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Direct determination of stresses from the stress equations of motion and wave propagation for a new class of elastic bodies

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Abstract
For a new class of elastic bodies, where the linearized strain tensor is given as a function of the Cauchy stress tensor, the problem of considering unsteady motions is studied. A system of partial differential equations that only depends on the stress tensor is found from the equation of motion, which is a system of six partial differential equations for the six components of the stress tensor. A simple boundary value problem is solved for a 1D bar using exact and numerical methods.

Keywords
Equation of motion, unsteady motions, strain limiting behaviour, finite element method

1. Introduction

In the recent years Rajagopal and his coworkers have studied new classes of elastic bodies [1–8], which cannot be classified as either Green or Cauchy elastic bodies [9]. One new class corresponds to a body where the norm of the gradient of the displacement field is small, as a consequence the strains are small and are given in general as nonlinear functions of the Cauchy stress tensor [6–8, 10]. For this class of constitutive relations, different boundary value problems have been solved considering steady motions [11–14].

This new class of elastic bodies could be used in order to study some problems such as the behaviour of brittle bodies containing cracks, where near the tips of the cracks we have that the stresses can be large but strains remain small [6, 10–13]. Other possible future applications of these new constitutive relations could be found in the modelling of some metal alloys, for which a nonlinear behaviour for the relation between stresses and strains can be observed for strains up to an order of 2.5% (see, for example, [15–17]). In Saito et al., [15] for the metal alloys Ti–12Ta–9Nb–3V–6Zr–O and 23Nb–0.7Ta–2Zr–O under a cold swaging process, a clear nonlinear behaviour can be observed in Figure 1 therein, where for that range of strains (up to an order of 2.5%) the material behaves approximately in an elastic manner. Similar results are presented in Figure 1 of Withey et al. [16] and for a titanium alloy in Figure 2b of Zhang et al. [17].[1]

Dedicated to Kumbakonam R Rajagopal.

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Another possible field of application of the constitutive relations addressed in the present communication lies in the modelling of the mechanical behaviour of rock, concrete and ceramic materials. In the case of rock, although such material can show some small inelastic and hysteresis phenomena even for small external loads, if in a first approximation such effects are neglected, we can find plenty of experimental data in the literature, which suggests that for the elastic range the material behaves in a nonlinear manner (see, for example, [18–23] and Chapter 1 of [24]). In particular see Figures 4, 5, 11 and 12 of Johnson and Rasolfosaon [18] for Lavoux limestone. In Figure 5 we see some experimental results for the wave speed versus the axial applied stress on a sample of rock (under compression), while in Figure 7 results are presented for the 'shear' modulus also versus the axial stress applied on the sample. In Figures 11 and 12 we can see the comparison of the acceleration waves as a function of the frequency for a material that can be considered to behave in a linear way, and a class of rock that behaves in a nonlinear way in Figure 12. Other interesting experimental results for the nonlinear mechanical behaviour of rock are presented in Figures 3 and 4 of McCall and Guyer [20] and Figures 2(a), 2(b) and 3 of Guyer and Johnson [19]. In Ostrovsky [22], a theoretical treatment is presented in order to study such nonlinear phenomena in rocks, which is based on the assumption that stresses are given as nonlinear functions of the linearized strains (see equations (4)–(6) and Figure 1 therein). These are some of the classes of problems that is necessary to study using the theories developed by Rajagapal and his collaborators (see [25]).

Regarding the mechanical behaviour of concrete, see, for example, Figures 9–11 of [26], where the experimental results presented also suggest that such materials show a nonlinear behaviour for small strains, in the case we assume they behave approximately as elastic bodies.

In the present work we study the problem where the displacement field (and as a result the strain field) and the stress tensor can change in time. Starting from the equations of motion and the new constitutive relations presented in the works by Rajagopal and Bustamante [6–8, 10], a new system of equations in terms of the stresses is found, which is a generalization for nonlinear bodies of equations presented in the works by Iacovache [27], Vâlcovici [28] and Ignaczak [29, 30] for linearized elastic bodies.

Some solutions are presented for the case of unsteady motion of a 1D bar. Solutions are obtained exactly and using the finite element method.

2. Basic equations

2.1. Kinematics

Let $\mathbf{X} \in \kappa_R(\mathcal{B})$ denote a particle belonging to a body $\mathcal{B}$ in the reference configuration $\kappa_R(\mathcal{B})$, and let $\mathbf{x} \in \kappa_t(\mathcal{B})$ denote the position of the same particle in the current configuration $\kappa_t(\mathcal{B})$, at time t. We shall assume that the mapping $\boldsymbol{\chi}$ which assigns the position $\mathbf{x}$ at time t, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is sufficiently smooth so as to make all the derivatives that are taken meaningful. The displacement $\mathbf{u}$, the deformation gradient $\mathbf{F}$, the left Cauchy–Green stretch tensors $\mathbf{B}$ and the linearized strain tensor $\boldsymbol{\varepsilon}$ are defined through:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right]. \tag{1}$$

More details concerning kinematics can be found, for example, in Chadwick [31] and Truesdell and Toupin [32].

2.2. Constitutive relations

Recently, it has been shown that the class of elastic bodies is much more general than previously thought (see [1–8, 10, 13, 33]). The generalization that has been put into place in such works, allows for an elastic body to be defined through implicit constitutive relations between the nonlinear Cauchy–Green stretch tensor $\mathbf{B}$ and the Cauchy stress tensor $\mathbf{T}$ by relations, for example, of the form:

$$\mathfrak{f}(\mathbf{B}, \mathbf{T}, \rho) = \mathbf{0}, \tag{2}$$

where ρ is the mass density. By virtue of the balance of mass, we can recast the dependence on the density with the dependence on the determinant of $\mathbf{B}$ and so express the relationship (2) only in terms of $\mathbf{B}$ and $\mathbf{T}$.

A special sub-class of the above implicit models is the following class that provides an expression for $\mathbf{B}$ in terms of $\mathbf{T}$, namely (see [2, 3, 33, 34]):

$$\mathbf{B} = \mathfrak{g}(\mathbf{T}). \tag{3}$$

In the case of isotropic bodies $\mathfrak{g}(\mathbf{T})$ becomes $\mathfrak{g}(\mathbf{T}) = \alpha_0\mathbf{I}+\alpha_1\mathbf{T}+\alpha_2\mathbf{T}^2$, where α_0, α_1 and α_2 are scalar functions, which depend on the three invariants of $\mathbf{T}$. A similar representation can be found for (2) (see, for example, [2]). It is easy to see that such expressions for $\mathfrak{g}$ and $\mathfrak{f}$ satisfy the objectivity condition (in the case of $\mathfrak{g}(\mathbf{T})$ that condition means the equation $\mathbf{Q}\mathfrak{g}(\mathbf{T})\mathbf{Q}^{\mathrm{T}} = \mathfrak{g}(\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}})$ has to be satisfied).
In the case $\|\nabla\mathbf{u}\| \sim O(\delta)$ with $\delta \ll 1$, considering that $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$, from (2) or (3) it is possible (under certain conditions we do not discuss for brevity here) to find a constitutive relation of the form [7, 8, 10]:

$$\boldsymbol{\varepsilon} = \mathfrak{h}(\mathbf{T}). \tag{4}$$

For isotropic bodies $\mathfrak{h}(\mathbf{T})$ becomes $\mathfrak{h}(\mathbf{T}) = \gamma_0\mathbf{I} + \gamma_1\mathbf{T} + \gamma_2\mathbf{T}^2$, where γ_0, γ_1 and γ_2 are scalar functions that depend on the invariants of the Cauchy stress tensor $\mathbf{T}$. This expression does not satisfy exactly the objectivity condition, but it does satisfy it approximately if $\|\nabla\mathbf{u}\| \sim O(\delta)$ with $\delta \ll 1$, (see, for example, the remarks in pp. 1379–1380 of [7]).

3. Equation of motion

Consider the equation of motion:

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\ddot{\mathbf{u}}. \tag{5}$$

Since $\rho \neq 0$ this equation can be written as $\frac{1}{\rho}\operatorname{div}\mathbf{T} + \mathbf{b} = \ddot{\mathbf{u}}$. By applying the gradient operator to this equation we obtain:[2]

$$\operatorname{grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \operatorname{grad}(\mathbf{b}) = \operatorname{grad}(\ddot{\mathbf{u}}) = \frac{\partial^2}{\partial t^2}(\operatorname{grad}\mathbf{u}). \tag{6}$$

Considering the definition of the linearized strain tensor $(1)_4$, taking the time derivative twice we have

$$\ddot{\boldsymbol{\varepsilon}} = \frac{1}{2}\frac{\partial^2}{\partial t^2}[\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^{\mathrm{T}}]. \tag{7}$$

Therefore, using (6) in (7) we obtain:

$$\operatorname{grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \left[\operatorname{grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right)\right]^{\mathrm{T}} + \operatorname{grad}(\mathbf{b}) + [\operatorname{grad}(\mathbf{b})]^{\mathrm{T}} = 2\ddot{\boldsymbol{\varepsilon}}. \tag{8}$$

For a constitutive equation of the form (4), equation (8) becomes:

$$\operatorname{grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \left[\operatorname{grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right)\right]^{\mathrm{T}} + \operatorname{grad}(\mathbf{b}) + [\operatorname{grad}(\mathbf{b})]^{\mathrm{T}} = 2\frac{\partial^2}{\partial t^2}[\mathfrak{h}(\mathbf{T})]. \tag{9}$$

This equation is a generalization of a similar case already considered for linearized bodies[3] (see, for example [27–30] and Section 59 of [35]).

Notice that (9) is a system of six second order nonlinear partial differential equations for the six independent components of the Cauchy stress tensor $\mathbf{T}$. It is interesting to see that these equations can be solved directly for the components of the stress tensor $\mathbf{T}$, and there is no need to use a stress potential. It is also interesting to compare this with the quasi-static case, where $\mathbf{T} = \mathbf{T}(\mathbf{x})$ (see Appendix A). In the present work, for simplicity, we only find solutions for the case ρ constant.
If $\mathbf{t}$ is the external traction on $\partial\kappa_t(\mathcal{B})$, for (9) the only boundary condition with physical meaning would be:

$$\mathbf{T}(\mathbf{x}, t)\mathbf{n} = \mathbf{t}(\mathbf{x}, t) \quad \text{on} \quad \partial\kappa_t(\mathcal{B}). \tag{10}$$

4. Solutions for a boundary value problem

Let us study a simple 1D problem in order to explore the possibilities of (4) when considering time effects. We are interested in calculating how stresses behave for the cylindrical bar:

$$0 \leq x \leq L, \quad 0 \leq r \leq r_o, \quad \text{if} \quad r_o \ll L. \tag{11}$$

This problem can be solved considering a one-dimensional approximation. Let us assume that the body is under the effect of the axial stress distribution $\mathbf{T} = \sigma(x,t)\mathbf{e}_1 \otimes \mathbf{e}_1$;[4] similarly, the displacement field is assumed to be of the form $\mathbf{u} = u(x,t)\mathbf{e}_1$, and as a result the linearized strain tensor field is given by $\boldsymbol{\varepsilon} = \varepsilon(x,t)\mathbf{e}_1 \otimes \mathbf{e}_1$. If we use the notation $\mathfrak{h} = \mathfrak{h}_{11}$, the simplified form of the boundary value problem (9) to be solved (if we assume for simplicity there is no body force and ρ is constant) is:

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho \frac{\partial^2}{\partial t^2}[\mathfrak{h}(\sigma)]. \tag{12}$$

This problem requires initial and boundary conditions. For the initial conditions let us consider:

$$\sigma(x,0) = \breve{\sigma}(x), \quad \frac{\partial}{\partial t}\sigma(x,t)\bigg|_{t=0} = \breve{\sigma}_t(x), \quad 0 \leq x \leq L; \tag{13}$$

whereas for the boundary conditions let us assume:

$$\sigma(0,t) = \tilde{\sigma}_a(t), \quad \sigma(L,t) = \tilde{\sigma}_b(t), \quad t > 0, \tag{14}$$

where $\breve{\sigma}(x)$, $\breve{\sigma}_t(x)$, $\tilde{\sigma}_a(t)$ and $\tilde{\sigma}_b(t)$ are given functions.

4.1. Some exact solutions

An exact solution for (12) for a general expression for $\mathfrak{h}$ is presented in the first part of this section. This solution does not necessarily satisfy (13), (14) for any given functions $\breve{\sigma}(x)$, $\breve{\sigma}_t(x)$, $\tilde{\sigma}_a(t)$ and $\tilde{\sigma}_b(t)$.

Let us define $z = kx + \lambda t$, where k and $\lambda \neq 0$ are real constants, and let us set

$$\sigma(x,t) = \varphi(z). \tag{15}$$

Equation (12) is equivalent to

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho \left[\frac{\partial^2 \mathfrak{h}}{\partial \sigma^2} \left(\frac{\partial \sigma}{\partial t} \right)^2 + \frac{\partial \mathfrak{h}}{\partial \sigma} \frac{\partial^2 \sigma}{\partial t^2} \right]. \tag{16}$$

From (15) we have that $\frac{\partial^2 \sigma}{\partial x^2} = \varphi''(z)k^2$, $\frac{\partial \sigma}{\partial t} = \varphi'(z)\lambda$ and $\frac{\partial^2 \sigma}{\partial t^2} = \varphi''(z)\lambda^2$, where $\varphi'(z) = \frac{\mathrm{d}\varphi}{\mathrm{d}z}$; therefore, (16) becomes $\varphi''(z)k^2 = \rho\lambda^2 \left(\frac{\partial^2 \mathfrak{h}}{\partial \sigma^2}\varphi'(z)^2 + \frac{\partial \mathfrak{h}}{\partial \sigma}\varphi''(z) \right)$, which is equivalent to the equation:

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}z^2} \frac{k^2}{\rho \lambda^2} = \frac{\mathrm{d}^2}{\mathrm{d}z^2}[\mathfrak{h}(\varphi(z))], \tag{17}$$

whose solution can be found (in general implicitly) by solving the algebraic equation:

$$\varphi(z)\frac{k^2}{\rho\lambda^2} = \mathfrak{h}(\varphi(z)) + \aleph_1 z + \aleph_0, \tag{18}$$

where $\aleph_0$, $\aleph_1$ are constants.

It is easy to verify that a function $\sigma(x,t) = \varphi(z)$ found from the above equation may not satisfy the boundary and initial conditions (13), (14) for *any* given functions $\breve{\sigma}(x)$, $\breve{\sigma}_t(x)$, $\tilde{\sigma}_a(t)$ and $\tilde{\sigma}_b(t)$. On the other hand, for given values of k, λ, $\aleph_0$ and $\aleph_1$, a particular function $\varphi(z)$ can be found from (18), and in such a case the initial and boundary conditions could be adjusted such that that solution is possible for the bar, i.e. if $\breve{\sigma}(x)$, $\breve{\sigma}_t(x)$, $\tilde{\sigma}_a(t)$ and $\tilde{\sigma}_b(t)$ are given as:

$$\breve{\sigma}(x) = \varphi(kx), \quad \breve{\sigma}_t(x) = \lambda\varphi'(kx), \quad \tilde{\sigma}_a(t) = \varphi(\lambda t), \quad \tilde{\sigma}_b(t) = \varphi(kL + \lambda t), \tag{19}$$

then $\varphi(z)$ obtained from (18) is a solution for the boundary value problem.

Some additional exact solutions can be found for (12) for some special expressions for $\mathfrak{h}(\sigma)$. One of such solutions is presented now; however, this solution has been obtained for an expression for $\mathfrak{h}(\sigma)$ that may not be necessarily interesting from the physical point of view.

Let us assume that[5]

$$\mathfrak{h}(\sigma) = \sigma^{m+1}, \quad m \geq 1. \tag{20}$$

The case $m = 0$ leads to an equation similar to the equation obtained from a linear stress–strain relation, and the solution for σ has the classical wave form.

With the assumption (20), equation (12) becomes

$$\frac{\partial^2 \sigma}{\partial x^2} = \rho(m+1)\frac{\partial}{\partial t}\left(\sigma^m \frac{\partial \sigma}{\partial t}\right). \tag{21}$$

Let us look for particular solutions of (21), which are of multiplicative form:

$$\sigma(x,t) = w(x)\psi(t). \tag{22}$$

Under this assumption and considering the special case $w(x) \neq 0$ and $\psi(t) \neq 0$, the differential equation (21) becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{1}{w^{m+1}} = \rho(m+1)\frac{1}{\psi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi^m \frac{\mathrm{d}\psi}{\mathrm{d}t}\right). \tag{23}$$

Therefore, two Sturm–Liouville problems are obtained by separating the variables x and t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi^m \frac{\mathrm{d}\psi}{\mathrm{d}t}\right) - \frac{\vartheta}{\rho(m+1)}\psi = 0, \tag{24}$$

where ϑ is a constant. The last equation under the substitution $\Phi(t) = \psi(t)^{m+1}$ can be written as

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} - \frac{\vartheta}{\rho}\Phi^{\frac{1}{m+1}} = 0, \tag{25}$$

Let us seek the solutions for the particular case $m = 1$, where we have:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} - \frac{\vartheta}{\rho}\Phi^{\frac{1}{2}} = 0, \tag{26}$$

These equations are special cases of the Emden–Fowler equation whose solutions (in implicit forms) are (see Section 2.3.1–2 of [36]):

$$x = \pm\int\left(\frac{2\vartheta}{3}w^3 + C_1\right)^{-1/2}\mathrm{d}w + C_2, \tag{27}$$

$$t = \pm\int\left(\frac{4\vartheta}{3\rho}\Phi^{3/2} + D_1\right)^{-1/2}\mathrm{d}\Phi + D_2 \tag{28}$$

where C_1, C_2, D_1 and D_2 are constants. Similar solutions can be found for other values for $m \geq 1$ and a similar procedure can be developed to find solutions for the case $m < 1$, which for brevity are not presented here.

4.2. Some numerical solutions

The problem (12) can be solved numerically for a more interesting expression for $\mathfrak{h}(\sigma)$. From [11, 12, 37] we consider:

$$\mathfrak{h}(\sigma) = \alpha\left[-1 + \frac{1}{1+\beta\sigma} + \frac{\gamma\sigma}{\sqrt{1+\iota\sigma^2}}\right], \tag{29}$$

where α, β and γ and ι are constants. We use the particular values [37]:

$$\alpha = 10^{-9}, \quad \beta = 10^{-3}\frac{1}{\mathrm{Pa}}, \quad \gamma = 10\frac{1}{\mathrm{Pa}}, \quad \iota = 10^{-11}\frac{1}{\mathrm{Pa}^2}. \tag{30}$$

With these values for the constants we can obtain a behaviour for $\varepsilon(\sigma)$, which in a first part is approximately linear in σ, and thereafter shows a strain limiting behaviour (see Figure 1 in [37]).

Regarding the initial conditions, it is assumed that:

$$\breve{\sigma}(x) = 0, \quad \breve{\sigma}_t(x) = 0, \quad 0 \leq x \leq L. \tag{31}$$

About the boundary conditions, we assume that the right side of the semi-infinite body is free of external traction, i.e. $\tilde{\sigma}_b(t) = 0$. For the left side we consider two cases:

$$\tilde{\sigma}_a(t) = \sigma_0, \quad \tilde{\sigma}_a(t) = \sigma_0 \sin(ft), \quad t > 0, \tag{32}$$

where σ_0 and f are constants. We consider the following values for such constants:

$$\sigma_0 = 10^6\,\mathrm{Pa}, \quad f = 10\,\mathrm{s}^{-1}. \tag{33}$$

Regarding ρ in (12) we assume $\rho = 7500\,\mathrm{Kg/m^3}$.

The boundary value problem (12), considering (31), (32) and the previous values for the different constants, was solved using the finite element method, working with the PDE module of the program Comsol 3.4 [38]. The influence of the mesh density was studied, but for brevity such analysis is not presented here. For the results shown in this section we assume $L = 1000$ m and 7681 degrees of freedom. In Figures 1–4 results are presented for $\sigma(x,t)$ and $\varepsilon(x,t)$ for the two types of boundary conditions discussed in (32). In Figures 1 and 2 we have

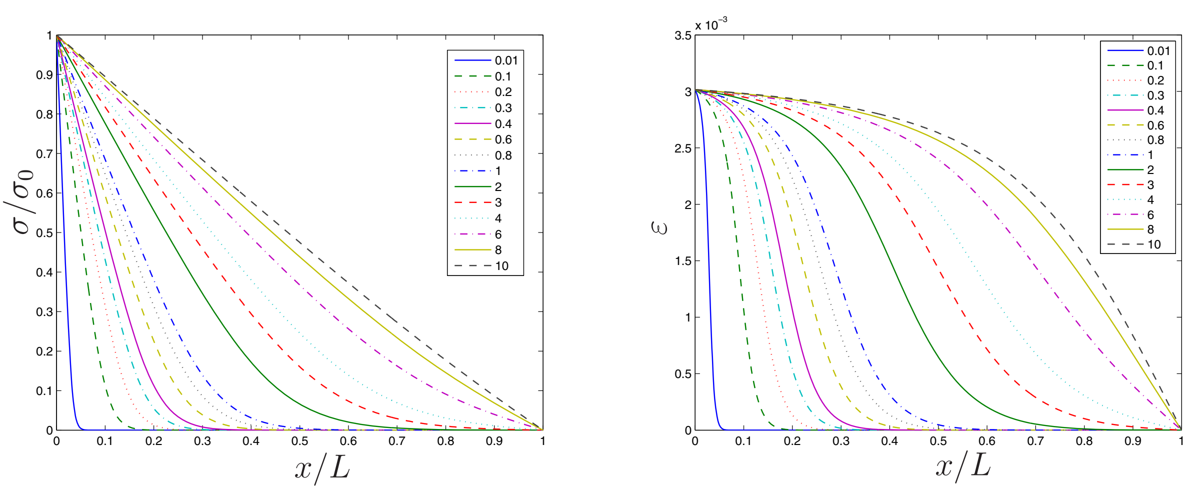

Figure 1. Results for the dimensionless stress σ/σ_0 and the strain ε in the case $\tilde{\sigma}_a(t) = \sigma_0$. The results are presented in terms of the dimensionless position x/L for different instants t (in seconds).

provided results for the case $\tilde{\sigma}_a(t) = \sigma_0$. In Figure 1 we have plots for the stress and the strain as functions of the position along the bar for different times. In Figure 2 the same information is presented considering contour plots for $\sigma(x/L,t)/\sigma_0$ and $\varepsilon(x/L,t)$. In Figures 3 and 4 we have results for the case $\tilde{\sigma}_a(t) = \sigma_0 \sin(ft)$.

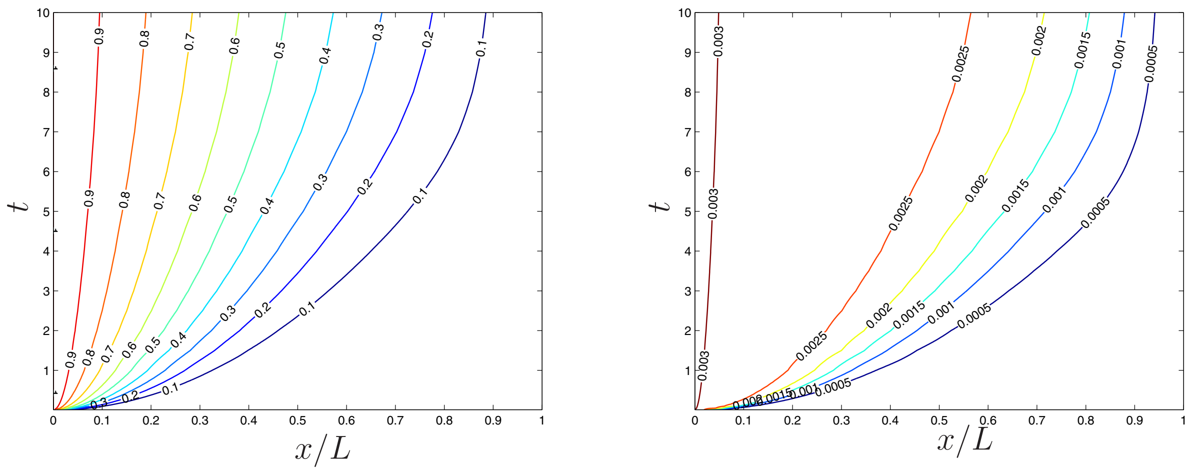

Figure 2. Contour plots for the dimensionless stress σ/σ_0 and the strain ε in the case $\tilde{\sigma}_a(t) = \sigma_0$. The time t is in seconds. (*Left*) Contour plot for σ/σ_0. (*Right*) contour plot for ε.

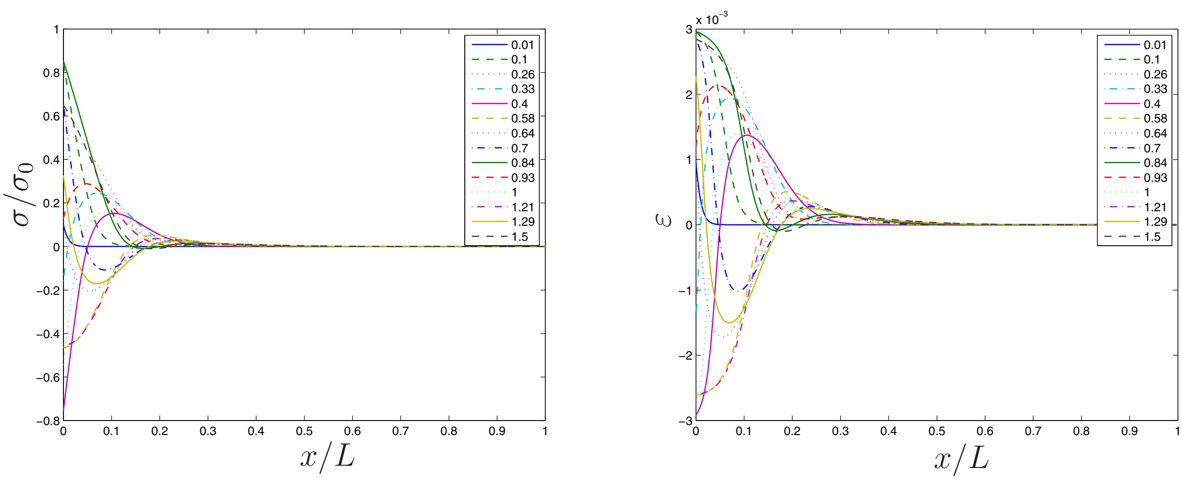

Figure 3. Results for the dimensionless stress σ/σ_0 and the strain ε in the case $\tilde{\sigma}_a(t) = \sigma_0 \sin(ft)$. The results are presented in terms of the dimensionless position x/L for different instants t (in seconds).

5. Further remarks

In the present work we have obtained generalizations for the results presented in the works by Iacovache [27], Vâlcovici [28] and Ignaczak [29, 30], in the case where the equilibrium equations are expressed only in terms of the components of the stress tensor. Such generalization to the case of nonlinear behaviour for an elastic body (see Section 59 of [35] for the special case of linearized constitutive equations) is possible if one considers one of the new classes of elastic bodies, which has been presented recently in the literature [2, 7].

We have an equation that can be used, for example, to study wave propagation phenomena for elastic bodies which can present small strains and displacements, but that can show a nonlinear behaviour. As remarked in [3, 13] such new classes of elastic bodies can be of interest in order to study some problems, such as the behaviour of brittle bodies with cracks, where we observe the presence of large stresses, but where strains remain small. Now, with the theory and the results presented in this work, we could study the properties of such bodies considering its dynamic response, which is a usual non-destructive technique used to analyze the behaviour of bodies with internal defects. The present work complements the series of results presented already in the literature for this new class of elastic bodies [10–13], now by taking into account the time effect (see also [39]).

This is a first work on a topic that can be of great importance in the future. There are several open questions, which will be addressed in future communications, such as considering an analysis of existence of solution, and

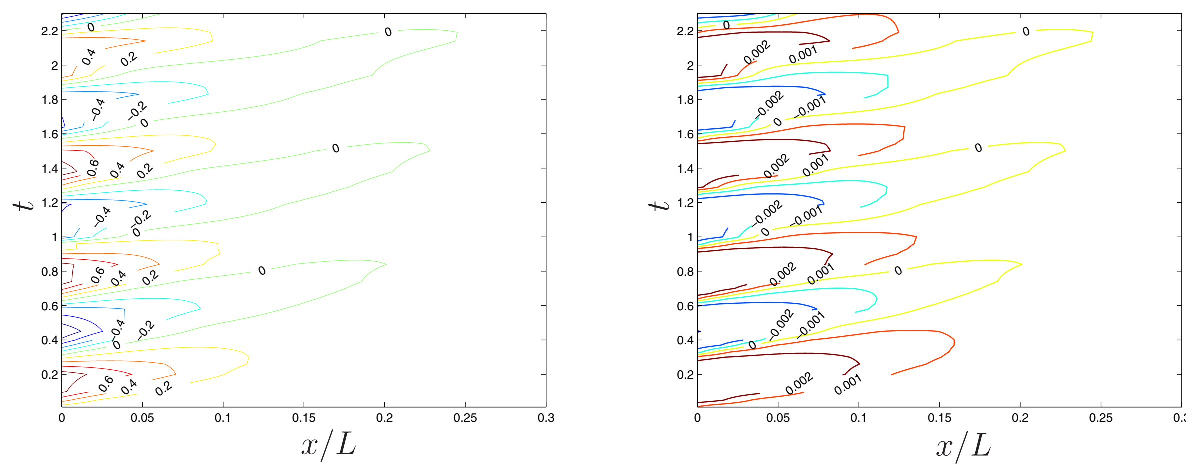

Figure 4. Contour plots for the dimensionless stress σ/σ_0 and the strain ε in the case $\tilde{\sigma}_a(t) = \sigma_0 \sin(ft)$. The time t is in seconds. (*Left*) Contour plot for σ/σ_0. (*Right*) Contour plot for ε.

of the uniqueness (or non-uniqueness) of solutions for some general classes of boundary value problems and constitutive relations (see [29] for similar studies for the case of linearized bodies).

Evidently, it is necessary to solve more boundary value problems, considering as well more realistic expressions for the function $\mathfrak{h}(\mathbf{T})$. One problem that can be considered is the case of wave propagation for the semi-infinite body $0 \leq x \leq L$, $-\infty \leq y \leq \infty$, $-\infty \leq z \leq \infty$, assuming a stress distribution of the form

$$\mathbf{T} = \sigma_A(x)\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_B(x)\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_C(x)\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{34}$$

which is assumed to be associated to the displacement field $\mathbf{u} = u(x)\mathbf{e}_1$, which produces the strain distribution

$$\boldsymbol{\varepsilon} = \varepsilon(x)\mathbf{e}_1 \otimes \mathbf{e}_1, \tag{35}$$

i.e. stresses and strains only depend on the position x (the direction of propagation of the stress and strain waves). In the directions y, z we could assume that since the body is infinite it cannot deform. From (4) we have

$$\varepsilon = \mathfrak{h}_{11}(\mathbf{T}), \quad 0 = \mathfrak{h}_{22}(\mathbf{T}), \quad 0 = \mathfrak{h}_{33}(\mathbf{T}), \tag{36}$$

while from (9) we need to solve (if body forces are not considered and for simplicity ρ is constant):

$$\frac{\partial^2 \sigma_A}{\partial x^2} = \rho \frac{\partial^2 \mathfrak{h}_{11}}{\partial t^2}(\mathbf{T}). \tag{37}$$

From $(36)_{2,3}$ we can find σ_B and σ_C in terms of σ_A, and by replacing these expressions in (37) we have a partial differential equation only for $\sigma_A(x, t)$.

Another interesting extension of this work could be the case where we study large elastic deformations, for which (3) has to be considered. However, in such a case it is possible to show that the equations equivalent to (9) correspond to a highly nonlinear system of integro-differential equations (see equation (48) in Appendix B).

Regarding the numerical results presented in Figures 1–4, the value for σ_0 in (33) is such that $\varepsilon(\sigma)$ behaves in a nonlinear manner, and is close to the region where it becomes approximately constant in terms of the axial stress, see Figure 1 of [37]. Despite such ‘high’ value for σ_0, from Figure 1 we notice that strains remain small. From Figure 4 we can see that the wave for the stresses and the strains changes with time, showing an interesting effect that cannot be obtained if we consider the standard constitutive equation for linearized elastic bodies.

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Note

1. In Figure 2(a) of [17] we observe a small hysteresis phenomena for that material, which in a first approximation could be neglected in the case that we would be interested in applying our theories to that problem.
2. The operators div and grad are the divergence and the gradient operators with respect to the current configuration. In the case $\|\nabla\mathbf{u}\| \sim O(\delta)$ with $\delta \ll 1$ there is no need to distinguish between the current and reference configurations, but in Appendix B, where we give some remarks about the case of large strains and displacement, the operators are defined with respect to these two configurations.
3. For linearized isotropic bodies we have $\mathfrak{h}(\mathbf{T}) = \frac{1+\nu}{E}\mathbf{T} - \frac{\nu}{E}\mathrm{tr}(\mathbf{T})\mathbf{I}$, where E is the Young modulus and ν is the Poisson ratio.
4. The subscript 1 denotes the axial direction x in the cylinder.
5. A more general form could be $\mathfrak{h}(\sigma) = c\sigma^{m+1}$, where c is a constant, which should be adjusted such that the magnitude of the strains are small (actually the gradient of the displacement field should be small). That constant could be included in the definition of ρ in (12).

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Appendix A: Some remarks about a decomposition for a solution of equation (9) and the compatibility equations

Let us consider equation (9), it could be possible to assume a decomposition for the stress tensor into a transient part plus a stationary part of the form:

$$\mathbf{T}(\mathbf{x},t)=\mathbf{T}_t(\mathbf{x},t)+\mathbf{T}_s(\mathbf{x}).$$

However, since in general $\mathfrak{h}(\mathbf{T})$ is a nonlinear tensor function, it is not possible to decompose the linearized strain tensor $\boldsymbol{\varepsilon}$ into a transient plus a stationary part; nevertheless, it is interesting to discuss the special case $\mathbf{T}=\mathbf{T}(\mathbf{x})$. If $\mathbf{T}(\mathbf{x})$ satisfies (5), it is easy to see that (9) would also hold, where the right side of the equation would be identically equal to zero. Therefore, it would be necessary to solve an equation of the form (assuming for simplicity that $\mathbf{b}=\mathbf{0}$)

$$\operatorname{grad}(\operatorname{div}\mathbf{T})+[\operatorname{grad}(\operatorname{div}\mathbf{T})]^{\mathrm{T}}=\mathbf{0}.$$

In principle, we could solve this equation directly for the six independent components of the stress tensor $\mathbf{T}$; however, we notice that in such a case the material properties do not appear in the equation (unlike in the case (5)). Moreover, such a solution would not necessarily be associated with continuous stationary displacement field $\mathbf{u}$. A classical method to deal with (5) in the stationary case is to use a stress tensor potential such that (5) would be satisfied automatically, and then to replace $\boldsymbol{\varepsilon}=\mathfrak{h}(\mathbf{T})$ in the compatibility equations such that the associated displacement field would be continuous.

For a 3D problem, in index notation, considering for simplicity only Cartesian coordinates, the components of the stress tensor $\mathbf{T}(\mathbf{x})$ can be expressed in terms of a stress tensor potential $\mathbf{a}$ as (see, for example, [32])

$$T_{km}=e_{krp}e_{msq}\frac{\partial^2 a_{rs}}{\partial x_p\partial x_q}, \tag{38}$$

where e_{ijk} are the permutation symbols and $a_{rs}=a_{sr}$. A 'classical' approach used in this case (if there are no body forces and time effect, where equation (5) is satisfied automatically) in order for the components of the strain tensor $\varepsilon_{ij}=\mathfrak{h}_{ij}(T_{km})$ to be associated with a continuous displacement field $\mathbf{u}$, is to consider the compatibility equations (see, for example, [32, 40])

$$\frac{\partial^2\varepsilon_{jk}}{\partial x_i\partial x_l}+\frac{\partial^2\varepsilon_{il}}{\partial x_j\partial x_k}-\frac{\partial^2\varepsilon_{ik}}{\partial x_j\partial x_l}-\frac{\partial^2\varepsilon_{jl}}{\partial x_i\partial x_k}=0, \tag{39}$$

where only six of these 81 equations are independent. Using (38) in (4) and replacing in (39) the following six in general nonlinear partial differential equations (of fourth order) are obtained for the six independent components of the stress potential:

$$\frac{\partial^2}{\partial x_i \partial x_l}\left[\mathfrak{h}_{jk}\left(e_{mrp}e_{nsq}a_{rs,pq}\right)\right] + \frac{\partial^2}{\partial x_j \partial x_k}\left[\mathfrak{h}_{il}\left(e_{mrp}e_{nsq}a_{rs,pq}\right)\right] - \frac{\partial^2}{\partial x_j \partial x_l}\left[\mathfrak{h}_{ik}\left(e_{mrp}e_{nsq}a_{rs,pq}\right)\right] - \frac{\partial^2}{\partial x_i \partial x_k}\left[\mathfrak{h}_{jl}\left(e_{mrp}e_{nsq}a_{rs,pq}\right)\right] = 0, \tag{40}$$

where the notation $a_{ij,kl} = \frac{\partial^2 a_{ij}}{\partial x_k \partial x_l}$ has been used.

For the special bidimensional problem, the three independent components of the stress tensor satisfies the equilibrium equation if they are expressed in terms of the Airy stress potential Φ (in Cartesian coordinates) as

$$T_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}, \quad T_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}, \quad T_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}. \tag{41}$$

For a 2D problem there is only one compatibility equation that is necessary to consider, namely

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0. \tag{42}$$

If (41) is used in (4) and if that result is replaced in (42), a nonlinear fourth order partial differential equation for Φ is obtained, which has been presented already in [7].

Regarding the boundary conditions, the only condition with clear physical meaning can be written in terms of $\mathbf{t}(\mathbf{x})$ and is the external traction on $\partial\kappa_t(\mathcal{B})$, which is $\mathbf{Tn} = \mathbf{t}$, where the additional restriction $\int_{\partial\kappa_t(\mathcal{B})} \mathbf{t}\,\mathrm{d}a = \mathbf{0}$ is needed.

Remark: *It is not mandatory to consider the compatibility equations (39), (42) or equation (9) in order to be able to solve problems in the case of working with the constitutive relations (2)–(4). As remarked in [41] (see also [39]), the strain tensor* $\boldsymbol{\varepsilon}$ *or the Cauchy–Green tensor* $\mathbf{B}$ *are not 'primitive' quantities, since they only make sense after being defined in terms of the displacement field* $\mathbf{u}$ *(see, for example, $(1)_4$). Therefore, a boundary value problem (in the 3D case) could be solved directly for the six components of* $\mathbf{T}$ *and the three components of* $\mathbf{u}$*, considering the equilibrium equation (5) (that has three components), and the six components of equation (4) in the form* $\frac{1}{2}\left[\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right) + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathrm{T}}\right] = \mathfrak{h}(\mathbf{T})$*. We would have in total nine equations and nine unknowns, so the problem is well posed. In the 2D case from (4) we have three equations, which plus the two components of the equilibrium equations (5) gives us in total five equations, for the three components of* $\mathbf{T}$ *plus the two components of* $\mathbf{u}$ *and naturally the problem is also well posed.*

Despite the previous remarks, we do find it interesting to study (9) and the classes of solutions we can find from that new class of boundary value problem.

Appendix B: The case of large elastic deformations

From the definition of the left Cauchy–Green deformation tensor we have $\mathbf{B} = \mathbf{FF}^{\mathrm{T}} = \mathbf{I} + \mathrm{Grad}\,\mathbf{u} + (\mathrm{Grad}\,\mathbf{u})^{\mathrm{T}} + \mathrm{Grad}\,\mathbf{u}(\mathrm{Grad}\,\mathbf{u})^{\mathrm{T}}$, therefore from (3) we have

$$\mathbf{I} + \mathrm{Grad}\,\mathbf{u} + (\mathrm{Grad}\,\mathbf{u})^{\mathrm{T}} + \mathrm{Grad}\,\mathbf{u}(\mathrm{Grad}\,\mathbf{u})^{\mathrm{T}} = \mathfrak{g}(\mathbf{T}). \tag{43}$$

By taking the second time derivative of (43) we obtain

$$\mathrm{Grad}\,\ddot{\mathbf{u}} + (\mathrm{Grad}\,\ddot{\mathbf{u}})^{\mathrm{T}} + \mathrm{Grad}\,\ddot{\mathbf{u}}(\mathrm{Grad}\,\mathbf{u})^{\mathrm{T}} + \mathrm{Grad}\,\mathbf{u}(\mathrm{Grad}\,\ddot{\mathbf{u}})^{\mathrm{T}} + 2\mathrm{Grad}\,\dot{\mathbf{u}}(\mathrm{Grad}\,\dot{\mathbf{u}}^{\mathrm{T}}) = \frac{\partial^2}{\partial t^2}\left[\mathfrak{g}(\mathbf{T})\right]. \tag{44}$$

(The gradient operator with respect to the reference configuration is denoted as Grad.) In order to have an equation similar to (9) in the case of large elastic strains, from the equation of motion (5) we have

$$\operatorname{Grad}\ddot{\mathbf{u}} = \operatorname{Grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \operatorname{Grad}\mathbf{b}. \tag{45}$$

From (44) we notice that expressions for $\dot{\mathbf{u}}$ and $\mathbf{u}$ are needed, which can be obtained from (5) as integrals with respect to time as:

$$\dot{\mathbf{u}}(\mathbf{X}, t) = \int_0^t \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \zeta), \zeta)]\, \mathrm{d}\zeta + \int_0^t \mathbf{b}(\mathbf{X}, \zeta)\, \mathrm{d}\zeta + \mathbf{p}(\mathbf{X}), \tag{46}$$

where $\mathbf{p}$ is an arbitrary function of space that appeared due to the time integration. Taking one more integration with respect to time we have:

$$\mathbf{u}(\mathbf{X}, t) = \int_0^t \left\{\int_0^\zeta \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \eta), \eta)]\, \mathrm{d}\eta\right\} \mathrm{d}\zeta + \int_0^t \left\{\int_0^\zeta \mathbf{b}(\mathbf{X}, \eta)\, \mathrm{d}\eta\right\} \mathrm{d}\zeta + \mathbf{p}(\mathbf{X})t + \mathbf{q}(\mathbf{X}), \tag{47}$$

where $\mathbf{q}$ is another arbitrary function of space that appeared due to the time integration.

Using the same procedure presented in Section 3, by eliminating the displacement field from the equation of motion, the following nonlinear integro-differential equation for the stress components is obtained, which is the generalization of (9) for large strains:

$$\begin{aligned}
&\operatorname{Grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \left[\operatorname{Grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right)\right]^{\mathrm{T}} + \operatorname{Grad}\mathbf{b} + (\operatorname{Grad}\mathbf{b})^{\mathrm{T}} \\
&+ \left[\operatorname{Grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \operatorname{Grad}\mathbf{b}\right]\left[\operatorname{Grad}\left(\int_0^t \left\{\int_0^\zeta \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \eta), \eta)]\, \mathrm{d}\eta\right\} \mathrm{d}\zeta\right.\right. \\
&\left.\left.+ \int_0^t \left\{\int_0^\zeta \mathbf{b}(\mathbf{X}, \eta)\, \mathrm{d}\eta\right\} \mathrm{d}\zeta + \mathbf{p}(\mathbf{X})t + \mathbf{q}(\mathbf{X})\right)\right]^{\mathrm{T}} + \operatorname{Grad}\left(\int_0^t \left\{\int_0^\zeta \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \eta), \eta)]\, \mathrm{d}\eta\right\} \mathrm{d}\zeta\right. \\
&\left.+ \int_0^t \left\{\int_0^\zeta \mathbf{b}(\mathbf{X}, \eta)\, \mathrm{d}\eta\right\} \mathrm{d}\zeta + \mathbf{p}(\mathbf{X})t + \mathbf{q}(\mathbf{X})\right)\left[\operatorname{Grad}\left(\frac{1}{\rho}\operatorname{div}\mathbf{T}\right) + \operatorname{Grad}\mathbf{b}\right]^{\mathrm{T}} \\
&+ 2\operatorname{Grad}\left(\int_0^t \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \zeta), \zeta)]\, \mathrm{d}\zeta + \int_0^t \mathbf{b}(\mathbf{X}, \zeta)\, \mathrm{d}\zeta + \mathbf{p}(\mathbf{X})\right) \times \\
&\times \left[\operatorname{Grad}\left(\int_0^t \frac{1}{\rho}\operatorname{div}[\mathbf{T}(\mathbf{x}(\mathbf{X}, \zeta), \zeta)]\, \mathrm{d}\zeta + \int_0^t \mathbf{b}(\mathbf{X}, \zeta)\, \mathrm{d}\zeta + \mathbf{p}(\mathbf{X})\right)\right]^{\mathrm{T}} = \frac{\partial^2}{\partial t^2}\left[\mathfrak{g}(\mathbf{T})\right].
\end{aligned} \tag{48}$$

Equation (48) is highly complicated in comparison with (9) and may not be of much use in order to study unsteady motions for constitutive relations of the class presented in (3) (see the remarks in [25] regarding transforming the original problems (5) and (43) into, for example, equation (48)).