PUSHED TRAVELING FRONTS IN MONOSTABLE EQUATIONS WITH MONOTONE DELAYED REACTION

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(Communicated by Masaharu Taniguchi)

Abstract. We study the wavefront solutions of the scalar reaction-diffusion equations $u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x))$, with monotone reaction term $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h > 0$. We are mostly interested in the situation when the graph of g is not dominated by its tangent line at zero, i.e. when the condition $g(x) \leq g'(0)x$, $x \geq 0$, is not satisfied. It is well known that, in such a case, a special type of rapidly decreasing wavefronts (pushed fronts) can appear in non-delayed equations (i.e. with $h = 0$). One of our main goals here is to establish a similar result for $h > 0$. To this end, we describe the asymptotics of all wavefronts (including critical and non-critical fronts) at $-\infty$. We also prove the uniqueness of wavefronts (up to a translation). In addition, a new uniqueness result for a class of nonlocal lattice equations is presented.

1. **Introduction.** In this work, we focus our efforts on the study of the existence, uniqueness and asymptotics of positive monotone bounded traveling wave solutions $u(t,x) = \phi(\nu \cdot x + ct)$, $\phi(-\infty) = 0$, to the scalar reaction-diffusion equation

$$u_t(t,x) = \Delta u(t,x) - u(t,x) + g(u(t-h,x)), \ x \in \mathbb{R}^m. \tag{1}$$

It is assumed that $\nu \in \mathbb{R}^m, |\nu| = 1$, that the wave velocity c is positive and the continuous monotone nonlinearity $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following assumption

(H) g is strictly increasing and the equation $g(x) = x$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Moreover, g is differentiable at the equilibria with $g'(0) > 1$,

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$g'(\kappa) < 1$, and g is C^1-smooth in some neighborhood of κ. In addition, there exist $C > 0$, $\theta \in (0, 1]$, $\delta > 0$ such that

$$|g(u)/u - g'(0)| \leq Cu^\theta, \quad u \in (0, \delta]. \tag{2}$$

Perhaps, model (1) is one of the simplest and most studied monostable delayed reaction-diffusion equations. See [1, 5, 15, 16, 18, 21, 23, 24, 27, 35, 36, 37, 38, 41] and references therein for more detail regarding (1) and its non-local versions. In fact, the last decade of studies has lead to almost complete description of the existence, uniqueness and stability properties of wavefronts to (1) whenever g satisfies (**H**) and the following quite important sub-tangency condition

$$g(x) \leq g'(0)x, \quad x \geq 0. \tag{3}$$

The latter inequality was already used in the celebrated work by A. Kolmogorov, I. Petrovskii and N. Piskunov [20] , where it was assumed that $g'(x) < g'(0)$ for all $x \in (0, \kappa]$. Roughly speaking, inequality (3) amounts to the dominance of the 'linear component' within essentially non-linear model (1). From the technical point of view, (3) together with the monotonicity of g allows to simplify enormously the analysis of traveling waves. Below we will illustrate this point in greater detail by discussing such key issues as the minimal (critical) speed of propagation, the stability, existence and uniqueness of waves, the asymptotic properties of wave profiles. In fact, none of these issues has been satisfactory investigated in that strongly nonlinear case when (3) does not hold, g is not monotone and $h > 0$[1]. The situation becomes more encouraging if we assume the monotonicity of g. In such a case, it is possible to apply the general theory of spreading speeds for abstract monostable evolution systems developed recently by X. Liang and X.-Q. Zhao in [21, 22]. In particular, the following result can be easily deduced from [22, Theorems 4.3, 4.4 and Remark 4.1] (see also [21, Section 5]):

Proposition 1. *Assume that $g, g'(0) > 1$, is increasing and continuous and that equation $g(x) = x$ has exactly two nonnegative solutions: 0 and $\kappa > 0$. Then the spreading speed c_* exists and it coincides with the minimal wave speed for (1).*

The Liang and Zhao approach was developed within the general framework of monotone semiflows that makes it possible to substantially weaken the smoothness conditions on g, especially at the neighborhoods of $0, \kappa$. Actually conditions similar to (2) can be found in almost all works concerning the front existence problem.[2] On the other hand, equation (1) determines a semiflow even without the requirement of Lipschitz continuity of g. This is due to the special form of equation (1). Indeed, to find a solution of the initial value problem $u(\theta, x) = u_0(\theta, x)$, $\theta \in [-h, 0]$, $x \in \mathbb{R}$, for (1) on the interval $[0, h]$, it suffices to integrate a *linear* inhomogeneous equation.

In order to appreciate more the above Liang and Zhao result, we would like to stress the following fact: the existence of a positive c_* splitting $\mathbb{R}_+$ on subsets of admissible and non-admissible (semi-)wavefront[3] speeds remains an unsolved problem when g is non-monotone and non-subtangential.

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[1]If $h = 0$, the wavefront problem for (1) is essentially bi-dimensional in many aspects and it is rather well understood, cf. [2, 13, 17, 19, 31, 42]. Next, since $-u + g(v)$ is negative for some $u, v \geq 0$, Schaaf's results [30] can not be applied to (1). In any event, the question of pushed waves was not considered in [30].

[2]Works [18, 36, 37] show that, after using an appropriate regularization argument, (2) can be dropped even in non-monotone case. However, this trick works only if g is subtangential.

[3]For non-monotone g, it is necessary to introduce some adjustments to the definition of traveling front, replacing it with the concept of semi-wavefront solution, see [35, 37]

One of our main objectives here is to complement Proposition 1 by considering other important open questions: the uniqueness, the asymptotics of wavefronts as well as the continuous dependence of c_* on g when condition (3) is not assumed. Our approach is different from [21, 22] and requires more restrictive hypothesis **(H)**.

At this stage of discussion, it is instructive to raise the same questions but for a different family of delayed evolution equations

$$u_t(t,x) = [u(t,x+1) + u(t,x-1) - 2u(t,x)] - u(t,x) + g(u(t-h,x)), \ x \in \mathbb{R}. \ (4)$$

It is obtained from (1), $m = 1$, by a formal discretization of the Laplace operator. Equivalently, we can consider the lattice differential equations

$$u'_n(t) = [u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - u_n(t) + g(u_n(t-h)), \ n \in \mathbb{Z}. \qquad (5)$$

Equations (4), (5) are special cases of more general nonlocal lattice population model proposed in [40]. Fortunately, when g is as in Proposition 1, the Liang and Zhao theory applies to equation (5) and to its non-local version (13) as well. In particular, this guarantees the existence of the minimal speed of propagation of traveling fronts. The existence, uniqueness, monotonicity and stability of wavefronts for the above equations were also analyzed by Ma and Zou in [25]. Once again, (3) together with **(H)** were assumed in the cited work. One of the noteworthy features of [25] consists in its novel (and non-trivial) proof of the wave uniqueness. This proof does not impose any restriction on $\sup\{g'(x), x \in [0, \kappa]\}$ what is quite remarkable in the case of delayed equations, cf. Subsection 1.2.

On the other hand, starting from the pioneering work of Zinner, Harris and Hudson [44], significant progress has been achieved in the understanding of waves solutions to non-delayed versions of (4), (5). See [8, 9, 25, 44] for more information and further references. Non-delayed equation (4) can be also viewed as a particular case of the following differential equation with convolution

$$u_t(t,x) = (J * u)(t,x) - u(t,x) + g(u(t,x)), \ x \in \mathbb{R}, \qquad (6)$$

which was introduced by Kolmogorov *et al* in [20]. Equation (6) was thoroughly investigated during the past three decades using various techniques, see [1, 7, 10, 11, 32] and references therein. Remarkably, sub-tangency condition (3) was avoided in the recent important contributions by Chen *et al.* [8, 9] and by Coville *et al.* [10, 11]. Our present work was nourished in part by several ideas and approaches developed in the mentioned four papers. For example, our analysis of the dependence $c_* = c_*(g)$ is based on the lower-upper solution method. Once again, the main difficulty consists in finding a 'good' upper solution (which additionally has to dominate lower solution), cf. consonant ideas expressed in [8, pp. 125-126] and [33]. As in [8], we will construct a new upper solution (for some velocity c' close to a given velocity c) from a given wavefront $\phi(t, c)$. In this way, $\phi(t, c)$ is considered as a skeleton (we call it 'a base function') for creating an upper solution by its suitable modification.[4] However, in difference with [8], our upper solution is not only formal but also *true upper solution* appearing in pair with an appropriate lower solution. Next, two noteworthy differences appear while comparing (1) and (6). First of them is technical: the presence of the second derivatives in (1) complicates the construction of the lower and upper solutions for (1) (these solutions must be C^1-smooth or satisfy additional conjugacy relations at the discontinuity points of the derivative, cf. [3, 6, 33, 41]). The other difficulty is more essential: the presence of positive

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[4]A similar idea was successfully applied to a model of the Belousov-Zhabotinskii reaction [33].

delay h can lead to the non-monotonicity of traveling fronts [4, 16, 28, 35, 37] while such monotonicity seems to be crucial for the applicability of various approaches, e.g. of the sliding solution method [3, 9, 10, 11]. Precisely in order to avoid front oscillations around κ, we will consider strictly increasing g in **(H)**. It should be mentioned that monotonicity of g is not obligatory when $h = 0$: this is because function $g(u(t - h)) + ku(t)$ is monotone in $u(t)$ for $k \gg 1$, $h = 0$, cf. [1].

Before going back to more detailed analysis of the main problems addressed in this paper, we would like to state some useful results concerning the wavefronts to equation (1) considered under assumption **(H)**. Set $g'_+ := \sup_{x>0} g(x)/x \geq g'(0) > 1$ and define $c_\#$ [respectively, c^*] as the unique positive number c for which the characteristic equation

$$\chi(z, c) := z^2 - cz - 1 + pe^{-zch} = 0 \tag{7}$$

with $p = g'(0)$ [respectively, with $p = g'_+$] has a double positive root. It is easy to see that $c_\# \leq c^*$. Note that $c_\# = c^*$ coincides with the minimal speed of propagation c_* whenever (3) is satisfied. If $c > c_\#$ then the characteristic equation (7) with $p = g'(0)$ has exactly two real solutions $0 < \lambda_2 < \lambda_1$, $\lambda_j = \lambda_j(c)$.

Proposition 2. *Assume* **(H)**. *Then, for each $c \geq c^*$ equation (1) has at least one monotone positive traveling front $u(t, x) = \phi(\nu \cdot x + ct, c)$. Next, for $c < c_\#$ equation (1) does not possess any positive bounded wave $u(t, x) = \psi(\nu \cdot x + ct)$, $\psi(-\infty) = 0$. Moreover, each positive bounded wave to (1) (if exists) is in fact a monotone front with profile satisfying $\phi'(s) > 0, s \in \mathbb{R}$. Finally, if $c \neq c_\#$, then the following asymptotic representation is valid (for appropriate s_0, $j \in \{1, 2\}$ and some $\varsigma > 0$):*

$$(\phi, \phi')(t + s_0, c) = e^{\lambda_j t}(1, \lambda_j) + O(e^{(\lambda_j + \varsigma)t}), \ t \to -\infty. \tag{8}$$

If $c = c_\#$ then besides (8) it may happen that

$$(\phi, \phi')(t + s_0, c) = -te^{\lambda_j t}(1, \lambda_j) + O(e^{\lambda_j t}), \ t \to -\infty. \tag{9}$$

Proof. The existence of fronts for $c \geq c^*$ follows from [36, Theorem 4] while their non-existence for $c < c_\#$ is a well known fact (e.g. see [36, Theorem 1]). Due to [35, Corollary 12], the wave profiles ψ are monotone, with $\psi'(s) > 0$, $s \in \mathbb{R}$. The exponential convergence $\psi(t) \to 0, t \to -\infty$, is a consequence of the Diekmann-Kaper theory, see [12] and [1, Lemma 3]. Therefore there is $\delta > 0$ such that

$$g(\psi(t - ch)) = [g'(0) + r(t)] \psi(t - ch), \text{ where } r(t) := \frac{g(\psi(t - ch))}{\psi(t - ch)} - g'(0) = o(e^{\delta t}).$$

On the other hand, it is easy to see that the convergence $\psi(t) \to 0, t \to -\infty$, is not super-exponential, cf. [37, Theorem 5.4 and Remark 5.5]. Now we can proceed as in [37, Remark 5.5] (where [26, Proposition 7.2] should be used) to obtain asymptotic formulas (8), (9).   □

Remark 1. Assuming that $g'(\kappa) < 1$ and that $|g'(\kappa) - g'(x)| \leq C|x - \kappa|^\theta$ for all x from some left neighborhood of κ, we can also derive an asymptotic representation of $(\phi, \phi')(t)$ at $+\infty$. Indeed, it is a standard exercise to check that the characteristic function $\chi(z, c)$ with $p \in (0, 1)$ has exactly to real roots $\mu_1(c) < 0 < \mu_2(c)$ while $\Re\mu_j(c) < \mu_1(c)$ for each complex root of $\chi(z, c)$. Therefore the positive equilibrium κ is hyperbolic and we can apply arguments developed in [16, Lemma 16] to obtain

$$(\kappa - \phi, -\phi')(t + s_0, c) = e^{\mu_1 t}(1, \mu_1) + O(e^{(\mu_1 - \varsigma)t}), \ t \to +\infty. \tag{10}$$

1.1. **A continuity property of the minimal speed of propagation.** Due to Proposition 2, if we assume (3) then the minimal speed c_* can be computed from characteristic equation (7) taken with $p = g'(0)$. Without (3), the computation of c_* represents a very difficult task even for non-delayed models [2, 17, 42]. In such a case, the value of $c_* = c_*(g)$ depends not only on $g'(0)$ but also on the whole monotone nonlinearity g. It is easy to realize that this dependence is lower semi-continuous with respect to the uniform convergence. In addition, it is immediate to find a sequence of strictly monotone smooth functions g_n, $g_n(0) = 0$, $g_n(\kappa) = \kappa$, $\max_{x \in [0,\kappa]} |g_n(x) - g_0(x)| \to 0$, $n \to \infty$, such that $\lim c_*(g_n) > c_*(g_0)$. It is also known that the function $c_* = c_*(g)$ is monotone: $c_*(g_1) \leq c_*(g_2)$ if $g_1(x) \leq g_2(x)$, $x \in [0,\kappa]$, cf. [22, Lemma 3.5]. However, we are unaware of conditions sufficient for the continuity of $c_*(g)$ with respect to a reasonable convergence in the space of strictly increasing functions g. In this section, we provide a first result in this direction by considering sequences $\{g_n\}_{n \geq 0}$ of functions satisfying assumption (**H**) with (2) replaced with the slightly more restrictive inequality

$$|g'(\kappa - u) - g'(\kappa)| + |g'(u) - g'(0)| \leq Cu^\theta, \quad u \in [0, \delta_g]. \tag{11}$$

Theorem 1.1. *Suppose that each continuous g_n satisfies (11) (with a suitable δ_{g_n}) and* (**H**). *If, additionally, there exists some small $r > 0$ such that $g_n \to g_0$ in $C^1[0, r]$, in $C^1[\kappa - r, \kappa]$ and in $C[0, \kappa]$, then $\lim c_*(g_n) = c_*(g_0)$.*

Remark 2. 1. We believe that the convergence of strictly increasing functions $g_n \to g_0$ in $C^1[0, r]$ and in $C[0, \kappa]$ could be well enough to have $\lim c_*(g_n) = c_*(g_0)$. 2. The proof of Theorem 1.1 contains a new approach to the problem of the existence of the minimal speed c_*, see [34, Theorem 1.1] for more details. 3. For the reader convenience, the asymptotics of noncritical wavefronts obtained in the proof of Theorem 1.1 are presented in the first part of Theorem 1.4.

1.2. **Uniqueness of wavefronts.** More subtle aspects of uniqueness and stability of wavefronts in (1) were studied so far under the geometric conditions even more restrictive than (3). For example, $g''(s) \leq 0$ was required in the main stability theorem of [27]. Similarly, uniqueness (up to a shift) of each non-critical (i.e. $c \neq c_*$) monotone traveling front of equation (1) can be deduced from [38, Corollary 4.9] whenever g meets the conditions: (A1) $g \in C^2[0, \kappa]$, $g(x) > 0, x \in (0, \kappa)$; (A2) $g'(\kappa) < 1$ and (3) holds; (A3) For every $\delta \in (0, 1)$, there exist $a = a(\delta) > 0$, $\alpha = \alpha(\delta) \geq 0$ and $\beta = \beta(\delta) \geq 0$ with $\alpha + \beta > 0$ such that for any $\theta \in (0, \delta]$ and $v \in [0, \kappa]$,

$$(1 - \theta)g(v) - g((1 - \theta)v) \leq -a\theta\kappa^\alpha v^\beta.$$

Let us show that (A3) is stronger than (3). Indeed, after dividing the latter inequality by θ and taking limit as $\theta \to +0$, we find that

$$-g(v) + g'(v)v \leq -a\kappa^\alpha v^\beta < 0, \quad v \in [0, \kappa].$$

Therefore $g'(v) < g(v)/v$, $v \in (0, \kappa]$, that, after an easy integration, yields

$$0 \leq g'(v) < \frac{g(v)}{v} \leq \frac{g(u)}{u} \leq g'(0+) = \lim_{u \to +0} \frac{g(u)}{u}, \quad v \geq u.$$

It is clear that the above inequalities are stronger that the Lipshitz condition

$$|g(u) - g(v)| \leq g'(0)|u - v|, \quad u, v \in [0, \kappa], \tag{12}$$

which in turn is more restrictive than (3).

Inequality (12) is one of the basic conditions of the uniqueness theory developed by Diekmann and Kaper, see contributions [12] and [1]. Suppose, for instance,

that $g \in C^{1,q}$ in some neighborhood of 0. Then (12) implies the uniqueness of all non-critical [12] as well as critical [1] wavefronts to (1). Additionally, [1] establishes the uniqueness of all fronts propagating at the velocity $c > c_u$ where c_u can be computed (similarly to c_* in the sub-tangential case) from the equation

$$z^2 - cz - 1 + \text{ess} \sup_{v \in [0,\kappa]} g'(v)e^{-zch} = 0.$$

An alternative approach to the uniqueness problem is based on the sliding method developed by Berestycki and Nirenberg [3]. This technique was successfully applied in [8, 10, 11, 25] to prove the uniqueness of *monotone* wavefronts without imposing any Lipshitz condition on g.[5] In the present paper, inspired by a recent Coville's work [10], we use the sliding method to prove the following assertion:

Theorem 1.2. *Assume that* **(H)** *is satisfied. Fix some* $c \geq c_*$*, and suppose that* $u_1(t,x) = \phi(\nu \cdot x + ct)$*,* $u_2(t,x) = \psi(\nu \cdot x + ct)$ *are two traveling fronts of equation (1). Then* $\phi(s) = \psi(s + s_0)$*,* $s \in \mathbb{R}$*, for some* s_0*.*

We note that, when $h > 0$, we were not able to drop the condition of strict monotonicity on g imposed in Theorem 1.2 (even while considering only monotone wavefronts). If $h = 0$, the monotonicity of g is not obligatory.

The ideas behind the proof of Theorem 1.2 combined with asymptotic description of wavefronts given in [1] also allow to derive a new uniqueness result for the following nonlocal lattice system

$$u_n'(t) = D[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)] - u_n(t) + \sum_{k \in \mathbb{Z}} \beta(n-k)g(u_k(t-h)), \ n \in \mathbb{Z}, \ (13)$$

where $D > 0, \beta(k) \geq 0$, $\sum_{k \in \mathbb{Z}} \beta(k) = 1$. Let $\gamma^\#$ be an extended non-negative real number such that $B(z) := \sum_{k \in \mathbb{Z}} \beta(k)e^{-zk}$ is finite when $z \in [0, \gamma^\#)$ and is infinite when $z > \gamma^\#$. By Cauchy-Hadamard formula, $\gamma^\# = -\limsup_{k \to +\infty} k^{-1} \ln \beta(-k)$, where we adopt the convention that $\ln(0) = -\infty$. Our requirement is that such $\gamma^\#$ is positive and that $B(\gamma^\# -) = +\infty$.

Theorem 1.3. *Assume* **(H)** *except for the strict character of the monotonicity of* g*. Suppose that* $w_j(t) = \phi(j + ct)$*,* $v_j(t) = \psi(j + ct)$ *are traveling fronts to nonlocal lattice equation (13) and* $c \neq 0$*. Then there is* s_0 *such that* $\phi(s) = \psi(s + s_0)$*,* $s \in \mathbb{R}$*.*

Remark 3. In Theorem 1.2, the inequality $g'(\kappa) < 1$ is formally required. However, our proof uses more weak restriction $g'(s) \leq 1$, $s \in [\kappa - \sigma, \kappa]$ for some $\sigma > 0$.

Remark 4. In various aspects, Theorem 1.3 improves and generalizes on the non-local case the main uniqueness theorem from [25]. In contrast with the mentioned result, we do not impose sub-tangency condition (3) and we allow for critical (minimal) waves. Next, the uniqueness result of [25] is valid only for profiles having prescribed asymptotic behavior at $-\infty$. Note also that our proof is rather short and does not use the monotonicity of profiles. Now, condition $c \neq 0$ seems to be essential, as Proposition 6.7 of work [11] suggests the possibility of infinitely many wave solutions (perhaps, discontinuous) for $c = 0$. On the other hand, Theorem 1.3 complements the main result of [14] (which is valid only for non-critical waves), where (12) was assumed together with the symmetry $\beta(k) = \beta(-k)$. Even though [14] (see also [1, 43] for several improvements) allows to consider non-monotone nonlinearity g.

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[5]In difference, the Diekmann-Kaper theory can be applied to the non-monotone waves and nonlinearities.

1.3. **Asymptotic formulas for the wave profiles.** It is well known [13] that in non-delayed case each critical wavefront which propagates at the velocity $c_* > c_\#$ (i.e. so called *pushed wavefront*) has its profile converging to 0 more rapidly than the near (i.e. propagating with the speeds $c \approx c_*$) non-critical wavefront profiles. This contrasts with the case $c_* = c_\#$, when the profile of the critical front (so called *pulled wavefront*) converges to 0 approximately at the same rate as the profile of each near wavefront does. Similar asymptotics were also established for wavefront solutions of lattice equation (4) without delay, see [9, Theorem 3]. Our third main result shows that the pushed fronts to (1) obey the same principle:

Theorem 1.4. *Assume* **(H)** *and let* $u(t, x) = \phi(\nu \cdot x + ct)$ *be a traveling front to equation* (1). *Then the following asymptotic represantions are valid (for an appropriate* s_0 *and some* $\sigma > 0$):

  1) if $c > c_*$ *then* $\phi(s + s_0) = e^{\lambda_2 t} + O(e^{(\lambda_2 + \sigma)t})$, $t \to -\infty$,
  2) if $c = c_* > c_\#$, *then* $\phi(s + s_0) = e^{\lambda_1 t} + O(e^{(\lambda_1 + \sigma)t})$, $t \to -\infty$.

The proof of the second formula is the most difficult part of this theorem. In order to establish that the pushed fronts to (1) satisfy 2), it suffices to show that each wavefront having asymptotic behavior as in 1) is 'robust' with respect to small perturbations of the velocity c. This would imply the existence of wavefronts propagating at the velocity $c' < c_*$ provided that the critical front behaves as in 1). The necessary perturbation result is demonstrated here with the use of upper-lower solutions method. Note that, due to the use of a discontinuous upper solution, the application of this method in the paper is different from a standard procedure.

2. **Proof of Theorem 1.1.** Fix some $h > 0$ and let functions $\{\tilde{g}_n\}_{n \geq 0}$ satisfy conditions of Theorem 1.1. Then the sequence $g'_{+,n} := \sup_{x \in (0,\kappa]} \tilde{g}_n(x)/x$, $n \geq 0$, is bounded and therefore, by Proposition 2, the sequence of positive numbers $c_*(\tilde{g}_n), n \geq 0$, is also bounded. Alternatively, the boundedness of $c_*(\tilde{g}_n), n \geq 0$, can be deduced from the convergence $\tilde{g}_n \to \tilde{g}_0$ and the monotonicity property of $c_*(\tilde{g})$, see Subsection 1.1.

Hence, without loss of generality, we can suppose that $\lim c_*(\tilde{g}_n)$ exists. Then $\lim c_*(\tilde{g}_n) \geq c_*(\tilde{g}_0)$ because of the lower semi-continuity property of $c_*(g)$. Suppose, for a contradiction, that $\lim c_*(\tilde{g}_n) > c_*(\tilde{g}_0)$. Set $g_n(x) = \max\{\tilde{g}_0(x), \tilde{g}_n(x)\}$, $n \geq 0$, then $g_n(x) \geq \tilde{g}_n(x)$ and therefore, without restricting the generality, we can assume that, with some intermediate c_0, it holds $c'' := \lim c_*(g_n) > c_0 > c_*(g_0)$. By Proposition 1, equation (1) has a wavefront $u(t, x) = \phi(\nu \cdot x + c_0 t)$ with ϕ satisfying

$$\phi''(t) - c_0 \phi'(t) - \phi(t) + g(\phi(t - c_0 h)) = 0.$$

Here and subsequently we are dropping the index 0 in $g_0 = \tilde{g}_0$. Next, $\lambda_{2,n}(c)$ will denote the smallest positive zero of the characteristic function $\chi_n(z, c) := z^2 - cz - 1 + g'_n(0)e^{-zch}$. Since $\lim g'_n(0) = g'(0)$, the value $\lambda_{2,n}(c)$ is well defined for all large n and $c \geq c_0$. Observe also that $\lambda_{2,n}(c)$ is a decreasing continuous function of c and that $\lim \lambda_{2,n}(c) = \lambda_2(c)$. To simplify the notation, we will write $\lambda' := \lambda_{2,n}(c')$, $\lambda := \lambda_2(c_0)$.

Now, due to Proposition 2 and Remark 1, there exist $S_1 < S_2$ such that $\phi(S_1) < r$, $\phi(S_2 - c_0 h) = \kappa - r$ and $\phi'(t)/\phi(t) \geq \lambda/2$ for all $t \leq S_1$, while $\phi'(t)/(\kappa - \phi(t - c_0 h)) > -0.5\mu_1(c_0)e^{c_0 h \mu_1(c_0)}$ if $t \geq S_2$. The eigenvalue $\mu_1(c_0) < 0$ was defined in Remark 1. By our assumptions, $G(u) := g(u)/u$ is C^1-smooth function within some connected left neighborhood of κ. Since $G'(\kappa) = (g'(\kappa) - 1)/\kappa < 0$, we can suppose that

$$G(u) - G(v) = G'(\omega)(u - v) < 0, \ u > v, \tag{14}$$

for $u, v, \omega \in [\kappa - r, \kappa]$ (after choosing appropriately small r).

Fix m large enough and $c' - c_0 > 0$ small enough to satisfy $c' < c''$, $(1+\theta)\lambda' > \lambda$, $\lambda' < \lambda$, $c_*(g_m) > c'$, $\chi_m(\lambda, c') < 0$, $|\tilde{g}_m - g|_{C[0,\kappa]} + (c_0 - c')\phi'(t) < 0, t \in [S_1, S_2]$,

$$c' - c_0 \gg 2\lambda^{-1}|\tilde{g}_m - g|_{C^1[0,r]} - 2\mu_1^{-1}(c_0)|\tilde{g}_m - g|_{C^1[\kappa-r,\kappa]}e^{-c_0 h \mu_1(c_0)}.$$

Then, for each $t \in \mathbb{R}$, we obtain that

$$\mathfrak{G}_m(t) := g_m(\phi(t - c_0 h)) - g(\phi(t - c_0 h)) + (c_0 - c')\phi'(t) \leq$$

$$\leq \begin{cases} \phi(t)|\tilde{g}_m - g|_{C^1[0,r]} + (c_0 - c')\phi'(t) < 0, & \text{if } t \leq S_1; \\ |\tilde{g}_m - g|_{C[0,\kappa]} + (c_0 - c')\phi'(t) < 0, & \text{if } t \in [S_1, S_2]; \\ (\kappa - \phi(t - c_0 h))|\tilde{g}_m - g|_{C^1[\kappa-r,\kappa]} + (c_0 - c')\phi'(t) < 0, & \text{if } t \geq S_2. \end{cases}$$

The remainder of the proof is divided in several parts.

Step I (Construction of a base function). Set $\phi_\sigma(t) := \sigma\phi(t)$, where $\sigma \in (1, 2)$ is close to 1. We have

$$E_m(t, \sigma) := \phi_\sigma''(t) - c'\phi_\sigma'(t) - \phi_\sigma(t) + g_m(\phi_\sigma(t - c'h)) \leq \phi_\sigma''(t) - c_0\phi_\sigma'(t) - \phi_\sigma(t) +$$

$$\sigma g(\phi(t - c_0 h)) + [(c_0 - c')\phi_\sigma'(t) + g_m(\phi_\sigma(t - c_0 h)) - \sigma g(\phi(t - c_0 h))] =$$

$$(c_0 - c')\phi_\sigma'(t) + g_m(\phi_\sigma(t - c_0 h)) - \sigma g(\phi(t - c_0 h)).$$

By our assumptions, $g_m(x) = g_m'(0)x + o(x)$, $x \to 0$. Therefore, for $t \to -\infty$,

$$|g_m(\phi_\sigma(t - c_0 h)) - \sigma g(\phi(t - c_0 h))| \leq \sigma|g_m(\phi(t - c_0 h)) - g(\phi(t - c_0 h))| +$$

$$|g_m(\phi_\sigma(t - c_0 h)) - \sigma g_m(\phi(t - c_0 h))| \leq \sigma\phi(t - c_0 h)|\tilde{g}_m - g|_{C^1[0,r]} + o(\phi(t - c_0 h)).$$

On the other hand, we infer from Proposition 2 that

$$(c_0 - c')\phi_\sigma'(t) = (c_0 - c')\zeta\sigma\phi(t)(1 + o(1)) = (c_0 - c')\zeta\sigma e^{\zeta c_0 h}\phi(t - c_0 h)(1 + o(1)).$$

for $\zeta \in \{\lambda_1(c_0), \lambda_2(c_0)\}$. As a consequence, there exists S_0 (which does not depend on σ) such that, for all σ close to 1,

$$E_m(t, \sigma) < 0, \ t \leq S_0.$$

Due to (14) and Remark 1, we obtain, for $\phi_\sigma(t - c_0 h)$, $\phi(t - c_0 h) \in [\kappa - r, \kappa]$, $t \geq S_2$, that

$$(c_0 - c')\phi_\sigma'(t) + g_m(\phi_\sigma(t - c_0 h)) - \sigma g(\phi(t - c_0 h)) = g_m(\phi_\sigma(t - c_0 h)) - g(\phi_\sigma(t - c_0 h)) +$$

$$(c_0 - c')\phi_\sigma'(t) + \sigma\phi(t - c_0 h)(G(\phi_\sigma(t - c_0 h)) - G(\phi(t - c_0 h))) <$$

$$(\kappa - \phi(t - c_0 h))|\tilde{g}_m - g|_{C^1[\kappa-r,\kappa]} + (c_0 - c')\phi'(t) < 0.$$

Hence, if $S_2'(\sigma)$ is defined as a unique point at which $\phi_\sigma(S_2'(\sigma) - c_0 h) = \kappa$, then for all σ close to 1,

$$E_m(t, \sigma) < 0, \ t \in [S_2, S_2'(\sigma)].$$

Finally, since uniformly on $[S_0, S_2]$

$$\lim_{\sigma \to 1}((c_0 - c')\phi_\sigma'(t) + g_m(\phi_\sigma(t - c_0 h)) - \sigma g(\phi(t - c_0 h))) = \mathfrak{G}_m(t) < 0,$$

we conclude that $E_m(t, \sigma) < 0$ for all σ close to 1 and all t such that $\sigma\phi(t - c_0 h) \leq 1$.

Step II (Construction of an upper solution). Fix σ such that $E_m(t, \sigma) < 0$ and for $a := b^2$, $b \in (0, 1]$, set $\phi_b(t) := \phi_\sigma(t) + ae^{\lambda' t} + be^{\lambda t}$. Let $S_3 = S_3(b)$ be that

unique point where $\phi_b(S_3(b)) = \kappa$. It is clear that $\phi_b'(S_3) > 0$ and $S_3 < S_2'(\sigma) - c_0 h$. Next,

$$E_+(t,b) := \phi_b''(t) - c'\phi_b'(t) - \phi_b(t) + g_m(\phi_b(t - c'h)) = E_m(t,\sigma) + b\chi_m(\lambda, c')e^{\lambda t} +$$

$$g_m(\phi_\sigma(t - c'h) + ae^{\lambda'(t-c'h)} + be^{\lambda(t-c'h)}) - g_m(\phi_\sigma(t - c'h)) -$$

$$g_m'(0)(ae^{\lambda'(t-c'h)} + be^{\lambda(t-c'h)}) \le E_m(t,\sigma) + b\chi_m(\lambda, c')e^{\lambda t} +$$

$$C(ae^{\lambda'(t-c'h)} + be^{\lambda(t-c'h)})(\phi_\sigma(t - c'h) + ae^{\lambda'(t-c'h)} + be^{\lambda(t-c'h)})^\theta \le E_m(t,\sigma) +$$

$$be^{\lambda t}\left(\chi_m(\lambda, c') + 3Ce^{-\lambda c'h}(be^{(\lambda'-\lambda)(t-c'h)} + 1)(\phi_\sigma^\theta(t - c'h) + 2b^\theta e^{\lambda'\theta(t-c'h)})\right) \le$$

$$E_m(t,\sigma) + be^{\lambda t}\left(\chi_m(\lambda, c') + C_1\phi_\sigma^\theta(t - c'h) + C_2 b^\theta e^{(\lambda'(1+\theta)-\lambda)(t-c'h)} +\right.$$

$$\left. C_3 b\phi_\sigma^\theta(t - c'h)e^{(\lambda'-\lambda)t}\right) \le E_m(t,\sigma) + be^{\lambda t}\left(\chi_m(\lambda, c') + C_4 e^{\nu t}\right), \quad t \le S_4,$$

for some positive ν, C_j and negative S_4 (which does not depend on b). Since $\chi_m(\lambda, c') < 0$, we may choose S_4 is such a way that $E_+(t,b) < 0$ for all $t \le S_4$, $b \in (0,1]$. On the other hand, we know that, uniformly on each compact interval, $E_+(t,b) \to E_m(t,\sigma), b \to 0+$. Therefore $E_+(t,b) < 0$, $t < S_2'(\sigma) = c_0 h + S_3(0)$ for all sufficiently small b.

Now, let us define an upper solution ϕ_+ by $\phi_+(t) := \min\{\kappa, \phi_b(t)\}$. It is clear that $\phi_+(t)$ is continuous and piece-wise C^1 on $\mathbb{R}$, being $t_0 := S_3(b)$ the unique point of discontinuity of the derivative where $\Delta\phi_+'|_{t_0} := \phi_+'(t_0+) - \phi_+'(t_0-) = -\phi_b'(t_0-) < 0$.

Step III (Construction of a lower solution). Consider the following concave monotone linear rational function

$$p(x) := \frac{g_m'(0)x}{1 + Ax} \le g_m'(0)x, \ x \ge 0, \ A := 2\frac{g_m'(0) - 1}{\kappa}, \ p(0) = 0, \ p(\frac{\kappa}{2}) = \frac{\kappa}{2},$$

and set $g_-(x) := \min\{g_m(x), p(x)\}$. Clearly, g_- is continuous and increasing, and

$$g_-'(0) = g_m'(0), \ g_-(0) = 0, \ g_-(\kappa/2) = (\kappa/2), \quad g_-(x) \le g_m'(0)x, \ x \ge 0.$$

Moreover, after some straightforward but tedious computations, one can check that function $g_-(x) = \min\{p(x), \max\{\tilde{g}_m(x), g(x)\}\}$ meets the smoothness condition of **(H)** in some right neighborhood of 0. This implies the existence of a monotone positive function ϕ_-, $\phi_-(-\infty) = 0$, $\phi_-(+\infty) = \kappa/2$, satisfying the equation

$$\phi_-''(t) - c'\phi_-'(t) - \phi_-(t) + g_-(\phi_-(t - c'h)) = 0,$$

e.g., see [36, Theorem 4]. Due to the property $g_-(x) \le g_m'(0)x$, $x \ge 0$, we also know that, for a small $\zeta > 0$,

$$(\phi_-, \phi_-')(s + s_0, c) = e^{\lambda' t}(1, \lambda') + O(e^{(\lambda'+\zeta)t}), \ t \to -\infty.$$

Finally, since $g_-(x) \le g_m(x)$ we obtain that

$$\phi_-''(t) - c'\phi_-'(t) - \phi_-(t) + g_m(\phi_-(t - c'h)) \ge 0.$$

Step IV (Iterations). Comparing asymptotic representations of monotone functions $\phi_-(t)$ and $\phi_+(t)$ at $-\infty$, we find easily that

$$\phi_-(t + s_1) \le \phi_+(t), \quad t \in \mathbb{R},$$

for some appropriate s_1. Simplifying, we will suppose that $s_1 = 0$. In the next stage of the proof, we need the following simple result:

Lemma 2.1. *Let $\psi : \mathbf{R} \to \mathbf{R}$ be a bounded classical solution of the second order impulsive equation*

$$\psi'' - c\psi' - \psi = f(t), \quad \Delta\psi|_{t_j} = \alpha_j, \quad \Delta\psi'|_{t_j} = \beta_j, \tag{15}$$

where $\{t_j\}$ is a finite increasing sequence, $f : \mathbf{R} \to \mathbf{R}$ is bounded and continuous at every $t \neq t_j$ and the operator Δ is defined by $\Delta w|_{t_j} := w(t_j+) - w(t_j-)$. Assume that equation $z^2 - cz - 1 = 0$ has two real roots $\xi_1 < 0 < \xi_2$, $\xi_j = \xi_j(c)$. Then

$$\psi(t) = \frac{1}{\xi_1 - \xi_2}\left(\int_{-\infty}^{t} e^{\xi_1(t-s)}f(s)ds + \int_{t}^{+\infty} e^{\xi_2(t-s)}f(s)ds\right) \tag{16}$$

$$+\frac{1}{\xi_2 - \xi_1}\left[\sum_{t<t_j} e^{\xi_2(t-t_j)}(\xi_1\alpha_j - \beta_j) + \sum_{t>t_j} e^{\xi_1(t-t_j)}(\xi_2\alpha_j - \beta_j)\right], \quad t \neq t_j.$$

Proof. See [33]. Alternatively, it can be checked by a straightforward substitution that ψ defined by (16) verifies equation (15). □

Similarly to [41], we also consider the monotone integral operator

$$(\mathcal{A}\phi)(t) := \frac{1}{\xi'_{12}}\left(\int_{-\infty}^{t} e^{\xi'_1(t-s)}g_m(\phi(s - c'h))ds + \int_{t}^{+\infty} e^{\xi'_2(t-s)}g_m(\phi(s - c'h))ds\right)$$

where $\xi'_{12} := \xi'_2 - \xi'_1$, $\xi'_j := \xi_j(c')$. Using properties of functions $\phi_-(t)$ and $\phi_+(t)$, we deduce from Lemma 2.1 that

$$\phi_-(t) \leq (\mathcal{A}\phi_-)(t) \leq (\mathcal{A}^2\phi_-)(t) \leq \cdots \leq (\mathcal{A}^2\phi_+)(t) \leq (\mathcal{A}\phi_+)(t) \leq \phi_+(t), \quad t \in \mathbb{R}.$$

The latter implies (see [41] for more detail) the existence of a monotone function $\phi(t)$ such that

$$(\mathcal{A}\phi)(t) = \phi(t), \quad \phi_-(t) \leq \phi(t) \leq \phi_+(t), \quad t \in \mathbb{R}.$$

This amounts to the existence of a wavefront for equation (1) (considered with the birth function g_m) propagating at velocity c'. Moreover, the latter estimations shows that, for some s_0 and positive δ,

$$\phi(s + s_0) = e^{\lambda' t} + O(e^{(\lambda'+\delta)t}), \ t \to -\infty. \tag{17}$$

Hence, inequality $c_*(g_m) > c'$ can not be true and therefore $\lim c_*(g_n) = c_*(g_0)$. □

3. **Proof of Theorem 1.2.** In our proof which was inspired by Coville work [10], we invoke the sliding method developed by Berestycki and Nirenberg [3, 8, 10, 11].

Lemma 3.1. *Fix some $c \geq c_*$ and suppose that ϕ, ψ are two wavefront profiles such that, for some finite T,*

$$\phi(t) < \psi(t), \quad t < T. \tag{18}$$

Then $\phi(t) < \psi(t)$ for all $t \in \mathbb{R}$.

Proof. Set $a_* = \inf \mathcal{A}$ where

$$\mathcal{A} := \{a \geq 0 : \psi(t) + a \geq \phi(t), \ t \in \mathbb{R}\}.$$

Note that $\mathcal{A} \neq \emptyset$ since $[\kappa, +\infty) \subset \mathcal{A}$. Moreover, $a_* \in \mathcal{A}$.

Now, if $a_* = 0$ then $\psi(t) \geq \phi(t)$, $t \in \mathbb{R}$. We claim that, in fact, $\psi(t) > \phi(t)$, $t \in \mathbb{R}$. Indeed, otherwise we can suppose that T is such that $\phi(T) = \psi(T)$. In this way,

the difference $\psi(t) - \phi(t) \geq 0$ reaches its minimal value 0 at T, while $\psi(T - ch) > \phi(T - ch)$. But then we get a contradiction:

$$0 = (\psi''(T) - \phi''(T)) - c(\psi'(T) - \phi'(T)) - (\psi(T) - \phi(T)) +$$
$$(g(\psi(T - ch)) - g(\phi(T - ch))) > 0. \qquad (19)$$

In this way, Lemma 3.1 is proved when $a_* = 0$ and consequently we may assume that $a_* > 0$. Let $\sigma > 0$ be small enough to satisfy

$$\max_{s \in [\kappa - \sigma, \kappa]} g'(s) \leq 1.$$

Case I. First, we assume that T is such that, additionaly

$$\phi(t), \psi(t) \in (\kappa - \sigma, \kappa), \ t \geq T - ch. \qquad (20)$$

In such a case non-negative function

$$w(t) := \psi(t) + a_* - \phi(t), \quad w(\pm\infty) = a_* > 0,$$

reaches its minimal value 0 at some leftmost point t_m, where

$$\psi(t_m) - \phi(t_m) = -a_*, \quad \psi'(t_m) - \phi'(t_m) = 0, \quad \psi''(t_m) - \phi''(t_m) \geq 0.$$

Since $\psi(t_m) < \phi(t_m)$, we have that $t_m > T$, so that

$$\psi_m := \psi(t_m - ch), \phi_m := \phi(t_m - ch) \in (\kappa - \sigma, \kappa).$$

In consequence, for some $\theta \in (\kappa - \sigma, \kappa)$,

$$0 = (\psi''(t_m) - \phi''(t_m)) - c(\psi'(t_m) - \phi'(t_m)) - (\psi(t_m) - \phi(t_m)) +$$
$$(g(\psi_m) - g(\phi_m)) \geq a_* + g(\psi_m) - g(\phi_m) \geq \qquad (21)$$

$$\begin{cases} a_* > 0, & \text{if } \psi_m \geq \phi_m; \\ a_* + g'(\theta)(\psi_m - \phi_m) > 0, & \text{if } \psi_m - \phi_m \in [-a_*, 0), \end{cases}$$

a contradiction. Observe that the strict inequality in the last line can be explained in the following way. The sign "$\geq$" can be replaced with "$=$" in

$$a_* + g(\psi(t_m - ch)) - g(\phi(t_m - ch)) = a_* + g'(\theta)(\psi(t_m - ch) - \phi(t_m - ch)) \geq 0,$$

if and only if $g'(\theta) = 1$ and $\psi(t_m - ch) - \phi(t_m - ch) = -a_*$. This, however, is impossible due to the definition of t_m as the leftmost point where $w(t_m) = 0$.

Case II. If (20) does not hold, then, due to the convergence of profiles at $+\infty$, we can find large $\tau > 0$ and $T_1 > T$ such that

$$\psi(t + \tau) > \phi(t), \ t < T_1, \quad \phi(t), \psi(t + \tau) \in (\kappa - \sigma, \kappa), \ t \geq T_1 - ch.$$

Therefore, in view of the result established in Case I, we obtain that

$$\psi(t + \tau) > \phi(t), \quad t \in \mathbb{R}. \qquad (22)$$

Define now τ_* by

$$\tau_* := \inf\{\tau \geq 0 : \text{ inequality (22) holds}\}.$$

It is clear that $\psi(t + \tau_*) \geq \phi(t), \ t \in \mathbb{R}$. Since, in addition,

$$\psi(t + \tau_*) \geq \psi(t) > \phi(t), \ t < T,$$

we conclude that $\psi(t + \tau_*) > \phi(t), \ t \in \mathbb{R}$, cf. (19). Now, if $\tau_* = 0$, then Lemma 3.1 is proved. Otherwise, $\tau_* > 0$ and for each $\varepsilon \in (0, \tau_*)$ there exists a unique $T_\varepsilon > T$ such that

$$\psi(t + \tau_* - \varepsilon) > \phi(t), \ t < T_\varepsilon, \ \psi(T_\varepsilon + \tau_* - \varepsilon) = \phi(T_\varepsilon).$$

It is immediate to see that $\lim T_\varepsilon = +\infty$ as $\epsilon \to 0+$. Indeed, if $T_{\varepsilon_j} \to T'$ for some finite T' and $\varepsilon_j \to 0+$, then we get a contradiction: $\psi(T' + \tau_*) = \phi(T')$. Therefore, if ε is small, then

$$\psi(t + \tau_* - \varepsilon), \phi(t) \in (\kappa - \sigma, \kappa), \ t \geq T_\varepsilon - ch,$$

that is $\psi(t + \tau_* - \varepsilon)$ and $\phi(t)$ satisfy condition (20) required in Case I. Thus we get $\psi(t + \tau_* - \varepsilon) > \phi(t)$ for all $t \in \mathbb{R}$, a contradiction to the definition of τ_*. This means that $\tau_* = 0$ and the proof of Lemma 3.1 is completed.   □

Corollary 1. *For a fixed $c \geq c_*$, both ϕ and ψ have the same type of asymptotic behaviour at $-\infty$ described in Proposition 2.*

Proof. For example, suppose that $\phi(t) \sim e^{\lambda_1 t}$ and $\psi(t) \sim e^{\lambda_2 t}$ as $t \to -\infty$. Then for every fixed $\tau \in \mathbb{R}$ there exists $T(\tau)$ such that $\psi(t + \tau) > \phi(t)$ for all $t < T(\tau)$. Applying Lemma 3.1, we obtain that $\psi(s) > \phi(t)$ for every $s := t + \tau, t \in \mathbb{R}$, what is clearly false.   □

Proof of Theorem 1.2. By Corollary 1, we can suppose that $\psi(t)$ and $\phi(t)$ have the same type (described in Proposition 2) of asymptotic behavior at $-\infty$. Consequently, $\psi(t + \tau), \phi(t)$ satisfy condition (18) of Lemma 3.1 for every small $\tau > 0$. But then $\psi(t + \tau) > \phi(t)$ for every small $\tau > 0$ that yields $\psi(t) \geq \phi(t), \ t \in \mathbb{R}$. By symmetry, we also find that $\phi(t) \geq \psi(t), \ t \in \mathbb{R}$, and Theorem 1.2 is proved.   □

4. **Proof of Theorem 1.3.** It is easy to see that each wave profile φ verifies

$$c\varphi'(t) = D[\varphi(t+1) + \varphi(t-1) - 2\varphi(t)] - \varphi(t) + \sum_{k \in \mathbb{Z}} \beta(k)g(\varphi(t - k - ch)). \quad (23)$$

First we note that $\varphi(t)$ takes its value in $(0, \kappa)$. Indeed, suppose for a moment that s_0 is the leftmost point where $M := \varphi(s_0) = \sup_{s \in \mathbb{R}} \varphi(s) \geq \kappa$. Then $\varphi'(s_0) = 0$ and $\varphi(s_0 + 1) + \varphi(s_0 - 1) - 2\varphi(s_0) < 0$, $g(\varphi(s_0 - k - ch)) \leq g(M)$. Consequently, $M < g(M), \ M \geq \kappa$, a contradiction.

Second, we claim that $\varphi(t)$ is strictly increasing at $-\infty$ (we believe that φ is monotone on $\mathbb{R}$, cf. [25]: however, for our purpose it suffices to establish the monotonicity of $\varphi(t)$ on some of intervals $(-\infty, \rho)$). Consider the characteristic function

$$\tilde{\chi}(z, c) := 1 + 2D + cz - D(e^z + e^{-z}) - g'(0)e^{-chz} \sum_{k \in \mathbb{Z}} \beta(k)e^{-kz}$$

and the bilateral Laplace transform $\Phi(z) := \int_\mathbb{R} e^{-zs}\varphi(s)ds$. For each fixed $c \neq 0$ function $\tilde{\chi}(z, c)$ is analytic in the region $\Pi_1 = \{0 < \Re z < \gamma_\#\}$ of the complex plane $\mathbb{C}$ and has a finite number of roots in any subregion $\{0 < \epsilon < \Re z < \gamma_\# - \epsilon\}$, see [14, Lemma 3.1]. Next, it was proved in [1] that, under the conditions of Theorem 1.3, $\Phi(z)$ is analytic in some maximal vertical strip $\Pi = \{0 < \Re z < \lambda\} \subset \Pi_1$ where $\lambda < \gamma_\#$ is a positive root (in difference with [14], not necessarily minimal and simple) of the equation $\tilde{\chi}(z, c) = 0$. Again using [14, Lemma 3.1] (or, alternatively, [1, Lemma 2]), we obtain that there exists $r > 0$ such that

$$\{\lambda\} = \{z \in \mathbb{C} : \tilde{\chi}(z, c) = 0, \ \lambda - r < \Re z < \lambda + r\}. \quad (24)$$

Moreover, $\varphi(t) = O(e^{\gamma t}), \ t \to -\infty$, for each $\gamma \in (0, \lambda)$. See Corollaries 1,3 and Theorem 6 in [1] for more detail. Yet we will need a stronger result:

Lemma 4.1. *Under assumptions of Theorem 1.3, we have that*

$$\varphi(s + s_0, c) = (a - t)^j e^{\lambda t} + O(e^{(\lambda + \sigma)t}), \ \varphi'(t) = \lambda\varphi(t)(1 + o(1)), \ t \to -\infty.$$

for appropriate $a, s_0, \ j \in \{0, 1\}$ and some $\sigma > 0$. As a consequence, φ is strictly increasing on some maximal open interval $(-\infty, \rho)$.

Proof. Here, we follow the proof of Theorem 3 (Step I) in [1]. Set

$$\mathcal{D}(t) := \sum_{k \in \mathbb{Z}} \beta(k)(g'(0)\varphi(t - k - ch) - g(\varphi(t - k - ch))),$$

Take C, δ, θ as in **(H)**. Observe that without restricting the generality, we can assume that $(1 + \theta)\lambda < \gamma_\#$. Since equation (23) is translation invariant, we can suppose that $\varphi(t) < \delta$ for $t \leq 0$. Applying the bilateral Laplace transform to (23), we obtain that

$$\tilde{\chi}(z, c)\Phi(z) = \int_{\mathbb{R}} e^{-zt}\mathcal{D}(t)dt =: \mathbf{D}(z), \ z \in \Pi.$$

We claim that, in fact, function $\mathbf{D}$ is analytic in the region $\Pi_\alpha = \{z : \Re z \in (0, (1 + \theta)\lambda)\}$. Indeed, we have

$$\mathbf{D}(x + iy) = \int_{\mathbb{R}} e^{-iyt}[e^{-xt}\mathcal{D}(t)]dt.$$

Given $x := \Re z \in (0, (1 + \theta)\lambda)$, we choose x' sufficiently close from the left to λ to satisfy $-x + (1 + \theta)x' > 0$. Then $\varphi(t) \leq C_x e^{x't}, \ t \in \mathbb{R}$, for some positive C_x and

$$|\mathcal{D}(t)| \leq C \sum_{k \geq t - ch} \beta(k)|\varphi(t - k - ch)|^{1+\theta} + \kappa(1 + g'(0)) \sum_{k < t - ch} \beta(k) \leq$$

$$e^{(1+\theta)x't}C_1 \sum_{k \geq t - ch} \beta(k)e^{-x'(1+\theta)(k+ch)} + \kappa(1 + g'(0)) \sum_{k < t - ch} \beta(k)e^{-x'(1+\theta)(k+ch-t)} \leq$$

$$e^{(1+\theta)x't}[C_2 + \kappa(1 + g'(0))]\sum_{k \in \mathbb{Z}} \beta(k)e^{-x'(1+\theta)k} \leq C_* e^{(1+\theta)x't}, \ t \in \mathbb{R}.$$

Since clearly $\mathcal{D}(t)$ is bounded on $\mathbb{R}$, we find that $e^{-xt}\mathcal{D}(t)$ belongs to $L^k(\mathbb{R})$, for each $k \in [1, \infty]$ and $x \in (0, (1 + \theta)\lambda)$. In consequence, $\mathbf{D}$ is analytic in Π_α. In addition, for each $x \in (0, (1 + \theta)\lambda)$ the function $\mathbf{d}_x(y) := \mathbf{D}(x + iy)$ is bounded and square integrable on $\mathbb{R}$. Also, for each vertical line $L_x := \{x + it, \ t \in \mathbb{R}\}$ where $\tilde{\chi}(x + it) \neq 0$, we have that $\tilde{\chi}(x + it) \sim cit, \ |t| \to \infty$. Thus $1/\tilde{\chi}(x + it)$ is square integrable on $\mathbb{R}$ as well. Consequently, for each fixed $x \in (0, (1 + \theta)\lambda)$ such that L_x does not contain zeros of $\tilde{\chi}(z)$, function $\mathbf{D}(x + iy)/\tilde{\chi}(x + iy)$ is integrable on $\mathbb{R}$.

As we have mentioned, $\tilde{\chi}(z, c)$ is analytic in the domain Π_α, while $\Phi(z) = \mathbf{D}(z)/\tilde{\chi}(z, c)$ is analytic in $\Re z \in (0, \lambda)$ and meromorphic in Π_α. In virtue of (24), we can suppose that $\Phi(z)$ has a unique singular point λ in Π_α which is either simple or double pole.

Now, for some $x'' \in (0, \lambda)$, using the inversion theorem for the Laplace transform, we obtain that

$$\varphi(t) = \frac{1}{2\pi i} \lim_{N \to +\infty} \int_{x'' - iN}^{x'' + iN} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z, c)}dz, \ t \in \mathbb{R}.$$

If $x \in (\lambda, (1 + \theta)\lambda)$ then

$$\int_{x''-iN}^{x''+iN} \frac{e^{zt}\mathbf{D}(z)dz}{\tilde{\chi}(z,c)}$$

$$= \left(\int_{x-iN}^{x+iN} + \int_{x''-iN}^{x-iN} - \int_{x''+iN}^{x+iN} \right) \frac{e^{zt}\mathbf{D}(z)dz}{\tilde{\chi}(z,c)} - 2\pi i \operatorname{Res}_{z=\lambda} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)}.$$

Since, by [1, Corollary 2],

$$\lim_{N \to +\infty} \max_{z \in [x'' \pm iN, x \pm iN]} (|\mathbf{D}(z)| + |1/\tilde{\chi}(z,c)|) = 0,$$

we conclude that, for each fixed $t \in \mathbb{R}$

$$\lim_{N \to +\infty} \int_{x'' \pm iN}^{x \pm iN} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)} dz = 0.$$

Observe also that function $\tilde{\chi}(z,c)$ does not have zero other than λ in a small strip centered at $\Re z = \lambda$. Therefore

$$\varphi(t) = -\operatorname{Res}_{z=\lambda} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)} + \frac{e^{xt}}{2\pi} \int_{\mathbb{R}} \frac{e^{iyt}\mathbf{d}_x(y)}{\tilde{\chi}(x+iy,c)} dy.$$

Since

$$\operatorname{Res}_{z=\lambda} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)} = \frac{e^{\lambda t}\mathbf{D}(\lambda)}{\tilde{\chi}'(\lambda,c)}, \quad \text{if } \chi'(\lambda,c) \neq 0,$$

$$\operatorname{Res}_{z=\lambda} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)} = \frac{2e^{\lambda t}}{\tilde{\chi}''(\lambda,c)} \left(t\mathbf{D}(\lambda) + \mathbf{D}'(\lambda) - \mathbf{D}(\lambda)\frac{\tilde{\chi}'''(\lambda,c)}{3\tilde{\chi}''(\lambda,c)} \right), \quad \text{if } \chi'(\lambda,c) = 0,$$

we get the desired representation. It should be noted here that $\tilde{\chi}''(\lambda,c) < 0$, that

$$\lim_{|t| \to +\infty} \int_{\mathbb{R}} \frac{e^{iyt}\mathbf{d}_x(y)}{\tilde{\chi}(x+iy,c)} dy = 0, \quad \operatorname{Res}_{z=\lambda} \frac{e^{zt}\mathbf{D}(z)}{\tilde{\chi}(z,c)} \neq 0.$$

Indeed, if the latter residue were equal to 0, then $\Phi(z)$ would not have a pole at λ.

Finally, it is easy to check that $c\varphi'(t) = D[\varphi(t+1) + \varphi(t-1) - 2\varphi(t)] - \varphi(t)$
$+ \sum_{k \in \mathbb{Z}} \beta(k)g'(0)\varphi(t-k-ch) + \mathcal{D}(t) = c\lambda\varphi(t)(1+o(1)), \ t \to -\infty.$ ☐

Next, we claim that the statement of Lemma 3.1 is also valid for solutions of (23). Regardless the fact that we do not know whether wavefronts are monotone on whole real line or they are not, the proof of Case II can be repeated almost literally. The monotonicity of wavefronts on $(-\infty, \rho)$ will be sufficient for this purpose. For instance, let us prove the following

Lemma 4.2. *Under the assumptions of Lemma 3.1, there are large $\tau > 0$ and $T_1 > T$ such that*

$$\psi(t+\tau) > \phi(t), \ t < T_1, \quad \phi(t), \psi(t+\tau) \in (\kappa - \sigma, \kappa), \ t \geq T_1 - ch.$$

Proof. Due to the monotonicity of ϕ and ψ at $-\infty$, we find that for every $\tau \geq 0$ there exists $T(\tau)$ such that

$$\psi(t+\tau) > \phi(t), \ t < T(\tau), \quad \phi(T(\tau)) = \psi(T(\tau) + \tau).$$

Let us prove that $T(\tau)$ is bounded from below on $\mathbb{R}_+$. Indeed, otherwise there exists a converging sequence τ_j such that $T(\tau_j) \to -\infty$. In turn, this forces $T(\tau_j) + \tau_j \to -\infty$. But then we can use the monotonicity properties of ϕ, ψ in order to get a contradiction:

$$\phi(T(\tau)) = \psi(T(\tau) + \tau) > \psi(T(\tau)).$$

Since $\phi(s) < \kappa$, $s \in \mathbb{R}$, we deduce in a similar way that the sequence $\{T(\tau_j)\}$ can not have a finite limit as $\tau_j \to +\infty$. Thus $T(\tau) \to +\infty$ as $\tau \to +\infty$. Since $\phi(+\infty) = \psi(+\infty) = \kappa$, the remainder of the proof is straightforward. $\qquad\square$

Now, the following main changes should be introduced in the proof of Lemma 3.1:

1. Set $\Delta(t) = \psi(t) - \phi(t)$. Instead of (19), we then have that $\Delta(T) = \Delta'(T) = 0$,

$$0 = D[\Delta(T+1) + \Delta(T-1) - 2\Delta(T)] - \Delta(T) +$$
$$\sum_{k \in \mathbb{Z}} \beta(k)\left(g(\psi(T-k-ch)) - g(\phi(T-k-ch))\right) > 0.$$

Here (non-strict) monotonicity of g is sufficient because of

$$\Delta(T+1) + \Delta(T-1) - 2\Delta(T) \geq \Delta(T-1) > 0, \ \ g(\psi(s)) \geq g(\phi(s)), \ \ s \in \mathbb{R}.$$

2. If $a_* > 0$, we take small positive $\sigma > 0$ and integer $N_1 > 0$ such that

$$\kappa \sum_{|k| \geq N_1} \beta(k) \leq 0.5 a_* (1 - \max_{s \in [\kappa - \sigma, \kappa]} g'(s))$$

and then we assume additionally that T is such that

$$\phi(t), \psi(t) \in (\kappa - \sigma, \kappa), \ \ t \geq T - N_1 - ch.$$

3. Similarly, in (21), the expression $g(\psi(t_m - ch)) - g(\phi(t_m - ch))$ should be replaced with

$$\sum_{k \in \mathbb{Z}} \beta(k)\left(g(\psi(t_m - k - ch)) - g(\phi(t_m - k - ch))\right) \geq -0.5 a_* (1 - \max_{s \in [\kappa - \sigma, \kappa]} g'(s)) +$$

$$\sum_{|k| < N_1} \beta(k) g'(\theta_k)(\psi(t_m - k - ch) - \phi(t_m - k - ch)) \geq -0.5 a_* (1 + \max_{s \in [\kappa - \sigma, \kappa]} g'(s)).$$

As a result, we get again a contradiction:

$$0 = D[\Delta(t_m + 1) + \Delta(t_m - 1) - 2\Delta(t_m)] - \Delta(t_m) +$$

$$\sum_{k \in \mathbb{Z}} \beta(k)\left(g(\psi(t_m - k - ch)) - g(\phi(t_m - k - ch))\right) > 0.5 a_* (1 - \max_{s \in [\kappa - \sigma, \kappa]} g'(s)) \geq 0.$$

To finalize the proof of Theorem 1.3, it suffices to repeat the last two paragraphs of the third section.

5. **Proof of Theorem 1.4.** In virtue of the front uniqueness, the first statement of Theorem 1.4 was already proved in the previous section (cf. (17) with $g_m = g$) so we have to consider the case $c = c_*$ only. Suppose, contrary to our claim, that

$$\phi(t + s_0) = e^{\lambda_2 t} + O(e^{(\lambda_2 + \delta)t}), \ t \to -\infty, \ \lambda_j := \lambda_j(c_*),$$

(without restricting the generality, we can assume that $s_0 = 0$), take some $c' < c_*$ close to c_* and consider the following piecewise continuous function

$$\phi_+(t) := \begin{cases} Me^{\rho t} + ae^{\lambda_2' t}, & \text{when } t \leq T_1, \\ \phi(t) + \epsilon, & \text{when } t \in (T_1, T_2], \\ \kappa, & \text{when } T > T_2, \end{cases}$$

where $\lambda_2' := \lambda_2(c') > \lambda_2$, $\rho = \lambda_2(1 + \theta) > \lambda_2'$, $M, a, \epsilon > 0$, $a \ll \epsilon \ll 1$, $M \gg 1$, and

$$Me^{\rho T_1} + ae^{\lambda_2' T_1} = \phi(T_1), \quad \phi(T_2) + \epsilon = \kappa.$$

For sufficiently large M and small $\epsilon > 0$, the above definitions yield large negative $T_1 = T_1(M, a)$ and large positive $T_2(\epsilon)$. Therefore, if M is sufficiently large and $a, c_* - c' > 0$ are sufficiently small, then we can suppose that, for all $t \leq T_1$,

$$E_+ := E_+(t, \epsilon, a, M, c') := \phi_+''(t) - c'\phi_+'(t) - \phi_+(t) + g(\phi_+(t - c'h)) =$$

$$Me^{\rho t}\left(\chi(\rho, c') + \left[\frac{g(\phi_+(t - c'h))}{\phi_+(t - c'h)} - g'(0)\right]e^{-\rho c'h}\right) +$$

$$ae^{\lambda_2' t}\left[\frac{g(\phi_+(t - c'h))}{\phi_+(t - c'h)} - g'(0)\right]e^{-\lambda_2' c'h} < 0.5Me^{\rho t}\chi(\rho, c') + C_1ae^{\lambda_2' t}\left[Me^{\rho t} + ae^{\lambda_2' t}\right]^\theta$$

$$\leq 0.5Me^{\rho t}\chi(\rho, c') + C_1ae^{\lambda_2'(1+\theta)t}\left[Me^{(\rho - \lambda_2')t} + a\right]^\theta \leq$$

$$0.5Me^{\rho t}\chi(\rho, c') + C_1aMe^{\lambda_2'(1+\theta)t} < Me^{\rho t}(0.5\chi(\rho, c') + C_1a) < 0.$$

Moreover, since $\rho > \lambda_2$, we also can choose $|T_1| \gg a$ in such a way that

$$\phi_+'(T_1-) \approx \rho\phi_+(T_1) > \phi_+'(T_1+) \approx \lambda_2\phi_+(T_1), \ Me^{\rho s} + ae^{\lambda_2' s} < \phi(s), \ s \in [T_1 - h, T_1].$$

Indeed, we can first determine (large negative) $\bar{T}_1$ as the leftmost root of equation $\phi(t) = Me^{\rho t}$ (with M large and positive). This corresponds to the limit case $a = 0$. The inequality $\phi_+'(\bar{T}_1-) > \phi_+'(\bar{T}_1+)$ is obvious in such a case. To prove the second inequality, suppose that for a moment that, for some $S \in [\bar{T}_1 - h, \bar{T}_1)$,

$$Me^{\rho S} = \phi(S), \quad Me^{\rho t} < \phi(t), \ t \in (S, \bar{T}_1).$$

Then $\rho Me^{\rho S} \leq \phi'(S)$ so that (assuming that M is large)

$$\rho \leq \phi'(S)/\phi(S) \approx \lambda_2,$$

a contradiction. Since $a \ll 1$ can considered as a small perturbation parameter, we deduce that the mentioned properties hold for all small a (where T_1 is close to $\bar{T}_1$).

Let $\sigma > 0$ be such that $\gamma := \max\{g'(s) : s \in [\kappa - \sigma, \kappa]\} < 1$. From now on, we fix M, T_1 chosen above and take $0 < a \ll c_* - c' \ll \epsilon < 1$ small enough to satisfy

$$\phi_+'(T_1+) - \phi_+'(T_1-) < \epsilon\frac{c_* - \sqrt{c_*^2 + 4}}{2} < 0, \ -\epsilon(1-\gamma) + (1+\gamma h)\max_{s\in\mathbb{R}}\phi'(s)(c_*-c') < 0.$$

If $t \in [T_1, T_1 + c'h]$, then

$$E_+(t, \epsilon, c') = (c_* - c')\phi'(t) - \epsilon + g(\phi_+(t - c'h)) - g(\phi(t - c_*h)).$$

Next, for $t \in [T_1 + c'h, T_2]$, we have

$$E_+(t, \epsilon, c') = \phi''(t) - c'\phi'(t) - \phi(t) - \epsilon + g(\phi(t - c'h) + \epsilon) =$$

$$(c_* - c')\phi'(t) - \epsilon + g(\phi(t - c'h) + \epsilon) - g(\phi(t - c_*h)).$$

Let us define T_1^+ from

$$\phi(T_1^+ - 2c^*h) = \kappa - \sigma.$$

Observe that T_1^+ does not depend on ϵ, c' (thus we may assume that $T_1^+ < T_2$) and that, for some $\theta_1 \in [\kappa - \sigma, \kappa]$ and $\theta_2 > T_1^+ - 2c^*h$,

$$-\epsilon + g(\phi(t - c'h) + \epsilon) - g(\phi(t - c_*h)) = g'(\theta_1)(\epsilon + \phi'(\theta_2)(c_* - c')h) - \epsilon \leq$$

$$-\epsilon(1 - \gamma) + \gamma h\max_{s\in\mathbb{R}}\phi'(s)(c_* - c'), \quad t \in [T_1^+, T_2].$$

As a consequence, we obtain that $E_+(t, \epsilon, c') < 0$ for all $t \in [T_1^+, T_2]$. On the other hand, if $t \geq T_2 + c'h$ then $E_+(t, \epsilon, c') = 0$, and if $t \in [T_2, T_2 + c'h)$, it holds

$$E_+(t, \epsilon, c') = -\kappa + g(\phi(t - c'h) + \epsilon) < 0.$$

Hence, if c' is close to c_* we find that

$$E_+(t, \epsilon, c') \leq 0, \ t \in \mathbb{R} \setminus [T_1, T_1^+],$$

$$\sup_{t \in [T_1, T_1^+]} E_+(t, \epsilon, c') = \omega(c', \epsilon), \quad \lim_{(c', \epsilon) \to (c_*, 0)} \omega(c', \epsilon) \leq 0. \tag{25}$$

Next, Lemma 2.1 assures that

$$\phi_+(t) = \frac{1}{\xi_2' - \xi_1'} \left(\int_{-\infty}^{t} e^{\xi_1'(t-s)} g(\phi_+(s - c'h)) ds + \int_{t}^{+\infty} e^{\xi_2'(t-s)} g(\phi_+(s - c'h)) ds \right.$$

$$\left. - \int_{-\infty}^{t} e^{\xi_1'(t-s)} E_+(s) ds - \int_{t}^{+\infty} e^{\xi_2'(t-s)} E_+(s) ds \right)$$

$$+ \frac{1}{\xi_2' - \xi_1'} \left[\sum_{t < T_j} e^{\xi_2'(t-T_j)} (\xi_1' \alpha_j - \beta_j) + \sum_{t > T_j} e^{\xi_1'(t-T_j)} (\xi_2' \alpha_j - \beta_j) \right],$$

where $\beta_2 < 0$, $\alpha_2 = 0$, $\alpha_1 = \epsilon$, and $\beta_1 < 0$ does not depend on ϵ. Consider

$$\mathcal{E}(t) := - \int_{-\infty}^{t} e^{\xi_1'(t-s)} E_+(s) ds - \int_{t}^{+\infty} e^{\xi_2'(t-s)} E_+(s) ds +$$

$$\sum_{t < T_j} e^{\xi_2'(t-T_j)} (\xi_1' \alpha_j - \beta_j) + \sum_{t > T_j} e^{\xi_1'(t-T_j)} (\xi_2' \alpha_j - \beta_j).$$

Since there exists $\nu > 0$ (independent on small $\epsilon, c_* - c'$) such that $\xi_1' \alpha_1 - \beta_1 > \nu$, $\xi_2' \alpha_1 - \beta_1 > \nu$, we infer from (25) that, for $t \leq T_1$ and small positive $\epsilon, c_* - c'$,

$$\mathcal{E}(t) > - \int_{T_1}^{T_1^+} e^{\xi_2'(t-s)} E_+(s) ds + e^{\xi_2'(t-T_1)} \nu \geq e^{\xi_2'(t-T_1)} \left(\nu - \frac{\omega(c', \epsilon)}{\xi_2'} \right) > 0.$$

Next, for $t \in [T_1, T_1^+]$ and small positive $\epsilon, c_* - c'$, we have that

$$\mathcal{E}(t) > - \int_{T_1}^{t} e^{\xi_1'(t-s)} E_+(s) ds - \int_{t}^{T_1^+} e^{\xi_2'(t-s)} E_+(s) ds + e^{\xi_1'(t-T_1)} \nu \geq$$

$$e^{\xi_1'(T_1^+ - T_1)} \nu - \omega(c', \epsilon) \sqrt{(c')^2 + 4} > 0.$$

Similarly, if $t \geq T_1^+$ then

$$\mathcal{E}(t) > - \int_{T_1}^{T_1^+} e^{\xi_1'(t-s)} E_+(s) ds + e^{\xi_1'(t-T_1)} \nu = e^{\xi_1'(t-T_1^+)} \left(e^{\xi_1'(T_1^+ - T_1)} \nu - \frac{\omega(c', \epsilon)}{|\xi_1'|} \right) > 0.$$

Therefore, for all $t \in \mathbb{R}$ and small $c_* - c', \epsilon > 0$,

$$\phi_+(t) > \frac{1}{\xi_2' - \xi_1'} \left(\int_{-\infty}^{t} e^{\xi_1'(t-s)} g(\phi_+(s - c'h)) ds + \int_{t}^{+\infty} e^{\xi_2'(t-s)} g(\phi_+(s - c'h)) ds \right).$$

To finalize the proof of Theorem 1.4, it suffices now to repeat Steps III and IV of Section 2. The construction of a lower solution is possible because of $c_* > c_\#$: this inequality assures the existence of two positive real roots $\lambda_2(c') < \lambda_1(c')$ for all c' close to c_*.

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