



## Short communication

## Lyapunov functions for fractional order systems

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## ABSTRACT

A new lemma for the Caputo fractional derivatives, when  $0 < \alpha < 1$ , is proposed in this paper. This result has proved to be useful in order to apply the fractional-order extension of Lyapunov direct method, to demonstrate the stability of many fractional order systems, which can be nonlinear and time varying.

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## 1. Introduction

Fractional calculus relates with the calculus of integrals and derivatives of orders that may be real or complex, and has become very popular in recent years due to its demonstrated applications in many fields of science and engineering [1].

The nature of many systems makes that they can be more precisely modeled using fractional differential equations. For instance it can be mentioned the diffusion process, such as those founded in batteries [2], some heat transfer process [3], the effect of the frequency in induction machines [4], amongst others. In that sense, the stability of these systems have to be proved using techniques developed for fractional order systems. More over, those models are often used inside classic control schemes, and for that reason the whole controlled system results in a fractional order one. In these cases, the stability of the whole controlled system have to be analyzed using the fractional order techniques as well.

Besides, sometimes fractional order controllers are used to control integer order systems, such as Fractional Order PID controllers [5]; high gain output feedback control schemes, where the feedback gain is estimated using a fractional differential equation [6,7]; Fractional Order Model Reference Adaptive Controllers (FOMRAC), where the adaptive law is given by a fractional differential equation [8–11], or where the adaptive law and the reference model are described by fractional differential equations [12–14]. In all these cases, the stability of the whole controlled system have to be analyzed using the fractional order techniques as well.

The stability of Fractional Order Linear Time Invariant systems (FOLTI) can be easily proved using the method proposed by Matignon [15]. However, for fractional order nonlinear time varying systems, this method can not be used. Diethelm [16] proved the stability of a fractional order nonlinear time varying system, under certain conditions, but this result is valid only for scalar fractional order systems.

So, in order to prove the stability of fractional order nonlinear and time varying systems in the vector case, some other techniques must be applied. One of these techniques is the fractional-order extension of Lyapunov direct method, proposed

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by Li et al. [17]. Using this technique, however, is often a really hard task, since finding a Lyapunov candidate function is more complex in the fractional order case.

Some authors have proposed Lyapunov functionals to prove the stability of fractional order systems. The two prominent works [18,19] can be cited, however the relation between the Lyapunov function and the fractional differential equation is not elementary nor simple. [20] proposes some other Lyapunov functionals, where the relation between them and the fractional differential system is more elementary, but these functionals are neither simple, and they are valid for fractional systems with specific characteristics.

Some Lyapunov functions have been proposed in works related to fractional sliding mode control [21–23]. Those Lyapunov functions have been used to prove the stability of the resulting fractional order system. However, the analysis have been possible using the classic Lyapunov direct method [24] and due to the possibility to define the sliding surface in a way which makes the corresponding derivative of the Lyapunov function negative definite.

This paper presents a new property for Caputo fractional derivatives when  $0 < \alpha < 1$ , which allows finding a simple Lyapunov candidate function for many fractional order systems, and the consequently stability proof for them, using the fractional-order extension of the Lyapunov direct method [17].

The paper is organized as follows: Section 2 presents some basic concepts about fractional calculus and the stability of fractional order systems, facilitating the understanding of the ideas presented in this work. Section 3 introduces the new lemma for Caputo fractional derivatives. Section 4 presents the usefulness of this property for the stability proof of some fractional order systems, through some examples. Finally, Section 5 presents the conclusions of the work.

## 2. Preliminaries

In this section, some basic definitions related to fractional calculus are presented. Some concepts and techniques related to the stability of fractional order systems are presented as well.

### 2.1. Fractional calculus

In fractional calculus, the traditional definitions of the integral and derivative of a function are generalized from integer orders to real orders. In the time domain, the fractional order derivative and fractional order integral operators are defined by a convolution operation.

Several definitions exist regarding the fractional derivative of order  $\alpha \geq 0$ , but the Caputo definition in (1) is used the most in engineering applications, since this definition incorporates initial conditions for  $f(\cdot)$  and its integer order derivatives, i.e., initial conditions that are physically appealing in the traditional way.

**Definition 1** (Caputo fractional derivative [1]). The Caputo fractional derivative of order  $\alpha \in \mathbb{R}^+$  on the half axis  $\mathbb{R}^+$  is defined as follows

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > a \quad (1)$$

with  $n = \min \{k \in \mathbb{N} / k > \alpha\}$ ,  $\alpha > 0$ .

One special property of the fractional derivatives is the generalization of the Leibniz rule, which is stated in the following property.

**Property 1** (Leibniz rule for fractional differentiation [25]). If  $f(t)$  and  $g(t)$  along with all its derivatives are continuous in  $[a, t]$ , then the Leibniz rule for fractional differentiation takes the form

$${}_a^C D_t^\alpha (f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(t) {}_a^C D_t^{\alpha-k} g(t) \quad (2)$$

### 2.2. Stability of fractional order systems

Using the Caputo derivative, a fractional order system (FOS) can be defined by

$${}_0^C D_t^\alpha x(t) = f(x, t) \quad (3)$$

where  $\alpha \in (0, 1)$  and  $t$  represents the time.

For stability analysis of fractional order nonlinear time-varying systems like (3), a fractional-order extension of Lyapunov direct method has been proposed [17], which is stated in Theorem 1.

**Definition 2.** A continuous function  $\gamma : [0, t) \rightarrow [0, \infty)$  is said to belong to class-K if it is strictly increasing and  $\gamma(0) = 0$  ([17]).

**Theorem 1** (Fractional-order extension of Lyapunov direct method [17]). *Let  $x = 0$  be an equilibrium point for the non autonomous fractional-order system (3). Assume that there exists a Lyapunov function  $V(t, x(t))$  and class-K functions  $\gamma_i (i = 1, 2, 3)$  satisfying*

$$\gamma_1(\|x\|) \leq V(t, x(t)) \leq \gamma_2(\|x\|) \tag{4}$$

$${}^C D_t^\beta V(t, x(t)) \leq -\gamma_3(\|x\|) \tag{5}$$

where  $\beta \in (0, 1)$ . Then the system (3) is asymptotically stable.

### 3. A new lemma for the Caputo fractional derivative

This section presents a new lemma, which allows to find Lyapunov candidate functions for demonstrating the stability of many fractional order systems, using the fractional-order extension of the Lyapunov direct method.

**Lemma 1.** *Let  $x(t) \in \mathbb{R}$  be a continuous and derivable function. Then, for any time instant  $t \geq t_0$*

$$\frac{1}{2} {}^C D_{t_0}^\alpha x^2(t) \leq x(t) {}^C D_{t_0}^\alpha x(t), \quad \forall \alpha \in (0, 1) \tag{6}$$

**Proof.** Proving that expression (6) is true, is equivalent to prove that

$$x(t) {}^C D_{t_0}^\alpha x(t) - \frac{1}{2} {}^C D_{t_0}^\alpha x^2(t) \geq 0, \quad \forall \alpha \in (0, 1) \tag{7}$$

Using Definition 1, it can be written that

$${}^C D_{t_0}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau \tag{8}$$

And in the same way

$$\frac{1}{2} {}^C D_{t_0}^\alpha x^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{x(\tau)\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau \tag{9}$$

So, Expression (7) can be written as

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{[x(t) - x(\tau)]\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau \geq 0 \tag{10}$$

Let us define the auxiliar variable  $y(\tau) = x(t) - x(\tau)$ , which implies that  $y'(\tau) = \frac{dy(\tau)}{d\tau} = -\frac{dx(\tau)}{d\tau}$ . In this way, Expression (10) can be written as

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{y(\tau)y'(\tau)}{(t-\tau)^\alpha} d\tau \leq 0 \tag{11}$$

Let us integrate by parts Expression (11), defining

$$du = y(\tau)y'(\tau)d\tau \quad u = \frac{1}{2}y^2$$

$$v = \frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha} \quad dv = \frac{\alpha}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha-1}$$

In that way, Expression (11) can be written as

$$-\left[ \frac{y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} \right] \Big|_{\tau=t} + \left[ \frac{y_0^2}{2\Gamma(1-\alpha)(t-t_0)^\alpha} \right] + \frac{\alpha}{2\Gamma(1-\alpha)} \int_{t_0}^t \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau \geq 0 \tag{12}$$

Let us check the first term of expression (12), which has an indetermination at  $\tau = t$ , so let us analyze the corresponding limit.

$$\lim_{\tau \rightarrow t} \frac{y^2(\tau)}{2\Gamma(1-\alpha)(t-\tau)^\alpha} = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \rightarrow t} \frac{[x(t) - x(\tau)]^2}{(t-\tau)^\alpha} = \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \rightarrow t} \frac{[x^2(t) - 2x(t)x(\tau) + x^2(\tau)]}{(t-\tau)^\alpha} \tag{13}$$

Given that the function is derivable, L'Hopital rule can be applied (because it results  $\frac{0}{0}$ ). Then

$$\begin{aligned} \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \rightarrow t} \frac{[x^2(t) - 2x(t)x(\tau) + x^2(\tau)]}{(t-\tau)^\alpha} &= \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \rightarrow t} \frac{[-2x(t)\dot{x}(\tau) + 2x(\tau)\dot{x}(\tau)]}{-\alpha(t-\tau)^{\alpha-1}} \\ &= \frac{1}{2\Gamma(1-\alpha)} \lim_{\tau \rightarrow t} \frac{[2x(t)\dot{x}(\tau) - 2x(\tau)\dot{x}(\tau)](t-\tau)^{1-\alpha}}{\alpha} = 0 \end{aligned} \quad (14)$$

So, Expression (12) is reduced to

$$\frac{y_0^2}{2\Gamma(1-\alpha)(t-t_0)^\alpha} + \frac{\alpha}{2\Gamma(1-\alpha)} \int_{t_0}^t \frac{y^2(\tau)}{(t-\tau)^{\alpha+1}} d\tau \geq 0 \quad (15)$$

Expression (15) is clearly true, and this concludes the proof.  $\square$

**Remark 1.** In the case when  $x(t) \in \mathbb{R}^n$ , Lemma 1 is still valid. That is,  $\forall \alpha \in (0, 1)$  and  $\forall t \geq t_0$

$$\frac{1}{2} {}^C D_t^\alpha x^T(t)x(t) \leq x^T(t) {}^C D_t^\alpha x(t) \quad (16)$$

The proof is straightforward, decomposing the expression (16) into a sum of scalar products and applying Lemma 1.

**Remark 2.** One can expect an equality in (6) when function  $x(t)$  is a constant or when  $\alpha = 1$ .

The case when  $\alpha = 1$  corresponds to the product rule for the integer order derivatives, which states that  $\frac{1}{2} \frac{dx^2(t)}{dt} = x(t) \frac{dx(t)}{dt}$ , so it can be considered as a particular case of the Lemma 1.

**Corollary 1.** For the fractional order system

$${}^C D_t^\alpha x(t) = f(x(t)) \quad (17)$$

where  $\alpha \in (0, 1)$ ,  $x = 0$  is the equilibrium point and  $x(t) \in \mathbb{R}$ , if the following condition is satisfied

$$x(t)f(x(t)) \leq 0, \quad \forall x \quad (18)$$

then the origin of the system (17) is stable. And if

$$x(t)f(x(t)) < 0, \quad \forall x \neq 0 \quad (19)$$

then the origin of the system (17) is asymptotically stable.

**Proof.** Let us propose the following Lyapunov candidate function, which is positive definite

$$V(x(t)) = \frac{1}{2} x^2(t) \quad (20)$$

Using Lemma 1 results

$${}^C D_t^\alpha V(x(t)) \leq x(t) {}^C D_t^\alpha x(t) \quad (21)$$

If  $x(t)f(x(t)) \leq 0$ , then  $x(t) {}^C D_t^\alpha x(t) \leq 0$ , and the fractional derivative (21) of the Lyapunov function results negative semi-definite. This implies, using the comparison principle [17] that  $V(x(t)) \leq V(x(0))$ ,  $\forall x$ .

According to the definition of the function  $V(x(t))$ , this implies that

$$\frac{1}{2} x^2(t) \leq \frac{1}{2} x^2(0), \quad \forall x \quad (22)$$

According to the definition of stability in the sense of Lyapunov [24], expression (22) allows concluding that the origin of the system (17) is stable in the sense of Lyapunov.

If  $x(t)f(x(t)) < 0$ ,  $\forall x \neq 0$ , then  $x(t) {}^C D_t^\alpha x(t) < 0$ , and the fractional derivative (21) of the Lyapunov function results negative definite. Given the relation between positive definite functions and class-K functions in [26], using Theorem 1 it can be concluded that the origin of the system (17) is asymptotically stable.  $\square$

**Remark 3.** In the case when system (17) is vectorial, that is  $x(t) \in \mathbb{R}^n$ , Corollary 1 is still valid. The proof is straightforward, using a Lyapunov candidate function given by  $V(x(t)) = \frac{1}{2} x^T(t)x(t)$  and applying Lemma 1.

**Remark 4.** The applicability of the lemma to the use of general quadratic Lyapunov functions  $x^T(t)Px(t)$ , with  $P$  being a positive definite matrix is currently under investigation.

#### 4. Usefulness of the lemma in the stability proof of fractional order systems

One of the most used Lyapunov candidate function to prove the stability of integer systems is the quadratic function. However, in the fractional case, it is no straightforward the use of those functions. Let us state the following example.

**Example 1.** Let us consider the following fractional order linear time-varying system, where  $0 < \alpha < 1$

$$\begin{aligned} {}_0^C D_t^\alpha x_1(t) &= -\sin^2(t)x_1(t) - \sin(t)\cos(t)x_2(t) \\ {}_0^C D_t^\alpha x_2(t) &= -\sin(t)\cos(t)x_1(t) - \cos^2(t)x_2(t) \end{aligned} \tag{23}$$

To prove the stability of system (23), let us use the classic Lyapunov direct method ([24]), proposing the quadratic function as a Lyapunov candidate, which is positive definite

$$V(x_1(t), x_2(t)) = \frac{1}{2}x_1^2(t) + \frac{1}{2}x_2^2(t) \tag{24}$$

Using the property of fractional derivatives in [1], which states that  $\dot{x}(t) = {}_0^C D_t^{1-\alpha} {}_0^C D_t^\alpha x(t)$ , it can be found that

$$\begin{aligned} \dot{x}_1(t) &= -{}_0^C D_t^{1-\alpha} [x_1(t)\sin^2(t) + x_2(t)\sin(t)\cos(t)] \\ \dot{x}_2(t) &= -{}_0^C D_t^{1-\alpha} [x_1(t)\sin(t)\cos(t) + x_2(t)\cos^2(t)] \end{aligned} \tag{25}$$

And then

$$\begin{aligned} \frac{dV(x_1(t), x_2(t))}{dt} &= x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t) \\ &= -x_1(t){}_0^C D_t^{1-\alpha} [x_1(t)\sin^2(t)] - x_1(t){}_0^C D_t^{1-\alpha} [x_2(t)\sin(t)\cos(t)] - x_2(t){}_0^C D_t^{1-\alpha} [x_2(t)\cos^2(t)] \\ &\quad - x_2(t){}_0^C D_t^{1-\alpha} [x_1(t)\sin(t)\cos(t)] \end{aligned} \tag{26}$$

As can be seen from Eq. (26), it is difficult to establish a definite sign for the first derivative of the Lyapunov function, and consequently to establish conclusions about stability.

If the fractional-order extension of the Lyapunov direct method is used instead, proposing the Lyapunov candidate function (24), and using the Property 1, it can be obtained that

$${}_0^C D_t^\alpha V(x_1(t), x_2(t)) = \frac{1}{2} \sum_{k=0}^{\infty} \binom{\alpha}{k} x_1^{(k)}(t) {}_0^C D_t^{\alpha-k} x_1(t) + \frac{1}{2} \sum_{k=0}^{\infty} \binom{\alpha}{k} x_2^{(k)}(t) {}_0^C D_t^{\alpha-k} x_2(t) \tag{27}$$

As can be seen from Eq. (27), evaluating the fractional derivative of the Lyapunov function implies evaluating an infinite sum, which includes higher order integer and fractional derivatives of the states of the system (23). It is evident that this is not an easy task.

However, if the Lemma 1 is used, using the Lyapunov candidate function (24), it is straightforward obtained that

$${}_0^C D_t^\alpha V(x_1(t), x_2(t)) = \frac{1}{2} {}_0^C D_t^\alpha x_1^2(t) + \frac{1}{2} {}_0^C D_t^\alpha x_2^2(t) \leq x_1(t) {}_0^C D_t^\alpha x_1(t) + x_2(t) {}_0^C D_t^\alpha x_2(t) = -[x_1(t)\sin(t) + x_2(t)\cos(t)]^2 \leq 0 \tag{28}$$

Eq. (28) shows that the fractional derivative of the Lyapunov function is negative semidefinite, so it can be concluded that the origin of the system (23) is stable.

Fig. 1 shows the evolution of the states of the system (23), using  $\alpha = 0.8$ . As expected from the analytical analysis already presented, which is valid for any bounded initial conditions, for this case when  $x_1(0) = 3$  and  $x_2(0) = 6$  the origin of the system is stable.

**Example 2.** Let us consider the following fractional order nonlinear system, with  $0 < \alpha < 1$

$$\begin{aligned} {}_0^C D_t^\alpha x_1(t) &= -x_1(t) + x_2^3(t) \\ {}_0^C D_t^\alpha x_2(t) &= -x_1(t) - x_2(t) \end{aligned} \tag{29}$$

Let us consider the following Lyapunov candidate function, which is positive definite.

$$V(x_1(t), x_2(t)) = \frac{1}{2}x_1^2(t) + \frac{1}{4}x_2^4(t) \tag{30}$$

Now, applying Lemma 1, it can be found that

$$\begin{aligned} {}_0^C D_t^\alpha V(x_1(t), x_2(t)) &= \frac{1}{2} {}_0^C D_t^\alpha x_1^2(t) + \frac{1}{4} {}_0^C D_t^\alpha x_2^4(t) \leq x_1(t) {}_0^C D_t^\alpha x_1(t) + \frac{1}{2} x_2^2(t) {}_0^C D_t^\alpha x_2^2(t) \leq x_1(t) {}_0^C D_t^\alpha x_1(t) + x_2^3(t) {}_0^C D_t^\alpha x_2(t) \\ &= -x_1^2(t) - x_2^4(t) < 0 \end{aligned} \tag{31}$$

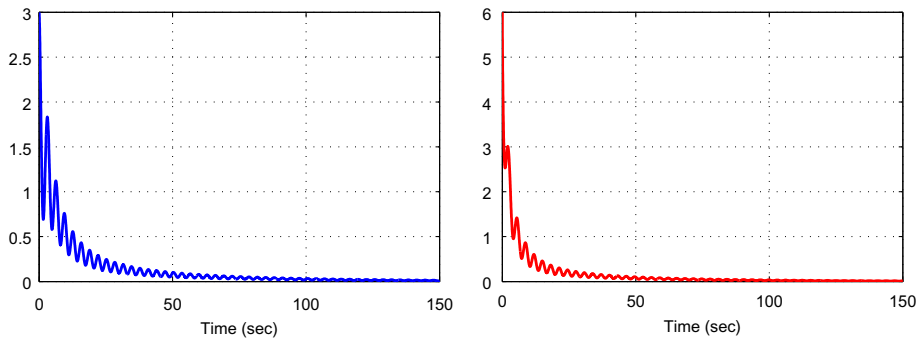


Fig. 1. Evolution of the state  $x_1(t)$  (left) and  $x_2(t)$  (right) of the system (23), using  $\alpha = 0.8$ .

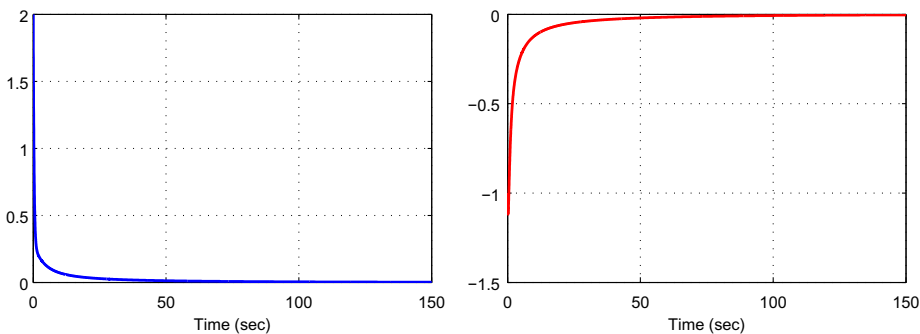


Fig. 2. Evolution of the state  $x_1(t)$  (left) and  $x_2(t)$  (right) of the system (29), using  $\alpha = 0.8$ .

As can be seen from Eq. (31), the fractional derivative of the Lyapunov function is negative definite, so it can be concluded from Corollary 1 that the origin of the system (29) is asymptotically stable.

Fig. 2 shows the evolution of the states of the system (29), using  $\alpha = 0.8$ . As expected from the analytical analysis already presented, which is valid for any bounded initial conditions, for this case when  $x_1(0) = 2$  and  $x_2(0) = -1$ , the origin of the system is asymptotically stable.

## 5. Conclusions

A new lemma related to the Caputo fractional derivative has been proposed in this paper. The result presented is valid for  $0 < \alpha < 1$ . The usefulness of this lemma for finding Lyapunov functions, and consequently proving the stability of many fractional order systems, using the fractional-order extension of the Lyapunov direct method, has been showed through some examples.

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