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A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem

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ABSTRACT

We consider Liouville-type and partial regularity results for the nonlinear fourth-order problem

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

where $p > 1$ and $n \geqslant 1$. We give a complete classification of stable and finite Morse index solutions (whether positive or sign changing), in the full exponent range. We also compute an upper bound of the Hausdorff dimension of the singular set of extremal solutions. Our approach is motivated by Fleming's tangent cone analysis technique for minimal surfaces and Federer's dimension reduction principle in partial regularity theory. A key tool is the monotonicity formula for biharmonic equations.

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1. Introduction

We study the following model biharmonic superlinear elliptic equation

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain or the entire space and $p > 1$ is a real number. Inspired by the tangent cone analysis in minimal surface theory, more precisely Fleming's key observation that the existence of an entire nonplanar minimal graph implies that of a singular area-minimizing cone (see his work on the Bernstein theorem [11]), we derive a monotonicity formula for solutions of (1.1) to reduce the non-existence of nontrivial entire solutions for the problem (1.1), to that of nontrivial homogeneous solutions. Through this approach we give a complete classification of stable solutions and those of finite Morse index, whether positive or sign changing, when $\Omega = \mathbb{R}^n$ is the whole Euclidean space. This in turn enables us to obtain partial regularity as well as an estimate of the Hausdorff dimension of the singular set of the extremal solutions in bounded domains.

Let us first describe the monotonicity formula. Eq. (1.1) has two important features. It is variational, with energy functional given by

$$\int \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}$$

and it is invariant under the scaling transformation

$$u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(\lambda x).$$

This suggests that the variations of the rescaled energy

$$r^{4\frac{p+1}{p-1}-n} \int\limits_{B_r(x)} \left[\frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}\right]$$

with respect to the scaling parameter r are meaningful. Augmented by the appropriate boundary terms, the above quantity is in fact nonincreasing. More precisely, take $u \in W^{4,2}_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$, fix $x \in \Omega$, let $0 < r < R$ be such that $B_r(x) \subset B_R(x) \subset \Omega$, and define

$$E(r; x, u) := r^{4\frac{p+1}{p-1}-n} \int\limits_{B_r(x)} \left[\frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}\right] + \frac{2}{p-1}\left(n - 2 - \frac{4}{p-1}\right) r^{\frac{8}{p-1}+1-n} \int\limits_{\partial B_r(x)} u^2$$

$$
\begin{aligned}
&+\frac{2}{p-1}\left(n-2-\frac{4}{p-1}\right)\frac{d}{dr}\left(r^{\frac{8}{p-1}+2-n}\int\limits_{\partial B_r(x)}u^2\right)\\
&+\frac{r^3}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+1-n}\int\limits_{\partial B_r(x)}\left(\frac{4}{p-1}r^{-1}u+\frac{\partial u}{\partial r}\right)^2\right]\\
&+\frac{1}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+4-n}\int\limits_{\partial B_r(x)}\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right)\right]\\
&+\frac{1}{2}r^{\frac{8}{p-1}+3-n}\int\limits_{\partial B_r(x)}\left(|\nabla u|^2-\left|\frac{\partial u}{\partial r}\right|^2\right),
\end{aligned}
$$

where derivatives are taken in the sense of distributions. Then, we have the following monotonicity formula.

Theorem 1.1. *Assume that*

$$n \geqslant 5, \quad p > \frac{n+4}{n-4}. \tag{1.2}$$

Let $u \in W^{4,2}_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$ be a weak solution of (1.1). Then, $E(r; x, u)$ is non-decreasing in $r \in (0, R)$. Furthermore there is a constant $c(n, p) > 0$ such that

$$\frac{d}{dr}E(r;0,u) \geqslant c(n,p)r^{-n+2+\frac{8}{p-1}}\int\limits_{\partial B_r}\left(\frac{4}{p-1}r^{-1}u+\frac{\partial u}{\partial r}\right)^2. \tag{1.3}$$

Remark 1.2. Monotonicity formulae have a long history that we will not describe here. Let us simply mention two earlier results that seem closest to our findings: the formula of Pacard [20] for the classical Lane–Emden equation and the one of Chang, Wang and Yang [2] for biharmonic maps.

Consider again Eq. (1.1) in the case where $\Omega = \mathbb{R}^n$, i.e.,

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n. \tag{1.4}$$

Let

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leqslant 4, \\ \frac{n+4}{n-4} & \text{if } n \geqslant 5, \end{cases}$$

denote the Sobolev exponent. When $1 < p \leqslant p_S(n)$, all positive solutions to (1.4) are classified: if $p < p_S(n)$, then $u \equiv 0$; if $p = p_S(n)$, then all solutions can be written in the form $u = c_n(\frac{\lambda}{\lambda^2+|x-x_0|^2})^{\frac{n-4}{2}}$ for some $c_n > 0, \lambda > 0, x_0 \in \mathbb{R}^n$, see the work of Xu and one of the authors [31]. However, there can be many sign-changing solutions to the equation (see the work by Guo, Li and one of the authors [15] for the critical case $p = p_S(n)$).

Here, we allow u to be sign-changing and p to be supercritical. Instead, we restrict the analysis to stable and finite Morse index solutions. A solution u to (1.4) is said to be stable if

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx, \quad \text{for all } \phi \in H^2(\mathbb{R}^n).$$

More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces E of $H^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx < p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx,$$

for any $\phi \in E \setminus \{0\}$. No assumption on the growth of u is needed in these definitions. Clearly, a solution is stable if and only if its Morse index is equal to zero. It is also standard knowledge that if a solution to (1.4) has finite Morse index, then there is a compact set $\mathcal{K} \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant p\int_{\mathbb{R}^n} |u|^{p-1}\phi^2\,dx, \quad \forall\phi \in H^2(\mathbb{R}^n\backslash\mathcal{K}).$$

Recall that if

$$\gamma = \frac{4}{p-1}, \quad K_0 = \gamma(\gamma+2)(\gamma-n+4)(\gamma-n+2), \tag{1.5}$$

then

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)} \tag{1.6}$$

is a singular solution to (1.4) in $\mathbb{R}^n \setminus \{0\}$. By the Hardy–Rellich inequality with best constant [25]

$$\int_{\mathbb{R}^n} |\Delta\phi|^2\,dx \geqslant \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{\phi^2}{|x|^4}\,dx, \quad \forall\phi \in H^2(\mathbb{R}^n),$$

the singular solution u_s is stable if and only if

$$pK_0 \leqslant \frac{n^2(n-4)^2}{16}. \tag{1.7}$$

Solving the corresponding quartic equation, (1.7) holds if and only if $p \geqslant p_c(n)$ where $p_c(n) > p_S(n)$ is the fourth-order Joseph–Lundgren exponent computed by Gazzola and Grunau [12]:

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leqslant 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geqslant 13. \end{cases}$$

Equivalently, for fixed $p > p_S(n)$, define n_p to be the smallest dimension such that (1.7) holds. Then,

$$(1.7) \quad \Leftrightarrow \quad p \geqslant p_c(n) \quad \Leftrightarrow \quad n \geqslant n_p.$$

The existence, uniqueness and stability of regular radial positive solutions to (1.4) is by now well understood (see the works of Gazzola–Grunau, of Guo and one of the authors, and of Karageorgis [12,16,18]): for each $a > 0$ there exists a unique entire radial positive solution $u_a(|x|)$ to (1.4) with $u_a(0) = a$. This radial positive solution is stable if and only if (1.7) holds.

In our second result, which is a Liouville-type theorem, we give a complete characterization of all finite Morse index solutions (whether radial or not, whether positive or not).

Theorem 1.3. *Let u be a smooth solution of (1.4) with finite Morse index.*

- *If $p \in (1, p_c(n))$, $p \neq p_S(n)$, then $u \equiv 0$;*
- *If $p = p_S(n)$, then u has finite energy i.e.*

$$\int_{\mathbb{R}^n} (\Delta u)^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition u is stable, then in fact $u \equiv 0$.

Remark 1.4. According to the preceding discussions, Theorem 1.3 is sharp: on the one hand, in the critical case $p = p_S(n)$, Guo, Li and one of the authors [15] have constructed a large class of solutions to (1.1) with finite energy. Since in this case $\frac{(p-1)n}{4} = p + 1$, by a result of Rozenblum [26], such solutions have finite Morse index. On the other hand, for $p \geqslant p_c(n)$, all radial solutions are stable (see [16,18]).

Remark 1.5. The above theorem generalizes a similar result of Farina [10] for the classical Lane–Emden equation.

Now consider (1.1) when Ω is a smoothly bounded domain of $\mathbb{R}^n$ and supplement it with Navier boundary conditions:

$$\begin{cases} \Delta^2 u = \lambda(u+1)^p & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $\lambda > 0$ is a parameter. It is well known that there exists a critical value $\lambda^* > 0$ depending on p and Ω such that

- If $\lambda \in (0, \lambda^*)$, (1.8) has a minimal and classical solution u_λ, which is positive and stable;
- If $\lambda = \lambda^*$, a unique weak solution, called the extremal solution u_{λ^*} exists for (P_{λ^*}). It is given as the pointwise limit $u_{\lambda^*} = \lim_{\lambda\uparrow} u_\lambda$;
- No weak solution of (1.8) exists whenever $\lambda > \lambda^*$.

An outstanding remaining problem is the regularity of the extremal solution u_{λ^*}. An application of Theorem 1.3 and standard blow-up analysis give

Theorem 1.6. *If $n < n_p$ (equivalently $p < p_c(n)$), the extremal solution u_{λ^*} is smooth.*

More generally,

Theorem 1.7. *Assume $p \neq \frac{n+4}{n-4}$ and $n < n_p$ (equivalently $p < p_c(n)$).*

- *Let Ω be a smoothly bounded domain and u be a smooth solution (1.8) of finite Morse index $k \in \mathbb{N}$. Then there exists a constant $C > 0$ depending only on k, N, Ω, p such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C.$$

- *Let Ω be any open set and u be a smooth solution of (1.1). Then, there exists a constant $C > 0$ depending only on k, N, Ω, p such that for every $i \leqslant 3$,*

$$\left|\nabla^i u\right| \leqslant C \operatorname{dist}(x, \partial\Omega)^{-\frac{4}{p-1}-i} \quad \text{a.e. in } \Omega.$$

In Theorem 1.7 one has the same results for $p = \frac{n+4}{n-4}$ if u is a stable solution. Next, we are interested in partial regularity for the extremal solution u_λ^*.

Definition 1.8. A point x belongs to the regular set of a function $u \in L^1_{loc}(\Omega)$ if there exists a neighborhood B of x such that $u \in L^\infty(B)$. Otherwise, x belongs to $\mathcal{S}$, the singular set of u.

By definition, the regular set is an open set. By elliptic estimates applied to (1.1), u is smooth in its regular set. Now, we state the interior partial regularity for u_{λ^*}.

Theorem 1.9. *Let $n \geqslant n_p$ and let u_{λ^*} be the extremal solution to (1.8). Then the Hausdorff dimension of its singular set $\mathcal{S}$ is no more than $n - n_p$. Moreover, when $n = n_p$, $\mathcal{S}$ is a discrete set.*

We now list some known results. We start with the analogous second order equation

$$\Delta u + |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^n. \tag{1.9}$$

As mentioned earlier, Farina completely classified finite Morse index solutions (positive or sign-changing) in his seminal paper [10]. His proof makes a delicate use of the classical Moser iteration method. More precisely, if one multiplies Eq. (1.9) by a power of u, say u^q, $q > 1$, Moser's iteration works because of the following simple identity

$$\int_{\mathbb{R}^n} u^q(-\Delta u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} \left|\nabla u^{\frac{q+1}{2}}\right|^2, \quad \forall u \in C_0^2(\mathbb{R}^n).$$

There have been many attempts to generalize Moser's iteration technique (or Farina's approach) to fourth order problems like (1.1). Unfortunately, this runs into problems: the corresponding identity reads

$$\int_{\mathbb{R}^n} u^q(\Delta^2 u) = \frac{4q}{(q+1)^2} \int_{\mathbb{R}^n} \left|\Delta u^{\frac{q+1}{2}}\right|^2 - \frac{q(q-1)^2}{4} \int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4, \quad \forall u \in C_0^4(\mathbb{R}^n),$$

and the additional term $\int_{\mathbb{R}^n} u^{q-3}|\nabla u|^4$ makes the Moser iteration argument difficult to use.

Another strategy is to use the test function $v = -\Delta u$. This allows to treat exponents less than $\frac{n}{n-8} + \epsilon_n$ for some $\epsilon_n > 0$, see the works of Cowan–Esposito–Ghoussoub [3] and Ye and one of the authors [32]. Another approach, obtained by Cowan and Ghoussoub[1] [4], and further exploited by Hajlaoui, Harrabi and Ye [17], is to derive the following interesting interpolated version of the inequality: for stable solutions to (1.1), there holds

$$\sqrt{p} \int_{\mathbb{R}^n} |u|^{\frac{p-1}{2}} \phi^2 \leqslant \int_{\mathbb{R}^n} |\nabla \phi|^2, \quad \forall \phi \in C_0^1(\mathbb{R}^n).$$

This approach improves the first upper bound $\frac{n}{n-8} + \epsilon_n$, but it again fails to catch the optimal exponent $p_c(n)$ (when $n \geqslant 13$). It should be remarked that by combining these two approaches one can show that stable positive solutions to (1.1) do not exist when $n \leqslant 12$ and $p > \frac{n+4}{n-4}$, see [17].

In the above references, only positive solutions to (1.1) are considered. One reason is their use of the following inequality, due to Souplet [29]

[1] A similar method was first announced in [7], and later published in the work by Farina, Sirakov and one of the authors [8].

$$\Delta u + \left(\frac{2}{p+1}\right)^{1/2} u^{\frac{p+1}{2}} \leqslant 0 \quad \text{in } \mathbb{R}^n. \tag{1.10}$$

As observed in [9] for a similar equation, the use of the above inequality can be completely avoided.

In this paper we take a completely new approach, which also avoids the use of (1.10) and requires minimal integrability. One of our motivations is Fleming's proof of the Bernstein theorem for minimal surfaces in dimension 3. Fleming used a monotonicity formula for minimal surfaces together with a compactness result to blow down the minimal surface. It turns out that the blow-down limit is a minimal cone. This is because the monotonic quantity is constant only for minimizing cones. Then, he proved that minimizing cones are flat, which implies in turn the flatness of the original minimal surface.

At last, let us sketch the proof of Theorem 1.3: we first derive a monotonicity formula for our equation (1.1). Then, we classify stable solutions: this is Theorem 4.1 in Section 4. To do this, we estimate solutions in the L^{p+1} norm, utilizing the aforementioned methods available in the literature, and then show that the *blow-down* limit $u^\infty(x) = \lim_{\lambda\to\infty} \lambda^{\frac{4}{p-1}} u(\lambda x)$ satisfies $E(r) \equiv const$. Then, Theorem 1.1 implies that u^∞ is a homogeneous stable solution, and we show in Theorem 3.1 that such solutions are trivial if $p < p_c(n)$. Then similar to Fleming's proof, the triviality of the *blow-down* limit implies that the original entire solution is also trivial. In Section 5, we extend our result to solutions of finite Morse index. Finally, in Section 6 we prove an ε-regularity result and use the Federer's dimension reduction principle to obtain the partial regularity of extremal solutions. This approach was used in [30] for (1.9), see also [6].

2. Proof of the monotonicity formula

In this section we derive a monotonicity formula for functions $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$ solving (1.1) in $B_R(0) \subset \Omega$. We assume that $p > \frac{n+4}{n-4}$.

Proof of Theorem 1.1. Since the boundary integrals in $E(r; x, u)$ only involve second order derivatives of u, the boundary integrals in $\frac{dE}{dr}(r; x, u)$ only involve third order derivatives of u. By our assumption $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$, for each $B_r(x) \subset B_R(0)$, $u \in W^{3,2}(\partial B_r(x))$. Thus, the following calculations can be rigorously verified. Assume that $x = 0$ and that the balls B_λ are all centered at 0. Take

$$\widetilde{E}(\lambda) := \lambda^{4\frac{p+1}{p-1}-n} \int_{B_\lambda} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}.$$

Define

$$v := \Delta u$$

and

$$u^\lambda(x) := \lambda^{\frac{4}{p-1}} u(\lambda x), \qquad v^\lambda(x) := \lambda^{\frac{4}{p-1}+2} v(\lambda x).$$

We still have $v^\lambda = \Delta u^\lambda$, $\Delta v^\lambda = |u^\lambda|^{p-1} u^\lambda$, and by differentiating in λ,

$$\Delta \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}.$$

Note that differentiation in λ commutes with differentiation and integration in x. A rescaling shows

$$\widetilde{E}(\lambda) = \int_{B_1} \frac{1}{2}\left(v^\lambda\right)^2 - \frac{1}{p+1}\left|u^\lambda\right|^{p+1}.$$

Hence

$$\begin{aligned}
\frac{d}{d\lambda}\widetilde{E}(\lambda) &= \int_{B_1} v^\lambda \frac{dv^\lambda}{d\lambda} - \left|u^\lambda\right|^{p-1} u^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{B_1} v^\lambda \Delta \frac{du^\lambda}{d\lambda} - \Delta v^\lambda \frac{du^\lambda}{d\lambda} \\
&= \int_{\partial B_1} v^\lambda \frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda} - \frac{\partial v^\lambda}{\partial r}\frac{du^\lambda}{d\lambda}. \qquad (2.1)
\end{aligned}$$

In what follows, we express all derivatives of u^λ in the $r = |x|$ variable in terms of derivatives in the λ variable. In the definition of u^λ and v^λ, directly differentiating in λ gives

$$\frac{du^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\left(\frac{4}{p-1}u^\lambda(x) + r\frac{\partial u^\lambda}{\partial r}(x)\right), \qquad (2.2)$$

$$\frac{dv^\lambda}{d\lambda}(x) = \frac{1}{\lambda}\left(\frac{2(p+1)}{p-1}v^\lambda(x) + r\frac{\partial v^\lambda}{\partial r}(x)\right). \qquad (2.3)$$

In (2.2), taking derivatives in λ once again, we get

$$\lambda\frac{d^2u^\lambda}{d\lambda^2}(x) + \frac{du^\lambda}{d\lambda}(x) = \frac{4}{p-1}\frac{du^\lambda}{d\lambda}(x) + r\frac{\partial}{\partial r}\frac{du^\lambda}{d\lambda}(x). \qquad (2.4)$$

Substituting (2.3) and (2.4) into (2.1) we obtain

$$\frac{d\widetilde{E}}{d\lambda} = \int_{\partial B_1} v^\lambda\left(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{p-5}{p-1}\frac{du^\lambda}{d\lambda}\right) - \frac{du^\lambda}{d\lambda}\left(\lambda\frac{dv^\lambda}{d\lambda} - \frac{2(p+1)}{p-1}v^\lambda\right)$$

$$= \int_{\partial B_1} \lambda v^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 3 v^\lambda \frac{d u^\lambda}{d\lambda} - \lambda \frac{d u^\lambda}{d\lambda} \frac{d v^\lambda}{d\lambda}. \tag{2.5}$$

Observe that v^λ is expressed as a combination of x derivatives of u^λ. So we also transform v^λ into λ derivatives of u^λ. By taking derivatives in r in (2.2) and noting (2.4), we get on ∂B_1,

$$\begin{aligned} \frac{\partial^2 u^\lambda}{\partial r^2} &= \lambda \frac{\partial}{\partial r} \frac{d u^\lambda}{d\lambda} - \frac{p+3}{p-1} \frac{\partial u^\lambda}{\partial r} \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \frac{p-5}{p-1} \lambda \frac{d u^\lambda}{d\lambda} - \frac{p+3}{p-1} \left(\lambda \frac{d u^\lambda}{d\lambda} - \frac{4}{p-1} u^\lambda \right) \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \frac{8}{p-1} \lambda \frac{d u^\lambda}{d\lambda} + \frac{4(p+3)}{(p-1)^2} u^\lambda. \end{aligned}$$

Then on ∂B_1,

$$\begin{aligned} v^\lambda &= \frac{\partial^2 u^\lambda}{\partial r^2} + \frac{n-1}{r} \frac{\partial u^\lambda}{\partial r} + \frac{1}{r^2} \Delta_\theta u^\lambda \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \frac{8}{p-1} \lambda \frac{d u^\lambda}{d\lambda} + \frac{4(p+3)}{(p-1)^2} u^\lambda + (n-1) \left(\lambda \frac{d u^\lambda}{d\lambda} - \frac{4}{p-1} u^\lambda \right) + \Delta_\theta u^\lambda \\ &= \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \left(n - 1 - \frac{8}{p-1} \right) \lambda \frac{d u^\lambda}{d\lambda} + \frac{4}{p-1} \left(\frac{4}{p-1} - n + 2 \right) u^\lambda + \Delta_\theta u^\lambda. \end{aligned}$$

Here Δ_θ is the Beltrami–Laplace operator on ∂B_1 and below ∇_θ represents the tangential derivative on ∂B_1. For notational convenience, we also define the constants

$$\alpha = n - 1 - \frac{8}{p-1}, \qquad \beta = \frac{4}{p-1} \left(\frac{4}{p-1} - n + 2 \right).$$

Now (2.5) reads

$$\begin{aligned} \frac{d}{d\lambda} \widetilde{E}(\lambda) = \int_{\partial B_1} & \lambda \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha \lambda \frac{d u^\lambda}{d\lambda} + \beta u^\lambda \right) \frac{d^2 u^\lambda}{d\lambda^2} \\ & + 3 \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha \lambda \frac{d u^\lambda}{d\lambda} + \beta u^\lambda \right) \frac{d u^\lambda}{d\lambda} \\ & - \lambda \frac{d u^\lambda}{d\lambda} \frac{d}{d\lambda} \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha \lambda \frac{d u^\lambda}{d\lambda} + \beta u^\lambda \right) \\ & + \int_{\partial B_1} \lambda \Delta_\theta u^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 3 \Delta_\theta u^\lambda \frac{d u^\lambda}{d\lambda} - \lambda \frac{d u^\lambda}{d\lambda} \Delta_\theta \frac{d u^\lambda}{d\lambda} \\ = {} & R_1 + R_2. \end{aligned}$$

Integrating by parts on ∂B_1, we get

$$
\begin{aligned}
R_2 &= \int_{\partial B_1} -\lambda \nabla_\theta u^\lambda \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2} - 3\nabla_\theta u^\lambda \nabla_\theta \frac{du^\lambda}{d\lambda} + \lambda \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&= -\frac{\lambda}{2} \frac{d^2}{d\lambda^2} \left(\int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right) - \frac{3}{2} \frac{d}{d\lambda} \left(\int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right) + 2\lambda \int_{\partial B_1} \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&= -\frac{1}{2} \frac{d^2}{d\lambda^2} \left(\lambda \int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right) - \frac{1}{2} \frac{d}{d\lambda} \left(\int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right) + 2\lambda \int_{\partial B_1} \left| \nabla_\theta \frac{du^\lambda}{d\lambda} \right|^2 \\
&\geqslant -\frac{1}{2} \frac{d^2}{d\lambda^2} \left(\lambda \int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right) - \frac{1}{2} \frac{d}{d\lambda} \left(\int_{\partial B_1} \left|\nabla_\theta u^\lambda\right|^2 \right).
\end{aligned}
$$

For R_1, after some simplifications we obtain

$$
\begin{aligned}
R_1 = \int_{\partial B_1} & \lambda \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \frac{d^2 u^\lambda}{d\lambda^2} \\
& + 3 \left(\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} + \alpha\lambda \frac{du^\lambda}{d\lambda} + \beta u^\lambda \right) \frac{du^\lambda}{d\lambda} \\
& - \lambda \frac{du^\lambda}{d\lambda} \left(\lambda^2 \frac{d^3 u^\lambda}{d\lambda^3} + (2+\alpha)\lambda \frac{d^2 u^\lambda}{d\lambda^2} + (\alpha+\beta) \frac{du^\lambda}{d\lambda} \right) \\
= \int_{\partial B_1} & \lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + \beta\lambda u^\lambda \frac{d^2 u^\lambda}{d\lambda^2} + 3\beta u^\lambda \frac{du^\lambda}{d\lambda} \\
& + (2\alpha - \beta)\lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 - \lambda^3 \frac{du^\lambda}{d\lambda} \frac{d^3 u^\lambda}{d\lambda^3} \\
= \int_{\partial B_1} & 2\lambda^3 \left(\frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + 4\lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda \left(\frac{du^\lambda}{d\lambda} \right)^2 \\
& + \frac{\beta}{2} \frac{d^2}{d\lambda^2} \left[\lambda \left(u^\lambda\right)^2 \right] - \frac{1}{2} \frac{d}{d\lambda} \left[\lambda^3 \frac{d}{d\lambda} \left(\frac{du^\lambda}{d\lambda} \right)^2 \right] + \frac{\beta}{2} \frac{d}{d\lambda} \left(u^\lambda\right)^2.
\end{aligned}
$$

Here we have used the relations (writing $f' = \frac{d}{d\lambda} f$ etc.)

$$
\lambda f f'' = \left(\frac{\lambda}{2} f^2 \right)'' - 2ff' - \lambda \left(f'\right)^2,
$$

and

$$
-\lambda^3 f' f''' = -\left[\frac{\lambda^3}{2} \left(\left(f'\right)^2 \right)' \right]' + 3\lambda^2 f' f'' + \lambda^3 \left(f''\right)^2.
$$

Since $p > \frac{n+4}{n-4}$, direct calculations show that

$$\alpha - \beta = \left(n - 1 - \frac{8}{p-1}\right) - \frac{4}{p-1}\left(\frac{4}{p-1} - n + 2\right) > 1. \tag{2.6}$$

Thus,

$$\begin{aligned}
&2\lambda^3\left(\frac{d^2u^\lambda}{d\lambda^2}\right)^2 + 4\lambda^2\frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + (2\alpha - 2\beta)\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\
&\quad = 2\lambda\left(\lambda\frac{d^2u^\lambda}{d\lambda^2} + \frac{du^\lambda}{d\lambda}\right)^2 + (2\alpha - 2\beta - 2)\lambda\left(\frac{du^\lambda}{d\lambda}\right)^2 \\
&\quad \geqslant 0.
\end{aligned} \tag{2.7}$$

Then,

$$R_1 \geqslant \int_{\partial B_1} \frac{\beta}{2}\frac{d^2}{d\lambda^2}\left[\lambda\left(u^\lambda\right)^2\right] - \frac{1}{2}\frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\frac{du^\lambda}{d\lambda}\right)^2\right] + \frac{\beta}{2}\frac{d}{d\lambda}\left(u^\lambda\right)^2.$$

Now, rescaling back, we can write those λ derivatives in R_1 and R_2 as follows.

$$\int_{\partial B_1}\frac{d}{d\lambda}\left(u^\lambda\right)^2 = \frac{d}{d\lambda}\left(\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda} u^2\right),$$

$$\int_{\partial B_1}\frac{d^2}{d\lambda^2}\left[\lambda\left(u^\lambda\right)^2\right] = \frac{d^2}{d\lambda^2}\left(\lambda^{\frac{8}{p-1}+2-n}\int_{\partial B_\lambda} u^2\right),$$

$$\int_{\partial B_1}\frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\frac{du^\lambda}{d\lambda}\right)^2\right] = \frac{d}{d\lambda}\left[\lambda^3\frac{d}{d\lambda}\left(\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u + \frac{\partial u}{\partial r}\right)^2\right)\right],$$

$$\frac{d^2}{d\lambda^2}\left(\lambda\int_{\partial B_1}\left|\nabla_\theta u^\lambda\right|^2\right) = \frac{d^2}{d\lambda^2}\left[\lambda^{1+\frac{8}{p-1}+2+1-n}\int_{\partial B_\lambda}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right],$$

$$\frac{d}{d\lambda}\left(\int_{\partial B_1}\left|\nabla_\theta u^\lambda\right|^2\right) = \frac{d}{d\lambda}\left[\lambda^{\frac{8}{p-1}+2+1-n}\int_{\partial B_\lambda}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right].$$

Substituting these into $\frac{d}{d\lambda}E(\lambda; 0, u)$ we finish the proof. $\square$

Denote $c(n,p) = 2\alpha - 2\beta - 2 > 0$. By (2.7), we have

Corollary 2.1.

$$\frac{d}{dr}E(r; 0, u) \geqslant c(n,p)r^{-n+2+\frac{8}{p-1}}\int_{\partial B_r}\left(\frac{4}{p-1}r^{-1}u + \frac{\partial u}{\partial r}\right)^2.$$

In particular, if $E(\lambda; 0, u) \equiv const.$ for all $\lambda \in (r, R)$, u is homogeneous in $B_R \setminus B_r$:

$$u(x) = |x|^{-\frac{4}{p-1}} u\left(\frac{x}{|x|}\right).$$

We end this section with the following observation: in the above computations we just need the inequality (2.6) to hold. In particular the formula can be easily extended to biharmonic equations with negative exponents. We state the following monotonicity formula for solutions of

$$\Delta^2 u = -\frac{1}{u^p}, \quad u > 0 \text{ in } \Omega \subset \mathbb{R}^n. \tag{2.8}$$

Lemma 2.2. *Assume that p satisfies*

$$n - 2 + \frac{8}{p+1} > \frac{4}{p+1}\left(\frac{4}{p+1} + n - 2\right). \tag{2.9}$$

Let u be a classical solution to (2.8) in $B_r(x) \subset B_R(x) \subset \Omega$. Then the following quantity

$$\begin{aligned}
\tilde{E}(r; x, u) := {} & r^{4\frac{p-1}{p+1}-n} \int\limits_{B_r(x)} \frac{1}{2}(\Delta u)^2 - \frac{1}{p-1}u^{1-p} \\
& - \frac{2}{p+1}\left(n-2+\frac{4}{p+1}\right) r^{-\frac{8}{p+1}+1-n} \int\limits_{\partial B_r(x)} u^2 \\
& - \frac{2}{p+1}\left(n-2+\frac{4}{p+1}\right) \frac{d}{dr}\left(r^{-\frac{8}{p+1}+2-n} \int\limits_{\partial B_r(x)} u^2\right) \\
& + \frac{r^3}{2}\frac{d}{dr}\left[r^{-\frac{8}{p+1}+1-n} \int\limits_{\partial B_r(x)} \left(-\frac{4}{p+1}r^{-1}u + \frac{\partial u}{\partial r}\right)^2\right] \\
& + \frac{1}{2}\frac{d}{dr}\left[r^{-\frac{8}{p+1}+4-n} \int\limits_{\partial B_r(x)} \left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)\right] \\
& + \frac{1}{2} r^{-\frac{8}{p+1}+3-n} \int\limits_{\partial B_r(x)} \left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)
\end{aligned}$$

is increasing in r. Furthermore there exists $c_0 > 0$ such that

$$\frac{d}{dr}E(r; 0, u) \geqslant c_0 r^{-n+2-\frac{8}{p+1}} \int\limits_{\partial B_r} \left(-\frac{4}{p+1}r^{-1}u + \frac{\partial u}{\partial r}\right)^2. \tag{2.10}$$

In the rest of the paper, sometimes we use $E(r; x)$ or $E(r)$ if no confusion occurs.

3. Homogeneous solutions

For the applications below, we give a non-existence result for homogeneous stable solution of (1.1). (This corresponds to the tangent cone analysis of Fleming.) By the Hardy–Rellich inequality, this result is sharp.

Theorem 3.1. *Let $u \in W^{2,2}_{loc}(\mathbb{R}^n \setminus \{0\})$ be a homogeneous, stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, for $p \in (\frac{n+4}{n-4}, p_c(n))$. Assume that $|u|^{p+1} \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$. Then $u \equiv 0$.*

Proof. There exists a $w \in W^{2,2}(\mathbb{S}^{n-1})$ such that in polar coordinates

$$u(r,\theta) = r^{-\frac{4}{p-1}} w(\theta).$$

Since $u \in W^{2,2}(B_2 \setminus B_1) \cap L^{p+1}(B_2 \setminus B_1)$, $w \in W^{2,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$.

Direct calculations show that w satisfies (in $W^{2,2}(\mathbb{S}^{n-1})$ sense)

$$\Delta_\theta^2 w - J_1 \Delta_\theta w + J_2 w = w^p, \tag{3.1}$$

where

$$\begin{aligned} J_1 &= \left(\frac{4}{p-1} + 2\right)\left(n - 4 - \frac{4}{p-1}\right) + \frac{4}{p-1}\left(n - 2 - \frac{4}{p-1}\right), \\ J_2 &= \frac{4}{p-1}\left(\frac{4}{p-1} + 2\right)\left(n - 4 - \frac{4}{p-1}\right)\left(n - 2 - \frac{4}{p-1}\right). \end{aligned}$$

Because $w \in W^{2,2}(\mathbb{S}^{n-1})$, we can test (3.1) with w, and we get

$$\int_{\mathbb{S}^{n-1}} |\Delta_\theta w|^2 + J_1 |\nabla_\theta w|^2 + J_2 w^2 = \int_{\mathbb{S}^{n-1}} |w|^{p+1}. \tag{3.2}$$

For any $\varepsilon > 0$, choose an $\eta_\varepsilon \in C_0^\infty((\frac{\varepsilon}{2}, \frac{2}{\varepsilon}))$, such that $\eta_\varepsilon \equiv 1$ in $(\varepsilon, \frac{1}{\varepsilon})$, and

$$r\left|\eta_\varepsilon'(r)\right| + r^2\left|\eta_\varepsilon''(r)\right| \leqslant 64 \quad \text{for all } r > 0.$$

Because $w \in W^{2,2}(\mathbb{S}^{n-1}) \cap L^{p+1}(\mathbb{S}^{n-1})$, $r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r)$ can be approximated by $C_0^\infty(B_{4/\varepsilon} \setminus B_{\varepsilon/4})$ functions in $W^{2,2}(B_{2/\varepsilon} \setminus B_{\varepsilon/2}) \cap L^{p+1}(B_{2/\varepsilon} \setminus B_{\varepsilon/2})$. Hence in the stability condition for u we are allowed to choose a test function of the form $r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r)$. Note that

$$\begin{aligned} \Delta\left(r^{-\frac{n-4}{2}} w(\theta)\eta_\varepsilon(r)\right) = {} & -\frac{n(n-4)}{4} r^{-\frac{n}{2}} \eta_\varepsilon(r) w(\theta) + r^{-\frac{n}{2}} \eta_\varepsilon(r) \Delta_\theta w(\theta) \\ & + 3r^{-\frac{n}{2}+1} \eta_\varepsilon'(r) w(\theta) + r^{-\frac{n}{2}+2} \eta_\varepsilon''(r) w(\theta). \end{aligned}$$

Substituting this into the stability condition for u, we get

$$
\begin{aligned}
&p\Big(\int_{\mathbb{S}^{n-1}} |w|^{p+1}\, d\theta\Big)\Big(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\, dr\Big)\\
&\quad\leqslant \Big(\int_{\mathbb{S}^{n-1}} \Big(|\Delta_\theta w|^2+\frac{n(n-4)}{2}|\nabla_\theta w|^2+\frac{n^2(n-4)^2}{16}w^2\Big)\, d\theta\Big)\Big(\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\, dr\Big)\\
&\qquad+O\Big[\Big(\int_0^{+\infty} r\eta_\varepsilon'(r)^2+r^3\eta_\varepsilon''(r)^2+\big|\eta_\varepsilon'(r)\big|\eta_\varepsilon(r)+r\eta_\varepsilon(r)\big|\eta_\varepsilon''(r)\big|\, dr\Big)\\
&\qquad\times\Big(\int_{\mathbb{S}^{n-1}} w(\theta)^2+\big|\nabla_\theta w(\theta)\big|^2\, d\theta\Big)\Big].
\end{aligned}
$$

Note that

$$
\int_0^{+\infty} r^{-1}\eta_\varepsilon(r)^2\, dr \geqslant |\log\varepsilon|,
$$

$$
\int_0^{+\infty} r\eta_\varepsilon'(r)^2+r^3\eta_\varepsilon''(r)^2+\big|\eta_\varepsilon'(r)\big|\eta_\varepsilon(r)+r\eta_\varepsilon(r)\big|\eta_\varepsilon''(r)\big|\, dr \leqslant C,
$$

for some constant C independent of ε. By letting $\varepsilon\to 0$, we obtain

$$
p\int_{\mathbb{S}^{n-1}} |w|^{p+1}\, d\theta \leqslant \int_{\mathbb{S}^{n-1}} |\Delta_\theta w|^2+\frac{n(n-4)}{2}|\nabla_\theta w|^2+\frac{n^2(n-4)^2}{16}w^2.
$$

Substituting (3.2) into this we get

$$
\int_{\mathbb{S}^{n-1}} (p-1)|\Delta_\theta w|^2+\Big(pJ_1-\frac{n(n-4)}{2}\Big)|\nabla_\theta w|^2+\Big(pJ_2-\frac{n^2(n-4)^2}{16}\Big)w^2 \leqslant 0.
$$

If $\frac{n+4}{n-4}<p<p_c(n)$, then $p-1>0$, $pJ_1-\frac{n(n-4)}{2}>0$ and $pJ_2-\frac{n^2(n-4)^2}{16}>0$ (cf. [13, p. 338]), so $w\equiv 0$ and then $u\equiv 0$. $\square$

For applications in Section 6, we record the form of $E(R;0,u)$ for a homogeneous solution u.

Remark 3.2. Suppose $u(r,\theta)=r^{-\frac{4}{p-1}}w(\theta)$ is a homogeneous solution, where $p>\frac{n+4}{n-4}$ and $w\in W^{2,2}(\mathbb{S}^{n-1})\cap L^{p+1}(\mathbb{S}^{n-1})$. In this case, for any $r>0$,

$$
\int_{B_r\setminus B_{r/2}} |\Delta u|^2+|u|^{p+1} \leqslant cr^{n-4\frac{p+1}{p-1}}.
$$

Because $n-4\frac{p+1}{p-1}>0$, by choosing $r=2^{-i}R$ and summing in i from 0 to $+\infty$, we see

$$\int\limits_{B_R}|\Delta u|^2+|u|^{p+1}\leqslant cR^{n-4\frac{p+1}{p-1}},$$

which converges to 0 as $R\to 0$. Hence for any $R>0$, $E(R;0,u)$ is well-defined and by the homogeneity, it equals $E(1;0,u)$. By definition

$$\begin{aligned}
E(1;0,u)&=\int\limits_{B_1}\frac{1}{2}(\Delta u)^2-\frac{1}{p+1}|u|^{p+1}\\
&\quad+\frac{4}{p-1}\left(n-2-\frac{4}{p-1}\right)\int\limits_{\partial B_1}u^2+\int\limits_{\partial B_1}|\nabla_\theta u|^2\\
&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int\limits_{B_1}|u|^{p+1}+\frac{1}{2}\int\limits_{\partial B_1}\left(\frac{\partial u}{\partial r}\Delta u-u\frac{\partial \Delta u}{\partial r}\right)\\
&\quad+\frac{4}{p-1}\left(n-2-\frac{4}{p-1}\right)\int\limits_{\partial B_1}u^2+\int\limits_{\partial B_1}|\nabla_\theta u|^2.
\end{aligned}$$

By noting that

$$\begin{aligned}
\frac{\partial u}{\partial r}&=-\frac{4}{p-1}r^{-1}u, &\qquad \frac{\partial^2 u}{\partial r^2}&=\frac{4}{p-1}\left(\frac{4}{p-1}+1\right)r^{-2}u,\\
\frac{\partial \Delta u}{\partial r}&=-\left(2+\frac{4}{p-1}\right)r^{-1}\Delta u, &\qquad \Delta u&=\frac{4}{p-1}\left(\frac{4}{p-1}+2-n\right)r^{-2}u+r^{-2}\Delta_\theta u,
\end{aligned}$$

we get

$$E(1;0,u)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int\limits_{B_1}|u|^{p+1}=\frac{1}{n-4\frac{p+1}{p-1}}\left(\frac{1}{2}-\frac{1}{p+1}\right)\int\limits_{\partial B_1}|w|^{p+1}.$$

Replacing $|u|^{p+1}$ by $(\Delta u)^2$, we also have

$$\begin{aligned}
E(1;0,u)&=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int\limits_{B_1}(\Delta u)^2+\frac{p-1}{p+1}\int\limits_{\partial B_1}|\nabla_\theta u|^2\\
&\quad+\frac{4}{p+1}\left(n-2-\frac{4}{p-1}\right)\int\limits_{\partial B_1}u^2.
\end{aligned}$$

4. The blow down analysis

In this section we use the blow-down analysis to prove the Liouville theorem for stable solutions. Throughout this section u always denotes a smooth stable solution of (1.1) in $\mathbb{R}^n$.

Theorem 4.1. *Let u be a smooth stable solution of (1.1) on $\mathbb{R}^n$. If $1 < p < p_c(n)$, then $u \equiv 0$.*

The following lemma appears in [32] for positive solution. It remains valid for sign-changing solutions, see also [17].

Lemma 4.2. *Let u be a smooth stable solution of (1.1) and let $v = \Delta u$. Then for some C we have*

$$\int_{\mathbb{R}^n} \left(v^2 + |u|^{p+1}\right)\eta^2 \leqslant C \int_{\mathbb{R}^n} u^2\left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2 + \left|\Delta\left(|\nabla\eta|^2\right)\right|\right) dx + C\int_{\mathbb{R}^n} |uv||\nabla\eta|^2\, dx \tag{4.1}$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$.

Proof. For completeness we give the proof. We have the identity

$$\int_{\mathbb{R}^n} \left(\Delta^2\xi\right)\xi\eta^2\, dx = \int_{\mathbb{R}^n} \left(\Delta(\xi\eta)\right)^2 + \int_{\mathbb{R}^n} \left(-4(\nabla\xi\cdot\nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\right) dx + \int_{\mathbb{R}^n} \xi^2\left(2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\right) dx,$$

for $\xi \in C^4(\mathbb{R}^n)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$, see for example [32, Lemma 2.3].

Taking $\xi = u$ yields

$$\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\, dx = \int_{\mathbb{R}^n} \left(\Delta(u\eta)\right)^2 + \int_{\mathbb{R}^n} \left(-4(\nabla u\cdot\nabla\eta)^2 + 2uv|\nabla\eta|^2\right) dx + \int_{\mathbb{R}^n} u^2\left(2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\right) dx.$$

Using the stability inequality with $u\eta$ yields

$$p\int_{\mathbb{R}^n} |u|^{p+1}\eta^2\, dx \leqslant \int_{\mathbb{R}^n} \left(\Delta(u\eta)\right)^2.$$

Therefore

$$\int_{\mathbb{R}^n} \left(|u|^{p+1}\eta^2 + \left(\Delta(u\eta)\right)^2\right) dx \leqslant C \int_{\mathbb{R}^n} \left(|\nabla u|^2 |\nabla \eta|^2 + |uv||\nabla \eta|^2\right) dx + C \int_{\mathbb{R}^n} u^2 \left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2\right) dx.$$

Using $\Delta(\eta u) = v\eta + 2\nabla\eta\cdot\nabla u + u\Delta\eta$ we obtain

$$\int_{\mathbb{R}^n} \left(|u|^{p+1} + v^2\right)\eta^2 \, dx \leqslant C \int_{\mathbb{R}^n} \left(|\nabla u|^2 |\nabla \eta|^2 + |uv||\nabla \eta|^2\right) dx + C \int_{\mathbb{R}^n} u^2 \left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2\right) dx.$$

But

$$\begin{aligned} 2\int_{\mathbb{R}^n} |\nabla u|^2 |\nabla\eta|^2 \, dx &= \int_{\mathbb{R}^n} \Delta\left(u^2\right)|\nabla\eta|^2 \, dx - 2\int_{\mathbb{R}^n} uv|\nabla\eta|^2 \, dx \\ &= \int_{\mathbb{R}^n} u^2 \Delta\left(|\nabla\eta|^2\right) dx - 2\int_{\mathbb{R}^n} uv|\nabla\eta|^2 \, dx, \end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} \left(|u|^{p+1} + v^2\right)\eta^2 \, dx \leqslant C \int_{\mathbb{R}^n} u^2 \left(\left|\nabla(\Delta\eta)\cdot\nabla\eta\right| + (\Delta\eta)^2 + \left|\Delta\left(|\nabla\eta|^2\right)\right|\right) dx + C\int_{\mathbb{R}^n} |uv||\nabla\eta|^2 \, dx.$$

This proves (4.1) □

Corollary 4.3. *There exists a constant C such that*

$$\int_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{-4} \int_{B_{2R}(x)\setminus B_R(x)} u^2 + CR^{-2} \int_{B_{2R}(x)\setminus B_R(x)} |uv|, \tag{4.2}$$

and

$$\int_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{n-4\frac{p+1}{p-1}}, \tag{4.3}$$

for all $B_R(x)$.

Proof. The first inequality is a direct consequence of (4.1), by choosing a cut-off function $\eta \in C_0^\infty(B_{2R}(x))$, such that $\eta \equiv 1$ in $B_R(x)$, and for $k \leqslant 3$, $|\nabla^k \eta| \leqslant \frac{1000}{R^k}$.

Exactly the same argument as in [32] or [17] provides the second estimate. For completeness, we record the proof here. Replace η in (4.1) by η^m, where m is a large integer and η is a cut-off function as before. Then

$$\int |uv||\nabla \eta^m|^2 = m^2 \int\limits_{B_{2R}(x)\setminus B_R(x)} |uv|\eta^{2m-2}|\nabla \eta|^2$$
$$\leqslant \frac{1}{2C}\int v^2\eta^{2m} + C\int u^2\eta^{2m-4}|\nabla \eta|^4.$$

Substituting this into (4.1), we obtain

$$\int \left(v^2 + |u|^{p+1}\right)\eta^{2m} \leqslant CR^{-4} \int\limits_{B_{2R}(x)} u^2\eta^{2m-4}$$
$$\leqslant CR^{-4}\Bigg(\int\limits_{B_{2R}(x)} |u|^{p+1}\eta^{(m-2)(p+1)}\Bigg)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})}.$$

This gives (4.3). Here we have used the fact $\eta^{2m} \geqslant \eta^{(m-2)(p+1)}$ because $0 \leqslant \eta \leqslant 1$, m is large, and $p > 1$. □

Proof of Theorem 4.1 for $1 < p \leqslant \frac{n+4}{n-4}$. For $p < \frac{n+4}{n-4}$, we can let $R \to +\infty$ in (4.3) to get $u \equiv 0$ directly. If $p = \frac{n+4}{n-4}$, this gives

$$\int\limits_{\mathbb{R}^n} v^2 + |u|^{p+1} < +\infty.$$

So

$$\lim_{R\to+\infty} \int\limits_{B_{2R}(x)\setminus B_R(x)} v^2 + |u|^{p+1} = 0.$$

Then by (4.2), and noting that now $n = 4\frac{p+1}{p-1}$,

$$\int\limits_{B_R(x)} v^2 + |u|^{p+1} \leqslant CR^{-4} \int\limits_{B_{2R}(x)\setminus B_R(x)} u^2 + C \int\limits_{B_{2R}(x)\setminus B_R(x)} |v|^2$$
$$\leqslant CR^{-4}\Bigg(\int\limits_{B_{2R}(x)\setminus B_R(x)} |u|^{p+1}\Bigg)^{\frac{2}{p+1}} R^{n(1-\frac{2}{p+1})} + C\int\limits_{B_{2R}(x)\setminus B_R(x)} |v|^2$$

$$\leqslant C\Big(\int\limits_{B_{2R}(x)\setminus B_R(x)} |u|^{p+1}\Big)^{\frac{2}{p+1}} + C\int\limits_{B_{2R}(x)\setminus B_R(x)} |v|^2.$$

This goes to 0 as $R\to+\infty$, and we still get $u\equiv 0$. □

Next we concentrate on the case $p>\frac{n+4}{n-4}$. We first use (4.3) to show

Lemma 4.4. $\lim_{r\to+\infty} E(r;0,u)<+\infty$.

Proof. Let us write $E(r)=E(r;0,u)$. Since $E(r)$ is non-decreasing in r, we have

$$E(r)\leqslant \frac{1}{r}\int\limits_r^{2r} E(t)\,dt \leqslant \frac{1}{r^2}\int\limits_r^{2r}\int\limits_t^{t+r} E(\lambda)\,d\lambda\,dt.$$

By (4.3),

$$\frac{1}{r^2}\int\limits_r^{2r}\int\limits_t^{t+r}\Big(\lambda^{4\frac{p+1}{p-1}-n}\int\limits_{B_\lambda}\frac{1}{2}(\Delta u)^2-\frac{1}{p+1}|u|^{p+1}\Big)\,d\lambda\,dt\leqslant C.$$

Next

$$\begin{aligned}
&\frac{1}{r^2}\int\limits_r^{2r}\int\limits_t^{t+r}\Big(\lambda^{\frac{8}{p-1}+1-n}\int\limits_{\partial B_\lambda} u^2\Big)\,d\lambda\,dt\\
&=\frac{1}{r^2}\int\limits_r^{2r}\int\limits_{B_{t+r}\setminus B_t}|x|^{\frac{8}{p-1}+1-n}u(x)^2\,dx\,dt\\
&\leqslant\frac{1}{r^2}\int\limits_r^{2r}\Big(\int\limits_{B_{3r}\setminus B_r}|x|^{(\frac{8}{p-1}+1-n)\frac{p+1}{p-1}}\Big)^{\frac{p-1}{p+1}}\Big(\int\limits_{B_{3r}}\big|u(x)\big|^{p+1}\Big)^{\frac{2}{p+1}}\,dt\\
&\leqslant C.
\end{aligned}$$

The same estimate holds for the term in $E(r)$ containing

$$\int\limits_{\partial B_\lambda}\Big(|\nabla u|^2-\Big|\frac{\partial u}{\partial r}\Big|^2\Big).$$

For this we need to note the following estimate

$$\int\limits_{B_r}|\nabla u|^2\leqslant Cr^2\int\limits_{B_{2r}}(\Delta u)^2+Cr^{-2+n\frac{p-1}{p+1}}\Big(\int\limits_{B_{2r}}|u|^{p+1}\Big)^{\frac{2}{p+1}}\leqslant Cr^{n-\frac{8}{p-1}-2}. \tag{4.4}$$

Now consider

$$\begin{aligned}
&\frac{1}{r^2}\int_r^{2r}\int_t^{t+r}\frac{\lambda^3}{2}\frac{d}{d\lambda}\left[\lambda^{\frac{8}{p-1}+1-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u+\frac{\partial u}{\partial r}\right)^2\right]d\lambda\,dt\\
&=\frac{1}{2r^2}\int_r^{2r}\left\{(t+r)^{\frac{8}{p-1}+4-n}\int_{\partial B_{t+r}}\left(\frac{4}{p-1}(t+r)^{-1}u+\frac{\partial u}{\partial r}\right)^2\right.\\
&\quad\left.-t^{\frac{8}{p-1}+4-n}\int_{\partial B_t}\left(\frac{4}{p-1}t^{-1}u+\frac{\partial u}{\partial r}\right)^2\right\}dt\\
&\quad-\frac{3}{2r^2}\int_r^{2r}\int_t^{t+r}\lambda^{\frac{8}{p-1}+3-n}\int_{\partial B_\lambda}\left(\frac{4}{p-1}\lambda^{-1}u+\frac{\partial u}{\partial r}\right)^2d\lambda\,dt\\
&\leqslant\frac{C}{r^2}\int_{B_{3r}\setminus B_r}|x|^{\frac{8}{p-1}+4-n}\left(\frac{4}{p-1}|x|^{-1}u+\frac{\partial u}{\partial r}\right)^2\\
&\leqslant C.
\end{aligned}$$

The remaining terms in $E(r)$ can be treated similarly. □

For any $\lambda>0$, define

$$u^\lambda(x):=\lambda^{\frac{4}{p-1}}u(\lambda x),\qquad v^\lambda(x):=\lambda^{\frac{4}{p-1}+2}v(\lambda x).$$

u^λ is also a smooth stable solution of (1.1) on $\mathbb{R}^n$.

By rescaling (4.3), for all $\lambda>0$ and balls $B_r(x)\subset\mathbb{R}^n$,

$$\int_{B_r(x)}\left(v^\lambda\right)^2+\left|u^\lambda\right|^{p+1}\leqslant Cr^{n-4\frac{p+1}{p-1}}.$$

In particular, u^λ are uniformly bounded in $L^{p+1}_{loc}(\mathbb{R}^n)$ and $v^\lambda=\Delta u^\lambda$ are uniformly bounded in $L^2_{loc}(\mathbb{R}^n)$. By elliptic estimates, u^λ are also uniformly bounded in $W^{2,2}_{loc}(\mathbb{R}^n)$. Hence, up to a subsequence of $\lambda\to+\infty$, we can assume that $u^\lambda\to u^\infty$ weakly in $W^{2,2}_{loc}(\mathbb{R}^n)\cap L^{p+1}_{loc}(\mathbb{R}^n)$. By compactness embedding for Sobolev functions, $u^\lambda\to u^\infty$ strongly in $W^{1,2}_{loc}(\mathbb{R}^n)$. Then for any ball $B_R(0)$, by interpolation between L^q spaces and noting (4.3), for any $q\in[1,p+1)$, as $\lambda\to+\infty$,

$$\left\|u^\lambda-u^\infty\right\|_{L^q(B_R(0))}\leqslant\left\|u^\lambda-u^\infty\right\|^t_{L^1(B_R(0))}\left\|u^\lambda-u^\infty\right\|^{1-t}_{L^{p+1}(B_R(0))}\to 0,\qquad(4.5)$$

where $t\in(0,1]$ satisfies $\frac{1}{q}=t+\frac{1-t}{p+1}$. That is, $u^\lambda\to u^\infty$ in $L^q_{loc}(\mathbb{R}^n)$ for any $q\in[1,p+1)$.

For any function $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \Delta u^\infty \Delta\varphi - \left(u^\infty\right)^p \varphi = \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} \Delta u^\lambda \Delta\varphi - \left(u^\lambda\right)^p \varphi = 0,$$

$$\int_{\mathbb{R}^n} (\Delta\varphi)^2 - p\left(u^\infty\right)^{p-1}\varphi^2 = \lim_{\lambda\to+\infty} \int_{\mathbb{R}^n} (\Delta\varphi)^2 - p\left(u^\lambda\right)^{p-1}\varphi^2 \geqslant 0.$$

Thus $u^\infty \in W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$ is a stable solution of (1.1) in $\mathbb{R}^n$.

Lemma 4.5. *u^∞ is homogeneous.*

Proof. For any $0 < r < R < +\infty$, by the monotonicity of $E(r;0,u)$ and Lemma 4.4,

$$\lim_{\lambda\to+\infty} E(\lambda R;0,u) - E(\lambda r;0,u) = 0.$$

Therefore, by the scaling invariance of E

$$\lim_{\lambda\to+\infty} E\left(R;0,u^\lambda\right) - E\left(r;0,u^\lambda\right) = 0.$$

We note that $E(r;0,u^\lambda)$ is absolutely continuous with respect to r, since we assume u^λ smooth. This still holds if we assume $u \in W^{4,2}(B_R(0)) \cap L^{p+1}(B_R(0))$, since boundary integrals only involve second order derivatives of u and so for each $B_r(0) \subset B_R(0)$, $u \in W^{3,2}(\partial B_r(0))$. Then by Corollary 2.1 we see that

$$\begin{aligned}
0 &= \lim_{\lambda\to+\infty} E\left(R;0,u^\lambda\right) - E\left(r;0,u^\lambda\right) \\
&\geqslant c(n,p) \lim_{\lambda\to+\infty} \int_{B_R\setminus B_r} \frac{\left(\frac{4}{p-1}|x|^{-1}u^\lambda(x) + \frac{\partial u^\lambda}{\partial r}(x)\right)^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx \\
&\geqslant c(n,p) \int_{B_R\setminus B_r} \frac{\left(\frac{4}{p-1}|x|^{-1}u^\infty(x) + \frac{\partial u^\infty}{\partial r}(x)\right)^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx.
\end{aligned}$$

Note that in the last inequality we only used the weak convergence of u^λ to u^∞ in $W^{1,2}_{loc}(\mathbb{R}^n)$. Now

$$\frac{4}{p-1} r^{-1} u^\infty + \frac{\partial u^\infty}{\partial r} = 0, \quad \text{a.e. in } \mathbb{R}^n.$$

Integrating in r shows that

$$u^\infty(x) = |x|^{-\frac{4}{p-1}} u^\infty\left(\frac{x}{|x|}\right).$$

That is, u^∞ is homogeneous. $\square$

By Theorem 3.1, $u^\infty \equiv 0$. Since this holds for the limit of any sequence $\lambda \to +\infty$, by (4.5) we get

$$\lim_{\lambda\to+\infty} u^\lambda = 0 \quad \text{strongly in } L^2\big(B_4(0)\big).$$

Now we show

Lemma 4.6. $\lim_{r\to+\infty} E(r;0,u) = 0$.

Proof. For all $\lambda \to +\infty$,

$$\lim_{\lambda\to+\infty} \int\limits_{B_4(0)} \left(u^\lambda\right)^2 = 0.$$

Because v^λ are uniformly bounded in $L^2(B_4(0))$, by the Cauchy inequality we also have

$$\lim_{\lambda\to+\infty} \int\limits_{B_4(0)} \left|u^\lambda v^\lambda\right| \leqslant \lim_{\lambda\to+\infty}\bigg(\int\limits_{B_4(0)} \left(u^\lambda\right)^2\bigg)^{\frac{1}{2}}\bigg(\int\limits_{B_4(0)} \left(v^\lambda\right)^2\bigg)^{\frac{1}{2}} = 0.$$

By (4.2),

$$\begin{aligned}\lim_{\lambda\to+\infty} \int\limits_{B_3(0)} \left(v^\lambda\right)^2 + \left|u^\lambda\right|^{p+1} &\leqslant C \lim_{\lambda\to+\infty}\bigg(\int\limits_{B_4(0)} \left(u^\lambda\right)^2 + \int\limits_{B_4(0)} \left|u^\lambda v^\lambda\right|\bigg) \\ &= 0. \end{aligned} \tag{4.6}$$

By the interior L^2 estimate, we get

$$\lim_{\lambda\to+\infty} \int\limits_{B_2(0)} \sum_{k\leqslant 2} \left|\nabla^k u^\lambda\right|^2 = 0.$$

In particular, we can choose a sequence $\lambda_i \to +\infty$ such that

$$\int\limits_{B_2(0)} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 \leqslant 2^{-i}.$$

By this choice we have

$$\int\limits_1^2 \sum_{i=1}^{+\infty} \int\limits_{\partial B_r} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 dr \leqslant \sum_{i=1}^{+\infty} \int\limits_1^2 \int\limits_{\partial B_r} \sum_{k\leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 dr \leqslant 1.$$

That is, the function

$$f(r) := \sum_{i=1}^{+\infty} \int_{\partial B_r} \sum_{k \leqslant 2} \left|\nabla^k u^{\lambda_i}\right|^2 \in L^1((1,2)).$$

There exists an $r_0 \in (1,2)$ such that $f(r_0) < +\infty$. From this we get

$$\lim_{i \to +\infty} \left\|u^{\lambda_i}\right\|_{W^{2,2}(\partial B_{r_0})} = 0.$$

Combining this with (4.6) and the scaling invariance of $E(r)$, we get

$$\lim_{i \to +\infty} E(\lambda_i r_0; 0, u) = \lim_{i \to +\infty} E\left(r_0; 0, u^{\lambda_i}\right) = 0.$$

Since $\lambda_i r_0 \to +\infty$ and $E(r; 0, u)$ is non-decreasing in r, we get

$$\lim_{r \to +\infty} E(r; 0, u) = 0. \qquad \square$$

By the smoothness of u, $\lim_{r \to 0} E(r; 0, u) = 0$. Then again by the monotonicity of $E(r; 0, u)$ and the previous lemma, we obtain

$$E(r; 0, u) = 0 \quad \text{for all } r > 0.$$

Then again by Corollary 2.1, u is homogeneous, and then $u \equiv 0$ by Theorem 3.1 (or by the smoothness of u). This finishes the proof of Theorem 4.1.

5. Finite Morse index solutions

In this section we prove Theorem 1.3 and we always assume that u is a smooth solution. First, by the doubling lemma [22] and our Liouville theorem for stable solutions, Theorem 4.1, we have

Lemma 5.1. *Let u be a smooth, finite Morse index (positive or sign changing) solution of (1.1). There exist a constant C and R_0 such that for all $x \in B_{R_0}(0)^c$,*

$$\left|u(x)\right| \leqslant C|x|^{-\frac{4}{p-1}}.$$

Proof. Assume that u is stable outside B_{R_0}. For $x \in B^c_{R_0}$, let $M(x) = |u(x)|^{\frac{p-1}{4}}$ and $d(x) = |x| - R_0$, the distance to B_{R_0}. Assume that there exists a sequence of $x_k \in B^c_{R_0}$ such that

$$M(x_k) d(x_k) \geqslant 2k. \tag{5.1}$$

Since u is bounded on any compact set of $\mathbb{R}^n$, $d(x_k) \to +\infty$.

By the doubling lemma [22], there exists another sequence $y_k \in B^c_{R_0}$, such that

(1) $M(y_k)d(y_k) \geqslant 2k$;
(2) $M(y_k) \geqslant M(x_k)$;
(3) $M(z) \leqslant 2M(y_k)$ for any $z \in B^c_{R_0}$ such that $|z - y_k| \leqslant \frac{k}{M(y_k)}$.

Now define

$$u_k(x) = M(y_k)^{-\frac{4}{p-1}} u\big(y_k + M(y_k)^{-1}x\big), \quad \text{for } x \in B_k(0).$$

By definition, $|u_k(0)| = 1$. By (3), $|u_k| \leqslant 2^{\frac{4}{p-1}}$ in $B_k(0)$. By (1), $B_{k/M(y_k)}(y_k) \cap B_{R_0} = \emptyset$, which implies that u is stable in $B_{k/M(y_k)}(y_k)$. Hence u_k is stable in $B_k(0)$.

By elliptic regularity, u_k are uniformly bounded in $C^5_{loc}(B_k(0))$. Up to a subsequence, u_k converges to u_∞ in $C^4_{loc}(\mathbb{R}^n)$. By the above conditions on u_k, we have

(1) $|u_\infty(0)| = 1$;
(2) $|u_\infty| \leqslant 2^{\frac{4}{p-1}}$ in $\mathbb{R}^n$;
(3) u_∞ is a smooth stable solution of (1.1) in $\mathbb{R}^n$.

By the Liouville theorem for stable solutions, Theorem 4.1, $u_\infty \equiv 0$. This is a contradiction, so (5.1) does not hold. $\square$

Corollary 5.2. *There exist a constant C_3 and R_0 such that for all $x \in B_{3R_0}(0)^c$,*

$$\sum_{k\leqslant 3} |x|^{\frac{4}{p-1}+k}\big|\nabla^k u(x)\big| \leqslant C_3. \tag{5.2}$$

Proof. For any x_0 with $|x_0| > 3R_0$, take $\lambda = \frac{|x_0|}{2}$ and define

$$\bar{u}(x) = \lambda^{\frac{4}{p-1}} u(x_0 + \lambda x).$$

By the previous lemma, $|\bar{u}| \leqslant C_1$ in $B_1(0)$. Standard elliptic estimates give

$$\sum_{k\leqslant 3} \big|\nabla^k \bar{u}(0)\big| \leqslant C_3.$$

Rescaling back we get (5.2). $\square$

Remark 5.3. By the same proof of Lemma 5.1 and Corollary 5.2, one easily obtains the second part of Theorem 1.7.

5.1. *The subcritical case* $1 < p < \frac{n+4}{n-4}$

We use the following Pohozaev identity. For its proof, see [23,24].

Lemma 5.4.

$$\int_{B_R} \frac{n-4}{2}(\Delta u)^2 - \frac{n}{p+1}|u|^{p+1} = \int_{\partial B_R} \frac{R}{2}(\Delta u)^2 + \frac{R}{p+1}|u|^{p+1} + R\frac{\partial u}{\partial r}\frac{\partial \Delta u}{\partial r} - \Delta u \frac{\partial(x\cdot\nabla u)}{\partial r}. \tag{5.3}$$

By taking $R \to +\infty$ and using (5.2), and noting that $p < \frac{n+4}{n-4}$, we see that

$$\int_{\partial B_R} \frac{R}{2}(\Delta u)^2 + \frac{R}{p+1}|u|^{p+1} + R\frac{\partial u}{\partial r}\frac{\partial \Delta u}{\partial r} - \Delta u \frac{\partial(x\cdot\nabla u)}{\partial r} \to 0.$$

By (5.2), we also have

$$(\Delta u)^2 + |u|^{p+1} \leqslant C\big(1+|x|\big)^{-4\frac{p+1}{p-1}}.$$

Since $p < \frac{n+4}{n-4}$, $4\frac{p+1}{p-1} > n$. Hence

$$\int_{\mathbb{R}^n} (\Delta u)^2 + |u|^{p+1} < +\infty.$$

Taking limit in (5.3), we get

$$\int_{\mathbb{R}^n} \frac{n-4}{2}(\Delta u)^2 - \frac{n}{p+1}|u|^{p+1} = 0. \tag{5.4}$$

Take an $\eta \in C_0^\infty(B_2)$, $\eta \equiv 1$ in B_1 and $\sum_{k\leqslant 2} |\nabla^k \eta| \leqslant 1000$, and denote $\eta_R(x) = \eta(x/R)$. By testing Eq. (1.1) with $u(x)\eta_R^2$, we get

$$\int_{\mathbb{R}^n} (\Delta u)^2\eta_R^2 - |u|^{p+1}\eta_R^2 = -\int_{\mathbb{R}^n} \big(2\nabla u \nabla \eta_R^2 + u\Delta\eta_R^2\big)\Delta u. \tag{5.5}$$

By the same reasoning as above, we get

$$\int_{\mathbb{R}^n} (\Delta u)^2 - |u|^{p+1} = 0.$$

Substituting (5.4) into this, we get

$$\left(\frac{n-4}{2} - \frac{n}{p+1}\right) \int_{\mathbb{R}^n} |u|^{p+1} = 0.$$

Since $\frac{n-4}{2} - \frac{n}{p+1} < 0$, $u \equiv 0$.

5.2. The critical case

Since u is stable outside B_{R_0}, Lemma 4.2 still holds if the support of η is outside B_{R_0}. Take $\varphi \in C_0^\infty(B_{2R} \setminus B_{2R_0})$, such that $\varphi \equiv 1$ in $B_R \setminus B_{3R_0}$ and $\sum_{k \leqslant 3} |x|^k |\nabla^k \varphi| \leqslant 100$. Then by choosing $\eta = \varphi^m$, where m is large, in (4.1), and by the same reasoning to derive (4.3), we get

$$\int_{B_R \setminus B_{3R_0}} (\Delta u)^2 + |u|^{p+1} \leqslant C.$$

Letting $R \to +\infty$, we get

$$\int_{\mathbb{R}^n} (\Delta u)^2 + |u|^{p+1} < +\infty.$$

Similar to (4.4), we have

$$R^{-2} \int_{B_{2R} \setminus B_R} |\nabla u|^2 \leqslant C \int_{B_{3R} \setminus B_{R/2}} (\Delta u)^2 + C\left(\int_{B_{3R} \setminus B_{R/2}} |u|^{p+1}\right)^{\frac{2}{p+1}}.$$

Then by applying the Hölder inequality to (5.5), we have

$$\left|\int_{\mathbb{R}^n} (\Delta u)^2 \eta_R^2 - |u|^{p+1} \eta_R^2\right|$$
$$\leqslant C\left[R^{-1}\left(\int_{B_{2R} \setminus B_R} |\nabla u|^2\right)^{\frac{1}{2}} + \left(\int_{B_{2R} \setminus B_R} |u|^{p+1}\right)^{\frac{1}{p+1}}\right]\left(\int_{B_{2R} \setminus B_R} (\Delta u)^2\right)^{\frac{1}{2}}.$$

After letting $R \to +\infty$ we obtain

$$\int_{\mathbb{R}^n} (\Delta u)^2 - |u|^{p+1} = 0.$$

5.3. The supercritical case

Now we consider the case $p > \frac{n+4}{n-4}$.

Lemma 5.5. *There exists a constant C_2, such that for all $r > 3R_0$, $E(r; 0, u) \leqslant C_2$.*

Proof. Expanding those boundary integrals in $E(r;0,u)$ into a full formulation involving the differentials of u up to second order, and substituting (5.2) into this formulation, we get

$$\begin{aligned}
E(r;0,u) &\leqslant Cr^{4\frac{p+1}{p-1}-n}\left(\int_{B_r}(\Delta u)^2+|u|^{p+1}\right)+Cr^{\frac{8}{p-1}+1-n}\int_{\partial B_r}u^2 \\
&\quad+Cr^{\frac{8}{p-1}+2-n}\int_{\partial B_r}|u||\nabla u|+Cr^{\frac{8}{p-1}+3-n}\int_{\partial B_r}|\nabla u|^2 \\
&\quad+Cr^{\frac{8}{p-1}+3-n}\int_{\partial B_r}|u|\left|\nabla^2 u\right|+Cr^{\frac{8}{p-1}+4-n}\int_{\partial B_r}|\nabla u|\left|\nabla^2 u\right| \\
&\leqslant C.
\end{aligned}$$

This constant only depends on the constant in (5.2). □

By Corollary 2.1, we get

Corollary 5.6.

$$\int_{B_{3R_0}^c}\frac{(\frac{4}{p-1}|x|^{-1}u(x)+\frac{\partial u}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx<+\infty.$$

As in the proof for stable solutions, define the blowing down sequence

$$u^\lambda(x)=\lambda^{\frac{4}{p-1}}u(\lambda x).$$

By Lemma 5.1, u^λ are uniformly bounded in $C^5(B_r(0)\setminus B_{1/r}(0))$ for any fixed $r>1$. u^λ is stable outside $B_{R_0/\lambda}(0)$. There exists a function $u^\infty\in C^4(\mathbb{R}^n\setminus\{0\})$, such that up to a subsequence of $\lambda\to+\infty$, u^λ converges to u^∞ in $C^4_{loc}(\mathbb{R}^n\setminus\{0\})$. u^∞ is a stable solution of (1.1) in $\mathbb{R}^n\setminus\{0\}$.

For any $r>1$, by Corollary 5.6,

$$\begin{aligned}
&\int_{B_r\setminus B_{1/r}}\frac{(\frac{4}{p-1}|x|^{-1}u^\infty(x)+\frac{\partial u^\infty}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx \\
&=\lim_{\lambda\to+\infty}\int_{B_r\setminus B_{1/r}}\frac{(\frac{4}{p-1}|x|^{-1}u^\lambda(x)+\frac{\partial u^\lambda}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx \\
&=\lim_{\lambda\to+\infty}\int_{B_{\lambda r}\setminus B_{\lambda/r}}\frac{(\frac{4}{p-1}|x|^{-1}u(x)+\frac{\partial u}{\partial r}(x))^2}{|x|^{n-2-\frac{8}{p-1}}}\,dx \\
&=0.
\end{aligned}$$

Hence u^∞ is homogeneous, and by Theorem 3.1, $u^\infty \equiv 0$ if $p < p_c(n)$. This holds for every limit of u^λ as $\lambda \to +\infty$, thus we have

$$\lim_{x\to\infty} |x|^{\frac{4}{p-1}} \big|u(x)\big| = 0.$$

Then as in the proof of Corollary 5.2, we get

$$\lim_{x\to\infty} \sum_{k\leqslant 4} |x|^{\frac{4}{p-1}+k} \big|\nabla^k u(x)\big| = 0.$$

For any $\varepsilon > 0$, take an R such that for $|x| > R$,

$$\sum_{k\leqslant 4} |x|^{\frac{4}{p-1}+k} \big|\nabla^k u(x)\big| \leqslant \varepsilon.$$

Then for $r \gg R$,

$$\begin{aligned} E(r;0,u) &\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_R(0)} \big[(\Delta u)^2 + |u|^{p+1}\big] + C\varepsilon r^{4\frac{p+1}{p-1}-n} \int_{B_r(0)\setminus B_R(0)} |x|^{-4\frac{p+1}{p-1}} \\ &\quad + C\varepsilon r^{4\frac{p+1}{p-1}+1-n} \int_{\partial B_r(0)} |x|^{-4\frac{p+1}{p-1}} \\ &\leqslant C(R)\big(r^{4\frac{p+1}{p-1}-n} + \varepsilon\big). \end{aligned}$$

Since $4\frac{p+1}{p-1} - n < 0$ and ε can be arbitrarily small, we get $\lim_{r\to+\infty} E(r;0,u) = 0$. Because $\lim_{r\to 0} E(r;0,u) = 0$ (by the smoothness of u), the same argument for stable solutions implies that $u \equiv 0$.

Remark 5.7. The monotonicity formula approach here is in some sense equivalent to the Pohozaev identity method (see for example [32]). The convergence of u^λ can also be seen by writing the equation in exponential polar coordinates.

6. Partial regularity in high dimensions

Here we study the partial regularity for the extremal solution to the problem (1.8), and prove Theorems 1.6 and 1.9. Recall that we defined n_p to be the smallest dimension such that Theorem 3.1 does not hold. This is also the smallest dimension such that the Liouville theorem for stable solutions, Theorem 4.1, and the classification result for stable homogeneous solutions, Theorem 3.1, do not hold.

6.1. Regularity when $n < n_p$

In this subsection we prove the full regularity when $n < n_p$.

Proof of Theorem 1.6. For $0 < \lambda < \lambda^*$ let $u_\lambda > 0$ be the minimal solution of (1.8). We claim that

$$\sup_{\lambda \in (0,\lambda^*)} \|u_\lambda\|_{L^\infty(\Omega)} < +\infty. \tag{6.1}$$

Then by elliptic estimates, as $\lambda \to \lambda^*$, u_λ are uniformly bounded in $C^5(\overline{\Omega})$. Because u_λ converges to u_{λ^*} pointwisely in Ω, $u_{\lambda^*} \in C^4(\overline{\Omega})$, and then we get $u_{\lambda^*} \in C^\infty(\overline{\Omega})$ by bootstrapping elliptic estimates.

To prove (6.1), we use the classical blow up method of Gidas and Spruck. Let x_λ attain $\max_{\overline{\Omega}} u_\lambda$, and assume that

$$L_\lambda = u_\lambda(x_\lambda) + 1 \to +\infty.$$

By the maximum principle, $x_\lambda \in \Omega$ is an interior point and

$$-\Delta u_\lambda > 0 \quad \text{in } \Omega. \tag{6.2}$$

Define

$$\bar{u}_\lambda = \lambda^{\frac{1}{p-1}} L_\lambda^{-1}\big(u_\lambda\big(x_\lambda + L_\lambda^{-\frac{p-1}{4}} x\big) + 1\big) \quad \text{in } \Omega_\lambda,$$

where $\Omega_\lambda = L_\lambda^{-\frac{p-1}{4}}(\Omega - x_\lambda)$. $\bar{u}_\lambda$ is a smooth stable solution of (1.1) in Ω_λ, satisfying

$$\bar{u}_\lambda(0) = \max_{\overline{\Omega_\lambda}} \bar{u}_\lambda = 1, \tag{6.3}$$

and the boundary condition

$$\bar{u}_\lambda = \lambda^{\frac{1}{p-1}} L_\lambda^{-1}, \qquad \Delta \bar{u}_\lambda = 0 \quad \text{on } \partial\Omega_\lambda.$$

From this, with the help of standard elliptic estimates, we see for any $R > 0$, $\bar{u}_\lambda$ are uniformly bounded in $C^5(\Omega_\lambda \cap B_R(0))$. By rescaling (6.2),

$$-\Delta \bar{u}_\lambda > 0 \quad \text{in } \Omega_\lambda. \tag{6.4}$$

Since Ω is a smooth domain, as $\lambda \to \lambda^*$, Ω_λ either converges to $\mathbb{R}^n$ or to a half space H. In the former case, $\bar{u}_\lambda$ converges (up to a subsequence) to a limit $\bar{u}$ in $C^4_{loc}(\mathbb{R}^n)$. Here $\bar{u}$ is a positive, stable, C^4 solution of (1.1) in $\mathbb{R}^n$. Then by Theorem 4.1, $\bar{u} \equiv 0$. However, by passing to the limit in (6.3), we obtain

$$\bar{u}(0) = 1.$$

This is a contradiction.

If Ω_λ converges to a half space $H = \{x_1 > -h\}$ for some $h > 0$, $\bar{u}_\lambda$ converges (up to a subsequence) to a limit $\bar{u}$ in $C^4_{loc}(\overline{H})$. Here $\bar{u}$ is a positive, stable, C^4 solution of (1.1) in H, with the boundary conditions

$$\bar{u} = \Delta\bar{u} = 0 \quad \text{on } \partial H.$$

By taking limits in (6.3) and (6.4), we obtain

$$\begin{cases} -\Delta\bar{u} = \bar{v} > 0, & \text{in } H, \\ -\Delta\bar{v} = \bar{u}^p > 0, & \text{in } H, \\ \bar{u} = \bar{v} = 0, & \text{on } \partial H, \\ \bar{u}(0) = \max_{\overline{H}} \bar{u} = 1. \end{cases}$$

By elliptic estimates, the last condition implies that $\bar{v}$ is bounded in H. Then by [Theorem 2, [5]] or [Theorem 10, [28]], $\frac{\partial\bar{u}}{\partial x_1} > 0$, $\frac{\partial\bar{v}}{\partial x_1} > 0$. Then the function $w(y) = \lim_{x_1\to+\infty} \bar{u}(x_1, y)$ exists for all $y \in \mathbb{R}^{n-1}$ and satisfies $\Delta^2 w = w^p$ in $\mathbb{R}^{n-1}$. By the arguments in [32, Section 3] this function w must be stable in $\mathbb{R}^{n-1}$ and nontrivial. By Theorem 1.3, $p \geqslant p_c(n-1) \geqslant p_c(n)$. This is impossible.

We conclude that $\bar{u} \equiv 0$, which is a contradiction. This finishes the proof of (6.1). □

6.2. An ε-regularity lemma

The remaining part is devoted to the proof of Theorem 1.9. In this subsection we prove an ε-regularity result, by establishing an improvement of decay estimate. First we need the following lemma.

Lemma 6.1. *There exists a constant C, such that, for any ball $B_{2r}(x) \subset \Omega$,*

$$r^{\frac{8p}{p-1}-n} \int_{B_r(x)} (u_{\lambda^*} + 1)^{2p} \leqslant C r^{4\frac{p+1}{p-1}-n} \int_{B_{2r}(x)} (\Delta u_{\lambda^*})^2. \tag{6.5}$$

Proof. Denote $w_\lambda = u_\lambda + 1$. By the maximum principle and Lemma 3.2 in [3], for any $\lambda \in (0, \lambda^*)$,

$$\Delta w_\lambda \leqslant -\sqrt{\frac{2\lambda}{p+1}} w_\lambda^{\frac{p+1}{2}} < 0 \quad \text{in } \Omega.$$

Since w_λ is smooth in Ω, we can follow the proof in [32] to get Eq. (2.15) in [32]. That is, for any $\eta \in C_0^\infty(\Omega)$,

$$\int_\Omega w_\lambda^{2p}\eta^2 \leqslant C \int_\Omega -\Delta w_\lambda w_\lambda^p \big(|\nabla\eta|^2 + |\Delta\eta^2|\big)$$

$$+C\int_{\Omega}(\Delta w_\lambda)^2\left[|\nabla\Delta\eta\nabla\eta|+\left|\Delta|\nabla\eta|^2\right|+|\Delta\eta|^2\right]. \tag{6.6}$$

Take $\varphi\in C_0^\infty(B_{2r}(x))$ such that $0\leqslant\varphi\leqslant 1$, $\varphi\equiv 1$ in $B_r(x)$ and

$$\sum_{k\leqslant 4} r^k\left|\nabla^k\varphi\right|\leqslant 1000.$$

Substituting $\eta=\varphi^m$ into (6.6) with m large, and then using Hölder's inequality (exactly as in the derivation of Eq. (2.16) of [32]), we get (6.5) for u_λ.

This implies that u_λ are uniformly bounded in $L^{2p}_{loc}(\Omega)$. By the interior L^2 estimate, u_λ are also uniformly bounded in $W^{4,2}_{loc}(\Omega)$. By the same proof of (4.5), as $\lambda\to\lambda^*$, $u_\lambda\to u_{\lambda^*}$ in $W^{3,2}_{loc}(\Omega)\cap L^{p+1}_{loc}(\Omega)$. Then

$$\begin{aligned}
r^{\frac{8p}{p-1}-n}\int_{B_r(x)}(u_{\lambda^*}+1)^{2p} &\leqslant \lim_{\lambda\to\lambda^*} r^{\frac{8p}{p-1}-n}\int_{B_r(x)}(u_\lambda+1)^{2p}\\
&\leqslant C\lim_{\lambda\to\lambda^*} r^{4\frac{p+1}{p-1}-n}\int_{B_{2r}(x)}(\Delta u_\lambda)^2\\
&\leqslant Cr^{4\frac{p+1}{p-1}-n}\int_{B_{2r}(x)}(\Delta u_{\lambda^*})^2.
\end{aligned}$$

Here we have used Fatou's lemma to deduce the first inequality. □

Below we denote $u=u_{\lambda^*}+1$. Inequality (6.5) implies that

$$\int_{B_r(x)}u^{2p}\leqslant Cr^{n-\frac{8p}{p-1}}, \tag{6.7}$$

for any ball $B_r(x)\subset\Omega$, with the constant C depending only on p and Ω. See for example the derivation of Eq. (2.16) in [32]. Similarly, u also satisfies (4.3) for any ball $B_R(x)\subset\Omega$. Estimate (6.5) will play a crucial role in our proof of the ε-regularity lemma. Note that both (6.5) and (6.7) are invariant under the scaling for (1.1). These two are also preserved under various limits (the precise notion of limit will be given below).

To prove the partial regularity of u, first we need the following improvement of decay estimate.

Lemma 6.2. *There exist two universal constants $\varepsilon_0>0$ and $\theta\in(0,1)$, such that if u is a positive stable solution of (1.1) satisfying the estimate (6.5), and*

$$(2R)^{4\frac{p+1}{p-1}-n}\int_{B_{2R}}\left[u^{p+1}+(\Delta u)^2\right]=\varepsilon\leqslant\varepsilon_0,$$

then

$$(\theta R)^{4\frac{p+1}{p-1}-n} \int_{B_{\theta R}} \left[u^{p+1} + (\Delta u)^2\right] \leqslant \frac{\varepsilon}{2}.$$

Proof. By rescaling, we can assume $R = 1$. By (6.5), we have

$$\int_{B_{3/2}} u^{2p} \leqslant C \int_{B_2} \left[u^{p+1} + (\Delta u)^2\right] \leqslant C\varepsilon. \tag{6.8}$$

By L^2 estimates applied to u,

$$\|u\|_{W^{4,2}(B_{4/3})} \leqslant C\left(\left\|u^p\right\|_{L^2(B_{3/2})} + \|u\|_{L^2(B_{3/2})}\right) \leqslant C\varepsilon^{\frac{1}{p+1}}.$$

We can choose an $r_0 \in (1, 4/3)$ so that

$$\|u\|_{W^{2,2}(\partial B_{r_0})} \leqslant C\varepsilon^{\frac{1}{p+1}}. \tag{6.9}$$

Now take the decomposition $u = u_1 + u_2$, where

$$\begin{cases} \Delta^2 u_1 = u^p, & \text{in } B_{r_0}, \\ u_1 = \Delta u_1 = 0, & \text{on } \partial B_{r_0}(0), \end{cases}$$

and

$$\begin{cases} \Delta^2 u_2 = 0, & \text{in } B_{r_0}, \\ u_2 = u, \quad \Delta u_2 = \Delta u, & \text{on } \partial B_{r_0}(0). \end{cases}$$

By the maximum principle, $\Delta u_1 < 0$ and $u_1 > 0$ in $B_{r_0}(0)$.

By this decomposition,

$$\int_{B_{r_0}} \Delta u_1 \Delta u_2 = 0.$$

Hence

$$\int_{B_{r_0}} (\Delta u)^2 = \int_{B_{r_0}} (\Delta u_1)^2 + \int_{B_{r_0}} (\Delta u_2)^2.$$

In particular,

$$\int_{B_{r_0}} (\Delta u_2)^2 \leqslant C\varepsilon. \tag{6.10}$$

By elliptic estimates for biharmonic functions and (6.9), we have

$$\sup_{B_{1/2}} |u_2| \leqslant C\Big(\int_{\partial B_{r_0}} u^2 + (\Delta u)^2\Big)^{1/2} \leqslant C\varepsilon^{\frac{1}{p+1}}.$$

Since Δu_2 is harmonic, $(\Delta u_2)^2$ is subharmonic in B_{r_0}. By the mean value inequality for subharmonic functions and (6.10), for any $r \in (0, r_0)$,

$$r^{4\frac{p+1}{p-1}-n}\int_{B_r}(\Delta u_2)^2 \leqslant r^{4\frac{p+1}{p-1}} r_0^{-n}\int_{B_{r_0}}(\Delta u_2)^2 \leqslant Cr^{4\frac{p+1}{p-1}}\varepsilon.$$

For u_1, first by the Green function representation (cf. [13, Section 4.2]), we have

$$\|u_1\|_{L^1(B_{r_0})} \leqslant C\big\|u^p\big\|_{L^1(B_{r_0})} \leqslant C\Big(\int_{B_2} u^{p+1}\Big)^{\frac{p}{p+1}} \leqslant C\varepsilon^{\frac{p}{p+1}}. \tag{6.11}$$

Then by L^2 estimates using (6.7), we have

$$\|u_1\|_{W^{4,2}(B_{r_0})} \leqslant C\big(\big\|u^p\big\|_{L^2(B_{r_0})} + \|u_1\|_{L^1(B_{r_0})}\big) \leqslant C\varepsilon^{\frac{1}{2}}.$$

By the Sobolev embedding theorem, we have

$$\|u_1\|_{L^{\frac{2n}{n-8}}(B_{r_0})} \leqslant C\varepsilon^{\frac{1}{2}}.$$

Then an interpolation between L^1 and $L^{\frac{2n}{n-8}}$ gives

$$\|u_1\|_{L^2(B_{r_0})} \leqslant C\varepsilon^{\frac{1}{2}+2\delta},$$

where $\delta > 0$ is a constant depending only on the dimension n.

Next, by interpolation between Sobolev spaces, we get

$$\|\Delta u_1\|_{L^2(B_{r_0})} \leqslant \varepsilon^{-\delta}\|u_1\|_{L^2(B_{r_0})} + C\varepsilon^{\delta}\big\|\Delta^2 u_1\big\|_{L^2(B_{r_0})} \leqslant C\varepsilon^{\frac{1}{2}+\delta}.$$

Multiplying the equation of u_1 by u_1 and integrating by parts, we get

$$\int_{B_{r_0}} u^p u_1 = \int_{B_{r_0}} (\Delta u_1)^2 \leqslant C\varepsilon^{1+2\delta}.$$

By convexity, there exists a constant depending only on p such that

$$u^{p+1} \leqslant C\big(u_1^{p+1} + |u_2|^{p+1}\big).$$

For $r \in (0, 1/2)$, which will be determined below,

$$\begin{aligned}
r^{4\frac{p+1}{p-1}-n} \int_{B_r} u^{p+1} &\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} u_1^{p+1} + Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} |u_2|^{p+1} \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} \left(u + |u_2|\right)^p u_1 + Cr^{4\frac{p+1}{p-1}} \sup_{B_r} |u_2|^{p+1} \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} u^p u_1 + Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} \varepsilon^{\frac{p}{p+1}} u_1 + Cr^{4\frac{p+1}{p-1}} \varepsilon \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_{r_0}} u^p u_1 + Cr^{4\frac{p+1}{p-1}-n} \int_{B_{r_0}} \varepsilon^{\frac{p}{p+1}} u_1 + Cr^{4\frac{p+1}{p-1}} \varepsilon \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \varepsilon^{1+2\delta} + Cr^{4\frac{p+1}{p-1}-n} \varepsilon^{\frac{2p}{p+1}} + Cr^{4\frac{p+1}{p-1}} \varepsilon.
\end{aligned}$$

For $(\Delta u)^2$, we have

$$\begin{aligned}
r^{4\frac{p+1}{p-1}-n} \int_{B_r} (\Delta u)^2 &\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} (\Delta u_1)^2 + Cr^{4\frac{p+1}{p-1}-n} \int_{B_r} (\Delta u_2)^2 \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \int_{B_{r_0}} (\Delta u_1)^2 + Cr^{4\frac{p+1}{p-1}} r_0^{-n} \int_{B_{r_0}} (\Delta u_2)^2 \\
&\leqslant Cr^{4\frac{p+1}{p-1}-n} \varepsilon^{1+2\delta} + Cr^{4\frac{p+1}{p-1}} \varepsilon.
\end{aligned}$$

Putting these two together, we get

$$r^{4\frac{p+1}{p-1}-n} \int_{B_r} (\Delta u)^2 + u^{p+1} \leqslant Cr^{4\frac{p+1}{p-1}-n} \varepsilon^{1+2\delta} + Cr^{4\frac{p+1}{p-1}-n} \varepsilon^{\frac{2p}{p+1}} + Cr^{4\frac{p+1}{p-1}} \varepsilon.$$

We first choose $r = \theta \in (0, 1/2)$ so that

$$C\theta^{4\frac{p+1}{p-1}} \leqslant \frac{1}{4}.$$

Then choose an ε_0 so that for every $\varepsilon \in (0, \varepsilon_0)$,

$$C\theta^{4\frac{p+1}{p-1}-n} \varepsilon^{1+2\delta} + C\theta^{4\frac{p+1}{p-1}-n} \varepsilon^{\frac{2p}{p+1}} \leqslant \frac{1}{4}\varepsilon.$$

By this choice we finish the proof. □

Remark 6.3. Lemma 6.2 also holds for a sign-changing solution u of (1.1) if it satisfies

$$r^{\frac{8p}{p-1}-n}\int_{B_r(x)}|u|^{2p}\leqslant Cr^{4\frac{p+1}{p-1}-n}\int_{B_{2r}(x)}\left[|u|^{p+1}+(\Delta u)^2\right],\tag{6.12}$$

for any ball $B_{2r}(x)\subset\Omega$. For the proof, we need to introduce a new function $\bar{u}_1$, which satisfies

$$\begin{cases}\Delta^2\bar{u}_1=|u|^p, & \text{in } B_{r_0},\\ \bar{u}_1=\Delta\bar{u}_1=0, & \text{on } \partial B_{r_0}(0),\end{cases}$$

By the maximum principle, $\bar{u}_1\geqslant|u_1|\geqslant 0$. By the same method for u_1, we have

$$\int_{B_{r_0}}|u|^p\bar{u}_1\leqslant C\varepsilon^{1+2\delta}.$$

We can use this to control $|u|^p|u_1|$.

Lemma 6.4. *There exist a universal constant $\varepsilon^*>0$ and $\theta\in(0,1)$, such that if u is a stable solution of (1.1) satisfying (6.12), and*

$$(2R)^{4\frac{p+1}{p-1}-n}\int_{B_{2R}(x_0)}\left[(\Delta u)^2+|u|^{p+1}\right]=\varepsilon\leqslant\varepsilon^*,$$

then u is smooth in B_R, and there exists a universal constant $C(\varepsilon^*)$ such that

$$\sup_{B_R(x_0)}|u|\leqslant C(\varepsilon^*)R^{-\frac{4}{p-1}}.$$

Proof. By choosing a small $\varepsilon^*>0$, we can apply Lemma 6.2 to any ball $B_r(x)$ with $x\in B_R(x_0)$ and $r\leqslant R/4$, which says

$$(\theta r)^{4\frac{p+1}{p-1}-n}\int_{B_{\theta r}(x)}(\Delta u)^2+|u|^{p+1}\leqslant\frac{1}{2}r^{4\frac{p+1}{p-1}-n}\int_{B_r(x)}\left[(\Delta u)^2+|u|^{p+1}\right].$$

Iterating the above implies

$$\int_{B_r(x)}(\Delta u)^2+|u|^{p+1}\leqslant Cr^{n-4\frac{p+1}{p-1}+\delta}$$

for any $x\in B_1$ and $r\leqslant 1/8$. Here $\delta>0$ is a constant depending only on ε_0 and θ in Lemma 6.2. In other words, u belongs to the homogeneous Morrey space

$L^{p+1,n-4\frac{p+1}{p-1}+\delta}(B_1)$. Then the Morrey space estimate for biharmonic operator gives the claim, since $L^{p+1,n-4\frac{p+1}{p-1}+\delta}(B_1) \subset L^{p,n-\frac{4p}{p-1}+\frac{\delta p}{p+1}}(B_1)$, see Appendix A. $\square$

This lemma implies the singular set of u,

$$\mathcal{S} \subset \left\{x\colon \liminf_{r\to 0} r^{4\frac{p+1}{p-1}-n} \int_{B_r(x)} \left[(\Delta u)^2 + |u|^{p+1}\right] \geqslant \varepsilon^*\right\}.$$

By a covering argument, this gives a bound on the Hausdorff dimension of the singular set of u $(= u_{\lambda^*} + 1)$

$$\dim \mathcal{S} \leqslant n - 4\frac{p+1}{p-1}.$$

In particular, u is smooth on an open dense set.

6.3. The Federer dimension reduction

In this section we use Federer's dimension reduction principle (see for example [27]) to prove the sharp dimension estimate on $\mathcal{S}$.

For any $x_0 \in \Omega$ and $\lambda \in (0,1)$, define the blowing up sequence

$$u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(x_0 + \lambda x), \quad \lambda \to 0,$$

which is also a stable solution of (1.1) in the ball $B_{1/\lambda}(0)$.

By rescaling (6.7), for all $\lambda \in (0,1)$ and balls $B_r(x) \subset B_{1/\lambda}$,

$$\int_{B_r(x)} \left(u^\lambda\right)^{2p} \leqslant C r^{n-\frac{8p}{p-1}}.$$

By elliptic estimates, u^λ is uniformly bounded in $W^{4,2}_{loc}(\mathbb{R}^n)$. Hence, up to a subsequence of $\lambda \to 0$, we can assume that $u^\lambda \to u^0$ in $W^{3,2}_{loc}(\mathbb{R}^n)$ and $L^{p+1}_{loc}(\mathbb{R}^n)$ (by the same proof of (4.5)). By testing the equation for u^λ (or the stability condition for u^λ) with smooth functions having compact support, and then taking the limit $\lambda \to 0$, we see that u^0 is a stable solution of (1.1) in $\mathbb{R}^n$.

We have

Lemma 6.5. *For any $r > 0$, $E(r;0,u^0) = \lim_{r\to 0} E(r;x_0,u)$. So u^0 is homogeneous.*

Proof. A direct rescaling shows $E(r;0,u^\lambda) = E(\lambda r;x_0,u)$. By the monotonicity of $E(r;x_0,u)$, we only need to show that, for every $r > 0$,

$$E\left(r;0,u^0\right) = \lim_{\lambda\to 0} E\left(r;0,u^\lambda\right).$$

Because u^λ is uniformly bounded in $W^{4,2}(B_r)$ and $L^{2p}(B_r)$, by the compactness results in the Sobolev embedding theorems and trace theorems, and interpolation between L^q spaces (see (4.5)), we have

$$\lim_{\lambda\to+\infty}\int_{B_r}\left(\Delta u^\lambda\right)^2=\int_{B_r}\left(\Delta u^0\right)^2,$$

$$\lim_{\lambda\to+\infty}\int_{B_r}\left(u^\lambda\right)^{p+1}=\int_{B_r}\left(u^0\right)^{p+1},$$

$$u^\lambda\to u^0\quad\text{in } W^{2,2}(\partial B_r).$$

The last claim implies that those boundary terms in $E(r;0,u^\lambda)$ converge to the corresponding ones in $E(r;0,u^0)$. Putting these together we get the convergence of $E(r;0,u^\lambda)$.

Since for any $r>0$, $E(r;0,u^0)=const.$, by Corollary 2.1, u^0 is homogeneous. □

Here we note that since u satisfies (4.3) for any ball $B_R(x)\subset\Omega$, so by the same argument as in the proof of Lemma 4.4, we can prove that $E(r;x,u)$ is uniformly bounded for all x and $r\in(0,1)$. Since $E(r;x,u)$ is non-decreasing in r, we can define the density function

$$\Theta(x,u):=\lim_{r\to0}E(r;x,u).$$

Lemma 6.6.

(1) *$\Theta(x,u)$ is upper semi-continuous in x;*
(2) *for all x, $\Theta(x,u)\geqslant0$;*
(3) *x is a regular point of u if and only $\Theta(x,u)=0$;*
(4) *there exist a universal constant $\varepsilon_0>0$, $x\in S(u)$ if and only if $\Theta(x,u)\geqslant\varepsilon_0$.*

Proof. By the $W^{4,2}$ regularity of u, for any $r>0$ fixed, $E(r;x,u)$ is continuous in x. $\Theta(x,u)$ is the decreasing limit of these continuous functions, thus is upper semi-continuous in x.

If u is smooth in a neighborhood of x, direct calculation shows $\Theta(x,u)=0$. Since regular points form a dense set, the upper semi-continuity of Θ gives $\Theta\geqslant0$.

By Lemma 6.4, if x is a singular point, for any $r>0$,

$$\int_{B_r(x)}(\Delta u)^2+u^{p+1}\geqslant\varepsilon^* r^{n-4\frac{p+1}{p-1}}.$$

In other words, for any $\lambda>0$, for the blowing up sequence u^λ at x_0,

$$\int_{B_1(0)} \left(\Delta u^\lambda\right)^2 + \left(u^\lambda\right)^{p+1} \geqslant \varepsilon^*.$$

Then because $u^\lambda \to u^0$ in $W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$ (see the proof of Lemma 6.5),

$$\begin{aligned}\int_{B_1(0)} \left(\Delta u^0\right)^2 + \left(u^0\right)^{p+1} &= \lim_{\lambda\to 0} \int_{B_1(0)} \left(\Delta u^\lambda\right)^2 + \left(u^\lambda\right)^{p+1} \\ &= \lim_{\lambda\to 0} \lambda^{-n+4\frac{p+1}{p-1}} \int_{B_\lambda(0)} (\Delta u)^2 + (u)^{p+1} \geqslant \varepsilon^*. \end{aligned} \tag{6.13}$$

Hence u^0 is nontrivial, and by Remark 3.2 and Lemma 6.5,

$$\Theta(x,u) = E\left(1;0,u^0\right) \geqslant c(n,p)\varepsilon^*.$$

Here $c(n,p)$ is a constant depending only on p and n.

On the other hand, if $\Theta(x,u) < c(n,p)\varepsilon^*$, then by Remark 3.2, for any blow up limit u^0 at x,

$$\int_{B_1(0)} \left(\Delta u^0\right)^2 + \left(u^0\right)^{p+1} < \varepsilon^*.$$

Then by the convergence of u^λ in $W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$, for λ sufficiently small,

$$\lambda^{4\frac{p+1}{p-1}-n} \int_{B_\lambda(x)} (\Delta u)^2 + u^{p+1} = \int_{B_1(0)} \left(\Delta u^\lambda\right)^2 + \left(u^\lambda\right)^{p+1} \leqslant \varepsilon_0.$$

By Lemma 6.4, u is smooth in $B_{\lambda/2}(x)$. Consequently, $\Theta(x,u) = 0$. These finish the proof of the last two claims. □

Remark 6.7. If $\lim_{\lambda\to 0} u^\lambda = u^0$ in some sense (for example, as in the above blowing up sequence) so that for any x and $r > 0$, $\lim_{\lambda\to 0} E(r;x,u^\lambda) = E(r;x,u^0)$, then

$$\lim_{\lambda\to 0} \Theta\left(x,u^\lambda\right) \leqslant \Theta\left(x;u^0\right).$$

That is, $\Theta(x;u)$ is also upper semi-continuous in u.

Remark 6.8. A direct consequence of this upper semi-continuity is the convergence of $\mathcal{S}(u^\lambda)$ for the blow up sequence u^λ. In fact, by combining the upper semi-continuity and the characterization of singular points using the density function Θ, we can show that given any $\delta > 0$,

$$\mathcal{S}\big(u^\lambda\big) \cap B_1 \subset \delta - \text{neighborhood of } \mathcal{S}\big(u^0\big),$$

for all λ small.

To prove Theorem 1.9, we argue by contradiction. So assume that the Hausdorff dimension of $\mathcal{S}(u)$ is strictly larger than $n - n_p$. Then by definition, there exists a $\delta > 0$ such that

$$H^{n-n_p+\delta}\big(\mathcal{S}(u) \cap B_1\big) > 0. \tag{6.14}$$

For a set $A \subset \mathbb{R}^n$, define

$$H_\infty^{n-n_p+\delta}(A) := \inf\left\{\sum_j (\operatorname{diam} S_j)^{n-n_p+\delta}, A \subset \bigcup_j S_j\right\}.$$

Then by [14, Lemma 11.2 and Proposition 11.3], (6.14) implies the existence of a density point $x_0 \in \mathcal{S}(u) \cap B_1$, that is,

$$\limsup_{r\to 0} \frac{H_\infty^{n-n_p+\delta}(\mathcal{S}(u) \cap B_r(x_0))}{r^{n-n_p+\delta}} > 0. \tag{6.15}$$

We can preform the blow up procedure at x_0 to obtain a homogeneous solution $u_{\infty,0}$ on $\mathbb{R}^n$. With the help of Remark 6.8, we can prove as in [14, Lemma 11.5] to show

$$H_\infty^{n-n_p+\delta}\big(\mathcal{S}(u_{\infty,0}) \cap B_1(0)\big) \geqslant \limsup_{r\to 0} \frac{H_\infty^{n-n_p+\delta}(\mathcal{S}(u) \cap B_r(x_0))}{r^{n-n_p+\delta}} > 0, \tag{6.16}$$

if we choose a suitable sequence $\lambda_i \to 0$ in the definition of $u_{\infty,0}$ to achieve the upper bound in (6.15).

Since $n \geqslant n_p$, (6.16) implies that $\mathcal{S}(u_{\infty,0}) \cap B_1(0)$ contains a point $x_1 \neq 0$, which can also be chosen to be a density point by [14, Proposition 11.3]. Note that the origin 0 always belongs to $\mathcal{S}(u_{\infty,0})$ because $u_{\infty,0}$ is homogeneous. This homogeneity also implies that the ray $\{tx_1\colon\ t \geqslant 0\} \subset \mathcal{S}(u_{\infty,0})$, and

$$\Theta(tx_1; u_{\infty,0}) \equiv \Theta(x_1; u_{\infty,0}) \quad \text{for } t > 0.$$

The main step in the dimension reduction procedure is to blow up once again at x_1. Assume that one limit function is $u_{\infty,1}$ and we have a sequence $\lambda_i \to 0$ so that

$$u_i := \lambda_i^{\frac{4}{p-1}} u_{\infty,0}(x_1 + \lambda_i x) \to u_{\infty,1},$$

where the convergence is understood as before.

We want to show that $u_{\infty,1}$ is in fact translation invariant in the direction x_1, thus can be viewed as a function defined on $\mathbb{R}^{n-1}$. This can be achieved by the following lemma, together with the fact that, for any $t \in \mathbb{R}$,

$$\begin{aligned}\Theta(tx_1; u_{\infty,1}) \geqslant \limsup_{i\to+\infty} \Theta(tx_1; u_i) &= \limsup_{i\to+\infty} \Theta\big((1+t\lambda_i)x_1; u_{\infty,0}\big)\\ &= \Theta(x_1; u_{\infty,0}) = \Theta(0; u_{\infty,1}),\end{aligned}$$

where we have used Lemma 6.5 and Remark 6.7.

Lemma 6.9. *Let $u \in W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$ be a homogeneous stable solution of (1.1) on $\mathbb{R}^n$, satisfying the monotonicity formula and the integral estimate (6.7). Then for any $x \neq 0$, $\Theta(x,u) \leqslant \Theta(0,u)$. Moreover, if $\Theta(x,u) = \Theta(0,u)$, u is translation invariant in the direction x, i.e. for all $t \in \mathbb{R}$,*

$$u(tx + \cdot) = u(\cdot) \quad a.e.\ in\ \mathbb{R}^n.$$

Proof. With the help of the integral estimate (6.7), similar to Lemma 4.4, for any $x_0 \in \mathbb{R}^n$,

$$\lim_{r\to+\infty} E(r; x_0, u) \leqslant C.$$

And we can define the blowing down sequence with respect to the base point x_0,

$$u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(x_0 + \lambda x), \quad \lambda \to +\infty.$$

Since u is homogeneous with respect to 0,

$$u^\lambda(x) = u\big(\lambda^{-1}x_0 + x\big),$$

which converges to $u(x)$ as $\lambda \to +\infty$ in $W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)$. Then Lemma 6.5 can be applied to deduce that

$$\begin{aligned}\Theta(0; u) = E(1; 0, u) &= \lim_{\lambda\to+\infty} E\big(1; 0, u^\lambda\big)\\ &= \lim_{\lambda\to+\infty} E(\lambda; x_0, u)\\ &\geqslant \Theta(x_0; u).\end{aligned}$$

Moreover, if $\Theta(x_0; u) = \Theta(0, u)$, the above inequality becomes an equality:

$$\lim_{\lambda\to+\infty} E(\lambda; x_0, u) = \Theta(x_0; u).$$

This then implies that $E(\lambda; x_0, u) \equiv \Theta(x_0; u)$ for all $\lambda > 0$. By Corollary 2.1, u is homogeneous with respect to x_0. Then for all $\lambda > 0$,

$$u(x_0+x)=\lambda^{\frac{4}{p-1}}u(x_0+\lambda x)=u\big(\lambda^{-1}x_0+x\big).$$

By letting $\lambda\to+\infty$ and noting that $u(\lambda^{-1}x_0+\cdot)$ are uniformly bounded in $W^{2,2}_{loc}(\mathbb{R}^n)$, we see

$$u(x_0+\cdot)=u(\cdot)\quad \text{a.e. on } \mathbb{R}^n.$$

Because u is homogeneous with respect to 0, a direct scaling shows that $\Theta(tx_0;u)=\Theta(x_0;u)$ for all $t>0$, so the above equality still holds if we replace x_0 by tx_0 for any $t>0$. A change of variable shows this also holds if $t<0$. □

We have shown that $u_{\infty,1}$ can be seen as a function defined on $\mathbb{R}^{n-1}$. It belongs to $W^{2,2}_{loc}(\mathbb{R}^{n-1})\cap L^{p+1}_{loc}(\mathbb{R}^{n-1})$, and it is still a weak solution of (1.1). Moreover, the estimates (6.7) and (6.12) hold for $u_{\infty,1}$. It can also be directly verified that $u_{\infty,1}$ is stable (by considering test functions $\varphi(x_1,\dots,x_{n-1})\eta(x_n)$ where $\varphi\in C^\infty_0(\mathbb{R}^{n-1})$ and $\eta\in C^\infty_0(\mathbb{R})$).

Similar to (6.16), when $u_{\infty,1}$ is viewed as a function defined on $\mathbb{R}^n$, we have

$$H^{n-n_p+\delta}_\infty\big(\mathcal{S}(u_{\infty,1})\cap B_1(0)\big)>0,$$

where $\mathcal{S}(u_{\infty,1})$ is a cylindrical set in $\mathbb{R}^n$. Then if we view u_1 as a function defined on $\mathbb{R}^{n-1}$, and by abusing notations, take $\mathcal{S}(u_{\infty,1})\subset\mathbb{R}^{n-1}$ as the base of the above cylindrical set, this means

$$H^{n-1-n_p+\delta}_\infty\big(\mathcal{S}(u_{\infty,1})\cap B_1(0)\big)>0.$$

We can repeat this reduction procedure until we get a solution $u_{\infty,n-n_p}$ on $\mathbb{R}^{n_p}$, which satisfies

$$H^{\delta}_\infty\big(\mathcal{S}(u_{\infty,n-n_p})\cap B_1(0)\big)>0.$$

In particular, $\mathcal{S}(u_{\infty,n-n_p})$ cannot be a singleton because $\delta>0$. By blowing up $u_{\infty,n-n_p}$ at a point $x\in\mathcal{S}(u_{\infty,n-n_p})$ with $x\neq 0$, we would get a homogeneous stable solution of $v\in W^{2,2}_{loc}(\mathbb{R}^{n_p-1})\cap L^{p+1}_{loc}(\mathbb{R}^{n_p-1})$, which is nontrivial by (6.13). However, this contradicts Theorem 3.1. Thus we disprove our initial assumption (6.14) and get the estimate

$$\dim\mathcal{S}(u)\leqslant n-n_p.$$

Finally, we prove the discreteness of $\mathcal{S}(u)$ when $n=n_p$.

Assume there exists $x_i\in\mathcal{S}(u)\cap B_1$, such that $x_i\to x_0$ but $x_i\neq x_0$. Take $r_i=|x_0-x_i|$ and define

$$u_i(x)=r_i^{\frac{4}{p-1}}u(x_0+r_ix).$$

After passing to a subsequence of i, we can assume that u_i converges uniformly to a stable homogeneous solution u_∞ in any compact set of $\mathbb{R}^{n_p}$. Since $z_i = (x_i - x_0)/r_i \in \mathbb{S}^{n_p-1}$, we can also assume that $z_i \to z_\infty \in \mathbb{S}^{n_p-1}$. By Remark 6.7, $z_\infty \in \mathcal{S}(u_\infty)$. As above, we can blow up u_∞ at z_∞ to get a stable homogeneous solution in $\mathbb{R}^{n_p-1}$, which contradicts Theorem 3.1. Thus $\mathcal{S}(u)$ must be a discrete set.

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Appendix A. Proof of estimate in Lemma 6.4

Let us use the notation

$$\|f\|_{q,\gamma,\Omega} = \sup_{x,r}\left(r^{-\gamma}\int\limits_{B(x,r)\cap\Omega}|f|^q\right)^{1/q},$$
$$L^{q,\gamma}(\Omega) = \left\{u \in L^q(\Omega)\colon\ \|u\|_{q,\gamma,\Omega} < \infty\right\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $0 < \gamma \leqslant n$, $1 \leqslant q < \infty$.

For completeness we give a proof of the following result, which is an adaptation of [19,21].

Lemma A.1. *Assume u is a weak solution of*

$$\Delta^2 u = |u|^{p-1}u \quad \text{in } B_1(0)$$

and $u \in L^{p,n-4\frac{p}{p-1}+\delta}(B_1(0))$ for some $\delta > 0$. Then u is bounded in $B_{1/2}(0)$.

We need some preliminaries. Let

$$I_\alpha(f)(x) = \int\limits_{\mathbb{R}^n}|x-y|^{-n+\alpha}f(y)\,dy.$$

Lemma A.2. *(See [19, Lemma 1].) If $f \in L^{1,\gamma}(\mathbb{R}^n)$, $0 < \epsilon < \gamma$ and $1 < p < \frac{n-\epsilon}{n-\epsilon-\alpha}$, then*

$$\int\limits_\Omega \left|I_\alpha(f)\right|^p(x)\,dx \leqslant C\,\mathrm{diam}(\Omega)^{n-\epsilon-(n-\alpha-\epsilon)p}\int\limits_\Omega |f|\,dx. \tag{A.1}$$

Lemma A.3. *(See Campanato [1].) Let $0 < \gamma < n$ and $c > 0$. Assume $\phi : (0, R] \to \mathbb{R}$ is a nonnegative nondecreasing function such that*

$$\phi(\rho) \leqslant c\left(\frac{\rho^n}{r^n}\phi(r) + r^\gamma\right) \quad \text{for all } 0 < \rho \leqslant r \leqslant R.$$

Then there is C depending only on n, γ, c such that

$$\phi(\rho) \leqslant C\rho^\gamma\left(\frac{\phi(r)}{r^\gamma} + 1\right) \quad \text{for all } 0 < \rho \leqslant r \leqslant R.$$

Lemma A.4. *Let v satisfy $\Delta^2 v = 0$ in $B_R(0)$. Then there is C such that*

$$\left|v(x)\right| \leqslant \frac{C}{R^n}\int_{B_R(0)} |v|\,dy \quad \text{for all } |x| \leqslant \frac{1}{2}R. \tag{A.2}$$

Proof. By scaling we can restrict to $R = 1$ and $v \in C^4(\overline{B}_1(0))$. Let $\eta \in C^\infty(\mathbb{R}^n)$ be a cut-off function with $\eta(x) = 1$ for $|x| \leqslant \frac{2}{3}$ and $\eta(x) = 0$ for $|x| \geqslant \frac{5}{6}$. Let $\Gamma(x) = c_n|x|^{4-n}$ be the fundamental solution of Δ^2 in $\mathbb{R}^n$, $c_n > 0$. Then

$$v(x) = \int_{B_1 \setminus B_{2/3}} v(y)\Delta^2\big(\Gamma(x-y)\eta(y)\big)\,dy \quad \text{for } |x| \leqslant \frac{1}{2}$$

and (A.2) follows. □

Proof of Lemma A.1. Let $R_1 < 1$ (close to 1), $|x| < R_1$ and $0 < r < \frac{1-R_1}{2}$. Let $u_1 = \Gamma * (|u|^{p-1}u\chi_{B_r(x)})$ where $\Gamma(x) = c_n|x|^{4-n}$ is the fundamental solution of Δ^2 in $\mathbb{R}^n$, $c_n > 0$, and $\chi_{B_r(x)}$ is the indicator function of $B_r(x)$. Let $u_2 = u - u_1$. Then $\Delta^2 u_2 = 0$ in $B_r(x)$. By (A.2)

$$\left|u_2(z)\right| \leqslant \frac{C}{r^n}\int_{B_r(x)} |u_2| \quad \text{for } z \in B_{r/4}(x).$$

Let $y \in B_{r/4}(x)$ and $0 < \rho < \frac{r}{4}$. Integrating in $B_\rho(y)$ and using Hölder's inequality

$$\int_{B_\rho(y)} |u_2|^p \leqslant C\left(\frac{\rho}{r}\right)^n \int_{B_r(x)} |u_2|^p.$$

Therefore

$$\int_{B_\rho(y)} |u|^p \leqslant C\int_{B_\rho(y)} |u_1|^p + C\left(\frac{\rho}{r}\right)^n \int_{B_r(x)} |u_2|^p$$

$$\leqslant C\left(\frac{\rho}{r}\right)^n \int\limits_{B_r(x)} |u|^p + C \int\limits_{B_r(x)} |u_1|^p. \tag{A.3}$$

Let $\gamma_0 = n - 4\frac{p}{p-1} + \delta$. Using (A.1) with $\alpha = 4$, $\gamma = \gamma_0$ and ϵ a number such that $n - 4\frac{p}{p-1} < \epsilon < \gamma_0$ we have

$$\int\limits_{B_r(x)} |u_1|^p \leqslant C r^{n-\epsilon-(n-4-\epsilon)p} \int\limits_{B_r(x)} |u|^p.$$

Then, combining with (A.3) we obtain

$$\begin{aligned}\int\limits_{B_\rho(y)} |u|^p &\leqslant C\left(\frac{\rho}{r}\right)^n \int\limits_{B_r(x)} |u|^p + C r^{n-\epsilon-(n-4-\epsilon)p} \int\limits_{B_r(x)} |u|^p \\ &\leqslant C\left(\frac{\rho}{r}\right)^n \int\limits_{B_r(x)} |u|^p + C r^{n-\epsilon-(n-4-\epsilon)p+\gamma_0}\end{aligned}$$

for any $y \in B_{r/4}(x)$, $0 < \rho < \frac{r}{4}$. We have the validity of the inequality for $0 < \rho \leqslant r$, possibly increasing C. Using the lemma of Campanato (Lemma A.3),

$$\int\limits_{B_\rho(y)} |u|^p \leqslant C \rho^{n-\epsilon-(n-4-\epsilon)p+\gamma_0}$$

for $0 < \rho \leqslant r$, which shows that $u \in L^{p,\gamma_1}(B_{R_1})$ where $R_1 < 1$ can be chosen arbitrarily close to 1, and $\gamma_1 = n-\epsilon-(n-4-\epsilon)p+\gamma_0$ can be chosen arbitrarily close to $n - \frac{4p}{p-1} + \delta p$. In particular we can choose $\gamma_1 > \gamma_0$. Repeating the process, we can find a decreasing sequence $R_i \to \frac{4}{5}$ and an increasing sequence $\gamma_i \to n-4$ such that $u \in L^{p,\gamma_i}(B_{R_i})$. Then by Lemma A.2 $u \in L^q(B_{3/4}(0))$ for all $q > 1$ and by standard elliptic regularity $u \in L^\infty(B_{1/2})$. $\square$

References

[1] S. Campanato, Equazioni ellittiche del II° ordine e spazi $\mathcal{L}^{(2,\lambda)}$, Ann. Mat. Pura Appl. (4) 69 (1965) 321–381.
[2] A. Chang, L. Wang, P. Yang, A regularity theory of biharmonic maps, Comm. Pure Appl. Math. (9) 52 (1999) 1113–1137.
[3] C. Cowan, P. Esposito, N. Ghoussoub, Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains, Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1033–1050.
[4] C. Cowan, N. Ghoussoub, Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains, Calc. Var. Partial Differential Equations 49 (2014) 291–293, http://dx.doi.org/10.1007/s00526-012-0582-4, in press.
[5] E.N. Dancer, Moving plane methods for systems on half spaces, Math. Ann. 342 (2) (2008) 245–254.
[6] J. Dávila, L. Dupaigne, A. Farina, Partial regularity of finite Morse index solutions to the Lane–Emden equation, J. Funct. Anal. 261 (2011) 218–232.
[7] L. Dupaigne, Variations elliptiques, Habilitation à Diriger des Recherches, 11 december 2011.

[8] L. Dupaigne, A. Farina, B. Sirakov, Regularity of the extremal solutions for the Liouville system, in: Geometric Partial Differential Equations, in: Publications of the Scuola Normale Superiore/CRM Series, vol. 15, 2013, pp. 139–144.
[9] L. Dupaigne, M. Ghergu, O. Goubet, G. Warnault, The Gelfand problem for the biharmonic operator, Arch. Ration. Mech. Anal. 208 (3) (2013) 725–752.
[10] A. Farina, On the classification of solutions of the Lane–Emden equation on unbounded domains of $\mathbb{R}^N$, J. Math. Pures Appl. 87 (2007) 537–561.
[11] W.H. Fleming, On the oriented Plateau problem, Rend. Circ. Mat. Palermo (2) 11 (1962) 69–90.
[12] F. Gazzola, H.-C. Grunau, Radial entire solutions for supercritical biharmonic equations, Math. Ann. 334 (2006) 905–936.
[13] F. Gazzola, H.-C. Grunau, G. Sweers, Polyharmonic boundary value problems, in: Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains, in: Lecture Notes in Math., vol. 1991, Springer-Verlag, Berlin, 2010.
[14] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monogr. Math., vol. 80, Birkhäuser, 1984.
[15] Y. Guo, B. Li, J. Wei, Large energy entire solutions for the Yamabe type problem of polyharmonic operator, J. Differential Equations 254 (1) (2013) 199–228.
[16] Z.M. Guo, J. Wei, Qualitative properties of entire radial solutions for a biharmonic equation with supcritical nonlinearity, Proc. Amer. Math. Soc. 138 (11) (2010) 3957–3964.
[17] H. Hajlaoui, A.A. Harrabi, D. Ye, On stable solutions of biharmonic problem with polynomial growth, arXiv:1211.2223v2, 2012.
[18] P. Karageorgis, Stability and intersection properties of solutions to the nonlinear biharmonic equation, Nonlinearity 22 (2009) 1653–1661.
[19] F. Pacard, A note on the regularity of weak solutions of $-\Delta u = u^\alpha$ in $\mathbb{R}^n$, $n \geqslant 3$, Houston J. Math. 18 (4) (1992) 621–632.
[20] F. Pacard, Partial regularity for weak solutions of a nonlinear elliptic equation, Manuscripta Math. 79 (2) (1993) 161–172.
[21] F. Pacard, Convergence and partial regularity for weak solutions of some nonlinear elliptic equation: the supercritical case, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (5) (1994) 537–551.
[22] P. Polácik, P. Quittner, P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems, Duke Math. J. 139 (3) (2007) 555–579.
[23] P. Pucci, J. Serrin, A general variational identity, Indiana Univ. Math. J. 35 (1986) 681–703.
[24] P. Pucci, J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, J. Math. Pures Appl. 69 (1990) 55–83.
[25] F. Rellich, Perturbation Theory of Eigenvalue Problems, Gordon and Breach Science Publisher, New York, 1969.
[26] G.V. Rozenblum, The distribution of the discrete spectrum for singular differential operators, Dokl. Akad. SSSR 202 (1972) 1012–1015.
[27] L. Simon, Lectures on geometric measure theory, in: Proceedings of the Centre for Mathematical Analysis, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[28] B. Sirakov, Existence results and a priori bounds for higher order elliptic equations and systems, J. Math. Pures Appl. 89 (2008) 114–133.
[29] P. Souplet, The proof of the Lane–Emden conjecture in four space dimensions, Adv. Math. 221 (2009) 1409–1427.
[30] K. Wang, Partial regularity of stable solutions to the supercritical equations and its applications, Nonlinear Anal. 75 (13) (2012) 5238–5260.
[31] J. Wei, X. Xu, Classification of solutions of high order conformally invariant equations, Math. Ann. 313 (2) (1999) 207–228.
[32] D. Ye, J. Wei, Liouville theorems for finite Morse index solutions of Biharmonic problem, Math. Ann. 356 (4) (2013) 1599–1612.