

Stress intensity factor for a semi-elliptical crack subjected to an arbitrary mode I loading

E Atroshchenko, S Potapenko and G Glinka

Mathematics and Mechanics of Solids 2014 19: 289 originally published online 31 October 2012

DOI: 10.1177/1081286512463573

The online version of this article can be found at:

<http://mms.sagepub.com/content/19/3/289>

Published by:



<http://www.sagepublications.com>

Additional services and information for *Mathematics and Mechanics of Solids* can be found at:

Email Alerts: <http://mms.sagepub.com/cgi/alerts>

Subscriptions: <http://mms.sagepub.com/subscriptions>

Reprints: <http://www.sagepub.com/journalsReprints.nav>

Permissions: <http://www.sagepub.com/journalsPermissions.nav>

Citations: <http://mms.sagepub.com/content/19/3/289.refs.html>

>> [Version of Record](#) - Apr 16, 2014

[OnlineFirst Version of Record](#) - Oct 31, 2012

[What is This?](#)

Stress intensity factor for a semi-elliptical crack subjected to an arbitrary mode I loading

Mathematics and Mechanics of Solids
2014, Vol. 19(3) 289–298
©The Author(s) 2012
Reprints and permissions:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/1081286512463573
mms.sagepub.com



E Atroshchenko

Department of Mechanical Engineering, University of Chile, Chile

S Potapenko

Department of Civil Engineering, University of Waterloo, Canada

G Glinka

Department of Mechanical Engineering, University of Waterloo, Canada

Received 29 August 2012; accepted 12 September 2012

Abstract

In this paper, we use the weight function for an elliptical crack embedded in an infinite elastic media in conjunction with the alternating method to derive the exact analytical solution for the stress intensity factor for a semi-elliptical surface crack subjected to an arbitrary mode I loading.

Keywords

Alternating method, fracture mechanics, stress intensity factor, semi-elliptical crack, weight function

1. Introduction

The problems of the accurate determination of stress intensity factors (SIFs) for cracks are receiving significant amounts of attention in the literature. This can be attributed to the fact that in many practical engineering problems, it is often required to investigate the durability of structural components weakened by a crack. In most practical situations, crack configurations can be either embedded or can be surface breaking planar cracks that are subjected to complex two-dimensional stress fields. The first analytical solution has been developed for the problem of a circular crack in an infinite elastic body by Sneddon [1], Kassir and Sih [2] and Fabrikant [3]. Recently, an exact analytical SIF solution has been derived for an elliptical crack subjected to an arbitrary applied stress field and embedded in an infinite elastic solid [4]. The method introduced in Atroshchenko et al. [4] was subsequently used in Atroshchenko et al. [5] to derive the SIF solution for an elliptical crack for a particular case of the applied concentrated force, i.e. *the weight function*. This exact analytical weight function enables an accurate determination of the SIF for an elliptical crack induced by any applied stress field. The SIF is obtained by integrating the product of the weight function and the applied stress field over the crack domain.

The problems formulated and solved for cracks embedded in an infinite elastic solid are based on the assumption that the crack is sufficiently far away from the boundaries of the body. However, in practical applications,

Corresponding author:

E Atroshchenko, Department of Mechanical Engineering, University of Chile, Av. Beauchef 850, Santiago 8370448, Chile.
Email: eatroshch@gmail.com

the effect of free boundaries may significantly affect the stress distribution near the crack front. In order to account for the free boundary effect, the alternating method can be used. The method includes the successive, iterative superposition of two solutions in order to satisfy the boundary conditions. The first solution is the SIF and the stress field in an infinite body containing a crack. The second solution is for the stress field in a semi-infinite or finite un-cracked body subjected to the stress field applied on the boundary. This solution for a semi-infinite body has been derived analytically (e.g. Love [6] and Kupradze [7]). At the same time, for finite bodies, the stress field can be obtained numerically using the finite element method (e.g. Smith and Sorensen [8] and Nishioka and Atluri [9]).

The alternating method was used in Kassir and Sih [2], Shah and Kobayashki [10] and Dhondt [11] to solve the problems of penny-shaped and elliptical cracks near the boundary of a semi-infinite elastic solid and in Smith and Sorensen [8], Nishioka and Atluri [9] and Hartranft and Sih [12] to solve the problems of a semi-circular and semi-elliptical surface cracks in semi-infinite and finite bodies. The main difficulty in using the alternating method for elliptical and semi-elliptical cracks was that the first solution – the SIF and the stress field for an elliptical crack in an infinite body – was restricted by the case of a polynomial applied pressure. Therefore, Kassir and Sih [2], Smith and Sorensen [8], Nishioka and Atluri [9] and Shah and Kobayashi [10] have used the polynomial approximation of the additional stress over the crack domain due to a free surface, which led to certain inaccuracies in the results.

In the present paper, the exact analytical weight function for an elliptical crack is employed in the alternating method to obtain an accurate SIF solution for a semi-elliptical surface crack in a semi-infinite body.

2. Formulation of a problem of a surface semi-elliptical crack in a semi-infinite body

The boundary value problem for a surface semi-elliptical crack is formulated as follows. Let (x, y, z) be the Cartesian system of coordinates. Suppose that a semi-elliptical crack (with semi-axes a and b , $a \geq b$) is embedded in a three-dimensional semi-infinite ($y > 0$) elastic body and occupies the open domain $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, y > 0$ in the plane $z = 0$ (Figure 1). The plane $y = 0$ represents the free surface (Figure 2). The crack is opened up by an arbitrary applied normal stress $P(x, y)$, symmetric with respect to the crack plane. Together with the standard equations of the elastic equilibrium [13], the following boundary conditions are imposed

$$\begin{aligned} \tau_{zx} = \tau_{zy} = 0, & \quad \text{for} \quad z = 0 \\ \sigma_{zz}(x, y, 0) = -P(x, y), & \quad \text{for} \quad (x, y) \in \Sigma \\ w(x, y, 0) = 0, & \quad \text{for} \quad (x, y) \in \Sigma_1 \\ \tau_{yx} = \sigma_{yy} = \tau_{yz} = 0, & \quad \text{for} \quad y = 0 \end{aligned} \quad (1)$$

where $\Sigma_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, y > 0$ is an open domain in the plane $z = 0$ outside of the crack, and τ_{zx} , τ_{zy} , $\sigma_{zz}(x, y, 0)$, τ_{yx} , τ_{yz} , σ_{yy} are the components of the elastic stress tensor.

The corresponding embedded elliptical crack problem is formulated as follows. The crack occupies the domain $S : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$ in the plane $z = 0$, and the stress field $P(x, y)$ is applied symmetrically with respect to the plane $y = 0$. The boundary conditions in this case are partially the same as equation (1) (i.e. the first three conditions)

$$\begin{aligned} \tau_{zx} = \tau_{zy} = 0, & \quad \text{for} \quad z = 0 \\ \sigma_{zz}(x, y, 0) = -p(x, y), & \quad \text{for} \quad (x, y) \in S \\ w(x, y, 0) = 0, & \quad \text{for} \quad (x, y) \in S_1 \end{aligned} \quad (2)$$

where $S_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} > 1$ is an open domain in the plane $z = 0$ outside of the crack, and

$$p(x, y) = \begin{cases} P(x, y), & y > 0 \\ P(x, -y), & y < 0 \end{cases} \quad (3)$$

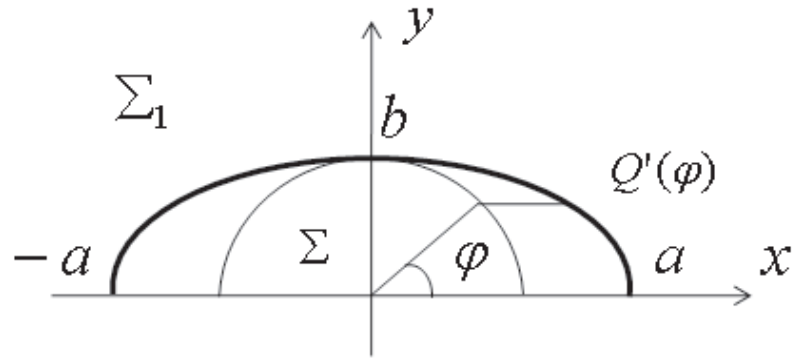


Figure 1. Semi-elliptical crack.

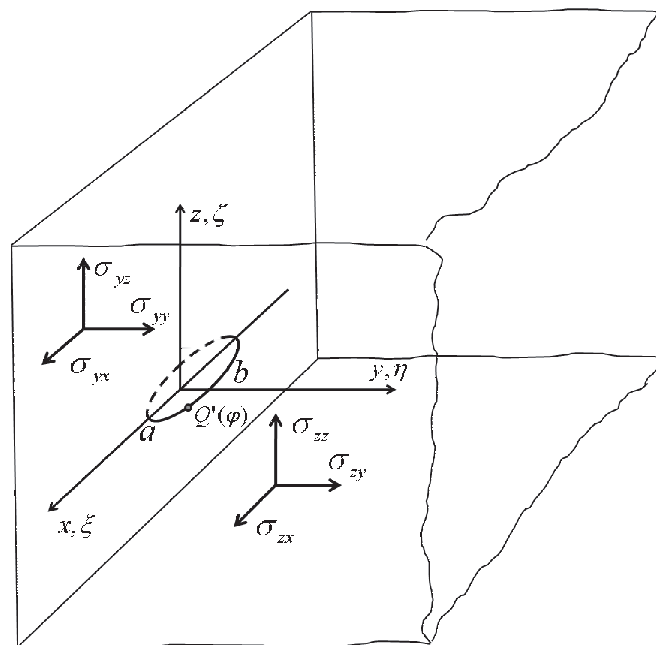


Figure 2. Coordinate systems associated with a surface semi-elliptical crack in a semi-infinite domain.

If the problem defined by equation (2) is considered instead of the problem of equation (1), the first three boundary conditions in equation (1) are satisfied by definition. The conditions $\tau_{yx} = \tau_{yz} = 0$ on the plane $y = 0$ are satisfied due to the symmetry of the applied load $p(x, y)$. In order to eliminate the normal stress σ_{yy} on the plane $y = 0$, the alternating method is employed.

Further, the following notation is used. Domains Σ and Σ_1 in elliptical coordinates ($x = ar \cos \theta, y = br \sin \theta$) are described by

$$\Sigma : 0 < r < 1, \quad 0 < \theta < \pi, \quad \Sigma_1 : r > 1, \quad 0 < \theta < \pi \tag{4}$$

An arbitrary point $Q'(x', y') = Q'(\varphi)$ on the crack contour is described by a parametric angle φ such that

$$x' = a \cos \varphi, \quad y' = b \sin \varphi \tag{5}$$

where for a semi-elliptical crack $0 < \varphi < \pi$, and for the corresponding elliptical crack $0 \leq \varphi \leq 2\pi$.

Together with the coordinates (x, y, z) , the system (ξ, η, ζ) will be used as the variables of integration.

3. Weight function for an embedded elliptical crack

In order to calculate the SIF for an elliptical crack, we use the weight function derived in Atroschenko et al. [5]. The weight function $W(Q', Q_0)$, corresponding to the point $Q_0(x_0, y_0)$ where the point force of magnitude $P = 1$ is applied (Figure 3), is obtained in the following form. Let $x_0 = ar_0 \cos \theta_0, y_0 = br_0 \sin \theta_0$, then $W(Q', Q_0) = W(\varphi; r_0, \theta_0)$ and for any fixed N the N th approximation $W^N(\varphi; r_0, \theta_0)$ of the weight function $W(\varphi; r_0, \theta_0)$ is given by

$$W^N(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2} \cos^2 \varphi + \sin^2 \varphi \right)^{1/4} \times \sum_{m=0}^N (\mathcal{X}_m^N(r_0, \theta_0) \cos m\varphi + \mathcal{Y}_m^N(r_0, \theta_0) \sin m\varphi) \quad (6)$$

The following notation is subsequently introduced. Let

$$\alpha_{nm}^c = \int_0^{2\pi} \sqrt{\frac{b^2}{a^2} \cos^2 \psi + \sin^2 \psi} \cos n\psi \cos m\psi d\psi$$

$$\alpha_{nm}^s = \int_0^{2\pi} \sqrt{\frac{b^2}{a^2} \cos^2 \psi + \sin^2 \psi} \sin n\psi \sin m\psi d\psi \quad (7)$$

For any fixed N ,

$$N1 = \begin{cases} N, & N \text{ is even} \\ N-1, & N \text{ is odd} \end{cases} \quad (8)$$

$$N2 = \begin{cases} N-1, & N \text{ is even} \\ N, & N \text{ is odd} \end{cases} \quad (9)$$

The matrix D_{2k} is defined as

$$D_{2k} = \begin{pmatrix} \alpha_{00}^c & \alpha_{02}^c & \alpha_{04}^c & \dots & \alpha_{0,2k}^c \\ \alpha_{20}^c & \alpha_{22}^c & \alpha_{24}^c & \dots & \alpha_{2,2k}^c \\ \alpha_{40}^c & \alpha_{42}^c & \alpha_{44}^c & \dots & \alpha_{4,2k}^c \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{2k,0}^c & \alpha_{2k,2}^c & \alpha_{2k,4}^c & \dots & \alpha_{2k,2k}^c \end{pmatrix}, \quad k = 0, 1, 2, \dots, \frac{N1}{2} \quad (10)$$

The parameter $\delta_{2k}^{2i,2j}$ is a cofactor obtained by removing the $i+1$ row and $j+1$ column from matrix D_{2k} if $i, j \leq k$ and 0 if $i, j > k$, the parameter $\Delta_{2k} = \det D_{2k}$. The even coefficients $\mathcal{X}_{2s}^N(r_0, \theta_0)$ of equation (6) are given by

$$\mathcal{X}_{2s}^N(r_0, \theta_0) = \sum_{i=0}^{N1/2} \left[\sum_{k=i}^{N1/2} (-1)^k \frac{\delta_{2k}^{2i,2s}}{\Delta_{2k}} b_{2i,k-i}^c + (-1)^i \frac{\delta_{N1}^{2i,2s}}{\Delta_{N1}} \left(f_{2i}^c - \sum_{l=0}^{N1/2-i} (-1)^l b_{2i,l}^c \right) \right] \quad (11)$$

where for any indexes n, i

$$f_n^c = \frac{2}{(-i)^n \sqrt{\pi}} \frac{r_0^n \cos n\theta_0}{\sqrt{1-r_0^2}} \quad (12)$$

$$b_{n,i}^c = \frac{2\Gamma(n+i+1)\Gamma(n+2i+3/2)}{\Gamma(n+1)\Gamma(i+3/2)(-i)^n} r_0^n \sqrt{1-r_0^2} \mathfrak{F}_i(n+\frac{3}{2}, n+1, r^2) \cos n\theta_0 \tag{13}$$

where $i = \sqrt{-1}$, Γ is the Gamma function and $\mathfrak{F}_i(n + \frac{3}{2}, n + 1, r^2)$ is the Jacobi polynomial of degree i in r^2 , defined by the hypergeometric function

$$\mathfrak{F}_i(n + \frac{3}{2}, n + 1, r^2) = {}_2F_1(-i, i + n + 3/2, n + 1, r^2) \tag{14}$$

The odd coefficients $\mathcal{X}_m^N(r_0, \theta_0)$ in expression (6) can be written in a similar way. Let

$$D_{2k+1} = \begin{pmatrix} \alpha_{11}^c & \alpha_{13}^c & \alpha_{15}^c & \dots & \alpha_{1,2k+1}^c \\ \alpha_{31}^c & \alpha_{33}^c & \alpha_{35}^c & \dots & \alpha_{3,2k+1}^c \\ \alpha_{51}^c & \alpha_{53}^c & \alpha_{55}^c & \dots & \alpha_{5,2k+1}^c \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{2k+1,1}^c & \alpha_{2k+1,3}^c & \alpha_{2k+1,5}^c & \dots & \alpha_{2k+1,2k+1}^c \end{pmatrix}, \quad k = 0, 1, 2, \dots, \frac{N-1}{2} \tag{15}$$

and $\delta_{2k+1}^{2i+1, 2s+1}$ be a cofactor obtained by removing the $i + 1$ row and $s + 1$ column from matrix D_{2k+1} . Then,

$$\begin{aligned} \mathcal{X}_{2s+1}^N(r_0, \theta_0) = & (-i) \sum_{i=0}^{(N-1)/2} \left[\sum_{k=i}^{(N-1)/2} (-1)^k \frac{\delta_{2k+1}^{2i+1, 2s+1}}{\Delta_{2k+1}} b_{2i+1, k-i}^c \right. \\ & \left. + (-1)^i \frac{\delta_{N2}^{2i+1, 2s+1}}{\Delta_{N2}} \left(f_{2i+1}^c - \sum_{l=0}^{(N-1)/2-i} (-1)^l b_{2i+1, l}^c \right) \right] \end{aligned} \tag{16}$$

Coefficients $\mathcal{Y}_m^N(r_0, \theta_0)$ are obtained by changing parameters α_{nm}^c into α_{nm}^s and $\cos n\theta_0$ into $\sin n\theta_0$ in equations (11) and (16).

After the coefficients $\mathcal{X}_m^N(r_0, \theta_0)$ and $\mathcal{Y}_m^N(r_0, \theta_0)$ have been derived and substituted in equation (6), equation (6) represents the weight function for an elliptical crack.

However, due to the symmetry of the applied stress field (3), only the part of expression (6) can be used to obtain the corresponding SIF, i.e.

$$W^N(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2} \cos^2 \varphi + \sin^2 \varphi \right)^{1/4} \sum_{m=0}^N \mathcal{X}_m^N(r_0, \theta_0) \cos m\varphi \tag{17}$$

The SIF $K(\varphi)$ induced by any arbitrary stress field $p(x, y)$ is obtained as

$$K(\varphi) = \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0) p(r_0, \theta_0) a b r_0 dr_0 d\theta_0 \tag{18}$$

where $W(\varphi; r_0, \theta_0) = W^N(\varphi; r_0, \theta_0)$ of equation (17) with sufficiently large number of terms N (which depends on the applied stress field $p(x, y)$).

4. Crack opening displacement for an embedded elliptical crack

In order to derive the stress field in the infinite body containing an elliptical crack, the crack opening displacement $w(x, y) = w(r, \theta)$, induced by the applied stress field $p(x, y)$, should be derived first. To achieve this goal, we use the procedure introduced in Atroshchenko et al. [4]. First, we expand the applied stress field in elliptical coordinates into a Fourier series. Due to the symmetry (equation (3)), the applied stress field $p(x, y)$ can be expanded into the cosine Fourier series in coordinates (r, θ) as

$$p(x, y) = p(r, \theta) = \frac{p_0(r)}{2} + \sum_{n=1}^{\infty} p_n^c(r) \cos n\theta \tag{19}$$

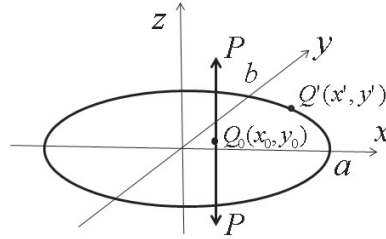


Figure 3. Elliptical crack subjected to the point load at the arbitrary location inside the crack domain.

Next, the following system of equations with unknown coefficients A_{kn}^c is solved

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} C_{nmki}^c A_{kn}^c = B_{ni}^c \quad (20)$$

where

$$C_{nmki}^c = \begin{cases} \alpha_{nm}^c, & m + 2k = n + 2i \\ 0, & \text{otherwise} \end{cases} \quad (21)$$

The right-hand side of (20) is given by

$$B_{ni}^c = \frac{2(n + 2i + \frac{3}{2})\Gamma(n + i + 1)}{(-i)^n \Gamma(1 + n)\Gamma(i + 3/2)} \int_0^1 p_n^c(r) r^{n+1} \sqrt{1 - r^2} \mathfrak{F}_i(n + \frac{3}{2}, n + 1, r^2) dr \quad (22)$$

After solving (20), the crack opening displacement can be obtained as

$$w(r, \theta) = \frac{2(1 - \nu^2) \pi b}{E} \sqrt{1 - r^2} \sum_{n=0}^{\infty} \frac{(-i)^n r^n}{\Gamma(n + 1)} \times \sum_{k=0}^{\infty} \frac{\Gamma(n + k + 1)}{\Gamma(k + \frac{3}{2})} \mathfrak{F}_k(n + \frac{3}{2}, n + 1, r^2) A_{kn}^c \cos n\theta \quad (23)$$

5. Alternating method

In order to use the alternating method to solve the problem of a semi-elliptical crack, the two following solutions are employed.

Solution 1 is the solution for the SIF and for the stress field component σ_{yy} obtained for an infinite body containing an elliptical crack. The SIF $K(\varphi)$ for an elliptical crack is obtained using equation (18). The stress field component σ_{yy} is obtained using the analytical solution presented in Panasyuk [13], together with the crack opening displacement obtained from equation (23).

Solution 2 is the one for the stress field components $(\tau_{zx}, \tau_{zy}, \sigma_{zz})$ in the un-cracked semi-infinite body, loaded on the boundary plane. Solution 2 has been given in an analytical form in Kupradze [7].

Next, the alternating method includes the successive approximations

$K^{(0)}(\varphi), K^{(1)}(\varphi), \dots$ of the SIF solution $K(\varphi)$ for a semi-elliptical crack. The iterating process is organized as follows.

In order to obtain the zero approximation $K^{(0)}(\varphi)$ of the SIF, the problem defined by equation (2) is considered instead of the problem of equation (1). It is also convenient to use the following notation

$$P(x, y) = P^{(0)}(x, y), \quad p(x, y) = p^{(0)}(x, y) \quad (24)$$

The SIF $K^{(0)}(\varphi)$ is obtained as

$$K^{(0)}(\varphi) = \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0) p^{(0)}(r_0, \theta_0) a b r_0 dr_0 d\theta_0 \quad (25)$$

The crack opening displacement $w(x, y) = w^{(0)}(x, y)$ is obtained from equation (23) with $p(x, y) = p^{(0)}(x, y)$.
 The stress field component $\sigma_{yy} = \sigma_{yy}^{(0)}$ is given in Panasyuk [13] as

$$\frac{2(1 - \nu^2)}{E} \sigma_{yy}^{(0)} = \frac{1}{2\pi} \iint_S \left[-2\nu \frac{\partial^2}{\partial z^2} + (1 - 2\nu) \frac{\partial^2}{\partial y^2} + z \frac{\partial^3}{\partial y^2 \partial z} \right] \frac{1}{r} w^{(0)}(\xi, \eta) d\xi d\eta \tag{26}$$

where

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2} \tag{27}$$

In order to eliminate the normal stress on the free boundary, the Neumann boundary value problem is considered, where the stress vector

$$(\tau_{yx}, \sigma_{yy}, \tau_{yz}) = (0, -\sigma_{yy}^{(0)}, 0) \tag{28}$$

is prescribed on the plane $y = 0$. The stress field solution $\sigma_{zz} = \sigma_{zz}^{(0)}$ to this problem is given in Kupradze [7] as

$$\sigma_{zz}^{(0)} = -\frac{1}{2\pi} \iint_{R^2} \left[(\lambda + 2\mu) \frac{\partial A_{13}}{\partial z} + \lambda \frac{\partial A_{23}}{\partial x} + \lambda \frac{\partial A_{33}}{\partial y} \right] \sigma_{yy}^{(0)}(\zeta, \xi) d\zeta d\xi \tag{29}$$

where the following notation is used

$$\begin{aligned} A_{13} &= \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial z \partial y} + \frac{1}{2(\lambda + \mu)} \frac{\partial^2 \Phi}{\partial z \partial y} \\ A_{23} &= \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial x \partial y} + \frac{1}{2(\lambda + \mu)} \frac{\partial^2 \Phi}{\partial x \partial y} \\ A_{33} &= -\frac{2\lambda + 3\mu}{2\mu(\lambda + \mu)} \frac{1}{\mathbf{R}} + \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial y^3} \\ \mathbf{R} &= \sqrt{(x - \xi)^2 + y^2 + (z - \zeta)^2} \\ \Phi &= y \ln(\mathbf{R} + y) - \mathbf{R} \\ \mu &= \frac{E}{2(1 + \nu)} \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \end{aligned} \tag{30}$$

After substituting $z = 0$ in equation (29), the additional stress due to the free boundary $\sigma_{zz}^{(0)} = \sigma_{zz}^{(0)}(x, y)$, $y > 0$ is obtained over the domain of the semi-elliptical crack. Due to the symmetry of the problem, shear stresses τ_{zx} and τ_{zy} vanish in the crack plane, i.e. $\tau_{zx}^{(0)} = 0$ and $\tau_{zy}^{(0)} = 0$ in the plane $z = 0$.

Next, the stress field $P^{(1)}(x, y) = P^{(0)}(x, y) - \sigma_{zz}^{(0)}(x, y)$ is applied over the crack domain and the first approximation $K^{(1)}(\varphi)$ for the SIF $K(\varphi)$ is obtained as

$$K^{(1)}(\varphi) = \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0) p^{(1)}(r_0, \theta_0) a b r_0 dr_0 d\theta_0 \tag{31}$$

where the stress field $p^{(1)}(r_0, \theta_0)$ is given in Cartesian coordinates by

$$p^{(1)}(x, y) = \begin{cases} P^{(1)}(x, y), & y > 0 \\ P^{(1)}(x, -y), & y < 0 \end{cases} \tag{32}$$

or

$$K^{(1)}(\varphi) = K^{(0)}(\varphi) - \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0) \tilde{\sigma}_{zz}^{(0)}(r_0, \theta_0) a b r_0 dr_0 d\theta_0 \tag{33}$$

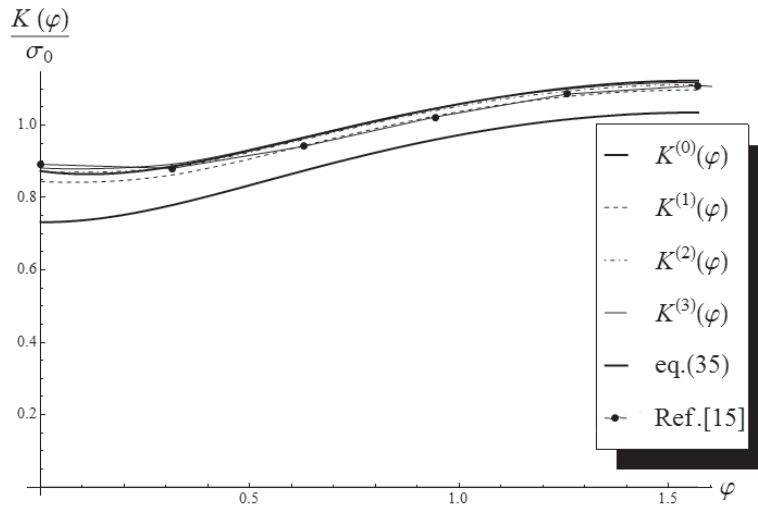


Figure 4. The variation of the SIF along the contour of a semi-elliptical crack subjected to a uniform tension.

where the stress field $\tilde{\sigma}_{zz}^{(0)}$ is given in Cartesian coordinates by

$$\tilde{\sigma}_{zz}^{(0)}(x, y) = \begin{cases} \sigma_{zz}^{(0)}(x, y), & y > 0 \\ \sigma_{zz}^{(0)}(x, -y), & y < 0 \end{cases} \quad (34)$$

The process is repeated until the sequence $K^{(0)}(\varphi), K^{(1)}(\varphi), \dots, K^{(n)}(\varphi)$ converges.

6. Numerical SIF data compared with existing literature solutions

The SIF solution for a semi-elliptical crack in a semi-infinite body subjected to a uniform pressure σ_0 has been determined by using the alternating method and compared with the Raju–Newman solution [14] and Nilsson's data [15]. The Raju–Newman SIF solution [14] was obtained by using the finite-element method for a semi-elliptical crack in a finite-thickness plate. Numerical data was subsequently fitted into an empirical equation. This equation in the case of a semi-infinite solid is given as

$$K(\varphi) = \sigma_0 \sqrt{\pi \frac{b}{Q^*}} M_1 f_\varphi g \quad (35)$$

where

$$\begin{aligned} M_1 &= 1.13 - 0.09 \frac{b}{a} \\ f_\varphi &= \left[\left(\frac{b}{a} \right)^2 \cos^2 \varphi + \sin^2 \varphi \right]^{1/4} \\ g &= 1 + 0.1(1 - \sin \varphi)^2 \\ Q^* &= 1 + 1.464 \left(\frac{b}{a} \right)^{1.65} \end{aligned} \quad (36)$$

Nilsson's solution [15] was also obtained by using the finite-element method for a semi-elliptical crack in a finite-thickness plate. Numerical data used for comparison is for the case when the crack depth is 10 times smaller than the plate width.

The variation of the first four approximations $K^{(0)}(\varphi), K^{(1)}(\varphi), K^{(2)}(\varphi), K^{(3)}(\varphi)$ of the SIF $K(\varphi)$ along the contour of a semi-elliptical crack with $a = 1, b = 0.5$ subjected to the uniform tension $p(x, y) = \sigma_0$ obtained by the alternating method and the SIF data from Newman and Raju [14] and Nilsson [15] are shown in Figure 4. The number of terms $N = 30$ was used in equation (17).

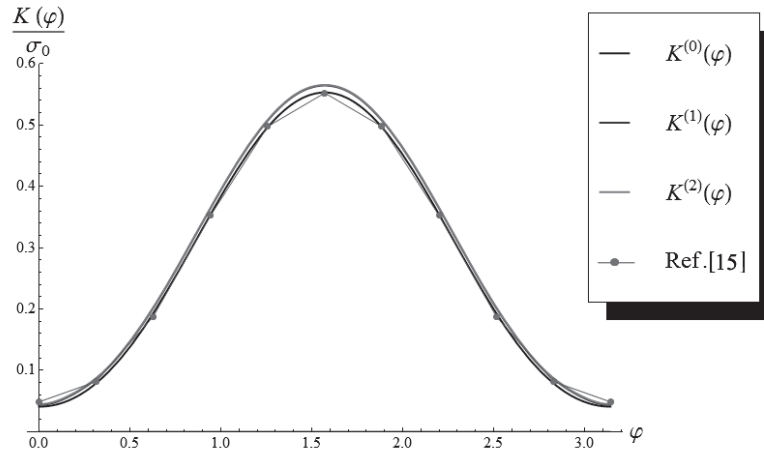


Figure 5. The variation of the SIF along the contour of a semi-elliptical crack subjected to a quadratic stress field.

Next, the SIF was obtained for a semi-elliptical crack subjected to the quadratic stress field

$$p(x, y) = \sigma_0 \left(\frac{y}{b} \right)^2 \quad (37)$$

The variation of the first three approximations $K^{(0)}(\varphi)$, $K^{(1)}(\varphi)$, $K^{(2)}(\varphi)$ of the SIF $K(\varphi)$ along the contour of a semi-elliptical crack with $a = 1$, $b = 0.5$ subjected to the applied stress field (37) is shown in Figure 5 in comparison with Nilsson's solution [15]. The number of terms $N = 20$ was used in equation (17). As seen from Figure 5, Nilsson's data coincide with the zero-th approximation of the actual SIF solution and do not show the effect of the free boundary.

7. Conclusion

In the present paper, the SIF for a surface semi-elliptical crack has been derived using the weight function for an embedded elliptical crack in conjunction with the alternating method. It has been demonstrated that the application of the exact weight function for an elliptical crack, obtained in Atroshchenko et al. [5], yields accurate SIF solutions for a semi-elliptical crack subjected to an arbitrary applied stress field. Numerical results have been compared with the known solutions for particular cases of applied uniform and quadratic stress fields, and have been found to be in good agreement.

Conflict of interest

None declared.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

References

- [1] Sneddon, IN. The distribution of stress in the neighbourhood of a crack in an elastic solid. *Proc R Soc Lond A* 1946; 187(1009): 229–620.
- [2] Kassir, MK, and Sih, GC. *Three-dimensional crack problems*. Leyden: Noordhoff International Publishing, 1975.
- [3] Fabrikant, VI. *Applications of potential theory in mechanics, a selection of new results*. London: Kluwer Academic Publishers, 1989.
- [4] Atroshchenko, E, Potapenko, S, and Glinka, G. Stress intensity factor for an embedded elliptical crack under arbitrary normal loading. *Int J Fatigue* 2009; 31(11–12): 1907–1910.
- [5] Atroshchenko, E, Potapenko, S, and Glinka, G. Weight function for an elliptical planar crack embedded in a homogeneous elastic medium. *Int J Fracture* 2010; 165(1): 39–45.

- [6] Love, AEH. The stress produced in a semi-infinite solid by pressure on part of the boundary. *Phil Trans R Soc A* 1929; 228: 377–420.
- [7] Kupradze, VD. *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*. Amsterdam: North-Holland Publishing Company, 1979.
- [8] Smith, FW, and Sorensen, DR. The semi-elliptical surface crack – a solution by the alternating method. *Int J Fracture* 1976; 12(1): 47–57.
- [9] Nishioka, T, and Atluri, SN. Analytical solution for embedded elliptical cracks, and finite element-alternating method for elliptical surface cracks, subjected to arbitrary loadings. *Eng Fracture Mech* 1983; 17: 247–268.
- [10] Shah, RC, and Kobayashi, AS. Stress intensity factors for an elliptical crack approaching the surface of a semi-elliptical solid. *Int J Fracture* 1973; 9(2): 133–146.
- [11] Dhondt, G. Green functions for a half circular crack in half infinite space under normal loading. *Int J Solids Structures* 1995; 32(16): 1807–1857.
- [12] Hartranft, RJ, and Sih, GC. Alternating method applied to the edge and surface crack problems. In: *Mechanics of fracture I: methods of analysis and solutions of crack problems*. Leyden: Noordhoff International Publishing, 1973.
- [13] Panasyuk, VV. *Strength and fracture of solids with cracks*. Lviv: National Academy of Sciences of Ukraine, 2002.
- [14] Newman Jr, JC, and Raju, IS. An empirical stress intensity factor equation for a surface crack. *Eng Fracture Mechanics* 1981; 15(1–2): 185–192.
- [15] Nilsson, N. Stress intensity factors for half-elliptical surface cracks in plates subjected to a complex stress field. SAQ/FoU-Report 98/10, 1998.