Knowledge-Based Systems 76 (2015) 127–138

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets

Huchang Liao [a,b], Zeshui Xu [a,*], Xiao-Jun Zeng [b], José M. Merigó [c]

[a] Business School, Sichuan University, Chengdu, Sichuan 610065, China
[b] School of Computer Science, University of Manchester, Manchester M13 9PL, United Kingdom
[c] Department of Management Control and Information Systems, University of Chile, Av. Diagonal Paraguay 257, 8330015 Santiago, Chile

ARTICLE INFO

Article history:
Received 10 May 2014
Received in revised form 27 September 2014
Accepted 7 December 2014
Available online 15 December 2014

Keywords:
Hesitant fuzzy linguistic term set
Correlation measure
Correlation coefficient
Qualitative decision making
Chinese medical diagnosis

ABSTRACT

The hesitant fuzzy linguistic term set (HFLTS) is a new and flexible tool in representing hesitant qualitative information in decision making. Correlation measures and correlation coefficients have been applied widely in many research domains and practical fields. This paper focuses on the correlation measures and correlation coefficients of HFLTSs. To start the investigation, the definition of HFLTS is improved and the concept of hesitant fuzzy linguistic element (HFLE) is introduced. Motivated by the idea of traditional correlation coefficients of fuzzy sets, intuitionistic fuzzy sets and hesitant fuzzy sets, several different types of correlation coefficients for HFLTSs are proposed. The prominent properties of these correlation coefficients are then investigated. In addition, considering that different HFLEs may have different weights, the weighted correlation coefficients and ordered weighted correlation coefficients are further investigated. Finally, an application example concerning the traditional Chinese medical diagnosis is given to illustrate the applicability and validation of the proposed correlation coefficients of HFLTSs in the process of qualitative decision making.

1. Introduction

Fuzzy knowledge based systems are based on the fact that experts usually rely on **common sense** from their domain knowledge when they solve problems. In addition, they also use ambiguous terms to express their cognition [1]. A simple example coming from the power system is like this: an expert who is in charge of the generator might say, "*Though the power transformer is* ***slightly*** *overloaded, I can keep this load for* ***a while***." In such a situation, it is impossible for the expert or his/her audiences to describe the term such as "slightly" or "a while" in crisp numerical value, but all of the audiences can understand what does that mean. In many real-life qualitative decision making problems, it is very common and straightforward for experts to express their opinions in terms of linguistic terms, such as "fast" speed, "high" price, "low" temperature, and "good" performance. Although linguistic terms are very close to human's cognitive process, computing with such linguistic terms is not easy. In 1975, Zadeh [2] proposed the fuzzy linguistic approach, which uses linguistic variables, whose values are not numbers but words or sentences in a natural or artificial language, to represent qualitative information of a person. In spite of being less precise than a number, the linguistic variable enhances the feasibility, flexibility and reliability of decision models and provides good results in different fields [3].

Nevertheless, as the fuzzy linguistic approach uses only one linguistic term to represent the value of a linguistic variable, it sometimes may not reflect exactly what the experts mean. In many cases with high degree of uncertainty, the experts might hesitant among several linguistic terms and need richer linguistic expressions to represent their opinions. For example, when evaluating the performance of a company, an expert may say "*it is* ***not too bad***"; another expert may say "its performance is ***between medium and high***." The traditional fuzzy linguistic approach cannot represent such comprehensive linguistic expressions. Recently, Rodríguez et al. [4] proposed a new proposal to improve the elicitation of linguistic information by using hesitant fuzzy linguistic term set (HFLTS) and context-free grammars. The HFLTS increases the flexibility and capability of elicitation of linguistic information by means of linguistic expressions. The context-free grammars fix the rules for the experts to build such flexible linguistic expressions, which can be transformed into HFLTS. With the use of HFLTS, the experts can provide their assessments by means of several linguistic terms or comparative linguistic expressions.

* Corresponding author. Tel.: +86 25 84483382.
E-mail addresses: liaohuchang@163.com (H. Liao), xuzeshui@263.net (Z. Xu).

http://dx.doi.org/10.1016/j.knosys.2014.12.009
0950-7051/© 2014 Elsevier B.V. All rights reserved.

Since the HFLTS provides a new and more powerful technique to represent experts' qualitative judgments, it has attracted more and more scholars' attention. Rodríguez et al. [4] introduced the concept of HFLTS, and investigated some basic operations and properties of HFLTS. A multi-criteria linguistic decision making model with linguistic expressions based on comparative terms was also given by them. Liao et al. [5] introduced a sort of distance and similarity measures for HFLTSs, based on which, a satisfactory-based decision making method was given for multi-criteria decision making (MCDM) under hesitant fuzzy linguistic circumstance. Wei et al. [6] developed some comparison methods and studied the aggregation theory for HFLTS. Chen and Hong [7] presented a new method for multi-criteria linguistic decision based on HFLTSs using the pessimistic attitude and the optimistic attitude of the decision maker. By means of a fuzzy envelope, Liu and Rodríguez [8] proposed a new representation of the HFLTS, which can be used to carry out the computing with words processes. Rodríguez et al. [9] also gave a fuzzy representation for the semantics of HFLTSs. Zhu and Xu [10] defined the hesitant fuzzy linguistic preference relation (HFLPR) and investigated its consistency. Liu et al. [11] investigated the additive consistency of HFLPR. Based on HFLTS and context-free grammars, Rodríguez et al. [12] developed a new linguistic group decision making model which deals with comparative linguistic expressions that are similar to those used by the experts in real-world decision making problems. Beg and Rashid [13] proposed a TOPSIS-based method for MCDM in which the opinion of the experts is represented by HFLTS. In order to handle hesitant fuzzy linguistic MCDM where some criteria conflict with each other, recently, Liao et al. [14] gave a step by step procedure of HFL-VIKOR method and validated it via some numerical examples.

All these above literatures show that HFLTS is a hot topic in both theoretical and practical fields. As HFLTS has been proposed for just a few years, much work needs to be done to enrich the framework of HFLTS theory. As it is well known, correlation measure is one of the most widely used indices in varying fields [15–28]. However, up to now, as far as we know, there is no research on the correlation measure of HFLTSs. Hence, in this paper, we focus on this issue and propose several important correlation measures and correlation coefficients for HFLTSs. To do so, the remainder of this paper is organized as follows: Section 2 gives some basic knowledge on fuzzy linguistic approach and HFLTS. The definition of HFLTS is improved and the HFLE is introduced. A short review on the correlation measures over fuzzy sets and its extensions is also given in this section. Section 3 proposes different forms of correlation measures and correlation coefficients for HFLTSs. The properties of these correlation coefficients are investigated in this section as well. In Section 4, the weighted correlation coefficients and ordered weighted correlation coefficients are investigated. An application example concerning the traditional Chinese medical diagnosis is given in Section 5 to show the applicability and validation of these correlation coefficients of HFLTSs. The paper ends with some concluding remarks in Section 6.

2. Preliminaries

2.1. Fuzzy linguistic approach

The fuzzy linguistic approach [2] was proposed to model linguistic information proposed by experts. In such an approach, the experts' opinions are taken as the values of a linguistic variable which is established by a linguistic descriptors and its semantics. Many different models were proposed to represent and calculate the values of a linguistic variable, such as the semantic model [3], the virtual linguist model [29], the 2-tuple linguistic model [30], and the proportional 2-tuple linguistic model [31]. The virtual linguist model is easy and straightforward, and it has been used by many scholars. A subscript-symmetric additive linguistic term set [32] is shown as:

$$S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\} \quad (1)$$

where the mid linguistic label s_0 represents an assessment of "indifference", and the rest of them are placed symmetrically around it. In particular, $s_{-\tau}$ and s_τ are the lower and upper bounds of linguistic labels used by experts in practical applications, τ is a positive integer, and S satisfies the following conditions:

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;
(2) The negation operator is defined: $\text{neg}(s_\alpha) = s_{-\alpha}$, especially, $\text{neg}(s_0) = s_0$.

Since the linguistic term set S is a discrete linguistic term set, it is not convenient for calculating and analyzing. In order to preserve all given linguistic information, Xu [29] extended the discrete linguistic term set into continuous linguistic term set $\bar{S} = \{s_\alpha | \alpha \in [-q, q]\}$, where $q(q > \tau)$ is a sufficiently large positive integer. In general, the linguistic term $s_\alpha (s_\alpha \in S)$ is determined by the experts, while the extended linguistic term (named virtual linguistic term), $\bar{s}_\alpha (\bar{s}_\alpha \in \bar{S})$, only appears in computation process.

For any two linguistic terms $s_\alpha, s_\beta \in \bar{S}$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, the following operational laws were introduced [29]:

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
(2) $\lambda s_\alpha = s_{\lambda\alpha}$;
(3) $(\lambda_1 + \lambda_2)s_\alpha = \lambda_1 s_\alpha \oplus \lambda_2 s_\alpha$;
(4) $\lambda(s_\alpha \oplus s_\beta) = \lambda s_\alpha \oplus \lambda s_\beta$.

2.2. Hesitant fuzzy linguistic term set

In quantitative settings, when an expert considers several values to determine the membership degree of an element to a set, the concept of hesitant fuzzy set (HFS) was introduced [33,34]. As for qualitative circumstances, when establishing the value of a linguistic variable, several linguistic terms may be elicited. Thus, motivated by the idea of HFS, Rodríguez et al. [4] introduced the concept of HFLTS.

Definition 1 [4]. Let $S = \{s_0, \ldots, s_\tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS), H_S, is an ordered finite subset of the consecutive linguistic terms of S.

The HFLTS can be used to elicit several linguistic values for a linguistic variable, but it is not similar to human way of thinking and reasoning. In order to make it more applicable, Rodríguez et al. [4] proposed a context-free grammar G_H to generate simple but elaborated linguistic expressions *ll* that are similar to the human's expressions. The expressions *ll* generated by the context-free grammar G_H may be either single valued linguistic terms or linguistic expressions. The transformation function E_{G_H} can be used to transform the expressions *ll* that are produced by G_H into HFLTS H_S (for more details, please refer to Refs. [4,12,14]). The way to obtain a HFLTS can be shown as Fig. 1.

It is noted that, regarding to the linguistic term set $S = \{s_0, \ldots, s_\tau\}$ given in Definition 1, when its subscripts are not symmetric, some problems will arise. For example, for a linguistic term set $S = \{s_0 = none, s_1 = very\ low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very\ high, s_6 = perfect\}$, according to the operational law, we have $s_2 \oplus s_3 = s_5$, which means, the aggregated result of linguistic terms "*low*" and "*medium*" is "*very high*". This is not coincident with our intuition. (for more details, see Refs. [5,10,14]). To overcome these problems, Liao et al. [5] replaced the linguistic

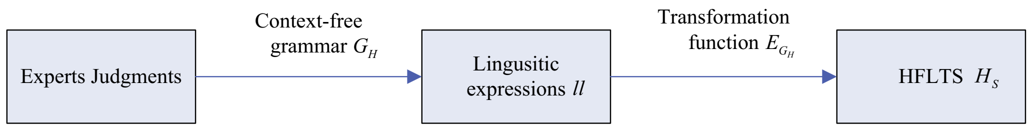

Fig. 1. The way to obtain a HFLTS.

term set $S = \{s_0, \ldots, s_\tau\}$ in Definition 1 by the subscript-symmetric linguistic term set $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$. In addition, it should also be noted that Definition 1 does not give any mathematical form for HFLTS. To overcome this incompleteness, the definition of HFLTS is refined as follows:

Definition 2. Let $x_i \in X,\ i = 1, 2, \ldots, N,$ be fixed and $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS) on X, H_S, is in mathematical terms of

$$H_S = \{\langle x_i, h_S(x_i)\rangle | x_i \in X\} \tag{2}$$

where $h_S(x_i)$ is a set of some values in the linguistic term set S and can be expressed as $h_S(x_i) = \{s_{\phi_l}(x_i) | s_{\phi_l}(x_i) \in S, l = 1, \ldots, L\}$ with L being the number of linguistic terms in $h_S(x_i)$. $h_S(x_i)$ denotes the possible degrees of linguistic variable x_i to the linguistic term set S. For convenience, $h_S(x_i)$ is called the hesitant fuzzy linguistic element (HFLE) and $\mathbb{H}_S$ is the set of all HFLEs.

Example 1. Quality management is more and more popular in our daily life. In the process of quality management, many aspects of certain products cannot be measured as crisp values but only qualitative values. Here we just consider a simple example that an expert evaluates the operational complexity of three automatic systems, represented as $x_1,\ x_2$ and x_3. Since this criterion is qualitative, it is impossible to give crisp values but only linguistic terms. The operational complexity of these automatic systems can be taken as a linguistic variable. The linguistic term set for the operational complexity can be set up as:

$$S = \{s_{-3} = \textit{very complex}, s_{-2} = \textit{complex}, s_{-1} = \textit{a little complex}, s_0 = \textit{medium}, s_1 = \textit{a little easy}, s_2 = \textit{easy}, s_3 = \textit{very easy}\}$$

With the linguistic term set and also the context-free grammar, the expert determines his/her judgments over these three automatic systems with linguistic expressions, which are $ll_1 = \textit{at least a little easy}$, $ll_2 = \textit{between complex and medium}$ and $ll_3 = \textit{great than easy}$. These linguistic expressions are similar to the human way of thinking and they can reflect the expert's hesitant cognition intuitively. Using the transformation function E_{G_H}, a HFLTS can be yielded as $H_S(x) = \{\langle x_1, h_S(x_1)\rangle, \langle x_2, h_S(x_2)\rangle, \langle x_3, h_S(x_3)\rangle\}$ with $h_S(x_1) = \{s_1, s_2, s_3\}$, $h_S(x_2) = \{s_{-2}, s_{-1}, s_0\}$, and $h_S(x_3) = \{s_3\}$ being three HFLEs.

Example 2. Consider a simple example that a Chief Information Officer (CIO) of a company evaluates the candidate ERP system in terms of three criteria, i.e., x_1 (potential cost), x_2 (function), and x_3 (operation complexity). Since the three criteria are qualitative, the CIO gives his evaluation values in linguistic expressions. Different criteria are associated with different linguistic term sets and different semantics. The linguistic term sets for these three criteria are set up as:

$$S_1 = \{s_{-3} = \textit{veryexpensive}, s_{-2} = \textit{expensive}, s_{-1} = \textit{a little expensive}, s_0 = \textit{medium}, s_1 = \textit{a little cheap}, s_2 = \textit{cheap}, s_3 = \textit{very cheap}\},$$

$$S_2 = \{s_{-3} = \textit{none}, s_{-2} = \textit{verylow}, s_{-1} = \textit{low}, s_0 = \textit{medium}, s_1 = \textit{high}, s_2 = \textit{veryhigh}, s_3 = \textit{perfect}\},$$

$$S_3 = \{s_{-3} = \textit{too complex}, s_{-2} = \textit{complex}, s_{-1} = \textit{a little complex}, s_0 = \textit{medium}, s_1 = \textit{a little easy}, s_2 = \textit{easy}, s_3 = \textit{every easy}\},$$

respectively. With these linguistic term sets and also the context-free grammar, the CIO provides his evaluation values in linguistic expressions for a ERP system as: $ll_1 = \textit{between cheap and very cheap}$, $ll_2 = \textit{at least high}$, $ll_3 = \textit{great than easy}$. Using the transformation function E_{G_H}, a HFLTS is obtained as $H(x) = \{\langle x_1, h_{S_1}(x_1)\rangle, \langle x_2, h_{S_2}(x_2)\rangle, \langle x_3, h_{S_3}(x_3)\rangle\}$ with $h_{S_1}(x_1) = \{s_2, s_3 | s_2, s_3 \in S_1\}$, $h_{S_2}(x_2) = \{s_1, s_2, s_3 | s_1, s_2, s_3 \in S_2\}$ and $h_{S_3}(x_3) = \{s_3 | s_3 \in S_3\}$. Furthermore, if we ignore the influence of different semantics over different linguistic term sets on criteria, i.e., let $S = \{s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3\}$, then the HFLTS $H(x)$ can be rewritten as $H_S(x) = \{\langle x_1, h_S(x_1)\rangle, \langle x_2, h_S(x_2)\rangle, \langle x_3, h_S(x_3)\rangle\}$ with $h_S(x_1) = \{s_2, s_3\}$, $h_S(x_2) = \{s_1, s_2, s_3\}$ and $h_S(x_3) = \{s_3\}$.

Note: From Examples 1 and 2, we can find that X, in Definition 2, could be either a set of objects on a linguistic variable or a set of linguistic variables of an object (in this case, the influence of different semantics over different linguistic term sets on different linguistic variable should be ignored).

There are several special HFLEs, such as:

(1) empty HFLE: $h_S = \{\}$,
(2) full HFLE: $h_S = S$,
(3) the complement of HFLE $h_S : h_S^c = S - h_S = \{s_{\phi_l} | s_{\phi_l} \in S$ *and* $s_{\phi_l} \notin h_S\}$.

It should be noted that the linguistic terms in a HFLE might be out of order. For a HFLE $h_S = \{s_{\phi_l} | l = 1, 2, \ldots, L\}$, in order to simplify the computation, we can arrange the linguistic terms $s_l (l = 1, \ldots, L)$ in any of the following orders:

- Ascending order: $\delta : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is a permutation satisfying $\delta_l \leqslant \delta_{l+1},\ l = 1, \ldots, L$;
- Descending order: $\eta : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is a permutation satisfying $\eta_l \geqslant \eta_{l+1},\ l = 1, \ldots, L$;
- Any order: $\phi : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is any permutation of the values in h_S.

For simplicity of presentation, in the following of this paper, we assume that the linguistic terms in each HFLE are arranged in ascending order.

It is also noted that different HFLEs have different number of linguistic terms. Thus, in order to operate correctly when comparing or computing with HFLEs, we always extend the short HFLEs by adding some linguistic terms in it till they have same length (for more information, refer to Ref. [5,10]). In this paper, we extend the short HFLEs by adding the linguistic term $\bar{s} = \frac{1}{2}(s^+ \oplus s^-)$ where s^+ and s^- are the maximal term and minimal term in the HFLE h_S.

After giving the fundamental axioms for distance measures, Liao et al. [5] developed a family of distance measures for HFLTSs (for more details, please refer to Ref. [5]), such as the hesitant fuzzy linguistic Hamming distance:

$$d_1\left(h_S^1, h_S^2\right) = \frac{1}{L}\sum_{l=1}^{L}\frac{\left|\delta_l^1 - \delta_l^2\right|}{2\tau+1} \quad (3)$$

and the hesitant fuzzy linguistic Euclidean distance:

$$d_2\left(h_S^1, h_S^2\right) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\frac{\left|\delta_l^1 - \delta_l^2\right|}{2\tau+1}\right)^2\right)^{1/2} \quad (4)$$

where $h_S^1 = \left\{s_{\delta_l^1} | l = 1, \ldots, L_1\right\}$ and $h_S^2 = \left\{s_{\delta_l^2} | l = 1, \ldots, L_2\right\}$ are two HFLEs with $L_1 = L_2 = L$ (otherwise, the shorter one can be extended by adding some linguistic terms). The linguistic terms $s_{\delta_l^k}$ in $h_S^k (k = 1, 2)$ are arranged in ascending order.

2.3. Review on correlation measures over fuzzy sets and its extensions

The correlation measures have been investigated in-depth by many scholars within the context of fuzzy sets (FSs), intuitionistic fuzzy sets, and hesitant fuzzy sets (HFSs). Different forms and formulas for correlation measures have been defined corresponding to different type of fuzzy sets and its extensions.

As to traditional fuzzy sets, motivated by the correlation coefficient in statistics, Murthy et al. [15] introduced the correlation measure between two fuzzy membership functions and developed a formula to calculate the correlation measure between two fuzzy membership functions. Later, Chaudhuri and Bhattacharya [16] extended Murthy et al.'s correlation formula by taking into account the rank correlation measure. Also adopting the concepts from conventional statistics, Chiang and Lin [17] derived another formula of correlation coefficient on the domain of fuzzy sets. All these three kinds of correlation coefficients over fuzzy sets lie in the interval $[-1, 1]$ and have similar meaning as that in conventional statistics. On the other hand, Yu [18] introduced quite different concepts of correlation and correlation coefficient to measure the interrelation of fuzzy numbers. The value of correlation coefficient he introduced is within interval $[0, 1]$. It is stated that all the above works calculate the correlation coefficient of fuzzy sets as a crisp number. By using the sup-min convolution, Liu and Gao [19] proposed a mathematical programming approach to calculate the correlation coefficient as a fuzzy number. After that, by applying the T_w-based extension principle, Hong [20] gave an exact solution of a fuzzy correlation coefficient without relying on programming.

Regarding to IFSs, many different forms of correlation measures have also been investigated. These distinct measures can be divided roughly into two sorts: one is from the classical statistics point of view, while the other is from the informational energy point of view. Hung [21] proposed the correlation measure for IFSs from statistic point of view by considering the membership degree and non-membership degree as two separate fuzzy sets. After that, Mitchell [22] extended his formula by taking the hesitant degree of IFSs into account. Szmidt and Kacprzyk [23] also proposed an improved version of correlation measure, in which he interpreted the IFSs as the ensembles of ordinary membership function. As these correlation measures are motivated from traditional statistics, the correlation coefficients of IFSs they developed are within interval $[-1, 1]$. On the other side, motivated by the information energy of a fuzzy set, Gerstenkorn and Manko [24] developed a quite different form of correlation measure for IFSs, which has the following form:

$$C(A, B) = \sum_{i=1}^{N}\left[\mu_A(x_i) \cdot \mu_B(x_i) + \upsilon_A(x_i) \cdot \upsilon_B(x_i)\right] \quad (5)$$

where $A = (\mu_A(x_i), \upsilon_A(x_i))$ and $B = (\mu_B(x_i), \upsilon_B(x_i))$, $i = 1, 2, \ldots, N$ are two sets of IFSs. Further, Hong and Hwang [25] extended this type of correlation measure into possibility space in which the set $\{x_i\}$ is an infinite universe of discourse. Moreover, Huang and Wu [26] improved the correlation measure and introduced the so-called centroid-method-based correlation measure for IFSs. As these correlation measures cannot guarantee the correlation coefficients of any two IFSs equals to one if and only if these two IFSs are the same, Xu [27] proposed a new form of correlation measure and circumvented this shortcoming. It should be stated that all the correlation coefficients derived by these information-energy-based correlation measures proposed in [24–27] lie in unit interval $[0, 1]$.

Recently, Chen et al. [28] investigated the correlation measure for HFSs and proposed a formula to calculate the correlation between two HFSs:

$$C_{HFS}(A, B) = \sum_{i=1}^{N}\left(\frac{1}{l_i}\sum_{j=1}^{l_i} h_{A\sigma(j)}(x_i) \cdot h_{B\sigma(j)}(x_i)\right) \quad (6)$$

where $A = \{h_A(x_i)\}$ and $B = \{h_B(x_i)\}$, $i = 1, 2, \ldots, N$ are two sets of HFSs and $h_{A\sigma(j)}(x_i)$ is the $\sigma(j)$th value in A.

3. Correlation and correlation coefficient of hesitant fuzzy linguistic term sets

In this section, we shall investigate the correlation and correlation coefficients between two HFLTSs. After reviewing the correlation measures of FSs, IFSs and HFSs, it is addressed that the correlation measures are motivated from two aspects, i.e., the statistics viewpoint and the information energy aspect. The statistic-based correlation measures restrict the correlation coefficient within the interval $[-1, 1]$, while the information-energy-based correlation measures set the correlation coefficient within the unit interval $[0, 1]$. It is hard to determine which type of measure is better for FSs. However, as HFLTS is proposed to depict the hesitant linguistic information from an expert, it should be more efficient to define the corresponding correlation coefficient based on the information energy of a HFLTS. In addition, traditional statistic analysis is on the basis of investigating a large number of, or sometime infinite, samples. Nonetheless, as to linguistic analysis, it is more common that we come across a finite and relatively small set of HFLTSs. Hence, in this paper, we study the correlation measure of HFLTSs from the information energy point of view.

Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. Motivated by the information energy of FSs [18], IFSs [24] and HFSs [28], the information energy of HFLTS H_S can be defined:

Definition 3. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. For a HFLTS $H_S = \{\langle x_i, h_S(x_i)\rangle | x_i \in X\}$ with $h_S(x_i) = \{s_{\delta_l}(x_i) | s_{\delta_l}(x_i) \in S, l = 1, \ldots, L\}$, the information energy of H_S is defined as:

$$E(H_S) = \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l(x_i)}{2\tau+1}\right)^2\right) \quad (7)$$

where $\delta_l(x_i)$ is the index of the lth smallest linguistic term in HFLE $h_S(x_i)$, L_i is the number of linguistic terms in $h_S(x_i)$, and N is the cardinality of X.

Based on the information energy of HFLTS, the correlation between two HFLTSs H_S^1 and H_S^2 is introduced:

Definition 4. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. For two HFLTSs $H_S^1 = \{\langle x_i, h_S^1(x_i)\rangle | x_i \in X\}$ and $H_S^2 = \{\langle x_i, h_S^2(x_i)\rangle | x_i \in X\}$ with $h_S^k(x_i) = \left\{s_{\delta_l^k}(x_i) | s_{\delta_l^k}(x_i) \in S, l = 1, \ldots, L_i\right\}$, $k = 1, 2$, the correlation between H_S^1 and H_S^2 is defined as:

$$C(H_S^1, H_S^2) = \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{|\delta_l^1(x_i)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_i)|}{2\tau+1}\right)\right) \tag{8}$$

where L_i is the maximum number of linguistic terms in $h_S^1(x_i)$ and $h_S^2(x_i)$ (the shorter one should be extended till equal length), and N is the cardinality of X.

Obviously, the correlation between HFLTSs H_S^1 and H_S^2 satisfies the following properties:

Property 1.1. $C\left(H_S^1, H_S^1\right) = E\left(H_S^1\right)$.

Property 1.2. $C\left(H_S^1, H_S^2\right) = C\left(H_S^2, H_S^1\right)$.

Based on the definitions of information energy and correlation of HFLTSs, the so-called correlation coefficient between HFLTSs H_S^1 and H_S^2 is further introduced:

Definition 5. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. For two HFLTSs $H_S^1 = \left\{\langle x_i, h_S^1(x_i)\rangle | x_i \in X\right\}$ and $H_S^2 = \left\{\langle x_i, h_S^2(x_i)\rangle | x_i \in X\right\}$ with $h_S^k(x_i) = \left\{s_{\delta_l^k}(x_i) | s_{\delta_l^k}(x_i) \in S, l = 1, \ldots, L_i\right\}$, $k = 1, 2$, the correlation coefficient between H_S^1 and H_S^2 is defined as:

$$\rho_1\left(H_S^1, H_S^2\right) = \frac{C\left(H_S^1, H_S^2\right)}{\left(E\left(H_S^1\right)\cdot E\left(H_S^2\right)\right)^{1/2}} = \frac{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{|\delta_l^1(x_i)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_i)|}{2\tau+1}\right)\right)}{\left(\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\cdot\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right)^{1/2}} \tag{9}$$

where L_i is the maximum number of linguistic terms in $h_S^1(x_i)$ and $h_S^2(x_i)$ (the shorter one should be extended till equal length), and N is the cardinality of X.

Furthermore, (9) can be simplified as

$$\rho_1'\left(H_S^1, H_S^2\right) = \frac{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(|\delta_l^1(x_i)|\cdot|\delta_l^2(x_i)|\right)\right)}{\left(\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\delta_l^1(x_i)\right)^2\right)\cdot\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\delta_l^2(x_i)\right)^2\right)\right)^{1/2}} \tag{10}$$

Analogously, the following properties are obvious for the correlation coefficient between HFLTSs H_S^1 and H_S^2:

Property 2.1. $\rho_1\left(H_S^1, H_S^1\right) = 1, \ \forall H_S^1 \in \mathbb{H}_S$.

Property 2.2. *If* $H_S^1 = H_S^2$, *then* $\rho_1\left(H_S^1, H_S^2\right) = 1, \forall H_S^1, H_S^2 \in \mathbb{H}_S$.

Property 2.3. $\rho_1\left(H_S^1, H_S^2\right) = \rho_1\left(H_S^2, H_S^1\right), \forall H_S^1, H_S^2 \in \mathbb{H}_S$.

Property 2.1 implies that the correlation coefficient is reflexive. Property 2.2 means that the correlation between two identical HFLTSs is always equal to 1. Property 2.3 denotes that the correlation coefficient is symmetric. These three properties are obvious according to Definition 5.

Theorem 1. *For HFLTSs* H_S^1 *and* H_S^2, $0 \leqslant \rho_1\left(H_S^1, H_S^2\right) \leqslant 1$ *holds.*

Proof. For the first part $\rho_1\left(H_S^1, H_S^2\right) \geqslant 0$, it is evident since $C\left(H_S^1, H_S^2\right) \geqslant 0$ and $E\left(H_S^1\right), E\left(H_S^2\right) \geqslant 0$.

$$\begin{aligned}C\left(H_S^1, H_S^2\right) &= \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{|\delta_l^1(x_i)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_i)|}{2\tau+1}\right)\right)\\ &= \frac{1}{L_1}\sum_{l=1}^{L_1}\left(\frac{|\delta_l^1(x_1)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_1)|}{2\tau+1}\right) + \frac{1}{L_2}\sum_{l=1}^{L_2}\left(\frac{|\delta_l^1(x_2)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_2)|}{2\tau+1}\right)\\ &\quad + \cdots + \frac{1}{L_n}\sum_{l=1}^{L_n}\left(\frac{|\delta_l^1(x_n)|}{2\tau+1}\cdot\frac{|\delta_l^2(x_n)|}{2\tau+1}\right)\\ &= \sum_{l=1}^{L_1}\left(\frac{|\delta_l^1(x_1)|}{(2\tau+1)\cdot\sqrt{L_1}}\cdot\frac{|\delta_l^2(x_1)|}{(2\tau+1)\cdot\sqrt{L_1}}\right)\\ &\quad + \sum_{l=1}^{L_2}\left(\frac{|\delta_l^1(x_2)|}{(2\tau+1)\cdot\sqrt{L_2}}\cdot\frac{|\delta_l^2(x_2)|}{(2\tau+1)\cdot\sqrt{L_2}}\right) + \cdots\\ &\quad + \sum_{l=1}^{L_n}\left(\frac{|\delta_l^1(x_n)|}{(2\tau+1)\cdot\sqrt{L_n}}\cdot\frac{|\delta_l^2(x_n)|}{(2\tau+1)\cdot\sqrt{L_n}}\right)\end{aligned}$$

Using the Cauchy–Schwarz inequality: $(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \cdots + a_n^2)\cdot\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)$ where $a_i, b_i \in R$, $i = 1, 2, \ldots, N$, it follows that:

$$\begin{aligned}C^2\left(H_S^1, H_S^2\right) &\leqslant \left(\frac{1}{L_1}\sum_{l=1}^{L_1}\left(\frac{\delta_l^1(x_1)}{2\tau+1}\right)^2 + \frac{1}{L_2}\sum_{l=1}^{L_2}\left(\frac{\delta_l^1(x_2)}{2\tau+1}\right)^2 + \cdots\right.\\ &\quad \left. + \frac{1}{L_n}\sum_{l=1}^{L_n}\left(\frac{\delta_l^1(x_n)}{2\tau+1}\right)^2\right)\cdot\left(\frac{1}{L_1}\sum_{l=1}^{L_1}\left(\frac{\delta_l^2(x_1)}{2\tau+1}\right)^2 + \frac{1}{L_2}\sum_{l=1}^{L_2}\left(\frac{\delta_l^2(x_2)}{2\tau+1}\right)^2 + \cdots\right.\\ &\quad \left. + \frac{1}{L_n}\sum_{l=1}^{L_n}\left(\frac{\delta_l^2(x_n)}{2\tau+1}\right)^2\right) = \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\\ &\quad \times\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right) = E\left(H_S^1\right)\cdot E\left(H_S^2\right)\end{aligned}$$

Thus, $C\left(H_S^1, H_S^2\right) \leqslant \left(E\left(H_S^1\right)\cdot E\left(H_S^2\right)\right)^{1/2}$. Then, it follows $\rho_1\left(H_S^1, H_S^2\right) \leqslant 1$.

Therefore, $0 \leqslant \rho_1\left(H_S^1, H_S^2\right) \leqslant 1$, which completes the proof of Theorem 1. □.

Property 2.4. $\rho_1\left(H_S^1, H_S^1\right)$ *is the supremum of all* $\rho_1\left(H_S^1, H_S^2\right)$, *which in other words,* $\rho_1\left(H_S^1, H_S^1\right) \geqslant \rho_1\left(H_S^1, H_S^2\right), \ \forall H_S^1, H_S^2 \in \mathbb{H}_S$.

Theorem 1 shows that the correlation coefficient between HFLTSs lies in unit interval [0, 1]. Property 2.4 is easy to yield from Property 2.1 and Theorem 1. This property implies the correlation coefficient between a HFLTS and itself is always greater than or equal to the correlation coefficient between it and any other HFLTS defined in the same universe.

Example 3. Consider a simple example where a buyer assesses three candidate houses. There are two linguistic variables x and y where x denotes the estimated price of a house and y represents the quality of a house. The linguistic term sets for x and y are as follows:

$$\begin{aligned}&S_1 = \{s_{-3} = \textit{veryexpensive}, s_{-2} = \textit{expensive}, s_{-1} = \textit{a little expensive},\\ &s_0 = \textit{medium}, s_1 = \textit{a little cheap}, s_2 = \textit{cheap}, s_3 = \textit{very cheap}\},\\ &S_2 = \{s_{-3} = \textit{verybad}, s_{-2} = \textit{bad}, s_{-1} = \textit{a little bad}, s_0 = \textit{medium},\\ &s_1 = \textit{a little good}, s_2 = \textit{good}, s_3 = \textit{very good}\}\end{aligned}$$

Suppose that the buyer evaluates three houses over these two linguistic variables and determines his/her judgments in linguistic expressions, such as

$ll_{11} = \textit{at least a little}$ expensive,
$ll_{12} = \textit{between medium and cheap}$ and $ll_{13} = \textit{lower than cheap}$;
$ll_{21} = \textit{great than good}$,
$ll_{22} = \textit{between medium and a little bad}$ and $ll_{23} = \textit{very bad}$.

Using the transformation function E_{G_H}, two HLFTSs are furnished, such as:

$$H_S^1 = \{\langle x_1, \{s_{-3}, s_{-2}, s_{-1}\}\rangle, \langle x_2, \{s_0, s_1, s_2\}\rangle, \langle x_3, \{s_2, s_3\}\rangle\},$$
$$H_S^2 = \{\langle x_1, \{s_2, s_3\}\rangle, \langle x_2, \{s_{-1}, s_0\}\rangle, \langle x_3, \{s_{-3}\}\rangle\}$$

To calculate the correlation between these two HFLTSs, firstly we calculate the information energy of each HFLTS. According to (7), it follows that

$$E\left(H_S^1\right) = \sum_{i=1}^{3}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{7}\right)^2\right) = \frac{1}{3}\left(\left(\frac{-3}{7}\right)^2 + \left(\frac{-2}{7}\right)^2 + \left(\frac{-1}{7}\right)^2\right) + \frac{1}{3}\left(0^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{2}{7}\right)^2\right) + \frac{1}{2}\left(\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2\right) = 0.2619;$$
$$E\left(H_S^2\right) = \sum_{i=1}^{3}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{7}\right)^2\right) = \frac{1}{2}\left(\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2\right) + \frac{1}{2}\left(\left(\frac{-1}{7}\right)^2 + \left(\frac{0}{7}\right)^2\right) + \left(\frac{-3}{7}\right)^2 = 0.3265.$$

Extending the HFLEs in H_S^2 to the equal length of those in H_S^1 by adding the corresponding averaging linguistic terms, it follows that

$$H_S^2 = \{\langle x_1, \{s_2, s_{2.5}, s_3\}\rangle, \langle x_2, \{s_{-1}, s_{-0.5}, s_0\}\rangle, \langle x_3, \{s_{-3}, s_{-3}\}\rangle\}.$$

via (8), the correlation between H_S^1 and H_S^2 is yielded:

$$\begin{aligned} C\left(H_S^1, H_S^2\right) &= \sum_{i=1}^{3}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{7}\cdot\frac{\left|\delta_l^2(x_i)\right|}{7}\right)\right) \\ &= \frac{1}{3}\left(\frac{|-3|}{7}\cdot\frac{2}{7} + \frac{|-2|}{7}\cdot\frac{2.5}{7} + \frac{|-1|}{7}\cdot\frac{3}{7}\right) + \frac{1}{3}\left(\frac{0}{7}\cdot\frac{|-1|}{7} + \frac{1}{7}\cdot\frac{|-0.5|}{7} + \frac{2}{7}\cdot\frac{0}{7}\right) + \frac{1}{2}\left(\frac{2}{7}\cdot\frac{|-3|}{7} + \frac{3}{7}\cdot\frac{|-3|}{7}\right) \\ &= 0.2517. \end{aligned}$$

Thus, the correlation coefficient is derived by (9):

$$\rho_1\left(H_S^1, H_S^2\right) = \frac{C\left(H_S^1, H_S^2\right)}{\left(E\left(H_S^1\right)\cdot E\left(H_S^2\right)\right)^{1/2}} = \frac{0.2517}{\sqrt{0.2619\cdot 0.3265}} = 0.8607$$

That is to say, the estimated price x and the quality of the house y have a high correlation coefficient.

Note: In the above example, we calculate the correlation for the case which is more general than Definition 5. In Definition 5, the two HFLTSs $H_S^1 = \left\{\langle x_i, h_S^1(x_i)\rangle | x_i \in X\right\}$ and $H_S^2 = \left\{\langle x_i, h_S^2(x_i)\rangle | x_i \in X\right\}$ have the same linguistic term set $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$. But in the above example, the two HFLTSs have the different linguistic term sets S_1 and S_2. In the process of computing with words or expressions, without taking any confusion, we usually ignore the influence of different semantics over different linguistic term sets on different linguistic variable. This would make the theoretical analysis of qualitative decision making more flexible and can be appropriate to handle more general cases.

As an alternative to Definition 5, a new type of the correlation coefficient for HFLTSs can be introduced:

Definition 6. Let $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set, and H_S^1 and H_S^2 be two HFLTSs defined as above. Then, the correlation coefficient of H_S^1 and H_S^2 is defined as:

$$\rho_2\left(H_S^1, H_S^2\right) = \frac{C\left(H_S^1, H_S^2\right)}{\max\left\{E\left(H_S^1\right), E\left(H_S^2\right)\right\}} = \frac{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right)}{\max\left\{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right), \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right\}} \quad (11)$$

where L_i is the maximum number of linguistic terms in $h_S^1(x_i)$ and $h_S^2(x_i)$ (the shorter one should be extended till equal length), and N is the cardinality of X.

Theorem 2. *The correlation coefficient $\rho_2\left(H_S^1, H_S^2\right)$ between two HFLTSs H_S^1 and H_S^2 satisfies:*

(1) $\rho_2\left(H_S^1, H_S^1\right) = 1, \ \forall H_S^1 \in \mathbb{H}_S$;
(2) $\rho_2\left(H_S^1, H_S^2\right) = \rho_2\left(H_S^2, H_S^1\right), \ \forall H_S^1, H_S^2 \in \mathbb{H}_S$;
(3) *If* $H_S^1 = H_S^2$, *then* $\rho_2\left(H_S^1, H_S^2\right) = 1, \ \forall H_S^1, H_S^2 \in \mathbb{H}_S$;
(4) $0 \leqslant \rho_2\left(H_S^1, H_S^2\right) \leqslant 1$

Proof. The proof of (1) (2) (3) is obvious according to Definition 6.

(4) It is evidence that $\rho_2\left(H_S^1, H_S^2\right) \geqslant 0$.

According to the proof of Theorem 1, it follows that

$$C\left(H_S^1, H_S^2\right) = \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right) \leqslant \sqrt{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\cdot\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)}.$$

Thus,

$$\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right) \leqslant \max\left\{\sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right), \sum_{i=1}^{N}\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right\}$$

which implies $\rho_2\left(H_S^1, H_S^2\right) \leqslant 1$. This completes the proof of Theorem 2. □

Theorem 2 reveals that the correlation coefficient ρ_2 is reflexive and symmetric, and its value varies within unit interval [0,1].

Example 4 (*Continued with Example 3*). The correlation coefficient ρ_2 between those two HFLTSs H_S^1 and H_S^2 in Example 3 is

$$\rho_2\left(H_S^1, H_S^2\right) = \frac{C\left(H_S^1, H_S^2\right)}{\max\left\{E\left(H_S^1\right), E\left(H_S^2\right)\right\}} = \frac{0.2517}{\max\{0.2619, 0.3265\}} = 0.7709,$$

which is smaller than the correlation coefficient $\rho_1\left(H_S^1, H_S^2\right)$.

The following theorem establishes the relationship between correlation coefficient $\rho_1\left(H_S^1, H_S^2\right)$ and $\rho_2\left(H_S^1, H_S^2\right)$.

Theorem 3. *For HFLTSs H_S^1 and H_S^2, $\rho_2\left(H_S^1, H_S^2\right) \leqslant \rho_1\left(H_S^1, H_S^2\right)$.*

Proof. Since

$$\sqrt{\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\cdot\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)}$$
$$\leqslant\max\left\{\sqrt{\left(\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\right)^2},\sqrt{\left(\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right)^2}\right\}$$
$$=\max\left\{\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right),\sum_{i=1}^N\left(\frac{1}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right\},$$

i.e.,

$$\left(E\left(H_S^1\right)\cdot E\left(H_S^2\right)\right)^{1/2}\leqslant\max\left\{E\left(H_S^1\right),E\left(H_S^2\right)\right\}$$

and thus,

$$\rho_2\left(H_S^1,H_S^2\right)=\frac{C\left(H_S^1,H_S^2\right)}{\max\left\{E\left(H_S^1\right),E\left(H_S^2\right)\right\}}\leqslant\frac{C\left(H_S^1,H_S^2\right)}{\left(E\left(H_S^1\right)\cdot E\left(H_S^2\right)\right)^{1/2}}$$
$$=\rho_1\left(H_S^1,H_S^2\right),$$

which completes the proof of Theorem 3. □

Furthermore, inspired by the idea of Xu [27], an enhanced correlation coefficient of HFLTSs H_S^1 and H_S^2 is obtained:

Definition 7. Let $S=\{s_t|t=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ be a linguistic term set, and H_S^1 and H_S^2 be two HFLTSs defined as above. Then, the enhanced correlation coefficient of H_S^1 and H_S^2 is defined as:

$$\rho_3\left(H_S^1,H_S^2\right)=\frac{1}{N}\sum_{i=1}^N\frac{\Delta\delta_{\min}+\Delta\delta_{\max}}{\Delta\delta_i+\Delta\delta_{\max}} \tag{12}$$

where

$$\Delta\delta_i=\frac{1}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1} \tag{13}$$

$$\Delta\delta_{\min}=\min_i\left\{\frac{1}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1}\right\} \tag{14}$$

$$\Delta\delta_{\max}=\max_i\left\{\frac{1}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1}\right\} \tag{15}$$

L_i is the maximum number of linguistic terms in $h_S^1(x_i)$ and $h_S^2(x_i)$ (the shorter one should be extended till the equal length), and N is the cardinality of X.

Theorem 2 shows that the correlation coefficient ρ_3 is also reflexive and symmetric, and lies in unit interval $[0,1]$. In addition, such kind of correlation coefficient also has its unique property.

Theorem 4. *The correlation coefficient $\rho_3\left(H_S^1,H_S^2\right)$ between two HFLTSs H_S^1 and H_S^2 satisfies:*

(1) $\rho_3\left(H_S^1,H_S^1\right)=1,\ \forall H_S^1\in\mathbb{H}_S$;
(2) $\rho_3\left(H_S^1,H_S^2\right)=\rho_3\left(H_S^2,H_S^1\right),\ \forall H_S^1,H_S^2\in\mathbb{H}_S$;
(3) *If* $H_S^1=H_S^2$, *then* $\rho_3\left(H_S^1,H_S^2\right)=1,\ \forall H_S^1,H_S^2\in\mathbb{H}_S$;
(4) $0\leqslant\rho_3\left(H_S^1,H_S^2\right)\leqslant1$.

The proof is omitted.

Theorem 5. *If $\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|=p$, where p is a constant value, then $\rho_3\left(H_S^1,H_S^2\right)=1$.*

Proof. If $\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|=p$, then $\Delta\delta_i=\frac{p}{2\tau+1}=\Delta\delta_{\min}=\Delta\delta_{\max}$. Thus,

$$\rho_3\left(H_S^1,H_S^2\right)=\frac{1}{N}\sum_{i=1}^N\frac{\Delta\delta_{\min}+\Delta\delta_{\max}}{\Delta\delta_i+\Delta\delta_{\max}}=1.$$

This completes the proof. □

Example 5 (*Continued with Example 3*). To calculate the correlation coefficient ρ_3 between the HFLTSs H_S^1 and H_S^2 in Example 3, we have

$$\Delta\delta_1=\frac{1}{3}\left(\left(\frac{-3-2}{7}\right)^2+\left(\frac{-2-2.5}{7}\right)^2+\left(\frac{-1-3}{7}\right)^2\right)=0.4167,$$
$$\Delta\delta_2=\frac{1}{3}\left(\left(\frac{0-(-1)}{7}\right)^2+\left(\frac{1-(-0.5)}{7}\right)^2+\left(\frac{2-0}{7}\right)^2\right)=0.0493,$$
$$\Delta\delta_3=\frac{1}{2}\left(\left(\frac{2-(-3)}{7}\right)^2+\left(\frac{3-(-3)}{7}\right)^2\right)=0.6224,$$
$$\Delta\delta_{\min}=\Delta\delta_2=0.0493,\quad\Delta\delta_{\max}=\Delta\delta_3=0.6224.$$

According to Definition 7, we have

$$\rho_3\left(H_S^1,H_S^2\right)=\frac{1}{3}\left(\frac{0.0493+0.6224}{0.4167+0.6224}+\frac{0.0493+0.6224}{0.0493+0.6224}+\frac{0.0493+0.6224}{0.6224+0.6224}\right)$$
$$=0.7287,$$

which is smaller than the correlation coefficients $\rho_1\left(H_S^1,H_S^2\right)$ and $\rho_2\left(H_S^1,H_S^2\right)$.

4. Weighted correlation and correlation coefficient of hesitant fuzzy linguistic term sets

4.1. Weighted correlation and correlation coefficient for HFLTSs

In the above section, three different types of correlation coefficients for HFLTSs are introduced. Each of them can be used to measure the relationship between two HFLTSs. However, all these correlation coefficients do not consider the relative importance of each HFLE in the HFLTSs. In many cases, such as multiple criteria decision making, different criteria may have different weights. Thus, when calculating the correlation coefficient for HFLTSs, the weights of each HFLE should be taken into account. In this subsection, we would propose some weighted forms of correlation coefficients for HFLTSs.

Suppose that $w=(w_1,w_2,\ldots,w_N)^T$ is the weighting vector of $x_i(i=1,2,\ldots,N)$ with $w_i\in[0,1]$, $i=1,2,\ldots,N$ and $\sum_{i=1}^N w_i=1$. Then, we can further obtain a sort of weighted correlation and correlation coefficient for HFLTSs. Firstly, the weighted correlation of any two HFLTSs H_S^1 and H_S^2 is given as:

$$C_w\left(H_S^1,H_S^2\right)=\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right) \tag{16}$$

and the weighed information energy of the set H_S is defined as:

$$E_w(H_S)=\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l(x_i)}{2\tau+1}\right)^2\right) \tag{17}$$

It should be stated that all the properties of correlation measure for HFLTSs in normal cases still hold in the weighted forms.

Analogously, different forms of correlation coefficients can be extended into weighted forms respectively, shown as:

$$\rho_4\left(H_S^1,H_S^2\right)=\frac{C_w\left(H_S^1,H_S^2\right)}{\left(E_w\left(H_S^1\right)\cdot E_w\left(H_S^2\right)\right)^{1/2}}=\frac{\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right)}{\left(\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right)\cdot\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right)^{1/2}} \tag{18}$$

$$\rho_5\left(H_S^1,H_S^2\right)=\frac{C_w\left(H_S^1,H_S^2\right)}{\max\left\{E_w\left(H_S^1\right),E_w\left(H_S^2\right)\right\}}=\frac{\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\left|\delta_l^1(x_i)\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_i)\right|}{2\tau+1}\right)\right)}{\max\left\{\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^1(x_i)}{2\tau+1}\right)^2\right),\sum_{i=1}^{N}\left(\frac{w_i}{L_i}\sum_{l=1}^{L_i}\left(\frac{\delta_l^2(x_i)}{2\tau+1}\right)^2\right)\right\}} \tag{19}$$

$$\rho_6\left(H_S^1,H_S^2\right)=\frac{1}{N}\sum_{i=1}^{N}\frac{\Delta\delta'_{\min}+\Delta\delta'_{\max}}{\Delta\delta'_i+\Delta\delta'_{\max}} \tag{20}$$

where

$$\Delta\delta'_i=\frac{w_i}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1}, \tag{21}$$

$$\Delta\delta'_{\min}=\min_i\left\{\frac{w_i}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1}\right\}, \tag{22}$$

$$\Delta\delta'_{\max}=\max_i\left\{\frac{w_i}{L_i}\sum_{l=1}^{L_i}\frac{\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|}{2\tau+1}\right\}. \tag{23}$$

It is obvious that when $w=(1/N,1/N,\ldots,1/N)^T$, then these weighted correlation and correlation coefficients reduce to the normal cases respectively. In addition, all of these weighted correlation coefficients satisfy the properties shown in the following Theorems:

Theorem 6. *The weighted correlation coefficients* $\rho_q\left(H_S^1,H_S^2\right)$, $q=4,5,6$ *between two HFLTSs* H_S^1 *and* H_S^2 *satisfy:*

(1) $\rho_q\left(H_S^1,H_S^1\right)=1,\ \forall H_S^1\in\mathbb{H}_S,\ q=4,5,6;$
(2) $\rho_q\left(H_S^1,H_S^2\right)=\rho_q\left(H_S^2,H_S^1\right),\ \forall H_S^1,H_S^2\in\mathbb{H}_S,\ q=4,5,6;$
(3) *If* $H_S^1=H_S^2$, *then* $\rho_q\left(H_S^1,H_S^2\right)=1,\ \forall H_S^1,H_S^2\in\mathbb{H}_S,\ q=4,5,6;$
(4) $0\leqslant\rho_q\left(H_S^1,H_S^2\right)\leqslant 1,\ q=4,5,6.$

Theorem 7. *For HFLTSs* H_S^1 *and* H_S^2, $\rho_5\left(H_S^1,H_S^2\right)\leqslant\rho_4\left(H_S^1,H_S^2\right)$.

Theorem 8. *If* $\left|\delta_l^1(x_i)-\delta_l^2(x_i)\right|=p$, *where p is a constant value, then* $\rho_6\left(H_S^1,H_S^2\right)=1$.

The proof of Theorems 6–8 is similar to that of Theorems 2, 3 and 5, respectively.

4.2. Ordered weighted correlation and correlation coefficient for HFLTSs

The ordered weighted averaging (OWA) operator was proposed by Yager [35] and the idea has been applied into many different research domains, such as fuzzy aggregation operators [36,37] and distance measures [5]. The prominent characteristic of the OWA operator is the reordering step in which the input data are arranged in descending order. It weights the ordered positions of the data rather than weights the input data themselves. The idea of the OWA operators also can be used for us to develop the correlation coefficient for HFLTSs. Inspired by this, the ordered weighted correlation of any two HFLTSs H_S^1 and H_S^2 can be introduced:

$$C_{ow}\left(H_S^1,H_S^2\right)=\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\left|\delta_l^1(x_{\xi(i)})\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right)\right) \tag{24}$$

where $\xi(1),\xi(1),\ldots,\xi(N)$ is any permutation of $(i=1,2,\ldots,N)$, such that

$$\frac{1}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\left|\delta_l^1(x_{\xi(i)})\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right)\geqslant\frac{1}{L_{\xi(i+1)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\left|\delta_l^1(x_{\xi(i+1)})\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_{\xi(i+1)})\right|}{2\tau+1}\right) \tag{25}$$

and $w=(w_1,w_2,\ldots,w_N)^T$ is the weighting vector of the ordered positions of elements $x_i(i=1,2,\ldots,N)$ with $w_i\in[0,1]$, $i=1,2,\ldots,N$ and $\sum_{i=1}^{N}w_i=1$.

Similarly, the ordered weighed information energy of set H_S is defined as:

$$E_{ow}(H_S)=\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\delta_l(x_{\xi(i)})}{2\tau+1}\right)^2\right) \tag{26}$$

In addition, the different forms of the ordered weighted correlation coefficients can be developed as follows:

$$\rho_7\left(H_S^1,H_S^2\right)=\frac{C'\left(H_S^1,H_S^2\right)}{\left(E'\left(H_S^1\right)\cdot E'\left(H_S^2\right)\right)^{1/2}}=\frac{\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\left|\delta_l^1(x_{\xi(i)})\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right)\right)}{\left(\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\delta_l^1(x_{\xi(i)})}{2\tau+1}\right)^2\right)\cdot\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\delta_l^2(x_{\xi(i)})}{2\tau+1}\right)^2\right)\right)^{1/2}} \tag{27}$$

$$\rho_8\left(H_S^1,H_S^2\right)=\frac{C'\left(H_S^1,H_S^2\right)}{\max\left\{E'\left(H_S^1\right),E'(H_S^2)\right\}}=\frac{\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\left|\delta_l^1(x_{\xi(i)})\right|}{2\tau+1}\cdot\frac{\left|\delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right)\right)}{\max\left\{\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\delta_l^1(x_{\xi(i)})}{2\tau+1}\right)^2\right),\sum_{i=1}^{N}\left(\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\left(\frac{\delta_l^2(x_{\xi(i)})}{2\tau+1}\right)^2\right)\right\}} \tag{28}$$

$$\rho_9\left(H_S^1,H_S^2\right)=\frac{1}{N}\sum_{i=1}^{N}\frac{\Delta\delta'_{\min}+\Delta\delta'_{\max}}{\Delta\delta'_i+\Delta\delta'_{\max}} \tag{29}$$

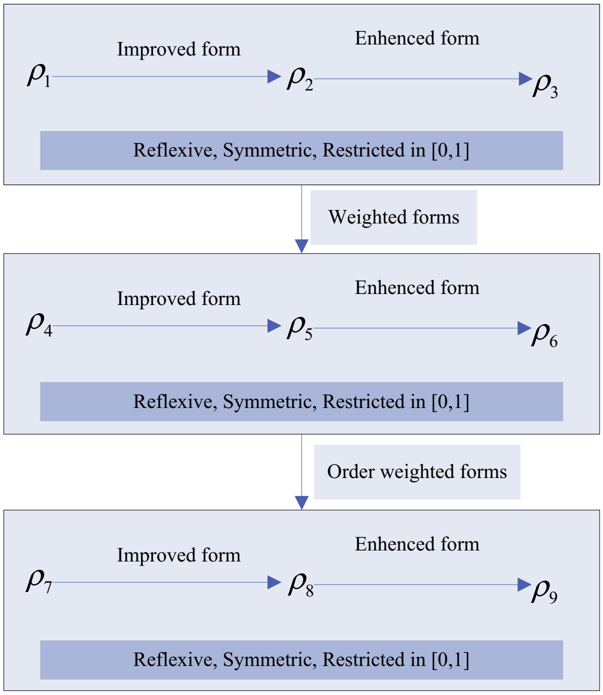

Fig. 2. Classification and relationship among different correlation coefficients for HFLTSs.

where

$$\Delta\delta'_i = \frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\frac{\left|\delta_l^1(x_{\xi(i)}) - \delta_l^2(x_{\xi(i)})\right|}{2\tau+1}, \tag{30}$$

$$\Delta\delta'_{\min} = \min_i\left\{\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\frac{\left|\delta_l^1(x_{\xi(i)}) - \delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right\}, \tag{31}$$

$$\Delta\delta'_{\max} = \max_i\left\{\frac{w_i}{L_{\xi(i)}}\sum_{l=1}^{L_{\xi(i)}}\frac{\left|\delta_l^1(x_{\xi(i)}) - \delta_l^2(x_{\xi(i)})\right|}{2\tau+1}\right\}. \tag{32}$$

In case of $w = (1/N, 1/N, \ldots, 1/N)^T$, then these weighted correlation and correlation coefficients reduce to the normal cases respectively.

The ordered weighted correlation coefficients also satisfy the following properties:

Theorem 9. *The ordered weighted correlation coefficients $\rho_q\left(H_S^1, H_S^2\right)$, $q = 7, 8, 9$ between two HFLTSs H_S^1 and H_S^2 satisfy:*

(1) $\rho_q\left(H_S^1, H_S^1\right) = 1,\ \forall H_S^1 \in \mathbb{H}_S,\ q = 7,8,9;$
(2) $\rho_q\left(H_S^1, H_S^2\right) = \rho_q\left(H_S^2, H_S^1\right),\ \forall H_S^1, H_S^2 \in \mathbb{H}_S,\ q = 7,8,9;$
(3) *If* $H_S^1 = H_S^2$, *then* $\rho_q\left(H_S^1, H_S^2\right) = 1,\ \forall H_S^1, H_S^2 \in \mathbb{H}_S,\ q = 7,8,9;$
(4) $0 \leqslant \rho_q\left(H_S^1, H_S^2\right) \leqslant 1$, $q = 7, 8, 9$.

Theorem 10. *For HFLTSs H_S^1 and H_S^2, $\rho_8\left(H_S^1, H_S^2\right) \leqslant \rho_7\left(H_S^1, H_S^2\right)$.*

Theorem 11. *If $\left|\delta_l^1(x_i) - \delta_l^2(x_i)\right| = p$, where p is a constant value, then $\rho_9\left(H_S^1, H_S^2\right) = 1$.*

The proof of Theorems 9–11 is similar to that of Theorems 2, 3 and 5, respectively.

The relationship among these different correlation coefficients between HFLTSs proposed above can be illustrated as Fig. 2.

5. Numerical example for decision making with correlation coefficient of hesitant fuzzy linguistic term sets

As linguistic expressions are much closer to human thinking than single linguistic term, the HFLTS turns out to be very efficient in representing hesitant qualitative information from experts. The correlation coefficient is proposed to measure the relationship between two data sets and the correlation coefficients of FSs, IFSs, IVIFSs and HFSs have already been applied widely in many domains. However, they cannot be used to handle HFLTSs. The correlation coefficients proposed in this paper can fill this gap well. These correlation coefficients can be implemented for many practical applications. In the following, a numerical example concerning the traditional Chinese medical diagnosis is given to illustrate the applicability and validation of the proposed correlation coefficients and the difference between them.

Example 6. In traditional Chinese medical diagnosis, a doctor always gets some imprecise information about a patient's symptoms, such as temperature, headache, cough, stomach pain, and so forth, through seeing, smelling, asking and touching. Suppose that a doctor wants to make a proper diagnosis D = {Viral fever, Typhoid, Pneumonia, Stomach problem} for a patient with the values of symptoms V = {temperature, headache, cough, stomach pain}. It is straightforward to represent those values of the symptoms in terms of linguistic expressions since traditional Chinese medical diagnosis cannot get crisp values. It should be stated that the power of Chinese medical diagnosis compared to west medical diagnosis is obtained partly due to the vagueness measurement of these symptoms. In Chinese medical diagnosis, there is no need to get very accurate values. Before starting the diagnosis, a medical knowledge-based data set involving symptom characteristic of the considered diagnoses is necessary to be constructed. Such a knowledge-based data set is described in terms of linguistic terms or linguistic expressions (see Table 1). Each symptom can be seen as a linguistic variable, whose corresponding linguistic term set are as follows:

$S_1 = \{s_{-3} = verylow, s_{-2} = low, s_{-1} = a\ little\ low, s_0 = normal,$
$s_1 = a\ little\ high, s_2 = high, s_3 = very\ high\},$

$S_2 = \{s_{-3} = none, s_{-2} = very\ slight, s_{-1} = slight,$
$s_0 = a\ little\ terrible, s_1 = terrible, s_2 = veryterrible,$
$s_3 = insufferable\},$

Table 1
Symptoms characteristic for the considered diagnosis in terms of linguistic expressions.

	Temperature (S_1)	Headache (S_2)	Cough (S_3)	Stomach pain (S_4)
Viral fever	*at least a little high*	*between a little terrible and very terrible*	*at least serious*	*none*
Typhoid	*greater than high*	*greater than terrible*	*at least serious*	*lower than very slight*
Pneumonia	*between normal and a little high*	*between slight and a little terrible*	*greater than very serious*	*none*
Stomach problem	*normal*	*none*	*none*	*greater than terrible*

Table 2
Symptoms characteristic for the considered diagnosis in terms of HFLTSs.

	Temperature (S_1)	Headache (S_2)	Cough (S_3)	Stomach pain (S_4)
Viral fever	$\{s_1, s_2, s_3\}$	$\{s_0, s_1, s_2\}$	$\{s_1, s_2, s_3\}$	$\{s_{-3}\}$
Typhoid	$\{s_2, s_3\}$	$\{s_1, s_2, s_3\}$	$\{s_1, s_2, s_3\}$	$\{s_{-3}, s_{-2}\}$
Pneumonia	$\{s_0, s_1\}$	$\{s_{-1}, s_0\}$	$\{s_2, s_3\}$	$\{s_{-3}\}$
Stomach problem	$\{s_0\}$	$\{s_{-3}\}$	$\{s_{-3}\}$	$\{s_1, s_2, s_3\}$

Table 3
Symptoms characteristic for the considered patients established by the Chinese doctor.

	Temperature (S_1)	Headache (S_2)	Cough (S_3)	Stomach pain (S_4)
Richard	*high*	*very terrible*	*between serious and very serious*	*none*
Catherine	*normal*	*none*	*none*	*between terrible and very terrible*
Nicle	*very high*	*terrible*	*very serious*	*none*
Kevin	*a little high*	*between slight and a little terrible*	*very serious*	*none*

Table 4
Symptoms characteristic for the considered patients in terms of HFLTSs.

	Temperature (S_1)	Headache (S_2)	Cough (S_3)	Stomach pain (S_4)
Richard	$\{s_2\}$	$\{s_2\}$	$\{s_1, s_2\}$	$\{s_{-3}\}$
Catherine	$\{s_0\}$	$\{s_{-3}\}$	$\{s_{-3}\}$	$\{s_1, s_2\}$
Nicle	$\{s_3\}$	$\{s_1\}$	$\{s_2\}$	$\{s_{-3}\}$
Kevin	$\{s_1\}$	$\{s_{-1}, s_0\}$	$\{s_2\}$	$\{s_{-3}\}$

Table 5
Correlation coefficient values of ρ_1 for each considered patient to the set of possible diagnosis.

	Viral fever	Typhoid	Pneumonia	Stomach problem
Richard	0.9281	**0.9504**	0.8083	0.7847
Catherine	0.7571	0.6666	0.7340	**0.9897**
Nicle	**0.9325**	0.9266	0.8213	0.6569
Kevin	0.8905	0.8150	**0.9698**	0.7446

Note: The bold values represent the highest correlation value for each patient with respect to different diagnosis.

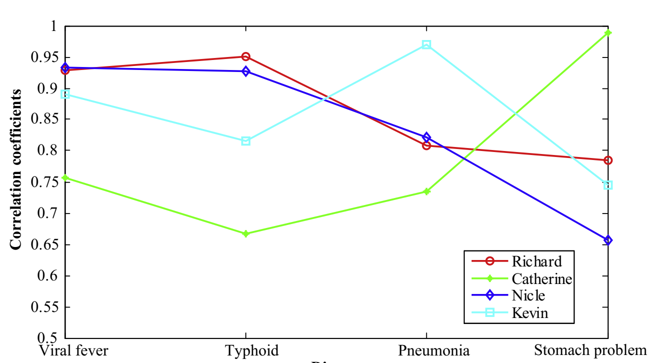

Fig. 3. Correlation coefficient values by using ρ_1.

Table 6
Correlation coefficient values of ρ_2 for each considered patient to the set of possible diagnosis.

	Viral fever	Typhoid	Pneumonia	Stomach problem
Richard	**0.9165**	0.8881	0.7345	0.7278
Catherine	0.7478	0.6044	0.6585	**0.9412**
Nicle	0.8696	**0.9131**	0.6956	0.6521
Kevin	0.7582	0.6566	**0.9091**	0.5955

Note: The bold values represent the highest correlation value for each patient with respect to different diagnosis.

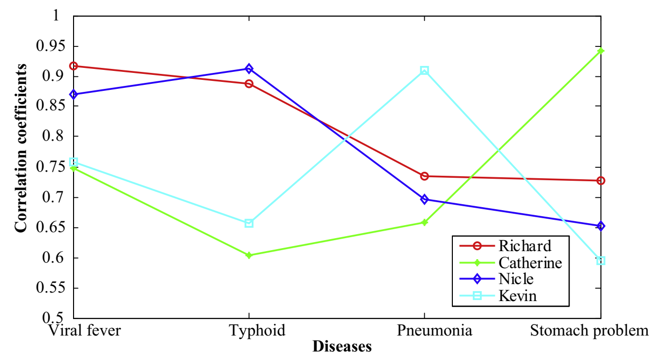

Fig. 4. Correlation coefficient values by using ρ_2.

$$S_3 = \{s_{-3} = none, s_{-2} = very\ slight, s_{-1} = slight, s_0 = a\ little\ serious, s_1 = serious, s_2 = veryserious, s_3 = insufferable\},$$

$$S_4 = \{s_{-3} = none, s_{-2} = veryslight, s_{-1} = slight, s_0 = a\ little\ terrible, s_1 = terrible, s_2 = veryterrible, s_3 = insufferable\}.$$

The linguistic expressions shown in Table 1 are similar to the human way of thinking and they can reflect the Chinese doctor's imprecise cognition against the patient's symptoms intuitively. Using the transformation function and ignoring the affection of different semantics of these linguistic term sets on different symptom characteristics, we can generate the following knowledge-based data set in terms of HFLTSs (see Table 2):

Suppose that there are four patients P = {Richard, Catherine, Nicle, Kevin}, whose symptoms are obtained by a Chinese doctor through seeing, smelling, asking and touching and are represented by the linguistic expressions shown in Table 3.

These linguistic expressions in Table 3 can be transformed into HFLTSs or linguistic terms according to the transformation function, and thus we obtain Table 4.

In order to diagnosis the disease for these four patients, we can calculate the correlation coefficient between the data set of each patient's symptoms and that of the diagnoses.

Firstly, according to Eq. (7), the information energy of each diagnosis is calculated as:

$$E(Viral\ fever) = 0.4082; \quad E(Typhoid) = 0.4558; \quad E(Pneumonia) = 0.3367; \quad E(Stomach\ problem) = 0.4626;$$

and the information energy of each patient are

$$E(Richard) = 0.3980; \quad E(Catherine) = 0.4184; \quad E(Nicle) = 0.4694; \quad E(Kevin) = 0.2959.$$

If we use the correlation coefficient ρ_1 to derive the relationship between each patient and the diseases, the resulting correlation coefficient values are listed in Table 5 and shown as Fig. 4.

The principle of the diagnosis is: the lager the value of correlation coefficient, the higher possibility of the diagnosis to the patient. From Table 5 and Fig. 3, we can find that Richard is suffering from typhoid; Catherine is suffering from stomach problem; Nicle is suffering from viral fever; while Kevin is suffering from pneumonia. According to the correlation coefficient values shown in Table 5 and Fig. 3, we can also find an interesting phenomenon, i.e., the correlation coefficient values of viral fever, typhoid and pneumonia are much closer than that of stomach problem. This is in accordance with our intuition because viral fever, typhoid and pneumonia are three diseases having very similar symptoms while the stomach problem is quite different from the former three diseases.

On the other hand, we can also use the other correlation coefficient formulas to derive the diagnosis for each patient. For example, if we use the correlation coefficient ρ_2 to derive the relationship between each patient and the diseases, the result of correlation coefficient values are listed in Table 6 and shown as Fig. 4.

Table 6 and Fig. 3 show that Richard is suffering from viral fever; Catherine is suffering from stomach problem; Nicle is suffering from typhoid and Kevin is suffering from pneumonia. It should be noted that the diagnosis result is a little different from that of Table 5 and Fig. 3. It results from the fact that different correlation coefficients are based on different linear relationships, and thus produce different results. In addition, this does also satisfy the actual feature of traditional Chinese medical diagnosis since the traditional Chinese medical diagnosis always cannot distinguish very similar diseases such as viral fever and typhoid.

Comparing Table 5 with Table 6, we can also find that all the values in Table 6 are small than those in Table 5. This just verifies Theorem 3.

6. Conclusions

Linguistic expressions are very close to human way of thinking. HFLTS is an efficient tool to represent the linguistic expressions given by experts. With the help of linguistic expressions and HFLTS, the theoretical investigation of common sense reasoning could be promoted forward. Considering that HFLTS was proposed for just a few years, in this paper, we have paid our attention to the basic characteristics of HFLTS and investigated several different forms of correlation measures and correlation coefficients for HFLTSs. The definition of HFLTS has been improved and the HFLE has been introduced. A number of different correlation measures and correlation coefficients for HFLTSs have been introduced. We have also made some in-depth study on the properties of these correlation coefficients. Furthermore, several weighted correlation coefficients and ordered weighted correlation coefficients for HFLTSs have been proposed and analyzed. An application example concerning the traditional Chinese medical diagnosis has been given to illustrate the applicability and validation of these correlation coefficients of HFLTSs.

In the future, we will apply these correlation coefficients to clustering analysis over hesitant fuzzy linguistic information. We will also implement these correlation coefficients to group decision making with hesitant qualitative information. As the OWA operator has many different forms of extensions [33], many different forms of correlation coefficients for HFLTSs, such as the induced ordered weighted correlation coefficient, the generalized ordered weighted correlation coefficient and other generalized forms of correlation coefficients for HFLTSs can be further investigated.

Acknowledgments

The authors would like to thank the editors and the anonymous reviewers for their insightful and constructive comments and suggestions that have led to this improved version of the paper. The work was supported in part by the National Natural Science Foundation of China (No. 61273209), the Excellent Ph.D. Thesis Foundation of Shanghai Jiao Tong University (No. 20131216), and the Scholarship from China Scholarship Council (No. 201306230047).

References

[1] M. Negnevitsky, Artificial Intelligence: A Guide to Intelligent Systems, third ed., Pearson Education, Edinburgh, 2011.
[2] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning-Part I, Inf. Sci. 8 (1975) 199–249.
[3] R.M. Rodríguez, L. Martínez, An analysis of symbolic linguistic computing models in decision making, Int. J. Gen. Syst. 42 (2013) 121–136.
[4] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic terms sets for decision making, IEEE Trans. Fuzzy Syst. 20 (2012) 109–119.
[5] H.C. Liao, Z.S. Xu, X.J. Zeng, Distance and similarity measures for hesitant fuzzy linguistic term sets and their application in multi-criteria decision making, Inf. Sci. 271 (2014) 125–142.
[6] C.P. Wei, N. Zhao, X.J. Tang, Operators and comparisons of hesitant fuzzy linguistic term sets, IEEE Trans. Fuzzy Syst. 22 (3) (2014) 575–585.
[7] S.M. Chen, J.A. Hong, Multicriteria linguistic decision making based on hesitant fuzzy linguistic term sets and the aggregation of fuzzy sets, Inf. Sci. 286 (2014) 63–74.
[8] H.B. Liu, R.M. Rodríguez, A fuzzy envelope of hesitant fuzzy linguistic term set and its application to multicriteria decision making, Inf. Sci. 258 (2014) 220–238.
[9] R.M. Rodríguez, H.B. Liu, L. Martínez, A fuzzy representation for the semantics of hesitant fuzzy linguistic term sets, in: Foundations of Intelligent Systems, Advances in Intelligent Systems and Computing, vol. 277, 2014, pp. 745–757.
[10] B. Zhu, Z.S. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, IEEE Trans. Fuzzy Syst. 22 (1) (2014) 35–45.
[11] H.B. Liu, J.F. Cai, L. Jiang, On improving the additive consistency of the fuzzy preference relations based on comparative linguistic expressions, Int. J. Intell. Syst. 29 (2014) 544–559.
[12] R.M. Rodríguez, L. Martínez, F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term set, Inf. Sci. 241 (2013) 28–42.
[13] I. Beg, T. Rashid, TOPSIS for hesitant fuzzy linguistic term sets, Int. J. Intell. Syst. 28 (2013) 1162–1171.
[14] H.C. Liao, Z.S. Xu, X.J. Zeng, Hesitant fuzzy linguistic VIKOR method and its application in qualitative multiple criteria decision making, IEEE Trans. Fuzzy Syst. (2014) (in press). http://dx.doi.org/10.1109/TFUZZ.2014.2360556.
[15] C.A. Murithy, S.K. Pal, D. Dutta-Majumder, Correlation between two fuzzy membership functions, Fuzzy Sets Syst. 17 (1985) 23–38.
[16] B.B. Chaudhuri, A. Bhattacharya, On correlation between two fuzzy sets, Fuzzy Sets Syst. 118 (2001) 447–456.
[17] D.A. Chiang, N.P. Lin, Correlation of fuzzy sets, Fuzzy Sets Syst. 102 (1999) 221–226.
[18] C.H. Yu, Correlation of fuzzy numbers, Fuzzy Sets Syst. 55 (1993) 303–307.
[19] S.T. Liu, C. Gao, Fuzzy measures for correlation coefficient of fuzzy numbers, Fuzzy Sets Syst. 128 (2002) 267–275.
[20] D.H. Hong, Fuzzy measures for a correlation coefficient of fuzzy numbers under T_w (the weakest t-norm)-based fuzzy arithmetic operations, Inf. Sci. 176 (2006) 150–160.
[21] W.L. Hung, Using statistical viewpoint in developing correlation of intuitionistic fuzzy sets, Int. J. Uncertain. Fuzz. 9 (2001) 509–516.
[22] H.B. Mitchell, A correlation coefficient for intuitionistic fuzzy sets, Int. J. Intell. Syst. 19 (2004) 483–490.
[23] E. Szmidt, J. Kacprzyk, Correlation of intuitionistic fuzzy sets, In: E. Hüllermeier, R. Kruse, F. Hoffmann (Eds.), IPMU 2010, LNAI, vol. 6178, 2010, pp. 169–177.
[24] T. Gerstenkorn, J. Manko, Correlation of intuitionistic fuzzy sets, Fuzzy Sets Syst. 44 (1991) 39–43.
[25] D.H. Hong, S.Y. Hwang, Correlation of intuitionistic fuzzy sets in probability spaces, Fuzzy Sets Syst. 75 (1995) 77–81.
[26] W.L. Hung, J.W. Wu, Correlation of intuitionistic fuzzy sets by centroid method, Inf. Sci. 144 (2002) 219–225.
[27] Z.S. Xu, On correlation measures of intuitionistic fuzzy sets, Lect. Notes Comput. Sci. 4224 (2006) 16–24.
[28] N. Chen, Z.S. Xu, M.M. Xia, Correlation coefficients of hesitant fuzzy sets and their application to clustering analysis, Appl. Math. Model. 37 (2013) 2197–2211.
[29] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004) 19–30.
[30] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (6) (2000) 746–752.
[31] J. Wang, J. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 14 (3) (2006) 435–445.
[32] Z.S. Xu, Deviation measures of linguistic preference relations in group decision making, Omega-Int. J. Manage. 33 (2005) 249–254.
[33] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (2010) 529–539.

[34] R.M. Rodriguez, L. Martinez, V. Torra, Z.S. Xu, F. Herrera, Hesitant fuzzy sets: state of the art and future directions, Int. J. Intell. Syst. 29 (2014) 495–524.
[35] R.R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1988) 183–190.
[36] H.C. Liao, Z.S. Xu, Intuitionistic fuzzy hybrid weighted aggregation operators, Int. J. Intell. Syst. 29 (2014) 971–993.
[37] H.C. Liao, Z.S. Xu, Some new hybrid weighted aggregation operators under hesitant fuzzy multi-criteria decision making environment, J. Intell. Fuzzy Syst. 26 (2014) 1601–1617.