{\beta}_j$, $j = 0, 1, 2$ depend on the invariants (2.14).

In what follows, we solve some boundary value problems that are governed by equations (2.12), (2.13) and (2.12), (2.17) in order to determine the efficacy of such models. We first consider problems wherein the stress and the electrical field are homogeneous and we consider problems within the confines of the constitutive relations (2.12) and (2.13). We follow this with a study of a problem wherein the stress and the electrical field within the body are inhomogeneous, within the context of constitutive equations (2.12) and (2.17), where we assume that the functions $\hat{\mathfrak{f}}$ and $\hat{\mathfrak{l}}$ are isotropic, although in many applications involving electro-active bodies, it is necessary to work with anisotropic bodies. As outlined in part I, applications such as piezoelectricity demand the use of such anisotropic bodies. However, there is a dearth of experimental data against which one can corroborate the predictions of the theory, especially in reasonably simple geometries wherein one can solve initial boundary value problems, as the equations are nonlinear and quite complicated. Moreover, it is important to first solve problems in simple enough geometries to assess the usefulness of such models before embarking on studies of boundary value problems in complicated geometries.

(c) Boundary value problems

We shall first give a short account of some important issues concerning the solution of boundary value problems within the context of nonlinear electro-elasticity. Unlike classical problems in elasticity (or electro-elasticity), wherein the expression for the stress is substituted into the balance of linear momentum that leads to an equation for the displacement field (and the electric field), we now have the situation wherein the constitutive relation has to be solved simultaneously with the balance of linear momentum. Thus, the stress is also a primitive in this approach. The basic variables that we work with are: the total stress tensor $\boldsymbol{\tau}$, the electric field $\mathbf{E}$, the electric displacement $\mathbf{D}$, the linearized strain tensor $\boldsymbol{\varepsilon}$ and the displacement field $\mathbf{u}$. These quantities have to satisfy the implicit constitutive relations (2.11): $\boldsymbol{\varepsilon} = \hat{\mathfrak{f}}(\boldsymbol{\tau}, \mathbf{E})$, $\mathfrak{l}(\boldsymbol{\tau}, \mathbf{E}, \mathbf{D}) = \mathbf{0}$ (or $(2.16)_2$ $\mathbf{D} = \hat{\mathfrak{l}}(\boldsymbol{\tau}, \mathbf{E})$), the equilibrium equation (no mechanical body forces and time dependence) (2.6): $\operatorname{div} \boldsymbol{\tau} = \mathbf{0}$, the simplified form of the Maxwell equations (2.2) (considering no time dependence): $\operatorname{curl} \mathbf{E} = \mathbf{0}$, $\operatorname{div} \mathbf{D} = 0$ and the kinematical relation $(2.1)_5$, namely $\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}]$.

Let us assume that the electric field is expressed through a scalar electric potential φ in the following manner:

$$\mathbf{E} = -\operatorname{grad} \varphi. \tag{2.18}$$

Such a potential would automatically satisfy $(2.2)_1$. To summarize, we need to find $\boldsymbol{\varepsilon}$, $\boldsymbol{\tau}$, φ, $\mathbf{D}$ by solving the equations (2.11), (2.6), $(2.2)_2$ and $(2.1)_5$

$$\boldsymbol{\varepsilon} = \hat{\mathfrak{f}}(\boldsymbol{\tau}, \mathbf{E}), \quad \mathfrak{l}(\boldsymbol{\tau}, \mathbf{E}, \mathbf{D}) = \mathbf{0}, \quad \operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad \operatorname{div} \mathbf{D} = 0, \quad \boldsymbol{\varepsilon} = \tfrac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}], \tag{2.19}$$

and considering $(2.16)_2$ instead of $(2.11)_2$ in the case $\|\mathbf{D}\| \sim O(\delta)$ with $\delta \ll 1$.

There are six components each for the strain and total stress tensors, three components for the electric displacement and the displacement field, and one component for the electric scalar potential; therefore there are 19 unknowns. If we count the number of equations in (2.19), then there are six equations from $\hat{\mathfrak{f}}$, three for $\mathfrak{l}$ (or $\hat{\mathfrak{l}}$ in the case of $(2.16)_2$), three equilibrium equations for the stress, one equation for the electric displacement and six for the strain–displacement kinematics relation; therefore we have in total 19 equations, and so the problem is determinate.

An alternative method to solve the boundary value problem is to introduce a stress tensor potential, where (see equation (227.10) in §227 of Truesdell & Toupin [9])

$$\tau^{km} = \mathrm{e}^{krp}\mathrm{e}^{msq}a_{rs,pq}, \tag{2.20}$$

where e^{krp} is the permutation symbol and $a_{rs} = a_{sr}$ are the components of the stress tensor potential. In such a case, $(2.19)_3$ would be satisfied automatically. If we assume again that (2.18) holds, then we would need to find the six components of a_{rs}, the six components of $\boldsymbol{\varepsilon}$, the three components of $\mathbf{D}$ and the scalar electric potential φ; therefore, in this alternative case, we would need to find 16 unknowns. These unknowns should be found by solving $(2.19)_1$ (six equations), $(2.19)_2$ (three equations), $(2.19)_4$ (one equation) and $(2.19)_5$ would be replaced by the compatibility equations (only six are independent) [9]:

$$\mathcal{R}^{(\boldsymbol{\varepsilon})}_{kmpq} = 0, \tag{2.21}$$

where the components of the Riemann tensor $\mathcal{R}^{(\boldsymbol{\varepsilon})}_{kmpq}$ are defined in terms of $\boldsymbol{\varepsilon}$, for example in equation (34.2) of Truesdell & Toupin [9] (the compatibility equations are necessary in order to obtain a unique displacement field solving $(2.19)_5$); therefore, we have, in total, 16 unknowns and again the problem is determinate.

If we are interested in solving the boundary value problem without using (2.18), we would have to consider the 19 equations in (2.19) plus $(2.2)_1$, which has three components, so in total we would have 22 equations to be solved. However, with regard to the unknowns, we would have the 18 independent components of $\boldsymbol{\varepsilon}$, $\boldsymbol{\tau}$, $\mathbf{D}$ and $\mathbf{u}$, plus the three components of $\mathbf{E}$, so in total there are 21 unknowns, which would make the problem ill-conditioned (unless we use (2.18)). This is an interesting fact about the simplified form of the Maxwell equations used in this work; we would not run into such a difficulty when using the full system of Maxwell's equations and the balance of linear momentum; for a detailed discussion, see [15] in particular §3 therein.

With regard to the boundary conditions, in most of the works published in electro-elasticity, the investigators have considered only the body and not the surrounding space for the analysis, but Maxwell's equations (and its specializations (2.2)) have to be satisfied not only for the electro-elastic body under consideration, but also for the whole surrounding space (see the discussion in Kovetz [10]). It is important to recognize this aspect to the problem in order to obtain a meaningful solution; however, if one is only interested in the response of the electro-elastic body, we enforce the usual boundary conditions for the displacement field and the stress

$$\mathbf{u} = \tilde{\mathbf{u}}(\mathbf{x}) \quad \mathbf{x} \in \partial\kappa^u_t(\mathcal{B}), \quad \boldsymbol{\tau}\mathbf{n} = \tilde{\mathbf{t}}(\mathbf{x}) \quad \mathbf{x} \in \partial\kappa^t_t(\mathcal{B}),$$

and for the electric variables (assuming φ is the basic variable, in virtue of its clear physical meaning in electrostatics [10,16]):

$$\varphi = \tilde{\varphi}(\mathbf{x}) \quad \mathbf{x} \in \partial\kappa^{\varphi}_t(\mathcal{B}), \quad \mathbf{D}\cdot\mathbf{n} = \tilde{\mathbf{D}}(\mathbf{x})\cdot\mathbf{n} \quad \mathbf{x} \in \partial\kappa^D_t(\mathcal{B}),$$

where $\tilde{\mathbf{u}}$, $\tilde{\mathbf{t}}$, $\tilde{\varphi}$ and $\tilde{\mathbf{D}}$ are known fields on the boundary of the body in the current configuration $\kappa_t(\mathcal{B})$, where $\partial\kappa_t(\mathcal{B}) = \partial\kappa^u_t(\mathcal{B}) \cup \partial\kappa^t_t(\mathcal{B}) = \partial\kappa^{\varphi}_t(\mathcal{B}) \cup \partial\kappa^u_D(\mathcal{B})$, and $\partial\kappa^u_t(\mathcal{B}) \cap \partial\kappa^t_t(\mathcal{B}) = \varnothing$, $\partial\kappa^{\varphi}_t(\mathcal{B}) \cap \partial\kappa^D_t(\mathcal{B}) = \varnothing$. If we assume for simplicity that the body is in free vacuum, it is possible to show that the presence of the surrounding space could have an important impact on the distribution of electric field, and therefore in the deformation of the body, because of the continuity conditions (2.5), (2.7) (for the equivalent magnetoelastic problem see [17,18]). The effect of considering the exterior free space depends on the geometry of the body, and in particular on its electromechanical behaviour and the magnitude of the electric susceptibility in vacuum ϵ_0 [19].

We make two additional remarks before turning our attention to presenting solutions to some simple boundary value problems:

- the fulfillment of the continuity conditions (2.5) for 'finite' geometries is not easy to achieve. In the classical theory of nonlinear electro-elasticity, there is one exact solution, to the best of our knowledge, for a boundary value problem that takes into account (2.5) for electro-active bodies of finite size; almost all exact solutions that have been established thus far have been obtained assuming infinite long tubes, slabs and cylinders (see [20]), the only exception being the problem of inflation of a sphere, see §5 of [13];
- regarding the boundary conditions for the traction, we must recognize that mechanical surface traction can be applied only by the interaction with the surface of another external body (see the discussion in Bustamante [21]). In the case of the traction associated with Maxwell stresses that appears in (2.8) (see (2.9)), for the sake of simplicity, we will assume that such Maxwell stresses can be incorporated into the definition of the external traction (see the discussion in McMeeking & Landis [22]); and
- when the bodies are in empty space, we consider the Maxwell stresses (2.9) as external traction loads in (2.8) [23].

3. Homogeneous stresses and electrical field

Let us consider two simple problems wherein we can assume that we have a homogeneous distribution of the total stress and the electric field.

(a) Slab under traction

Let us consider a slab defined through

$$-\frac{L_1}{2} \leq x_1 \leq \frac{L_1}{2} \quad -\frac{L_2}{2} \leq x_2 \leq \frac{L_2}{2}, \quad -\frac{L_3}{2} \leq x_3 \leq \frac{L_3}{2}. \tag{3.1}$$

Because we work under the assumption of small gradient for the displacement field, we do not make a distinction between the current and the reference configurations; therefore, in this particular problem and in the three problems that are described later, the body is described by using the coordinates in the current configuration.

Let us assume that $L_3 \gg L_1$ and $L_2 \gg L_1$. Let us further assume that the stress and the electric field are of the form:[1]

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_{0_i} \mathbf{e}_i \otimes \mathbf{e}_i, \quad \mathbf{E} = E_0 \mathbf{e}_1, \tag{3.2}$$

where τ_{0_i}, $i = 1, 2, 3$ and E_0 are constants; therefore (2.19)$_3$ and (2.2)$_1$ are automatically satisfied. In this case, it follows from (2.12) that

$$\varepsilon_{11} = \hat{\alpha}_0 + \hat{\alpha}_1 \tau_{0_1} + \hat{\alpha}_2 \tau_{0_1}^2 + \hat{\alpha}_3 E_0^2 + 2\hat{\alpha}_4 E_0^2 \tau_{0_1} + 2\hat{\alpha}_5 E_0^2 \tau_{0_1}^2 \tag{3.3}$$

and

$$\varepsilon_{22} = \hat{\alpha}_0 + \hat{\alpha}_1 \tau_{0_2} + \hat{\alpha}_2 \tau_{0_2}^2, \quad \varepsilon_{33} = \hat{\alpha}_0 + \hat{\alpha}_1 \tau_{0_3} + \hat{\alpha}_2 \tau_{0_3}^2, \tag{3.4}$$

and $\varepsilon_{ij} = 0$, $i \neq j$, $i, j = 1, 2, 3$. The scalar functions $\hat{\alpha}_q$, $q = 0, 1, \ldots, 5$ depend on the invariants (2.14)

$$I_1 = \tau_{0_1} + \tau_{0_2} + \tau_{0_3}, \quad I_2 = \tau_{0_1}^2 + \tau_{0_2}^2 + \tau_{0_3}^2, \quad I_3 = \tau_{0_1}^3 + \tau_{0_2}^3 + \tau_{0_3}^3 \tag{3.5}$$

and

$$I_4 = E_0^2, \quad I_5 = E_0^2 \tau_{0_1}, \quad I_6 = E_0^2 \tau_{0_1}^2. \tag{3.6}$$

[1]For problems where we consider uniform distribution of electric field, it is not necessary to use (2.18) because (2.2)$_1$ is satisfied trivially.

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Because τ_{0_i}, $i=1,2,3$ are constant, and because in virtue of (3.3)–(3.6), we can also conclude that the components of the strain are constant, it follows that the compatibility equations (2.21) are satisfied and from $(2.19)_5$ a unique $\mathbf{u}$ can be calculated.

From (3.2) and (2.13), it cannot be said immediately that $\mathbf{D}$ has only a component in the x_1-direction; therefore, we shall assume that in general $\mathbf{D}$ is of the form

$$\mathbf{D}=D_i\mathbf{e}_i. \tag{3.7}$$

From (3.2) and (2.13), we obtain that

$$\beta_0E_0+\beta_1\tau_{0_1}E_0+\beta_2\tau_{0_1}^2E_0+\beta_3D_1+\beta_4\tau_{0_1}D_1+\beta_5\tau_{0_1}^2D_1=0, \tag{3.8}$$

$$(\beta_3+\beta_4\tau_{0_2}+\beta_5\tau_{0_2}^2)D_2+\beta_6\tau_{0_2}E_0D_3=0 \tag{3.9}$$

and

$$\beta_6(\tau_{0_3}-\tau_{0_2})E_0D_2+(\beta_3+\beta_4\tau_{0_3}+\beta_5\tau_{0_3}^2)D_3=0, \tag{3.10}$$

where the functions β_r, $r=0,1,2,\ldots,6$ depend on the invariants (3.5) and (3.6) and the invariants (2.15)

$$I_7=D_1^2,\quad I_8=D_1^2\tau_{0_1},\quad I_9=D_1^2\tau_{0_1}^2,\quad I_{10}=D_1^2E_0^2,\quad I_{11}=D_1^2\tau_{0_1}^2E_0^2. \tag{3.11}$$

We note that (3.9) and (3.10) can be written as

$$\begin{pmatrix}\beta_3+\beta_4\tau_{0_2}+\beta_5\tau_{0_2}^2 & \beta_6\tau_{0_2}E_0\\ \beta_6(\tau_{0_3}-\tau_{0_2})E_0 & \beta_3+\beta_4\tau_{0_3}+\beta_5\tau_{0_3}^2\end{pmatrix}\begin{pmatrix}D_2\\ D_3\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{3.12}$$

If $\mathtt{M}$ denotes the matrix $\mathtt{M}=\begin{pmatrix}\beta_3+\beta_4\tau_{0_2}+\beta_5\tau_{0_2}^2 & \beta_6\tau_{0_2}E_0\\ \beta_6(\tau_{0_3}-\tau_{0_2})E_0 & \beta_3+\beta_4\tau_{0_3}+\beta_5\tau_{0_3}^2\end{pmatrix}$, we have two possible solutions for $\mathbf{D}$. One solution is obtained if we assume $D_2=D_3=0$, but there could be another possibility, which is to assume that in general $D_2\neq 0$ and $D_3\neq 0$, and $\det\mathtt{M}=0$. In that case, it follows from (3.11) that equation $\det\mathtt{M}=0$ would be, in general, a nonlinear relation for D_1. If D_1 is found such that both $\det\mathtt{M}=0$ and (3.8) are satisfied, then from (3.12) there would be other possibilities for D_2 and D_3 and one of the two would be arbitrary.

For the sake of simplicity, let us consider the solution $D_2=D_3=0$, then from (3.5), (3.6), (3.11) and (3.8), we would have an algebraic equation (in general nonlinear) to obtain D_1 in terms of E_0 and τ_{0_i}, $i=1,2,3$. Because such a value for D_1 would be constant, then $(2.19)_4$ would be satisfied automatically.

We thus have a solution for the set of equations (2.19); let us now turn our attention to a discussion of the boundary conditions. At the surfaces $x_1=\pm L_1/2$, if we want $(2.5)_2$ to be satisfied, then we need

$$D_1^{\mathrm{o}}=D_1, \tag{3.13}$$

where D_1^{o} is the electric displacement outside the body at the boundary. From (2.3), for vacuum we would have

$$D_1^{\mathrm{o}}=\epsilon_0E_1^{\mathrm{o}}. \tag{3.14}$$

Across the surfaces $x_2=\pm L_2/2$ and $x_3=\pm L_3/2$ in order for the continuity condition $(2.5)_1$ to be satisfied we would need $E_1^{\mathrm{o}}=E_0$, but this condition in general does not give the same value for E_1^{o} as that which is obtained from (3.13) to (3.14); however, if $L_3\gg L_1$ and $L_2\gg L_1$, the surfaces $x_2=\pm L_2/2$ and $x_3=\pm L_3/2$ are located far away, and so, as an approximation, we do not consider the electric continuity conditions for such surfaces.

Therefore, from the point of view of the electric field, from the exterior space we need (as an approximation) an electric field of the form

$$\mathbf{E}=E_1^{\mathrm{o}}\mathbf{e}_1, \tag{3.15}$$

where E_1^{o} is obtained from (3.14) and (3.13). Such an electric field and the electric displacement associated with it are constant and so they satisfy the simplified forms of Maxwell equations for vacuum.

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With regard to the mechanical boundary conditions, we assume that at the surfaces $x_2 = \pm L_2/2$ and $x_3 = \pm L_3/2$ external mechanical forces are applied such that the total stress inside the body is given by $(3.2)_1$. If we denote $\tilde{\mathbf{t}}$ to be this external mechanical traction, from $\boldsymbol{\tau}\mathbf{n} = \tilde{\mathbf{t}}$ we find that on the surfaces $x_2 \pm L_2/2$ and $x_3 \pm L_3/2$

$$\tilde{\mathbf{t}} = \pm\tau_{0_2}\mathbf{e}_2, \quad \tilde{\mathbf{t}} = \pm\tau_{0_3}\mathbf{e}_2, \tag{3.16}$$

respectively. As for the surface $x_1 = \pm L_1/2$, we assume that the body is exposed to free vacuum, and the traction will be found using the Maxwell stress (2.9). Using (3.15) and (2.3) for the exterior field, from (2.9) we have

$$\boldsymbol{\tau}_{\mathrm{m}} = \frac{\epsilon_0 (E_1^{\mathrm{o}})^2}{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3), \tag{3.17}$$

and so from (3.17) we find that $\boldsymbol{\tau}_{\mathrm{m}}\mathbf{n} = (\epsilon_0 (E_1^{\mathrm{o}})^2/2)\mathbf{e}_1$ on the surface $x_1 = L_1/2$ and $\boldsymbol{\tau}_{\mathrm{m}}\mathbf{n} = -(\epsilon_0 (E_1^{\mathrm{o}})^2/2)\mathbf{e}_1$ on the surface $x_1 = -L_1/2$; therefore, from (2.8) we have

$$\tau_{0_1} = \frac{\epsilon_0 (E_1^{\mathrm{o}})^2}{2} \quad \text{on} \quad x_1 = \pm L_1/2. \tag{3.18}$$

We note that in $(3.2)_1$ the stress τ_{0_1} is not an arbitrary quantity, but depends on the magnitude of the electric field.

(b) A slab in a state of shear

For the problem in question, we shall assume a solution for stress and electric field of the form

$$\boldsymbol{\tau} = \tau_{0_{12}}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad \mathbf{E} = E_0\mathbf{e}_1, \tag{3.19}$$

where $\tau_{0_{12}}$ and E_0 are constant. We consider the same geometry defined by (3.1), but for this problem, we assume that the external force is applied at the surfaces $x_1 = \pm L_1/2$. From (2.12), we obtain that

$$\varepsilon_{11} = \hat{\alpha}_0 + \hat{\alpha}_1\tau_{0_{12}}^2 + \hat{\alpha}_3 E_0^2 + \hat{\alpha}_5\tau_{0_{12}}^2 E_0^2, \quad \varepsilon_{22} = \hat{\alpha}_0 + \hat{\alpha}_2\tau_{0_{12}}^2, \quad \varepsilon_{33} = \hat{\alpha}_0 \tag{3.20}$$

and

$$\varepsilon_{12} = \hat{\alpha}_1\tau_{0_{12}} + \hat{\alpha}_4\tau_{0_{12}}E_0^2, \quad \varepsilon_{13} = \varepsilon_{23} = 0, \tag{3.21}$$

whereas from (2.13), we have

$$\beta_0 E_0 + \beta_2\tau_{0_{12}}^2 E_0 \beta_3 D_1 + \beta_4\tau_{0_{12}} D_2 + \beta_5\tau_{0_{12}}^2 D_1 - \beta_6 E_0 D_3 \tau_{0_{12}} = 0, \tag{3.22}$$

$$\beta_1\tau_{0_{12}}E_0 + \beta_3 D_2 + \beta_4\tau_{0_{12}} D_1 + \beta_5\tau_{0_{12}}^2 D_2 = 0 \tag{3.23}$$

and

$$\beta_3 D_3 - \beta_6 E_0 \tau_{0_{12}} D_1 = 0, \tag{3.24}$$

where I_i, $i = 1, 2, \ldots, 11$ are given from (2.14), (2.15)

$$I_1 = I_3 = I_5 = 0, \quad I_2 = 2\tau_{0_{12}}^2, \quad I_4 = E_0^2, \quad I_6 = \tau_{0_{12}}^2 E_0^2, \quad I_7 = D_1^2 + D_2^2 + D_3^2 \tag{3.25}$$

and

$$I_8 = 2\tau_{0_{12}} D_1 D_2, \quad I_9 = \tau_{0_{12}}^2 (D_1^2 + D_2^2), \quad I_{10} = D_1^2 E_0^2, \quad I_{11} = \tau_{0_{12}}^2 E_0^2 D_2^2. \tag{3.26}$$

From (3.23) and (3.24), in general, D_2 and D_3 are different from zero, unlike the previous problem studied in §3*a*.

Regarding the continuity conditions (2.7), the assumption is that the mechanical traction is applied at the surfaces $x_1 = \pm L_1/2$. For these surfaces, on using $(2.5)_2$, we find that $D_1 = D_1^{\mathrm{o}}$ and so from (2.3), we obtain $E_1^{\mathrm{o}} = \epsilon_0^{-1} D_1^{\mathrm{o}} = \epsilon_0^{-1} D_1$. Regarding $(2.5)_2$, from (3.19), we obtain $E_2^{\mathrm{o}} = E_3^{\mathrm{o}} = 0$ at $x_1 = \pm L_1/2$. Because $L_1 \ll L_2$ and $L_1 \ll L_3$, the surfaces $x_2 = \pm L_2/2$ and $x_3 = \pm L_3/2$ are far away, and thus we do not check the continuity conditions (2.5) at those surfaces.

4. Non-homogeneous distribution of stresses

In this section, for the sake of simplicity, only the constitutive relations given by (2.16) will be considered, which in the case of isotropic functions $\hat{\mathfrak{f}}$, $\hat{\mathfrak{l}}$ reduce to (2.12) and (2.17). For the class of problem discussed here, complex ordinary differential equations will be obtained, which are solved using numerical methods; therefore, some additional assumptions are needed regarding the constitutive equations (2.16) in order to obtain closed form solutions.

We assume there exists a scalar function $\Omega = \Omega(\boldsymbol{\tau}, \mathbf{E})$ such that

$$\boldsymbol{\varepsilon} = \frac{\partial \Omega}{\partial \boldsymbol{\tau}}, \quad \mathbf{D} = -\frac{\partial \Omega}{\partial \mathbf{E}}. \tag{4.1}$$

This assumption has been made with the purpose of facilitating the development of prototypical expressions for the functions $\hat{\alpha}_i$ and $\hat{\beta}_j$, $i = 0, 1, \ldots, 5$ and $j = 0, 1, 2$ in (2.11) and (2.17). The function Ω has not been obtained on the basis of thermodynamic arguments; such an analysis is beyond the scope of this work.

For an isotropic function $\Omega = \Omega(I_1, I_2, I_3, I_4, I_5, I_6)$, where I_k, $k = 1, 2, \ldots, 6$ are defined in (2.14). Using the chain rule for the derivative (in index notation for a Cartesian coordinate system) from (4.1), we have $\varepsilon_{ij} = \partial\Omega/\partial\tau_{ij} = \sum_{k=1}^{6}(\partial\Omega/\partial I_k)(\partial I_k/\partial\tau_{ij})$ and $D_i = -\sum_{k=1}^{6}(\partial\Omega/\partial I_k)(\partial I_k/\partial E_i)$, which after some algebraic manipulations become

$$\boldsymbol{\varepsilon} = \frac{\partial \Omega}{\partial I_1}\mathbf{I} + 2\frac{\partial \Omega}{\partial I_2}\boldsymbol{\tau} + 3\frac{\partial \Omega}{\partial I_3}\boldsymbol{\tau}^2 + \frac{\partial \Omega}{\partial I_5}\mathbf{E}\otimes\mathbf{E} + \frac{\partial \Omega}{\partial I_6}(\mathbf{E}\otimes\boldsymbol{\tau}\mathbf{E} + \boldsymbol{\tau}\mathbf{E}\otimes\mathbf{E}) \tag{4.2}$$

and

$$\mathbf{D} = -2\left(\frac{\partial \Omega}{\partial I_4}\mathbf{E} + \frac{\partial \Omega}{\partial I_5}\boldsymbol{\tau}\mathbf{E} + \frac{\partial \Omega}{\partial I_6}\boldsymbol{\tau}^2\mathbf{E}\right), \tag{4.3}$$

with the following relationships holding:

$$\hat{\alpha}_0 = \frac{\partial \Omega}{\partial I_1}, \quad \hat{\alpha}_1 = 2\frac{\partial \Omega}{\partial I_2}, \quad \hat{\alpha}_2 = 3\frac{\partial \Omega}{\partial I_3}, \quad \hat{\alpha}_3 = \frac{\partial \Omega}{\partial I_5}, \quad \hat{\alpha}_4 = \frac{\partial \Omega}{\partial I_6}, \quad \hat{\alpha}_5 = 0 \tag{4.4}$$

and

$$\hat{\beta}_0 = -2\frac{\partial \Omega}{\partial I_4}, \quad \hat{\beta}_1 = -2\frac{\partial \Omega}{\partial I_5}, \quad \hat{\beta}_2 = -2\frac{\partial \Omega}{\partial I_6}. \tag{4.5}$$

We shall consider the resolution of boundary value problems in electro-elasticity, wherein the class of bodies of interest exhibit two characteristics: the first is the strain limiting behaviour that is exhibited by a large class of electro-elastic bodies, the second is the saturation phenomenon for the polarization field **P** in terms of the intensity of the electric field. Thus, it is imperative that one picks constitutive relations that reflect such characteristics.

In Bustamante & Rajagopal [24] (see also Ortiz *et al.* [25]), a scalar function Ω was proposed so that the model exhibits 'strain-limiting' behaviour; the specific form chosen for Ω is $-\alpha[I_1 - (1/\beta)\ln(1+\beta I_1)] + (\alpha\gamma/\iota)\sqrt{1+\iota I_2}$, where α, β, γ and ι are constants. An interesting feature of such a function is that it is not only strain limiting but the model also exhibits different response with regard to compression and tension. From the point of view of numerical computations, some difficulties arise when $I_1 < 0$, especially when $\beta I_1 \to -1$; therefore, a modified version of the function presented in Bustamante & Rajagopal [24] is used here, that is Ω is of the form $-\alpha[I_1 - \int_0^{I_1}(1/(1+\beta(\varpi^2)^b))\,\mathrm{d}\varpi] + (\alpha\gamma/\iota)\sqrt{1+\iota I_2}$, where b is a constant with $b > \frac{1}{2}$.

In Bustamante [26], an expression for an energy function was proposed for the equivalent magnetoelastic problem, which produces the saturation phenomena (in that case for the magnetization) discussed within the context of fig. 1 in Bustamante & Rajagopal [1], such a function has the form (for isotropic bodies): (elastic part)$(g_0 + g_1 I_4) - \ln[\cosh(\sqrt{I_4}/m_1)]m_0 m_1 - (\zeta_0/2)I_4 + (\epsilon_0\zeta_1/2)I_5$, where g_0, g_1, m_0, m_1, ζ_0 and ζ_1 are constants.

Table 1. Values for the constants in (4.6).

α	10^{-8}
β	$10^{-3}\ \dfrac{1}{\mathrm{Pa}^{2b}}$
γ	$10\ \dfrac{1}{\mathrm{Pa}}$
ι	$5\times 10^{-7}\ \dfrac{1}{\mathrm{Pa}^2}$
g_0	1
g_1	$-10^{-10}\ \dfrac{\mathrm{V}}{\mathrm{m}}$
m_0	$10^{-2}\ \dfrac{\mathrm{VC}}{\mathrm{m}^3}$
m_1	$10^{3}\ \dfrac{\mathrm{V}}{\mathrm{m}}$
ζ_0	$10^{-7}\ \dfrac{\mathrm{Nm}^2}{\mathrm{C}^2}$
ζ_1	$10^{-8}\ \dfrac{1}{\mathrm{Pa}}$
ϵ_0	$8.854\times 10^{-12}\ \dfrac{\mathrm{Nm}^2}{\mathrm{C}^2}$
b	0.55

In view of the previous discussion concerning the function Ω, we propose

$$\Omega(I_1,I_2,I_4,I_5)=\left\{-\alpha\left[I_1-\int_0^{I_1}\frac{1}{(1+\beta(\varpi^2)^b)}\mathrm{d}\varpi\right]+\frac{\alpha\gamma}{\iota}\sqrt{1+\iota I_2}\right\}(g_0+g_1I_4)$$
$$-\ln\left[\cosh\left(\frac{\sqrt{I_4}}{m_1}\right)\right]m_0m_1-\frac{\zeta_0}{2}I_4+\frac{\epsilon_0\zeta_1}{2}I_5, \tag{4.6}$$

where the term $(g_0+g_1I_4)$ and $(\epsilon_0\zeta_1/2)I_5$ would be the coupling between the stresses and the electric fields.

For the different constants that appear in (4.6), the values presented in table 1 are used.

Consider the problem of the uniaxial extension of a bar, where a constant distribution of normal axial stress σ and electric field E are assumed. For this problem from (4.6), we obtain

$$\varepsilon=-\alpha\left\{1-\frac{1}{[1+\beta(\sigma^2)^b]}\right\}(g_0+g_1E^2)+\frac{\alpha\gamma}{\sqrt{1+\iota\sigma^2}}(g_0+g_1E^2)\sigma+\epsilon_0\zeta_1E^2 \tag{4.7}$$

and

$$D=2\left\{\alpha g_1\left[\sigma-\int_0^{\sigma}\frac{1}{(1+\beta(\varpi^2)^b)}\mathrm{d}\varpi-\frac{\gamma}{\iota}\sqrt{1+\iota\sigma^2}\right]+\tanh\left(\frac{E}{m_1}\right)\frac{m_0}{2E}+\frac{\zeta_0}{2}\right\}E-\epsilon_0\zeta_1\sigma E, \tag{4.8}$$

where ε and D are the axial component of the linearized strain and the electric displacement, respectively. In figures 1 and 2, the behaviour of ε and P as functions of the stress and electric field are depicted. A limiting strain behaviour for ε for higher values for the stress is observed, and in this case the application of an electric field produces the shrinking of the bar. In figure 2, a plot for $P(E)$ is presented (see (2.4)), where a behaviour similar to the saturation phenomenon can be observed.

While with the particular expression for Ω (see (4.6)), there is coupling between the stress and strains, and the electric field and the electric displacement, from figures 1 and 2 we see that with the values for the constants from table 1, the influence of the stress on the electric displacement

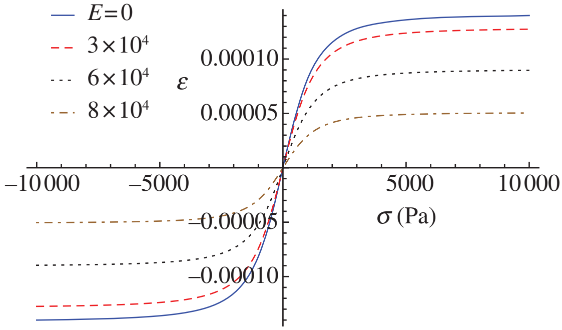

Figure 1. Behaviour of the axial strain as a function of the axial stress for different magnitudes of the electric field in V m^{-1}. (Online version in colour.)

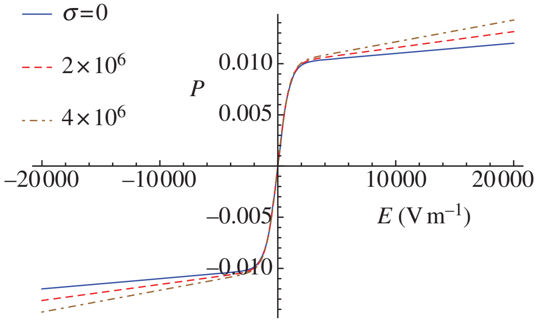

Figure 2. Behaviour of the polarization field as a function of the electric field for different magnitudes of the axial stress. The unit for the stress is Pa. (Online version in colour.)

is rather weak; in order for the curve $P(E)$ to be influenced by the stress, it is necessary to apply relatively high stresses.

(a) Radial inflation of a thin circular plate

Let us next study a boundary value problem in the following domain defined in cylindrical coordinates through

$$r_\mathrm{i} \le r \le r_\mathrm{o}, \quad 0 \le z \le L. \tag{4.9}$$

In this first case, we assume $L \ll r_\mathrm{i}$, i.e. we have thin circular plate, which is under the effect of the following stress distribution and electric field (in terms of the electric scalar potential (2.18) $\varphi = \varphi(r)$):

$$\boldsymbol{\tau} = \tau_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + \tau_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta, \quad \mathbf{E} = -\frac{\mathrm{d}\varphi}{\mathrm{d}r}\mathbf{e}_r, \tag{4.10}$$

i.e. this is a plane stress problem (described in this case in terms of the polar coordinates r, θ).

If $\boldsymbol{\tau}$ is of the form $(4.10)_1$, then in cylindrical coordinates, the only equilibrium equation $(2.19)_3$ that needs to be considered is

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \tag{4.11}$$

which has the solution

$$\tau_{\theta\theta} = \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rr}). \tag{4.12}$$

On using (4.6) in (4.2) by appealing to (4.10), we obtain

$$\varepsilon_{rr} = -\alpha \left\{1 - \frac{1}{[1+\beta(I_1^2)^b]}\right\}(g_0 + g_1 I_4) + \frac{\alpha\gamma}{\sqrt{1+\iota I_2}}(g_0 + g_1 I_4)\tau_{rr} + \epsilon_0 \zeta_1 E_r^2, \tag{4.13}$$

$$\varepsilon_{\theta\theta} = -\alpha \left\{1 - \frac{1}{[1+\beta(I_1^2)^b]}\right\}(g_0 + g_1 I_4) + \frac{\alpha\gamma}{\sqrt{1+\iota I_2}}(g_0 + g_1 I_4)\tau_{\theta\theta} \tag{4.14}$$

and

$$\varepsilon_{zz} = -\alpha \left\{1 - \frac{1}{[1+\beta(I_1^2)^b]}\right\}(g_0 + g_1 I_4), \tag{4.15}$$

where

$$I_1 = \tau_{rr} + \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rr}), \quad I_2 = \tau_{rr}^2 + \left[\frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rr})\right]^2, \quad I_4 = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)^2. \tag{4.16}$$

In order for the strain components (4.13)–(4.15) to have an associated continuous displacement field, certain compatibility conditions must be satisfied. Because in this problem $\varepsilon_{ij} = \varepsilon_{ij}(r)$, the relevant compatibility equations are (Saada [27, p. 142])

$$\frac{\mathrm{d}^2\varepsilon_{rr}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r} - \frac{\mathrm{d}\varepsilon_{rr}}{\mathrm{d}r} = 0 \quad \Leftrightarrow \quad r\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r} + \varepsilon_{\theta\theta} - \varepsilon_{rr} = c \tag{4.17}$$

and

$$\frac{1}{r}\frac{\mathrm{d}\varepsilon_{zz}}{\mathrm{d}r} = 0, \quad \frac{\mathrm{d}^2\varepsilon_{zz}}{\mathrm{d}r^2} = 0, \tag{4.18}$$

where c is a constant. These two last equations are satisfied if $(\mathrm{d}\varepsilon_{zz}/\mathrm{d}r) = 0$.

Because the plate is thin, that is because L is much smaller than r_{i}, we shall make the approximation that the normal strain in the z-direction ε_{zz} can be neglected. In virtue of this, we shall not solve equation (4.18) and thus solve only equation (4.17).

Regarding the electric displacement, in virtue of (4.3), on using (4.10) we obtain

$$D_r = 2\left\{\alpha g_1\left[I_1 - \int_0^{I_1}\frac{1}{(1+\beta(\varpi^2)^b)}\mathrm{d}\varpi - \frac{\gamma}{\iota}\sqrt{1+\iota I_2}\right] + \tanh\left(\frac{\sqrt{I_4}}{m_1}\right)\frac{m_0}{2\sqrt{I_4}} + \frac{\zeta_0}{2}\right\}E_r - \epsilon_0\zeta_1\tau_{rr}E_r. \tag{4.19}$$

and $D_\theta = D_z = 0$. Because $\mathbf{D} = D_r\mathbf{e}_r$, $(2.2)_2$ leads to

$$\frac{\mathrm{d}D_r}{\mathrm{d}r} + \frac{D_r}{r} = 0. \tag{4.20}$$

Therefore, it is necessary to solve the two (in general nonlinear) second-order ordinary differential equations (4.17) and (4.20) to obtain solutions for $\tau_{rr}(r)$ and $\varphi(r)$.

(i) Boundary conditions

Regarding the boundary conditions, because the equations to be solved are of second order in $\tau_{rr}(r)$ and $\varphi(r)$, four boundary conditions are needed so that the problem is well posed. There is an inherent difficulty with regard to the specification of traction boundary conditions as the traction caused by the mechanical interactions with the external world coexists with the traction caused by the interaction of the electrical field with the body (which leads to the Maxwell stresses). We do not have a compelling argument for how the total traction is apportioned between the mechanical and Maxwell traction. In view of this difficulty, the traction owing to the electric field that induces the Maxwell stresses is included in the definition of the external mechanical traction (see the comments at the end of §2*c*).

At the inner surface of the plate $r = r_{\mathrm{i}}$, we assume that

$$\varphi(r_{\mathrm{i}}) = \tilde{\varphi}_{\mathrm{i}}, \quad \tau_{rr}(r_{\mathrm{i}}) = -p_{\mathrm{i}}, \tag{4.21}$$

where $\tilde{\varphi}_{\mathrm{i}}$ is a given value for the electric potential, and p_{i} is a radial normal stress applied on the inner surface of the plate, which incorporates in its definition the force owing to the electromagnetic field (see (2.8) and (2.9)).

At the outer surface $r=r_{\mathrm{o}}$, for φ, the simplest condition that can be considered is

$$\varphi(r_{\mathrm{o}})=\tilde{\varphi}_{\mathrm{o}}, \tag{4.22}$$

where $\tilde{\varphi}_{\mathrm{o}}$ is a given value for the electric potential for the outer surface.

For τ_{rr} on the outer surface of the plate, it is assumed there is no external load; therefore

$$\tau_{rr}(r_{\mathrm{o}})=0. \tag{4.23}$$

From (2.8) and (2.9), we see that in the case the surface $r=r_{\mathrm{o}}$ would be in contact only with vacuum, then the Maxwell stresses have to be considered as the external load. From (2.9) using (2.3), the Maxwell stresses can be expressed as

$$\boldsymbol{\tau}_{\mathrm{m}}=\epsilon_0^{-1}[\mathbf{D}^{\mathrm{o}}\otimes\mathbf{D}^{\mathrm{o}}-\tfrac{1}{2}(\mathbf{D}^{\mathrm{o}}\cdot\mathbf{D}^{\mathrm{o}})\mathbf{I}], \tag{4.24}$$

where $\mathbf{D}^{\mathrm{o}}$ is the electric displacement calculated at $r=r_{\mathrm{o}}$ in vacuum. In virtue of the continuity condition $(2.5)_2$ $D_r^{\mathrm{o}}(r_{\mathrm{o}})=D_r(r_{\mathrm{o}})$, $D_\theta^{\mathrm{o}}(r_{\mathrm{o}})=0$ and from (4.24) and (2.8), we have

$$\tau_{rr}(r_{\mathrm{o}})=\frac{\epsilon_0^{-1}}{2}(D_r(r_{\mathrm{o}}))^2. \tag{4.25}$$

The boundary condition (4.25) is nonlinear because $D_r(r_{\mathrm{o}})$ must be obtained by solving (4.20). The use of (4.25) may cause some additional difficulties with regard to the convergence of the Newton method; therefore, the simpler condition presented in (4.23) has been used as a first approximation. In (4.23), we are assuming that the body is in contact with another external body so that the mechanical traction $\mathbf{t}_{\mathrm{a}}$ can be adjusted to eliminate $\boldsymbol{\tau}_{\mathrm{m}}\mathbf{n}$ (see (2.8)).

In §7 of Vu & Steinmann [19] (see also [28]), there is a discussion regarding the effect of considering the surrounding free space (in particular the Maxwell stresses), and in situations wherein such influence can be neglected from the analysis.

Regarding the continuity condition for the electric field (2.5), if the plate is very thin, then we can consider the approximation wherein we need to only impose (2.5) at $z=0$ and $z=L$ with $r_{\mathrm{i}}\leq r\leq r_{\mathrm{o}}$, and we can neglect the conditions at the other surfaces. If the electric field outside is denoted by $\mathbf{E}^{\mathrm{o}}=E_r^{\mathrm{o}}\mathbf{e}_r$, then $(2.5)_1$ is satisfied on the surfaces $z=0$ and $z=L$ if $E_r^{\mathrm{o}}=E_r$. Because the azimuthal component of the electric field inside is zero, by the continuity condition $E_\theta^{\mathrm{o}}=0$. Finally, because $\mathbf{D}=D_r\mathbf{e}_r$, in order for $(2.5)_2$ to be satisfied we need $D_z^{\mathrm{o}}=0$, where D_z^{o} is the radial component of the electric displacement outside the plate, because for free space (2.3) holds as a consequence $E_z^{\mathrm{o}}=0$. Therefore, one solution in free space for which (2.5) is satisfied is

$$\mathbf{E}^{\mathrm{o}}=E_r\mathbf{e}=-\frac{\mathrm{d}\varphi}{\mathrm{d}r}\mathbf{e}_r. \tag{4.26}$$

(ii) Numerical results

In this section, we present the numerical results obtained using the finite-element method to solve equations (4.17) and (4.20) subject to (4.13), (4.14) and (4.19). The systems of coupled equations (4.17) and (4.20) have been solved using the finite-element method and the program COMSOL v. 3.4 [29], where $\tau_{rr}(r)$ and $\varphi(r)$ are the functions to be found. A mesh sensitivity analysis was carried out, but for the sake of brevity it is not presented here. There are 15 363 degrees of freedom, and the elements are Lagrange cubic. Finally, $r_{\mathrm{i}}=0.1\,\mathrm{m}$ and $r_{\mathrm{o}}=0.2\,\mathrm{m}$. In figure 3, we have results for the plate for three different cases.

1. In figure 3*a*(i,ii), results for the plate under the effect of an internal radial normal stress $p_{\mathrm{i}}=1.3\times10^3$ Pa applied on the surface r_{i} are presented, in the case, there is no difference

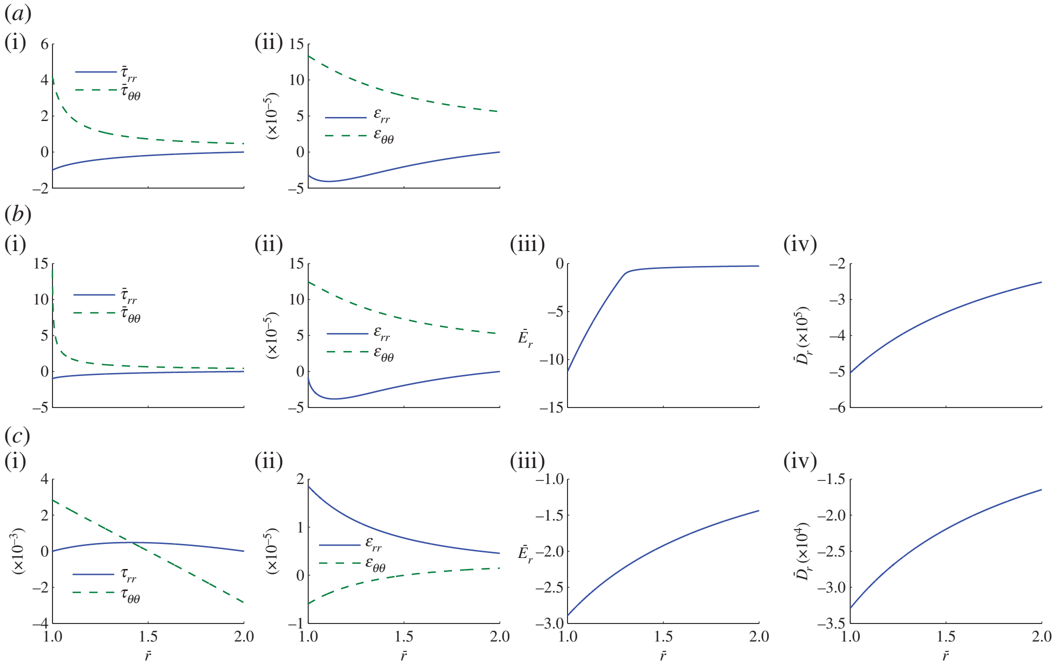

Figure 3. From top to bottom. (*a*) Case $p_i = 1.3 \times 10^3$ Pa and $\varphi_o = 0$ (there is no electric field), (i) normalized components of the stress (see equation (4.27)), (ii) components of the strain tensor. (*b*) Case $p_i = 1.3 \times 10^3$ Pa and $\varphi_o = 6 \times 10^2$ V, (i) normalized components of the stress, (ii) components of the strain, (iii) normalized electric field and (iv) normalized electric displacement (see equation (4.28)). (*c*) Case $p_i = 0$ (there is no external mechanical load) and $\varphi_o = 10^6$ V, (i) components of the stress in Pa (not normalized), (ii) components of the strain, (iii) normalized electric field and (iv) normalized electric displacement. (Online version in colour.)

of potential between the inner and the outer radii, i.e. $\varphi_i = \varphi_o = 0$. In figure 3*a*(i), we have the depiction of the normalized radial and azimuthal components of the total stress tensor in terms of the normalized radius, where

$$\bar{\tau}_{rr} = \frac{\tau_{rr}}{p_i}, \quad \bar{\tau}_{\theta\theta} = \frac{\tau_{\theta\theta}}{p_i}, \quad \bar{r} = \frac{r}{r_i}. \tag{4.27}$$

In figure 3*a*(ii), a similar plot is shown for the radial and azimuthal components of the strain tensor.

For the class of constitutive equations used in this work, if there is no external electric field (in this case due to the fact that there is no difference in the applied electric potential), then there is no electric displacement (see (2.17) or (4.19) and consider the case $\mathbf{E} = \mathbf{0}$).

The particular value for p_i used here was the maximum magnitude for the pressure for which convergence of the Newton method was achieved.

2. In figure 3*b*(i)–(iv), results are presented for the case of the same radial normal stress $p_i = 1.3 \times 10^3$ Pa applied on the surface r_i, and, additionally, an electric field appears owing to a difference in the electric potential $\varphi_i = 0$ on r_i and $\varphi_o = 6 \times 10^2$ V on r_o. In these figures, the same normalized radial position $\bar{r}$ defined in $(4.27)_3$ is used. In figure 3*b*(i,ii), results are shown for the normalized components of the stress tensor (defined in $(4.27)_{2,3}$) and the components of the strain tensor. One can note that the application of an electric field causes an increase in the magnitude of the azimuthal component of the total stress tensor, in particular in a narrow zone near $\bar{r} = 1$. Despite this rapid increment in the magnitude of the stress, from figure 3*b*(ii), it is observed that the components of the strain remain small.

In figure 3*b*(iii,iv), we have plots for the radial component of the normalized electric field and the electric displacement, which have been defined through

$$\bar{E}_r = \frac{E_r}{(\varphi_o - \varphi_i)/(r_o - r_i)}, \quad \bar{D}_r = \frac{D_r}{\epsilon_0(\varphi_o - \varphi_i)/(r_o - r_i)}, \tag{4.28}$$

where $E_r = -\mathrm{d}\varphi/\mathrm{d}r$.

The particular value $\varphi_o = 6 \times 10^2$ V on r_o was the maximum magnitude for the electric potential for which the numerical method converges.

3. In figure 3*c*(i–iv), results for the stresses, strain, electric field and electric displacement are plotted, when there is no external mechanical load applied, but an electric field is present due to a difference of electric potential, in this case $\varphi_i = 0$ on r_i and $\varphi_o = 10^6$ V on r_o. It is worth observing that a relatively high value of φ_o is used in this case in comparison with the value used to obtain the results shown in figure 3*b*(iii–iv). For the results depicted in figure 3*c*(i,iv), it was possible to apply a higher value for φ_o without difficulty with regard to the convergence of the Newton method.

   In figure 3*c*(i), a plot of the components of the total stress tensor (not normalized) is presented. The stresses are not normalized, because there is no internal radial normal stress that can be used to define such normalized quantities. From figure 3*c*(ii), we observe that the magnitude of the components of the strain is similar to the cases shown in figure 3*a*(ii) and *b*(ii), but the behaviour is rather different, because the radial component of the strain is positive, and the azimuthal component of the strain is negative.

(b) Inflation and extension of a very long cylindrical tube

Consider the boundary value problem corresponding to the same geometry defined previously, namely $r_i \le r \le r_o$, $0 \le z \le L$, but now in the limit $L \to \infty$, i.e. the tube is very long. As an approximation, the boundary conditions (2.5) are not required to be satisfied at the surfaces $z = 0$, $z = L$. We assume that the tube is under the effect of the stress distribution of the form

$$\boldsymbol{\tau} = \tau_{rr}(r)\mathbf{e}_r \otimes \mathbf{e}_r + \tau_{\theta\theta}(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \tau_{zz}(r)\mathbf{e}_z \otimes \mathbf{e}_z, \tag{4.29}$$

and an electric field of form $(4.10)_2$. If the total stress is of this form, the only equilibrium equation to be satisfied is (4.11), and the solution (4.12) is also valid here. Regarding $\tau_{zz}(r)$, this component of the total stress is not arbitrary as is shown later on.

If the same constitutive equation defined through (4.6) is used in this problem (with the values for the constants presented in table 1), using (4.29) and $(4.10)_2$ the same components for the strains ε_{rr}, $\varepsilon_{\theta\theta}$ as in (4.13) and (4.14) are obtained; however, in the present case, $(2.14)_{1,2}$ becomes

$$I_1 = \tau_{rr} + \frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rr}) + \tau_{zz}, \quad I_2 = \tau_{rr}^2 + \left[\frac{\mathrm{d}}{\mathrm{d}r}(r\tau_{rr})\right]^2 + \tau_{zz}^2. \tag{4.30}$$

Regarding the component ε_{zz}, from (4.2), we obtain that

$$\varepsilon_{zz} = -\alpha\left\{1 - \frac{1}{[1+\beta(I_1^2)^b]}\right\}(g_0 + g_1 I_4) + \frac{\alpha\gamma}{\sqrt{1+\iota I_2}}(g_0 + g_1 I_4)\tau_{zz}. \tag{4.31}$$

The rest of the components of the strain tensor are zero. As for $\mathbf{D}$, this is given by (4.19) using I_1 from $(4.30)_1$.

In order to have a continuous displacement field associated with the components of the strain tensor, considering that $\varepsilon_{ij} = \varepsilon_{ij}(r)$, the compatibility equations (4.17) and (4.18) have to be satisfied, and in the present problem (4.18) *cannot* be neglected.

To summarize, we need to solve equations (4.17) and (4.18) that are equivalent to $\mathrm{d}\varepsilon_{zz}/\mathrm{d}r = 0$ and equation (4.20). These equations are solved to find three functions: $\tau_{rr}(r)$, $\tau_{zz}(r)$ and $\varphi(r)$.

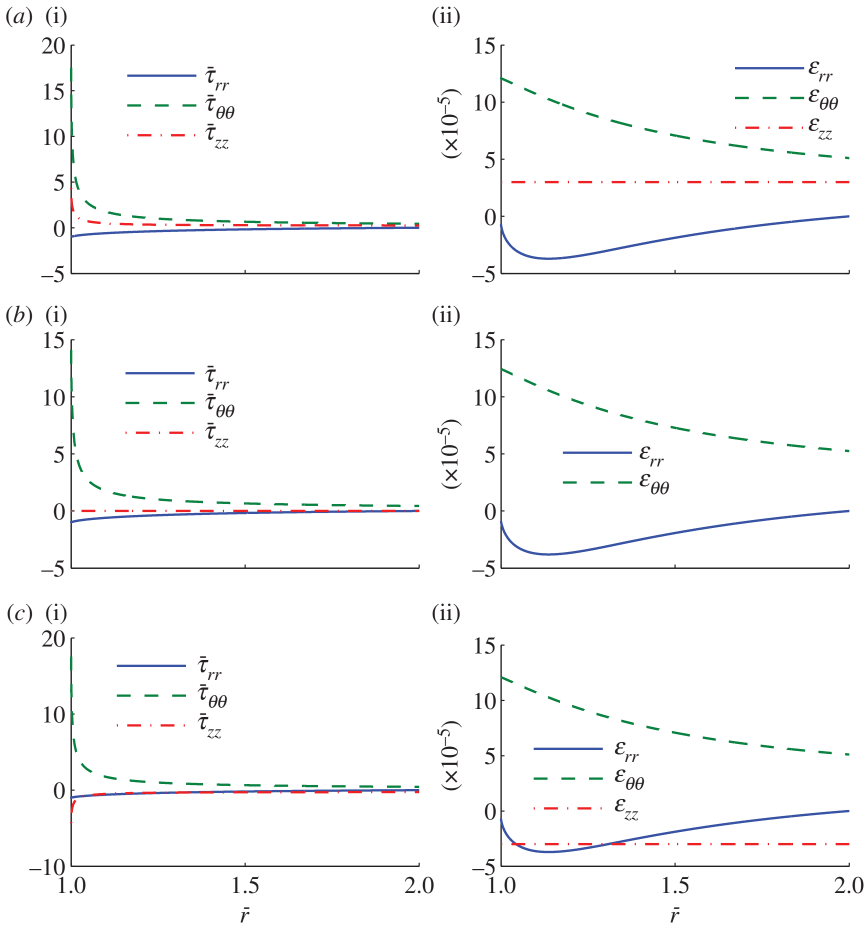

Figure 4. From top to bottom. (*a*) Case $p_i = 1.3 \times 10^3$ Pa, $\varphi_o = 6 \times 10^2$ V and $\varepsilon_{zz} = 3 \times 10^{-5}$, (i) normalized components of the stress, (ii) components of the strain tensor. (*b*) Case $p_i = 1.3 \times 10^3$ Pa, $\varphi_o = 6 \times 10^2$ V and $\varepsilon_{zz} = 0$, (i) normalized components of the stress, (ii) components of the strain. (*c*) Case $p_i = 1.3 \times 10^3$ Pa, $\varphi_o = 6 \times 10^2$ V and $\varepsilon_{zz} = -3 \times 10^{-5}$, (i) normalized components of the stress, (ii) components of the strain. (Online version in colour.)

Regarding the equation $d\varepsilon_{zz}/dr = 0$, after integrating, we find $\varepsilon_{zz}(r) = \varepsilon_{zz_0}$, where ε_{zz_0} is a constant. This is a nonlinear algebraic equation, which has to be solved along with (4.17) and (4.20). For the results presented in this section, ε_{zz_0} is given. Equations (4.17), (4.20) and $\varepsilon_{zz}(r) = \varepsilon_{zz_0}$ are solved using the finite-element program COMSOL [29], and the same statistic for the mesh is used as in the previous problem.

Results for five different cases are presented.

1. In figure 4, we depict the results for three cases.

   — In figure 4*a*(i,ii), results are presented for the normalized stresses (see equation (4.27)) with $\bar{\tau}_{zz} = \tau_{zz}/p_i$, in the case a radial normal stress $p_i = 1.3 \times 10^3$ Pa is applied on the surface $r = r_i$, if there is a difference in the electric potential $\varphi_i = 0$, $\varphi_o = 6 \times 10^2$ V, and the tube is uniformly stretched in the axial direction with $\varepsilon_{zz_0} = 3 \times 10^{-5}$.

   In figure 4*a*(ii), we provide a plot of the constant axial strain, in order to compare it with the behaviour of the radial and azimuthal components of the strain. In

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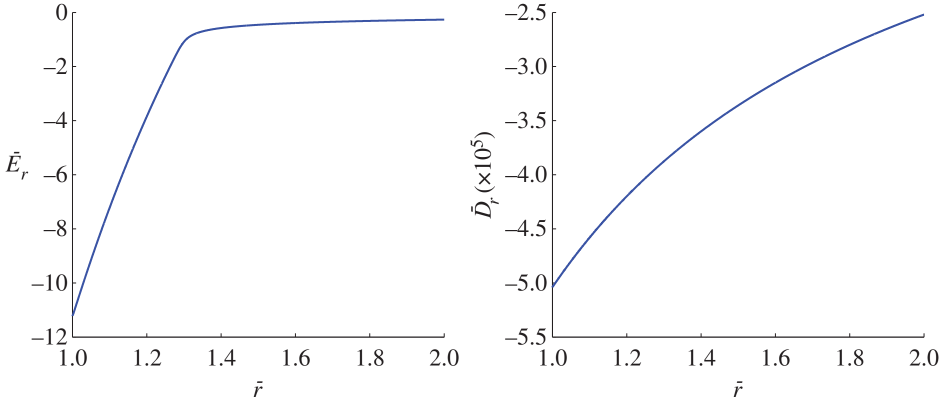

Figure 5. Normalized electric field and electric displacement for the cases presented in figure 4. (Online version in colour.)

figure 4*a*(i,ii) and in the other figures, one can observe that the azimuthal component of the stress presents a rapid increase in its value near $r = r_{\mathrm{i}}$, but the magnitude of the different components of the strain remain bounded.

- In figure 4*b*(i)(ii), results are presented for the normalized stresses and the components of the strain, for the same internal radial normal stress and difference in the electric potential, when $\varepsilon_{zz_0} = 0$.
- In figure 4*c*(i)(ii), results are presented for the case $p_{\mathrm{i}} = 1.3 \times 10^3$ Pa, $\varphi_{\mathrm{i}} = 0$, $\varphi_{\mathrm{o}} = 6 \times 10^2$ V and $\varepsilon_{zz_0} = -3 \times 10^{-5}$, i.e. the tube is being compressed.

Because the use of (4.6) with the constants from table 1 implies that the stresses have a weak influence in the distribution of electric field and electric displacement, for the electric variables only two plots are shown in figure 5.

2. In figure 6, results are presented for the case when an internal radial stress $p_{\mathrm{i}} = 2 \times 10^3$ Pa and an axial strain $\varepsilon_{zz_0} = 3 \times 10^{-5}$ V are applied. For the plot showing the normalized components of the stress (figure 6 upper side on the left), there is a very rapid increase in the azimuthal and axial components of the stress near $r = r_{\mathrm{i}}$. In the plot on the right, the same components of the stress are depicted for a narrower region near that point. The concentration for the stress in the case of the azimuthal component is of the order 200. The strains remain bounded and small.
3. Finally, in figure 7, results are portrayed for the case $p_{\mathrm{i}} = 0$, $\varphi_{\mathrm{i}} = 0$, $\varphi_{\mathrm{o}} = 3 \times 10^3$ V and $\varepsilon_{zz_0} = 3 \times 10^{-5}$. Because there is no internal radial normal stress, the components of the stress tensor are not shown normalized. The magnitude of the axial component of the stress is quite high compared with the other components; therefore, the different components of the total stress tensor are shown separately. The components of the electric field and the electric displacement are normalized as in equation (4.28).

5. Final remarks

In this paper, we studied boundary value problems, within the context of the electro-elastic constitutive relations presented in part I [1]. The constitutive relations allow for the nonlinear coupling between the stresses, linearized strain, electric field and the electric displacement, an impossibility within the context of theories for electro-elastic materials that are currently available. We studied several different boundary value problems with the view of determining the efficacy of the model developed in part I. We were particularly interested in depicting the strain limiting and the polarization saturation characteristics of electro-elastic behaviour that has

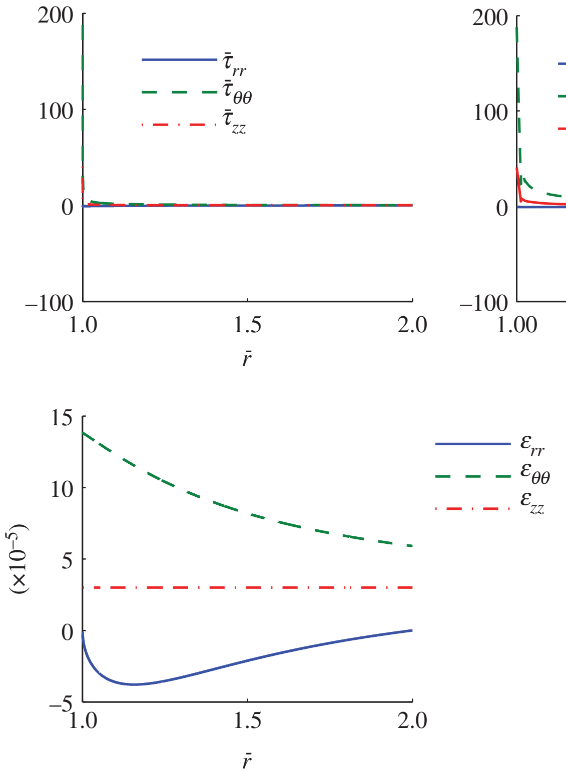

Figure 6. Normalized components of the stress and components of the strain, in the case there is no difference in the electric potential. (Online version in colour.)

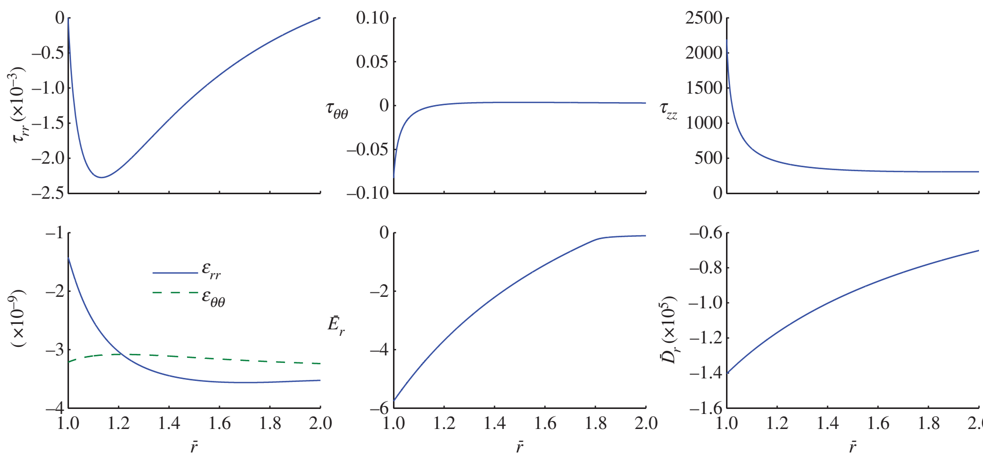

Figure 7. Results for the case there is no internal pressure applied on the tube. The components of the total stress tensor are in Pa. (Online version in colour.)

been observed and we were able to confirm such behaviour. Several boundary value problems were studied within the context of the constitutive relations. The first class of problems concerned homogeneous states of stress; and, in this case, the response of a slab in a state of uniform stress subject to traction, shear and an electric field was analysed. The second class of problems concerned inhomogeneous states of stress, and for such states of stress, we studied the response of a thin circular plate and a long cylindrical tube subject to inflation and an electric field. We were able to show that the bodies exhibited limiting strain and saturation of the polarization that is observed in such electro-elastic bodies.

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