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Solvability of a nonlinear Neumann problem for systems arising from a burglary model[☆]

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ABSTRACT

We study a one dimensional version of a problem that arises in mathematical modeling for burglary of houses. Our approach uses techniques inspired by coincidence degree, see Mawhin (1979). A priori estimates are obtained through a somewhat unusual combination of estimates based upon maximum or minimum properties and on L^1-estimates of the type introduced by Ward, see Ward (1981) in some periodic problems.

1. The problem

Let $L > 0$, $\eta > 0$, let $A^0 : [0, L] \to \mathbb{R}$ be a positive function of class C^2 such that

$$(A^0)'(0) = 0 = (A^0)'(L), \tag{1}$$

and let $A^1 : [0, L] \to \mathbb{R}$ be a positive continuous function. Consider the Neumann problem

$$\eta[A - A^0(x)]'' - A + A^0(x) + NA = 0, \qquad A'(0) = 0 = A'(L), \tag{2}$$

$$\left(N' - 2N\frac{A'}{A}\right)' - NA + A^1(x) - A^0(x) = 0, \qquad N'(0) = 0 = N'(L). \tag{3}$$

A solution of (2)–(3) is a couple of real functions $(A, N) \in C^2([0, T]) \times C^2([0, T])$ such that $A(x) > 0$ for all $x \in [0, L]$ which satisfies the system and the boundary conditions. We are interested in *positive solutions* of this problem, i.e. in solutions (A, N) such that $A(x) > 0$ and $N(x) > 0$ for all $x \in [0, L]$.

This problem is a one dimensional version of a problem that arises in the pioneering work of [1] where a very successful model for burglary of houses was obtained by Short et al. See also the related papers [2–7]. The model of [1] was derived by first considering an agent based statistical model to study the formation of hot spots taking two major sociological effects into account: the "broken window effect" and the "repeat near-repeat effect". In a second step, by taking a suitable limit of the equations for the discrete model derived in the first step, a continuous model for the two unknowns (A, N) was obtained:

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A representing attractiveness for a house to be burglarized, and N representing density of burglars. By the definitions of A and N, the restrictions $A > 0$ and $N > 0$ appear as natural.

When A^0, A^1 are positive constants, system (2)–(3) admits the unique positive constant solution

$$A = A^1, \qquad N = \frac{A^1 - A^0}{A^1}$$

under the condition that $A^1 > A^0$.

In [1] and for the PDE case, a linear stability analysis of the (corresponding) constant solution was performed, while in [3], a study of global bifurcation of solutions from this constant solution is done.

With a view towards extending some of these results, a natural question to start with is to know if a positive (non constant) solution still exists when A^0 and A^1 are no longer constant. For the one dimensional model that we consider here, this means A^0 and A^1 depend on x. This question is answered in this paper by a combination of estimates based upon maximum or minimum properties, on L^1-estimates of the type introduced by Ward [8] in some periodic problems, and by the use of the Leray–Schauder degree results.

A similar problem was recently considered in [4] for a variant of the burglary model in which the linear part of the differential system was invertible. This is not the case here, which makes the fixed point reduction more complicated and requires a more sophisticated version of the Leray–Schauder theory.

2. The homotopy and *a priori* estimates

Let us associate to (2)–(3) the homotopy, with $\lambda \in (0, 1]$,

$$\eta[A - A^0(x)]'' = \lambda\left[A - A^0(x) - NA\right], \qquad A'(0) = 0 = A'(L), \tag{4}$$

$$\left(N' - 2\lambda N\frac{A'}{A}\right)' = \lambda[NA - A^1(x) + A^0(x)], \qquad N'(0) = 0 = N'(L). \tag{5}$$

For $\lambda = 1$, (4)–(5) reduces to (2)–(3).

For any $B \in L^1(0, L)$ we denote by $\overline{B}$ its mean value $L^{-1}\int_0^L B(x)\,dx$, and for any $B \in C([0, L])$ we set $\max B := \max_{[0,L]} B$ and $\min B := \min_{[0,T]} B$.

Lemma 1. *If (A, N) is any possible solution of (4)–(5) for some $\lambda \in (0, 1]$, then*

$$\overline{A} = \overline{A^1}. \tag{6}$$

Proof. Add Eqs. (4) and (5), integrate both members over $[0, L]$ and use the boundary conditions. ∎

Lemma 2. *If (A, N) is any possible solution of (4)–(5) for some $\lambda \in (0, 1]$, then*

$$\overline{NA} = \overline{A^1} - \overline{A^0}. \tag{7}$$

Proof. Integrate Eq. (4) over $[0, L]$, use the boundary conditions and (6). ∎

Remark 1. Notice that (7) implies that a necessary condition for the existence of a positive solution (A, N) is that

$$\overline{A^1} > \overline{A^0}. \tag{8}$$

We shall assume, from now on, the stronger condition:

$$A^1(x) > A^0(x) \quad (x \in [0, L]), \tag{9}$$

which is necessary to show that N cannot have a minimum equal to zero.

Lemma 3. *If (A, N) is any possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$, then, for all $x \in [0, L]$,*

$$|A'(x)| \le L \max|(A^0)''| + \frac{2L\overline{A^1}}{\eta}. \tag{10}$$

$$A(x) \le L^2 \max|(A^0)''| + \left(1 + \frac{2L^2}{\eta}\right)\overline{A^1} := A_2. \tag{11}$$

Proof. Let (A, N) be a possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$. From Eq. (4) we obtain, for all $x \in [0, L]$,

$$\begin{aligned}|A''(x)| &\leq |(A^0)''(x)| + \frac{\lambda}{\eta}\left|A(x) - A^0(x) - N(x)A(x)\right| \\ &\leq |(A^0)''(x)| + \frac{1}{\eta}[A(x) + A^0(x) + N(x)A(x)].\end{aligned}$$

Hence, for any $x \in [0, L]$, using (6) and (7) and the boundary conditions,

$$\begin{aligned}|A'(x)| &= \left|\int_0^x A''(y)\,dy\right| \leq \int_0^x |A''(y)|\,dy \leq \int_0^L |A''(y)|\,dy \\ &\leq L \max|(A^0)''| + \frac{L}{\eta}\left(\overline{A^1} + \overline{A^0} + \overline{A^1} - \overline{A^0}\right) \\ &= L \max|(A^0)''| + \frac{2L\overline{A^1}}{\eta}. \qquad (12)\end{aligned}$$

On the other hand, there exists $\xi \in [0, L]$ such that

$$\overline{A} = A(\xi).$$

Therefore, using (6) and (12), we obtain, for any $x \in [0, L]$,

$$\begin{aligned}A(x) &= A(\xi) + \int_\xi^x A'(y)\,dy \leq \overline{A} + \int_0^L |A'(y)|\,dy \\ &\leq \overline{A^1} + L^2 \max|(A^0)''| + \frac{2L^2}{\eta}\overline{A^1}. \quad \blacksquare\end{aligned}$$

Define

$$A_0 := \min A^0. \qquad (13)$$

Lemma 4. *If (A, N) is any possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$, then, for all $x \in [0, L]$,*

$$A(x) \geq A_0. \qquad (14)$$

Proof. Let (A, N) be a possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$. If $A - A^0$ reaches its minimum at $\xi \in [0, L]$, then

$$0 \leq \eta[A''(\xi) - (A^0)''(\xi)] = \lambda\left[A(\xi) - A^0(\xi) - N(\xi)A(\xi)\right] \leq \lambda\left[A(\xi) - A^0(\xi)\right]$$

so that, for all $x \in [0, L]$,

$$A(x) \geq A^0(x)$$

and hence

$$A(x) \geq \min A \geq \min A^0 = A_0. \quad \blacksquare$$

Corollary 1. *If (A, N) is any possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$, then, for all $x \in [0, L]$,*

$$\left|\frac{A'(x)}{A(x)}\right| \leq \frac{L\eta \max|(A^0)''| + 2L\overline{A^1}}{\eta A_0} := A_3. \qquad (15)$$

We now obtain an upper bound for N.

Lemma 5. *If (A, N) is any possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$, then,*

$$\frac{\overline{A^1} - \overline{A^0}}{A_2} \leq \frac{1}{L}\int_0^L N(x)\,dx \leq \frac{\overline{A^1} - \overline{A^0}}{A_0}. \qquad (16)$$

Proof. It follows from (7), the inequalities

$$(\min A)\int_0^L N(x)\,dx \leq \int_0^L N(x)A(x)\,dx \leq (\max A)\int_0^L N(x)\,dx,$$

and inequalities (11) and (14). $\blacksquare$

Lemma 6. *If* (A, N) *is any possible positive solution of* (4)–(5) *for some* $\lambda \in (0, 1]$*, then, for any* $x \in [0, L]$*, one has*

$$\left| N'(x) - 2\lambda N(x)\frac{A'(x)}{A(x)} \right| \leq 2L[\overline{A^1} - \overline{A^0}]. \tag{17}$$

Proof. Let (A, N) be a possible positive solution of (4)–(5) for some $\lambda \in (0, 1]$. It follows from Eq. (5) that, for any $x \in [0, L]$,

$$\begin{aligned} \left| \left[N'(x) - 2\lambda N(x)\frac{A'(x)}{A(x)} \right]' \right| &\leq |N(x)A(x) + A^1(x) - A^0(x)| \\ &= N(x)A(x) + A^1(x) - A^0(x). \end{aligned}$$

Hence, using the boundary conditions and (7), we get, for any $x \in [0, L]$,

$$\begin{aligned} \left| N'(x) - 2\lambda N(x)\frac{A'(x)}{A(x)} \right| &= \left| \int_0^x \left[N'(y) - 2\lambda N(y)\frac{A'(y)}{A(y)} \right]' dy \right| \\ &\leq \int_0^L [N(x)A(x) + A^1(x) - A^0(x)]\, dx \\ &\leq L\left(\overline{A^1} - \overline{A^0} + \overline{A^1} - \overline{A^0}\right) \\ &= 2L\left(\overline{A^1} - \overline{A^0}\right). \quad \blacksquare \end{aligned}$$

Lemma 7. *If* (A, N) *is any possible positive solution of* (4)–(5) *for some* $\lambda \in (0, 1]$*, then, for any* $x \in [0, L]$*, one has*

$$|N(x) - \overline{N}| \leq 2(\overline{A^1} - \overline{A^0})\left(L^2 + \frac{A_3}{LA_0}\right) := A_4. \tag{18}$$

Proof. It follows from inequality (17) that, for any $x \in [0, L]$,

$$|N'(x)| \leq 2N(x)\left|\frac{A'(x)}{A(x)}\right| + 2L\left(\overline{A^1} - \overline{A^0}\right).$$

On the other hand, there exists $\xi \in [0, L]$ such that $\overline{N} = N(\xi)$. Consequently, using inequality (16), we have, for all $x \in [0, L]$,

$$\begin{aligned} |N(x) - \overline{N}| &= |N(x) - N(\xi)| = \left|\int_\xi^x N'(y)\, dy\right| \leq \int_0^L |N'(y)|\, dy \\ &\leq 2A_3\int_0^L N + 2L^2\left(\overline{A^1} - \overline{A^0}\right) \\ &\leq \left(\frac{2A_3L}{A_0} + 2L^2\right)(\overline{A}^1 - \overline{A}^0) := A_4. \quad \blacksquare \end{aligned}$$

Corollary 2. *If* (A, N) *is any possible positive solution of* (4)–(5) *for some* $\lambda \in (0, 1]$*, then, for any* $x \in [0, L]$*, one has*

$$N(x) \leq \frac{\overline{A^1} - \overline{A^0}}{A_0} + A_4 := A_5. \tag{19}$$

Lemma 8. *If* (A, N) *is any possible positive solution of* (4)–(5) *for some* $\lambda \in (0, 1]$*, then* N *cannot have its minimum equal to zero.*

Proof. If N reaches its minimum at ξ and $N(\xi) = 0$, then, $N'(\xi) = 0$ and $N''(\xi) \geq 0$, so that, using Eq. (5) and assumption (9),

$$0 \leq N''(\xi) = -\lambda(A^1(\xi) - A^0(\xi)) < 0,$$

a contradiction. $\blacksquare$

Lemma 9. *If* (A, N) *is any possible positive solution of* (4)–(5) *for some* $\lambda \in (0, 1]$*, then* $A - A^0$ *cannot have its minimum equal to zero.*

Proof. If $A - A^0$ reaches a zero minimum at ξ, then

$$0 \le (A - A^0)''(\xi) = -\frac{\lambda N(\xi)A(\xi)}{\eta} < 0,$$

a contradiction. ■

3. A frame for coincidence degree techniques

We now write the homotopy system (4)–(5) in a fixed point form using a slight modification of the construction in coincidence degree theory [9], due to the presence of the first term in Eq. (5).

Proposition 1. *For any $\lambda \in (0, 1]$, (A, N) is a solution of (4)–(5) if and only if (A, N) is a solution of the following system of equations*

$$\begin{aligned} A(x) \;&=\; A(0) + A^0(x) - A^0(0) - \overline{(A - A^0 - NA)} \\ &\quad + \frac{\lambda}{\eta}\int_0^x \left[\int_0^y \big(A(z) - A^0(z) - N(z)A(z)\big)\, dz\right] dy \end{aligned} \tag{20}$$

$$\begin{aligned} N(x) \;&=\; N(0) - \overline{(NA - A^1 + A^0)} + \lambda \int_0^x \left[2N(y)\frac{A'(y)}{A(y)}\right] dy \\ &\quad + \lambda \int_0^x \left[\int_0^y [N(z)A(z) - A^1(z) + A^0(z)]\, dz\right] dy. \end{aligned} \tag{21}$$

Proof. If (A, N) satisfies system (20)–(21), then, taking $x = 0$ in both equations we find

$$\overline{(A - A^0 - NA)} = 0, \qquad \overline{(NA - A^1 + A^0)} = 0. \tag{22}$$

Differentiating Eqs. (20) and (21), we obtain

$$A'(x) = A^{0\prime}(x) + \frac{\lambda}{\eta}\left[\int_0^x \big(A(z) - A^0(z) - N(z)A(z)\big)\, dz\right]$$

$$N'(x) = \lambda\left[2N(x)\frac{A'(x)}{A(x)} + \int_0^x [N(z)A(z) - A^1(z) + A^0(z)]\, dz\right].$$

In particular, taking $x = 0$, we obtain $A'(0) = 0 = N'(0)$, and, taking $x = L$ and using (22), we obtain $A'(L) = 0 = N'(L)$. So the Neumann boundary conditions are satisfied. Finally, differentiating one more time, we obtain

$$(A - A^0)''(x) = \frac{\lambda}{\eta}\left[A(x) - A^0(x) - N(x)A(x)\right]$$

$$N''(x) = \lambda\left[\left(2N(x)\frac{A'(x)}{A(x)}\right)' + N(x)A(x) - A^1(x) + A^0(x)\right],$$

which is equivalent to system (4)–(5). The proof of the converse is similar and left to the reader. ■

Let us now take $R_2 > A_2 \ge A_0 > R_0 > 0$, $R_3 > A_3A_2$ and $R_5 > A_5$, where $A_0,\ A_2,\ A_3,\ A_5$ are respectively given by (13), (11), (15) and (19).

Next let us consider the open bounded subset of the Banach space

$$E := C^1([0, L]) \times C([0, L])$$

(with the usual norm $\|(A, N)\|_E = \|A\|_\infty + \|A'\|_\infty + \|N\|_\infty$), defined by

$$\Omega := \{(A, N) \in E : R_0 < A(x) < R_2,\ |A'(x)| < R_3, 0 < N(x) < R_5 \text{ for all } x \in [0, L]\}. \tag{23}$$

Define the mapping $\mathcal{T} : \overline{\Omega} \times [0, \lambda] \to E$ by

$$\begin{aligned} \mathcal{T}(A, N, \lambda) = \Bigg(& A(0) + A^0(x) - A^0(0) - \overline{(A - A^0 - NA)} + \frac{\lambda}{\eta}\int_0^x \left[\int_0^y \big(A(z) - A^0(z) - N(z)A(z)\big)\, dz\right] dy, \\ & N(0) - \overline{(NA - A^1 + A^0)} + \lambda \int_0^x \left[2N(y)\frac{A'(y)}{A(y)}\right] dy + \lambda\int_0^x \left[\int_0^y [N(z)A(z) - A^1(z) + A^0(z)]\, dz\right] dy\Bigg). \end{aligned}$$

It is standard to show, using the Arzela–Ascoli theorem, that $\mathcal{T}$ is compact on Ω and, by Proposition 1, its fixed points are the solutions of (4)–(5).

Proposition 2. *$(A, N) \in \overline{\Omega}$ is a fixed point of $\mathcal{T}(\cdot, 0)$ if and only if $A - A^0$ and N are constant and $A = A^0 + B$ with (B, N) satisfies the algebraic system*

$$B - N\overline{A^0} - NB = 0, \qquad N\overline{A^0} + NB - \overline{A^1} + \overline{A^0} = 0, \tag{24}$$

whose unique solution is given by

$$B = \overline{A^1} - \overline{A^0}, \qquad N = \frac{\overline{A^1} - \overline{A^0}}{\overline{A^1}}. \tag{25}$$

Proof. $(A, N) \in \overline{\Omega}$ is a fixed point of $\mathcal{T}(\cdot, 0)$ if and only if

$$A(x) = A^0(x) - A^0(0) + A(0) + \overline{(A - A^0 - NA)}, \qquad N(x) = N(0) + \overline{(NA - A^1 + A^0)},$$

i.e. if and only if $B = A - A^0$ and N are constant and

$$\overline{(A - A^0 - NA)} = 0, \qquad \overline{(NA - A^1 + A^0)} = 0.$$

Then (24) follows immediately and furthermore implies $\overline{A} = \overline{A^1}$. ■

4. The existence theorem

We are now in a position to state and prove our existence theorem.

Theorem 1. *Let $L > 0, \eta > 0$, let $A^0 : [0, L] \to \mathbb{R}$ be a positive function of class C^2 such that $(A^0)'(0) = (A^0)'(L) = 0$, and let $A^1 : [0, L] \to \mathbb{R}$ be a positive continuous function such that $A^1(x) > A^0(x)$ for all $x \in [0, L]$. Then the Neumann problem (2)–(3) has at least one positive solution.*

Proof. We apply Leray–Schauder's continuation theorem to the operator $\mathcal{T}$. It follows from Lemmas 3, 4, and Corollaries 1 and 2 that, for any $\lambda \in (0, 1]$ and any possible fixed point (A, N) of $\mathcal{T}(\cdot, \lambda)$, one has $(A, N) \notin \partial\Omega$. Indeed, any possible solution in $\overline{\Omega}$ belongs to Ω. On the other hand, by Proposition 2, any fixed point of $\mathcal{T}(\cdot, 0)$ has the form $(A^0 + B, N)$ with B and N constants satisfying the algebraic system (24). This system has the unique solution (25), so that $R_0 - A^0 < A - A^0 < R_2 - A^0$ and $0 < N < R_5$. Consequently, using the homotopy invariance of the Leray–Schauder degree d_{LS}, we obtain

$$d_{LS}[I - \mathcal{T}(\cdot, 1), \Omega, 0] = d_{LS}[I - \mathcal{T}(\cdot, 0), \Omega, 0].$$

But $\mathcal{T}(\cdot, 0)$ maps E into the 2-dimensional manifold made of (A, N) such that $(A - A^0, N)$ is a constant function in E. Letting $A = A^0 + B$ and using the invariance of the Leray–Schauder degree by translation, we obtain

$$d_{LS}[I - \mathcal{T}(\cdot, 0), \Omega, 0] = d_{LS}[(I - \mathcal{T})(A^0 + \cdot, \cdot, 0), \Omega - (A^0, 0), 0]$$

where

$$\begin{aligned}(I - \mathcal{T})(A^0 + B, N, 0) &= (A^0 + B - A^0 - B(0) + \overline{(B - NA^0 - NB)}, N - N(0) + \overline{(NA^0 + NB - A^1 + A^0)}) \\ &= (B - B(0) + \overline{(B - NA^0 - NB)}, N - N(0) + \overline{(NA^0 + NB - A^1 + A^0)}) \\ &= [I - \widetilde{\mathcal{T}}](B, N),\end{aligned}$$

where $\widetilde{\mathcal{T}}$ takes values in the 2-dimensional vector space of constant functions of E, which is isomorphic to $\mathbb{R}^2$. Consequently, using the reduction theorem for the Leray–Schauder degree, we obtain, with d_B the Brouwer degree,

$$\begin{aligned}d_{LS}[I - \widetilde{\mathcal{T}}, \Omega - (A^0, 0), 0] &= d_B[(I - \widetilde{\mathcal{T}})|_{\mathbb{R}^2}, (\Omega - (A^0, 0)) \cap \mathbb{R}^2, 0] \\ &= d_B[\mathcal{F}, (R_0 - \max A^0, R_2 - \min A^0) \times (0, R_5), 0],\end{aligned}$$

where $\mathcal{F} : [R_0 - \max A^0, R_2 - \min A^0] \times [0, R_5] \to \mathbb{R}$ is defined by

$$\mathcal{F}(B, N) = (B - N\overline{A^0} - NB, N\overline{A^0} + NB - \overline{A^1} + \overline{A^0}).$$

At the unique zero of $\mathcal{F}$ given by (25), the Jacobian is easily computed and is positive. Hence

$$d_B[\mathcal{F}, (R_0 - \max A^0, R_2 - \min A^0) \times (0, R_5), 0] = 1.$$

Thus

$$d_{LS}[I - \mathcal{T}(\cdot, 1), \Omega, 0] = 1$$

and the existence of a fixed point of $\mathcal{T}(\cdot, 1)$, i.e. of a solution of (2)–(3) contained in Ω follows. ■

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