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A principal axis formulation for nonlinear magnetoelastic deformations: Isotropic bodies

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ABSTRACT

In this work a new set of principal axis invariants is proposed in order to study the problem of considering large magneto-elastic deformations, for bodies that are isotropic in the un-deformed configuration when no external magnetic induction is applied. The new set of invariants has clear physical meanings and may have an experimental advantage over the standard invariants used in many previous works in this area. The principal axis invariant formulation is also shown to be more general. Some simple boundary value problems are solved, such as the simple shear, and the biaxial extension of a slab, where with the use of these new invariants, it is possible to study in a much simpler manner the effect of different types of deformations on the response of the material. An illustrated simple specific constitutive equation is proposed which compares well with experiment.

1. Introduction

Magneto-sensitive elastomers correspond to a class of rubber-like material filled with magneto-active particles, which can react to the presence of magnetic fields, see for example (Bednarek, 1999; Boczkowska and Awietjan, 2009; Boczkowska et al., 2012; Böse, 2007; Ginder et al., 2002; Gordaninejad et al., 2012; Jolly et al., 1996; Lokander and Stenberg, 2003; Mitsumata, 2009; Mitsumata and Ohori, 2011; Varga et al., 2006). These materials have attracted the attention of different researchers due to the possible applications, for example, in vibration control and in the design of flexible robots, see, for example (Albanese et al., 2003; Böse et al., 2012; Deng and Gong, 2008; Farshad and Le Roux, 2004; Ghafoorianfar et al., 2013; Ginder et al., 1999, 2000, 2001; Kashima et al., 2012; Li and Zhang, 2008; Yalcintas and Dai, 2004; Zhu et al., 2012). The theory of nonlinear magneto-elastic interactions was developed many decades ago, we can mention, for example, the monograph by Brown (1966) and Maugin (1988) and Eringen and Maugin (1990); Hutter et al. (2006); Tiersten (1964). In the recent years several new communications have been published in this area, presenting some new theoretical formulations (Brigadnov and Dorfmann, 2003; Bustamante et al., 2008; Chatzigeorgiou et al., 2014; Dorfmann and Ogden, 2003; Dorfmann et al., 2004a, 2004b; Dorfmann and Ogden, 2005; Galipeau and Ponte-Castañeda, 2013; Kankanala and Triantafyllidis, 2004; Ponte-Castañeda and Galipeau, 2011; Saxena et al., 2014a, 2014b; Steigmann, 2004; Vu and Steinmann, 2010, Ogden and Steigmann, 2010.), some numerical results (Bustamante et al., 2011; Salas and Bustamante, 2014) and some experimental works (Bellan and Bossis, 2002; Danas et al., 2003; Ginder et al., 1999). We mention, in particular the formulation by Dorfmann and Ogden (Dorfmann and Ogden, 2003; Dorfmann et al., 2004a, 2004b; Dorfmann and Ogden, 2005), which due to its mathematical simplicity is used as a basis of the present work. That formulation is based on the concept of the total energy function, which is a scalar function depending on the deformation gradient and one of the magnetic variables, from where we can obtain the stresses and the other magnetic variable in a rather simple manner (Dorfmann et al., 2004b).

At the present moment, there is a need for more experimental work and also for better numerical methods to solve more realistic boundary value problems (see, for example, the final remarks in Salas and Bustamante, 2014). In several of the theoretical formulations presented in the literature the standard set of invariants by Rivlin and Spencer (see, for example, Spencer and Eringen, 1971; Zheng, 1994) have been used in order to model the behaviour of isotropic (Dorfmann and Ogden, 2003; Dorfmann et al., 2004a, 2004b; Dorfmann and Ogden, 2005) and transversely isotropic magneto-sensitives bodies (Bustamante, 2010; Saxena et al., 2014b, 2014a). Such set of invariants may not be convenient in order to try

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to fit experimental data to obtain more realistic expressions for the total energy function, due to the lack of clear physical meaning of some of them (see, for example, Shariff, 2008); additionally, the standard set of invariants may create some problems for the fitting when working with small strains, due to error propagation from experimental measurements (see, for example, Criscione, 2014 and the works cited therein).

Isotropic magneto-active bodies behave as transversely isotropic bodies when an external magnetic field is applied, because such field creates magnetic bodies forces in the direction of the field, therefore, in the deformed configuration such materials behave as if they have a preferred direction. In the case of magneto-sensitive elastomers, if the particles are randomly distributed inside the elastomeric matrix, it is said that is an isotropic magneto-sensitive (MS) elastomer.

Considering the above remarks, in this communication a set of principal axis invariants is presented in order to model the behaviour of isotropic MS bodies. This new set is based on the principal axis invariants considered by Shariff (2008) for purely elastic deformations in transversely isotropic materials. The new invariants proposed for isotropic MS bodies have clear physical meanings, as a result, these invariants can be more attractive in order to look for expressions for the total energy function by fitting experimental data, and also in order to design a rational program of experiments for MS elastomers. In addition to this, the classical invariants (and most of their variants) can be explicitly expressed in terms of the principal axis invariants, and hence if the constitutive equation is initially written in terms of the classical invariants, the relevant formulations can be easily obtained in terms of both classical and principal axis invariants. However, if on the onset the constitutive equation is written in terms of principal axis invariants it is generally impossible to convert it explicitly in terms of classical invariants and we cannot write the relevant formulations in terms of the classical invariants. Hence, expressing the energy function in terms of the principal axis invariants is more general and the need to have principal formulations in MS elastomer problems is crucial, and we do this here. A numerical procedure, such as the finite element method, can be easily constructed based on the principal axis formulations developed in this paper.

This work is divided in 7 sections: In Section 2 some relations for the kinematics of the deformation of a body are presented, and some of the basic expressions of the theory of Dorfmann and Ogden (Dorfmann et al., 2004b) for nonlinear MS solids are shown. In Section 3 the new set of invariants for isotropic MS bodies is presented, and the expressions for the stresses and the magnetic variable are derived. In Section 4 some simple boundary value problems are solved using the expressions for the stresses and field in terms of the new set of invariants. In Section 5 an alternative constitutive equation is given, where the magnetic field (instead of the magnetic flux) is treated as an independent variable. In Section 6, we make some comments on constitutive equations and compare our theory with an experimental data using a simple constitutive form. The conclusion is given in Section 7.

2. Basic equations

2.1. Kinematics

Let $\mathscr{B}$ denotes the MS body, $\boldsymbol{x}\in\mathscr{B}_t$ denotes the position of a particle $X\in\mathscr{B}$ in the current configuration $\mathscr{B}_t$. The position of the same particle in the reference configuration is denoted as $\boldsymbol{X}\in\mathscr{B}_r$, where $\mathscr{B}_r$ is the body in the reference configuration, which we assume is undeformed. It is assumed that there exists a one-to-one mapping $\chi=\chi(\boldsymbol{X},t)$ such that $\boldsymbol{x}=\chi(\boldsymbol{X},t)$ for any time $t>0$. The deformation gradient and the left $\boldsymbol{b}$ and right $\boldsymbol{C}$ Cauchy–Green deformation tensors are defined, respectively, by

$$\boldsymbol{F}=\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{X}},\quad \boldsymbol{b}=\boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}=\boldsymbol{V}^2,\quad \boldsymbol{C}=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}=\boldsymbol{U}^2, \tag{1}$$

where we assume χ is such that $J=\det\boldsymbol{F}>0$, $(\ )^{\mathrm{T}}$ is the transpose of a second order tensor and, $\boldsymbol{V}$ and $\boldsymbol{U}$ are the left and the right stretch tensors, respectively. We have used small case characters in order to denote the left Cauchy–Green deformation tensor, in order to avoid problems with the notation used in magnetostatics. More details about kinematics can be found, for example, in Ogden (1997); Truesdell and Toupin (1960). In this work we only consider quasi-statics deformations. The principal values for the deformation are

$$\lambda_i=\boldsymbol{e}_i\cdot\boldsymbol{U}\boldsymbol{e}_i,\quad i=1,2,3,\quad \text{no sum in } i, \tag{2}$$

where $\boldsymbol{e}_i$ is the principal direction of $\boldsymbol{U}$. In this communication all subscripts i, j and k take the values 1, 2 and 3, unless stated otherwise.

2.2. The equations of magnetostatics and the balance equations

In this section we review briefly some elements of the theory of magnetostatics (Kovetz, 2000) and of the theory for nonlinear MS bodies by Dorfmann and Ogden (Dorfmann et al., 2004b and Ogden and Steigmann, 2010).

Let $\boldsymbol{H}$ and $\boldsymbol{B}$ denote the magnetic field and the magnetic induction in the current configuration. In the absence of electric interactions and time effects, the magnetic field and the magnetic induction have to satisfy the simplified form of the Maxwell equations.

$$\operatorname{div}\ \boldsymbol{B}=0,\quad \operatorname{curl}\boldsymbol{H}=\boldsymbol{0}. \tag{3}$$

Using the global form of (3), it is possible to define the following Lagrangian counterparts in the reference configuration of the magnetic field and the magnetic induction $\boldsymbol{H}_l$ and $\boldsymbol{B}_l$ (see, for example, Dorfmann and Ogden, 2003; Dorfmann et al., 2004a, 2004b):

$$\boldsymbol{H}_l=\boldsymbol{F}^{\mathrm{T}}\boldsymbol{H},\quad \boldsymbol{B}_l=J\boldsymbol{F}^{-1}\boldsymbol{B}. \tag{4}$$

In vacuum, the magnetic field and the magnetic induction are related by the equation

$$\boldsymbol{B}=\mu_0\boldsymbol{H}, \tag{5}$$

where μ_0 is the magnetic permeability in vacuo. For condensed matter an additional field is required, which is the magnetization field $\boldsymbol{M}$, which is related to $\boldsymbol{B}$ and $\boldsymbol{H}$ through

$$\boldsymbol{B}=\mu_0\boldsymbol{H}+\boldsymbol{M}. \tag{6}$$

The presence of a magnetic field in a MS body creates magnetic body loads and as a result the mechanical Cauchy stress tensor is not symmetric (Maugin, 1988). One of the key points of the theory of Dorfmann and Ogden (Dorfmann et al., 2004b) was to define a total Cauchy-like stress tensor, which is denoted as $\boldsymbol{\tau}$ that incorporates in its definition the body forces, which are written as the divergence of a tensor field. Since from the practical point of view it is not possible to differentiate the different loads inside a MS body, the above assumption would be valid.

Let $\boldsymbol{T}$ denote the total nominal stress tensor, which is related to $\boldsymbol{\tau}$ through (Dorfmann et al., 2004b):

$$\boldsymbol{T}=J\boldsymbol{F}^{-1}\boldsymbol{\tau}. \tag{7}$$

Another key ingredient of the theory of Dorfmann and Ogden (Dorfmann et al., 2004b) was the assumption that there exists a total energy function $\Omega_M = \Omega_F(\boldsymbol{F}, \boldsymbol{B}_l)$, which incorporates in its definitions the elastic and magnetic energy stored by the MS body.[1] It has been proved that (see Eqs. (3.11) and (3.13) of Dorfmann et al., 2004b)

$$\boldsymbol{T} = \frac{\partial \Omega_F}{\partial \boldsymbol{F}}, \quad \boldsymbol{H}_l = \frac{\partial \Omega_F}{\partial \boldsymbol{B}_l}. \tag{8}$$

Using (7) and $(4)_1$ we have

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\frac{\partial \Omega_F}{\partial \boldsymbol{F}}, \quad \boldsymbol{H} = \boldsymbol{F}^{-\mathrm{T}}\frac{\partial \Omega_F}{\partial \boldsymbol{B}_l}, \tag{9}$$

and for incompressible bodies Eq. $(9)_1$ becomes

$$\boldsymbol{\tau} = \boldsymbol{F}\frac{\partial \Omega_F}{\partial \boldsymbol{F}} - p\boldsymbol{I}. \tag{10}$$

The total stress tensor $\boldsymbol{\tau}$ must satisfy the balance equation (in absence of mechanical body forces)

$$\operatorname{div} \ \boldsymbol{\tau} = \boldsymbol{0}. \tag{11}$$

Through the surface $\partial \mathscr{B}_t$ of the body $\mathscr{B}_t$, the magnetic variables and the total stress tensor must satisfy the following continuity conditions (see, for example, Kovetz, 2000; Dorfmann et al., 2004b):

$$\boldsymbol{n}\cdot[\![\boldsymbol{B}]\!] = 0, \quad \boldsymbol{n}\times[\![\boldsymbol{H}]\!] = \boldsymbol{0}, \quad \boldsymbol{\tau}\boldsymbol{n} = \hat{\boldsymbol{t}} + \boldsymbol{\tau}_M\boldsymbol{n}, \tag{12}$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial \mathscr{B}_t$, $\hat{\boldsymbol{t}}$ is the external mechanical traction, $[\![\,]\!]$ denotes the difference of a quantity from outside and inside a body, and $\boldsymbol{\tau}_M$ is the Maxwell stress tensor defined as (Kovetz, 2000)

$$\boldsymbol{\tau}_M = \boldsymbol{H}\otimes\boldsymbol{B} - \frac{1}{2}(\boldsymbol{H}\cdot\boldsymbol{B})\boldsymbol{I}. \tag{13}$$

3. A new set of invariants for isotropic magneto-sensitive bodies

The function Ω_F must satisfy the objectivity condition, therefore that function is usually written as $\Omega(\boldsymbol{C},\boldsymbol{B}_l) = \Omega_F(\boldsymbol{F},\boldsymbol{B}_l)$. In order to use some of the concepts presented in Shariff (2008), let us define the unit vector $\boldsymbol{a}$ as

$$\boldsymbol{a} = \frac{1}{B}\boldsymbol{B}_l \quad \text{if} \quad B \neq 0. \tag{14}$$

where we have defined $B = |\boldsymbol{B}_l|$.

As in Shariff (2008), we define the invariant ζ_i as

$$\zeta_i = (\boldsymbol{a}\cdot\boldsymbol{e}_i)^2. \tag{15}$$

We propose the following set $\{\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, B\}$ of 7 invariants for Ω (of which only 6 are independent) for isotropic MS solids. We note that only two of the scalars ζ_i are independent, since $\zeta_3 = 1 - \zeta_1 - \zeta_2$. Therefore, for an isotropic MS body we can express.

$$\Omega(\boldsymbol{C}, \boldsymbol{B}_l) = \hat{\Omega}(\mathrm{B}, \lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3). \tag{16}$$

We list the connections of these invariants with the set of invariants normally considered in the literature[2]:

$$I_1 = \mathrm{tr}(\boldsymbol{C}) = \sum_{i=1}^{3}\lambda_i^2, \ \ I_2 = \frac{I_1^2 - \mathrm{tr}(\boldsymbol{C}^2)}{2} = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2,$$
$$I_3 = \det(\boldsymbol{C}) = J^2 = (\lambda_1\lambda_2\lambda_3)^2, \ \ I_4 = \boldsymbol{B}_l\cdot\boldsymbol{B}_l = B^2,$$
$$I_5 = \boldsymbol{B}_l\cdot\boldsymbol{C}\boldsymbol{B}_l = B^2\sum_{i=1}^{3}\zeta_i\lambda_i^2, \ \ I_6 = \boldsymbol{B}_l\cdot\boldsymbol{C}^2\boldsymbol{B}_l = B^2\sum_{i=1}^{3}\zeta_i\lambda_i^4. \tag{17}$$

From $(9)_1$ and (10) if we have $\Omega_M = \Omega(\boldsymbol{C},\boldsymbol{B}_l)$ then for compressible and incompressible bodies we have, respectively:

$$\boldsymbol{\tau} = 2J^{-1}\boldsymbol{F}\frac{\partial \Omega}{\partial \boldsymbol{C}}\boldsymbol{F}^{\mathrm{T}}, \quad \boldsymbol{\tau} = 2\boldsymbol{F}\frac{\partial \Omega}{\partial \boldsymbol{C}}\boldsymbol{F}^{\mathrm{T}} - p\boldsymbol{I}. \tag{18}$$

From (16) it is possible to obtain the following relations (Shariff, 2008)

$$\left(\frac{\partial \Omega}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial \hat{\Omega}}{\partial \lambda_i}(i \text{ not summed}) \tag{19}$$

and

$$\left(\frac{\partial \Omega}{\partial \boldsymbol{C}}\right)_{ij} = \frac{\left(\frac{\partial \hat{\Omega}}{\partial \zeta_i} - \frac{\partial \hat{\Omega}}{\partial \zeta_j}\right)}{\left(\lambda_i^2 - \lambda_j^2\right)}\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j \ \ i \neq j, \tag{20}$$

where $(\partial\Omega/\partial\boldsymbol{C})_{ij}$ are the components of $\partial\Omega/\partial\boldsymbol{C}$ relative to the basis $\{\boldsymbol{e}_i\}$ and $\boldsymbol{A} = \boldsymbol{a}\otimes\boldsymbol{a}$. It is assumed that Ω has sufficient regularity to ensure that, as λ_i approaches λ_j, $i \neq j$, equation (20) has a limit.

Using (19), (20) in $(18)_1$ we obtain the principal axis components for the total stress $\boldsymbol{\tau}$ tensor for a compressible solid

$$\tau_{ii} = \frac{\lambda_i}{J}\frac{\partial \hat{\Omega}}{\partial \lambda_i}(i \text{ not summed}), \tag{21}$$

$$\tau_{ij} = \frac{2\lambda_i\lambda_j}{J}\frac{\left(\frac{\partial \hat{\Omega}}{\partial \zeta_i} - \frac{\partial \hat{\Omega}}{\partial \zeta_j}\right)}{\left(\lambda_i^2 - \lambda_j^2\right)}\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j, \tag{22}$$

while for an incompressible solid since $J = 1$, from $(18)_2$, the components take the form

$$\tau_{ii} = \lambda_i\frac{\partial \hat{\Omega}}{\partial \lambda_i} - p, \quad i \text{ not summed}, \tag{23}$$

$$\tau_{ij} = 2\lambda_i\lambda_j\frac{\left(\frac{\partial \hat{\Omega}}{\partial \zeta_i} - \frac{\partial \hat{\Omega}}{\partial \zeta_j}\right)}{\left(\lambda_i^2 - \lambda_j^2\right)}\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j, \quad i \neq j, \tag{24}$$

where $\lambda_1\lambda_2\lambda_3 = 1$.

Regarding $(8)_2$ for $\boldsymbol{H}_l$ it is possible to prove that

$$\boldsymbol{H}_l = \frac{\partial \Omega}{\partial \boldsymbol{B}_l} = \frac{\partial \hat{\Omega}}{\partial B}\boldsymbol{a} + \frac{2}{B}\sum_{k=1}^{3}\frac{\partial \hat{\Omega}}{\partial \zeta_k}[(\boldsymbol{a}\cdot\boldsymbol{e}_k)\boldsymbol{e}_k - \zeta_k\boldsymbol{a}], \tag{25}$$

[1] For simplicity and brevity in most of this work we only consider $\boldsymbol{B}_l$ as the independent magnetic variable, but it is possible to reformulate everything considering instead $\boldsymbol{H}_l$ as the independent magnetic variable, see §3.7 of Dorfmann et al. (2004b) and Section 5 of the present work.

[2] See, for example: Eqs. (4.4) and (4.9) of Brigadnov and Dorfmann (2003); Eq. (4.9) of Dorfmann and Ogden (2003); Eqs. (16) and (17) of Dorfmann et al. (2004a); Eqs. (3.21) and (3.22) of Dorfmann et al. (2004b) and Eq. (88) of Steigmann (2004).

and from $(9)_2$ the expression for $\boldsymbol{H}$ is given as

$$\boldsymbol{H} = \sum_{i=1}^{3}\left\{\frac{1}{\lambda_i}\left[\frac{\partial\widehat{\Omega}}{\partial B}+\frac{2}{B}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i}-\sum_{k=1}^{3}\frac{\partial\widehat{\Omega}}{\partial\zeta_k}\zeta_k\right)\right](\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{v}_i\right\}, \tag{26}$$

where $\boldsymbol{v}_i$ is the principal direction of the left stretch tensor $\boldsymbol{V}$. It is possible to show that (21), (22) and (26) are equivalent to the expressions for the total stress and magnetic field considering the usual invariants (17); such connections are presented in the Appendix A.

3.1. The undeformed configuration

An interesting special case to consider is when there is an external magnetic induction applied on a body, but there is no deformation, i.e. $\boldsymbol{F}=\boldsymbol{I}$, which implies $\lambda_1=\lambda_2=\lambda_3=1$. Let us assume that $\boldsymbol{B}_l$ is constant and that $\boldsymbol{a}=\boldsymbol{e}_3$ such that $\zeta_3=1$, $\zeta_1=\zeta_2=0$. From (21), (22) for compressible bodies and for incompressible bodies (23), (24) we can see that is necessary to apply an external traction such that the body remains undeformed. For example, from (23)–(25) for incompressible bodies we have

$$\boldsymbol{\tau}_0 = \sum_{i=1}^{3}\frac{\partial\widehat{\Omega}}{\partial\lambda_i}(B,1,1,1,0,0,1)\boldsymbol{e}_i\otimes\boldsymbol{e}_i - p\boldsymbol{I}, \tag{27}$$

$$\boldsymbol{H}_0 = \Omega_0'(B)\boldsymbol{e}_3, \tag{28}$$

where

$$\Omega_0'(B) = \frac{\partial\widehat{\Omega}}{\partial B}(B,1,1,1,0,0,1). \tag{29}$$

4. Solution of some boundary value problems

In this section we solve some boundary value problems considering the expressions for the constitutive equations using the principal axis invariants. For brevity we only look for solutions for incompressible bodies. Three problems are studied: the triaxial uniform extension of a slab, the uniform shear and the extension and inflation of a thick-walled tube. These three problems have been treated in the literature using the classical set of invariants given in[3] (17), see for example, §3.1, 3.2 and 4 of Dorfmann and Ogden (2005).

4.1. Triaxial extension of a slab

We consider the pure homogeneous deformation defined by

$$x_1 = \lambda_1 X_1,\; x_2 = \lambda_2 X_2,\; x_3 = \lambda_3 X_3, \tag{30}$$

where x_i and X_i are the Cartesian components of $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively.

For this deformation $\boldsymbol{F} = \boldsymbol{U} \equiv \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$ and the principal axes of the deformation coincide with the Cartesian coordinate directions and are fixed as the values of the stretches change. Here we consider a slab of material of uniform finite thickness h, with (plane) faces normal to the X_3 direction and very large in the X_1 and X_2 directions, i.e. the slab is defined by $-h/2 \le X_3 \le h/2$, $-L \le X_1 \le L$, $-L \le X_2 \le L$, where $h \ll L$. We assume the slab is subject to a magnetic induction field $\boldsymbol{B}_l$, where, for simplicity, we consider only when both B and $\boldsymbol{a}$ are independent of $\boldsymbol{X}$ so that (3) are satisfied automatically.

For an incompressible MS body we define

$$\tilde{\Omega}(B,\lambda_1,\lambda_2,\zeta_1,\zeta_2,\zeta_3) = \widehat{\Omega}(B,\lambda_1,\lambda_2,\lambda_3=1/(\lambda_1\lambda_2),\zeta_1,\zeta_2,\zeta_3), \tag{31}$$

with the symmetrical property $\tilde{\Omega}(B,\lambda_1,\lambda_2,\zeta_1,\zeta_2,\zeta_3) = \tilde{\Omega}(B,\lambda_2,\lambda_1,\zeta_2,\zeta_1,\zeta_3)$.

From (23), (24) it is possible to show that the stress components take the form (see, for example, Shariff, 2008)

$$\tau_{11}-\tau_{33} = \lambda_1\frac{\partial\tilde{\Omega}}{\partial\lambda_1},\quad \tau_{22}-\tau_{33} = \lambda_2\frac{\partial\tilde{\Omega}}{\partial\lambda_2}, \tag{32}$$

$$\tau_{12} = \frac{2\left(\frac{\partial\tilde{\Omega}}{\partial\zeta_1}-\frac{\partial\tilde{\Omega}}{\partial\zeta_2}\right)}{\left(\lambda_1^2-\lambda_2^2\right)}\lambda_1\lambda_2\boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_2,$$

$$\tau_{13} = \frac{2\left(\frac{\partial\tilde{\Omega}}{\partial\zeta_1}-\frac{\partial\tilde{\Omega}}{\partial\zeta_3}\right)}{\left(\lambda_1^2-\lambda_3^2\right)}\lambda_1\lambda_3\boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_3,\; \tau_{23} = \frac{2\left(\frac{\partial\tilde{\Omega}}{\partial\zeta_2}-\frac{\partial\tilde{\Omega}}{\partial\zeta_3}\right)}{\left(\lambda_2^2-\lambda_3^2\right)}\lambda_2\lambda_3\boldsymbol{e}_2\cdot\boldsymbol{A}\boldsymbol{e}_3. \tag{33}$$

The components of the total stress tensor are constant and so (11) is satisfied automatically.

A special case: When the direction of the magnetic induction is such that $\boldsymbol{a}=\boldsymbol{e}_3$ (i.e. the field is applied in the direction of the thickness of the plate), from (15) we have $\zeta_3=1$, $\zeta_1=\zeta_2=0$, and from (32)–(33) we obtain

$$\tau_{11}-\tau_{33} = \lambda_1\frac{\partial\tilde{\Omega}}{\partial\lambda_1},\; \tau_{22}-\tau_{33} = \lambda_2\frac{\partial\tilde{\Omega}}{\partial\lambda_2},\; \tau_{12}=\tau_{23}=\tau_{13}=0 \tag{34}$$

while from (14), $(4)_1$, (6) and (26) we have

$$\boldsymbol{B} = \lambda_3 B\boldsymbol{e}_3,\; \boldsymbol{H} = \frac{1}{\lambda_3}\frac{\partial\tilde{\Omega}}{\partial B}\boldsymbol{e}_3,\; \boldsymbol{M} = \left(\lambda_3 B - \frac{\mu_0}{\lambda_3}\frac{\partial\tilde{\Omega}}{\partial B}\right)\boldsymbol{e}_3. \tag{35}$$

We assume the slab is surrounded by vacuum, considering the continuity conditions $(12)_{1,2}$ for the surfaces $X_3=\pm h/2$, the magnetic induction in vacuum must be the same as within the slab (see Eq. $(12)_1$), therefore from (5) and (13) the non-zero Cartesian components of the Maxwell stress tensor are

$$\tau_{M_{11}} = \tau_{M_{22}} = -\frac{1}{2\mu_0}B^2\lambda_3^2 = -\tau_{M_{33}}. \tag{36}$$

If no mechanical traction is supplied to the plane faces of the slab $X_3=\pm h/2$, then from $(12)_3$ we have $\tau_{33}=\tau_{M_{33}}$ and from (34) we have

$$\tau_{11} = \frac{1}{2\mu_0}B^2\lambda_3^2 + \lambda_1\frac{\partial\tilde{\Omega}}{\partial\lambda_1},\; \tau_{22} = \frac{1}{2\mu_0}B^2\lambda_3^2 + \lambda_1\frac{\partial\tilde{\Omega}}{\partial\lambda_2}. \tag{37}$$

Since $L \gg h$ as an approximation we do not require (12) to be satisfied across the surfaces $X_1=\pm L$ and $X_2=\pm L$, i.e. for a very large plate in the directions 1 and 2 we are neglecting 'edge' effects, and assuming then that the uniform field (30), (35) is a solution of the boundary value problem.

Let us specialize to the case of equibiaxial deformation, where $\lambda_1=\lambda_2=\lambda$ and $\lambda_3=\lambda^{-2}$, from (37) we obtain

$$\tau_{11} = \tau_{22} = \frac{1}{2\mu_0\lambda^4}B^2 + \lambda\frac{\partial\tilde{\Omega}}{\partial\lambda_1}(B,\lambda,\lambda,0,0,1), \tag{38}$$

taking note that due the symmetry in $\tilde{\Omega}$

[3] See additionally, for example: Pucci and Saccomandi (1993); §5 of Brigadnov and Dorfmann (2003) for some results on the shear problem; §5 of Dorfmann and Ogden (2003); Dorfmann et al. (2004a) and §4 of Dorfmann et al. (2004b) for some problems considering cylindrical geometries.

$$\frac{\partial\tilde{\Omega}}{\partial\lambda_1}(B,\lambda,\lambda,0,0,1)=\frac{\partial\tilde{\Omega}}{\partial\lambda_2}(B,\lambda,\lambda,0,0,1). \quad (39)$$

In the reference configuration we have

$$\tau_{11}=\tau_{22}=\frac{1}{2\mu_0}B^2+\frac{\partial\tilde{\Omega}}{\partial\lambda_1}(B,1,1,0,0,1) \quad (40)$$

which is the initial lateral stress needed to prevent deformation due to the magnetic field.

4.2. Simple shear

In this section we present some results for a simple shear deformation, in this case for the slab $-L\le X_1\le L$, $-h/2\le X_2\le h/2$, $-L\le X_3\le L$ with $h\ll L$. In this problem the principal directions of $\boldsymbol{U}$ change continuously during deformation. Let the axes of $\boldsymbol{x}$ and $\boldsymbol{X}$ to coincide such that the deformation can be described by the equations:

$$x_1=X_1+\gamma X_2,\ x_2=X_2,\ x_3=X_3, \quad (41)$$

where $0\le\gamma$ is commonly called *the amount of shear*. Let θ denote the orientation (in the anticlockwise sense relative to the X_1 axis) of the in plane Lagrangian principal axes. The angle θ is restricted according by the following

$$\frac{\pi}{4}\le\theta<\frac{\pi}{2}. \quad (42)$$

The Cartesian matrix for the deformation tensor $\boldsymbol{F}$ is

$$\boldsymbol{F}\equiv\begin{pmatrix}1&\gamma&0\\0&1&0\\0&0&1\end{pmatrix}. \quad (43)$$

The principal directions have components:

$$\boldsymbol{e}_1\equiv\begin{pmatrix}c\\s\\0\end{pmatrix}\ \boldsymbol{e}_2\equiv\begin{pmatrix}-s\\c\\0\end{pmatrix}\ \boldsymbol{e}_3\equiv\begin{pmatrix}0\\0\\1\end{pmatrix}, \quad (44)$$

where $c=\cos(\theta)$ and $s=\sin(\theta)$. It can be easily shown that the principal stretches take the values

$$\lambda_1=\frac{\gamma+\sqrt{\gamma^2+4}}{2}\ge 1,\ \lambda_2=\frac{1}{\lambda_1}=\frac{\sqrt{\gamma^2+4}-\gamma}{2}\le 1,\ \lambda_3=1, \quad (45)$$

where we have the connections

$$c=\frac{1}{\sqrt{1+\lambda_1^2}},\ s=\frac{\lambda_1}{\sqrt{1+\lambda_1^2}},\ c^2-s^2=-\gamma cs. \quad (46)$$

Without loss of generality, we consider $\tau_{33}=0$, since incompressibility allows the superposition of an arbitrary hydrostatic stress without effecting the deformation.

From (21), (22) the Cartesian components of the total stress take the form:

$$\begin{aligned}\tau_{11}&=2\Big[l_1c^2+l_2s^2-2l_4cs+2\gamma\Big((l_1-l_2)cs+l_4\big(c^2-s^2\big)\Big)\\&\quad+\gamma^2\big(l_1s^2+l_2c^2+2l_4cs\big)-l_3\Big],\\ \tau_{12}&=2\Big((l_1-l_2)cs+l_4\big(c^2-s^2\big)+\gamma\big(l_1s^2+l_2c^2+2l_4cs\big)\Big),\\ \tau_{22}&=2\big(l_1s^2+l_2c^2+2l_4cs-l_3\big),\\ \tau_{13}&=2(l_5c-l_6s+\gamma(l_5s+l_6c)),\\ \tau_{23}&=2(l_5s+l_6c),\end{aligned} \quad (47)$$

where

$$\begin{aligned}&l_1=\frac{1}{2\lambda_1}\frac{\partial\widehat{\Omega}}{\partial\lambda_1},\ l_2=\frac{1}{2\lambda_2}\frac{\partial\widehat{\Omega}}{\partial\lambda_2},\ l_3=\frac{1}{2\lambda_3}\frac{\partial\widehat{\Omega}}{\partial\lambda_3},\\ &l_4=\frac{\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_1}-\frac{\partial\widehat{\Omega}}{\partial\zeta_2}\right)}{\left(\lambda_1^2-\lambda_2^2\right)}\boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_2,\ l_5=\frac{\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_1}-\frac{\partial\widehat{\Omega}}{\partial\zeta_3}\right)}{\left(\lambda_1^2-\lambda_3^2\right)}\boldsymbol{e}_1\cdot\boldsymbol{A}\boldsymbol{e}_3,\\ &l_6=\frac{\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_2}-\frac{\partial\widehat{\Omega}}{\partial\zeta_3}\right)}{\left(\lambda_2^2-\lambda_3^2\right)}\boldsymbol{e}_2\cdot\boldsymbol{A}\boldsymbol{e}_3.\end{aligned} \quad (48)$$

In general, the Poynting relation of the isotropic theory, i.e. $\tau_{11}-\tau_{22}=\gamma\tau_{12}$, does not hold, however, when $\boldsymbol{a}$ is parallel to $\boldsymbol{e}_3$ (fixed), we have, $l_4=l_5=l_6=0$, the Cauchy stress is coaxial with the left stretch tensor $\boldsymbol{V}$ and the response of the transversely strain energy is similar to an isotropic material, and we can easily show that the Poynting relation holds.

In the case when the direction of the magnetic induction is perpendicular to the direction[4] $\boldsymbol{e}_3$, from (47) the shear components of the total stress tensor are $\tau_{13}=\tau_{23}=0$, and from (44), (45) and (46) we have the relations

$$\frac{\partial\lambda_1}{\partial\gamma}=s^2,\ \frac{\partial\lambda_2}{\partial\gamma}=-c^2,\ \frac{\partial\zeta_1}{\partial\gamma}=2\lambda_1sc^3(\boldsymbol{a}\cdot\boldsymbol{e}_1)(\boldsymbol{a}\cdot\boldsymbol{e}_2),\ \frac{\partial\zeta_2}{\partial\gamma}=-\frac{\partial\zeta_1}{\partial\gamma}. \quad (49)$$

In, general, the Poynting relation does not hold.

Since a simple shear deformation depends on γ, the strain energy function can be considered as a function of γ and B, i.e. $\Omega_M=\Omega_S(\gamma,B)$. We can easily deduce that, for $\boldsymbol{a}\bullet\boldsymbol{e}_3=0$ we have

$$\tau_{12}=\frac{\partial\Omega_S}{\partial\gamma}. \quad (50)$$

It is important to discuss about the magnetic field produced in this deformation and the continuity conditions (12). For this problem, as an approximation, we only consider (12) for the surfaces $X_2=\pm h/2$. Since $\boldsymbol{B}_l=B\boldsymbol{a}$ with $\boldsymbol{a}\bullet\boldsymbol{e}_3=0$, from $(4)_2$ and (43) we have for the normal component of the magnetic induction to the surfaces $X_2=\pm h/2$ in the current configuration, which we denote $B_\perp$, is given as $B_\perp=Ba_2$. On the other hand, from (26) we have for the components of the magnetic field that are tangent to the surface $X_2=\pm h/2$ are given as $H_1=\sum_{i=1}^2\left\{\frac{\sqrt{\zeta_i}}{\lambda_i}\left[\frac{\partial\widehat{\Omega}}{\partial B}+\frac{2}{B}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i}-\sum_{k=1}^3\frac{\partial\widehat{\Omega}}{\partial\zeta_k}\zeta_k\right)\right]v_{i_1}\right\}$, and $H_3=\sum_{i=1}^2\left\{\frac{\sqrt{\zeta_i}}{\lambda_i}\left[\frac{\partial\widehat{\Omega}}{\partial B}+\frac{2}{B}\left(\frac{\partial\widehat{\Omega}}{\partial\zeta_i}-\sum_{k=1}^3\frac{\partial\widehat{\Omega}}{\partial\zeta_k}\zeta_k\right)\right]v_{i_3}\right\}$, where v_{i_1} and v_{i_3}, $i=1,2$ are the components of $\boldsymbol{v}_i$ in the directions 1, 3, respectively. If $\boldsymbol{B}^{(o)}$ denotes the magnetic field in vacuum, considering the previous results and $(12)_{1,2}$, the components of $\boldsymbol{B}^{(o)}$ are given as

$$B_1^{(o)}=\frac{1}{\mu_0}H_1,\quad B_2^{(o)}=Ba_2,\quad B_3^{(o)}=\frac{1}{\mu_0}H_3. \quad (51)$$

For the previous external magnetic induction from (5), (13) for the Maxwell stress tensor we have the components:

[4] In this problem we also assume that B and $\boldsymbol{a}$ do not depend on $\boldsymbol{X}$, i.e. the field is homogeneous.

$$\tau_{M_{11}} = \frac{1}{2\mu_0}\left(B_1^{(o)2} - B_2^{(o)2}\right), \quad \tau_{M_{22}} = \frac{1}{2\mu_0}\left(B_2^{(o)2} - B_1^{(o)2}\right), \tag{52}$$

$$\tau_{M_{33}} = -\frac{1}{2\mu_0}\left(B_1^{(o)2} + B_2^{(o)2}\right), \quad \tau_{M_{12}} = \frac{1}{\mu_0} B_1^{(o)} B_2^{(o)}. \tag{53}$$

Now, as mentioned previously, for the case $\boldsymbol{a}\bullet\boldsymbol{e}_3 = 0$ we have $\tau_{13}=\tau_{23}=0$, then for the surface $X_2 = \pm h/2$ considering (12)$_3$, (52)$_2$ and (53)$_2$ we have

$$\tau_{12} = \tau_{M_{12}} + \hat{t}_1, \quad \tau_{22} = \tau_{M_{22}} + \hat{t}_2. \tag{54}$$

As a summary, for given γ, B and $\boldsymbol{a}$ (such that $\boldsymbol{a}\bullet\boldsymbol{e}_3 = 0$), Eqs. (51) and (54) give the expressions for the external magnetic field $\boldsymbol{B}^{(o)}$ and the external mechanical traction $\hat{\boldsymbol{t}}$, which are necessary to produce the deformation (41). This is only valid if $h \ll L$, i.e. neglecting the continuity conditions (12) for the surfaces $X_1 = \pm L$ and $X_2 = \pm L$. This point is very important in order to apply these results when fitting experimental data, since in reality we can only work with finite size blocks.

4.3. Extension and inflation of a thick-walled tube

In this last example we study the problem of an incompressible MS thick-walled circular cylindrical tube, under inflation and extension. The tube initial geometry is defined by

$$R_i \le R \le R_o, \ 0 \le \Theta \le 2\pi, \ 0 \le Z \le L, \tag{55}$$

where R_i, R_o, L are positive constants and R, Θ, Z are cylindrical polar coordinates. We assume the tube is deformed as

$$r^2 - r_i^2 = \frac{1}{\lambda_z}\left(R^2 - R_i^2\right), \ \theta = \Theta, \ z = \lambda_z Z, \tag{56}$$

where r_i is the internal radius of the deformed tube, r, θ and z are cylindrical polar coordinates in the deformed configuration and $\lambda_z > 0$ (constant) is the axial stretch.

The principal stretches for this deformation are given by (see, for example, Shariff, 2008)

$$\lambda_1 = \frac{1}{\lambda\lambda_z}, \ \lambda_2 = \lambda = \frac{r}{R}, \ \lambda_3 = \lambda_z. \tag{57}$$

It can be easily shown that $\boldsymbol{F} \equiv \mathrm{diag}(1/(\lambda\lambda_z), \lambda, \lambda_z)$ and the principal directions are

$$\boldsymbol{e}_1 = \boldsymbol{E}_R, \ \boldsymbol{e}_2 = \boldsymbol{E}_\Theta, \ \boldsymbol{e}_3 = \boldsymbol{E}_Z, \tag{58}$$

where $\boldsymbol{E}_R$, $\boldsymbol{E}_\Theta$, $\boldsymbol{E}_Z$ are the base vectors for the R, Θ, Z cylindrical coordinate system.

Consider the case when the magnetic induction $\boldsymbol{B}_l = B\boldsymbol{a}$ is such that

$$\boldsymbol{a} = \cos(\alpha)\boldsymbol{E}_\Theta + \sin(\alpha)\boldsymbol{E}_Z = \cos(\alpha)\boldsymbol{e}_2 + \sin(\alpha)\boldsymbol{e}_3, \tag{59}$$

where $0 \le \alpha \le \pi/2$ and $\zeta_1 = 0$ and let us assume that in general $B = B(r)$.

Considering that λ and λ_z are the independent variables, the strain energy function $\hat{\Omega}$ can be written as

$$\Omega_E(B, \lambda, \lambda_z, \zeta_2) = \hat{\Omega}(B, 1/(\lambda\lambda_z), \lambda, \lambda_z, 0, \zeta_2, 1 - \zeta_2). \tag{60}$$

From (23) and (24) it is possible to prove that the components of the total stress $\boldsymbol{\tau}$ in the cylindrical coordinate system are given by:

$$\tau_{\theta\theta} - \tau_{rr} = \lambda\frac{\partial\Omega_E}{\partial\lambda}, \ \tau_{zz} - \tau_{rr} = \lambda_z\frac{\partial\Omega_E}{\partial\lambda_z}, \tag{61}$$

$$\tau_{\theta z} = \frac{2\frac{\partial\Omega_E}{\partial\zeta_2}}{\left(\lambda^2 - \lambda_z^2\right)}\lambda\lambda_z cs, \ \tau_{r\theta} = \tau_{rz} = 0, \tag{62}$$

where $c = \cos(\alpha)$ and $s = \sin(\alpha)$. It is clear that when α=0 or α=$\pi/2$ the shear stress $\tau_{\theta z}$ is zero.

By considering the symmetry of the problem, the equation of equilibrium (11) with negligible body forces reduces to

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = 0, \tag{63}$$

and from (63) and (61) we have the solution

$$\tau_{rr} = \int_{r_i}^{r} \lambda(\xi)\frac{\partial\Omega_E}{\partial\lambda}(\xi)\mathrm{d}\xi + \tau_{rr_i}, \tag{64}$$

where $\lambda(\xi) = \xi/R(\xi)$, where $R(\xi)$ is obtained from (56)$_1$ replacing r by the auxiliary variable ξ, and τ_{rr_i} is the value of τ_{rr} at $r = r_i$.

Let us discuss now about the magnetic variables and the continuity conditions (12). Using the expression for $\mathbf{F}$ and $\boldsymbol{B}_l$ from (56) and (59) we obtain

$$\boldsymbol{B} \equiv B\begin{pmatrix} 0 \\ \lambda\cos\alpha \\ \lambda_z\sin\alpha \end{pmatrix}, \quad \boldsymbol{H} \equiv \begin{pmatrix} 0 \\ H_\theta(r) \\ H_z(r) \end{pmatrix}, \tag{65}$$

where, unlike the problems presented in Sections 4.1 and 4.2, here B is not constant and it can depend on r. For brevity we do not present the full expressions for the components of $\boldsymbol{H}$, but for this problems from (26) it is easy to see that they depend on r and $B = B(r)$. For this expression for $\boldsymbol{B}$ Eq. (3)$_1$ (in cylindrical coordinates) is satisfied automatically, but (3)$_2$ becomes $1/r(\mathrm{d}(rH_\theta)/\mathrm{d}r) = 0$ and $\mathrm{d}H_z/\mathrm{d}r = 0$. In general both equations cannot be satisfied at the same time for the same $B = B(r)$. As an illustration let us assume in (59) two cases: $\alpha = 0$ and $\alpha = \pi/2$.

4.3.1. Case $\alpha = 0$

In the first case $\alpha = 0$ from (15) we have that $\zeta_2 = 1$ and $\zeta_3 = 0$, therefore from (26) we have $H_\theta(r) = 1/\lambda(\partial\Omega_E/\partial B)$, and B must be found by solving the in general nonlinear equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{R}{r}\frac{\partial\Omega_E}{\partial B}\right) = 0 \Leftrightarrow \frac{R}{r}\frac{\partial\Omega_E}{\partial B} = c_o, \tag{66}$$

where c_o is a constant. Since[5] $\boldsymbol{B} = B\boldsymbol{E}_\Theta$ and $\boldsymbol{H} = \frac{1}{\lambda}\frac{\partial\Omega_E}{\partial B}\boldsymbol{E}_\Theta$, the continuity conditions (12)$_{1,2}$, which in the case $R_o \ll L$ we consider them only for the surfaces $R = R_i$, $R = R_o$, are satisfied if

$$B_\theta^{(o)}(r_o) = \mu_0\frac{R(r_o)}{r_o}\frac{\partial\Omega_E}{\partial B}(r = r_o), \quad B_\theta^{(o)}(r_i) = \mu_0\frac{R(r_i)}{r_i}\frac{\partial\Omega_E}{\partial B}(r = r_i), \tag{67}$$

where $B_\theta^{(o)}(r)$ is the θ component of the magnetic induction in vacuum, which for continuity is the only nonzero component of the magnetic induction outside the tube. For vacuum, since $H_\theta^{(o)}(r) = \mu_0 B_\theta^{(o)}(r)$, $B_\theta^{(o)}(r)$ should be found from solving $\mathrm{d}(rB_\theta^{(o)}(r))/\mathrm{d}r = 0$, from where we have $B_\theta^{(o)}(r) = c_1/r$, where c_1 is a

[5] In this problem the base vector for R, Θ, Z is the same as for r, θ, z.

constant. Two different such expressions can be found for the vacuum space for $r \leq r_i$ and $r \geq r_o$, and we have two constants that can be determined from the aforementioned continuity conditions (67).

Regarding the continuity condition $(12)_3$, let us assume at the surface $r = r_i$ $(R = R_i)$ there is a pressure P being applied, while on the surface $r = r_o$ $(R = R_o)$ there is no mechanical load. From (67) and (13) for the radial component of the Maxwell stress tensor evaluated at $r = r_i$ and $r = r_o$ we have

$$\tau_{M_{rr}}(r = r_i) = -\frac{\mu_0}{2}\left[\frac{R(r_i)}{r_i}\frac{\partial \Omega_E}{\partial B}(r = r_i)\right]^2$$
$$\tau_{M_{rr}}(r = r_o) = -\frac{\mu_0}{2}\left[\frac{R(r_o)}{r_o}\frac{\partial \Omega_E}{\partial B}(r = r_o)\right]^2, \tag{68}$$

then from (64) and $(68)_1$ for the surface $r = r_i$ we obtain $\tau_{rr_i} = -P - \mu_0/2[(R(r_i)/r_i)(\partial\Omega_E/\partial B)(r = r_i)]^2$, and from above and $(68)_2$ and (64) for the surface $r = r_o$ we obtain the relation

$$\int_{r_i}^{r_o} \lambda(\xi)\frac{\partial \Omega_E}{\partial \lambda}(\xi)\mathrm{d}\xi - P - \frac{\mu_0}{2}\left[\frac{R(r_i)}{r_i}\frac{\partial \tilde{\Omega}}{\partial B}(r = r_i)\right]^2 = -\frac{\mu_0}{2}\left[\frac{R(r_o)}{r_o}\frac{\partial \Omega_E}{\partial B}(r = r_o)\right]^2. \tag{69}$$

As a summary, given R_i, R_o, λ_z, P and[6] c_1 we need to solve in parallel the, in general nonlinear algebraic equations $(66)_2$ and (69), in order to find r_i and $B = B(r)$. Using the above results, from (64) we can obtain τ_{zz} and the total traction to stretch the tube $\mathcal{N}$ can be obtained from $\mathcal{N} = 2\pi\int_{r_i}^{r_o} r\tau_{zz}(r)\,\mathrm{d}r$.

4.3.2. Case $\alpha = \pi/2$

When $\alpha = \pi/2$ from (59) and (15) we have that $\zeta_2 = 0$ and $\zeta_3 = 1$ and the only nonzero components of $\boldsymbol{B}$ and $\boldsymbol{H}$ are components in the axial direction $B_z(r) = \lambda_z B(r)$ and from (26) $H_z(r) = 1/\lambda_z(\partial\Omega_E/\partial B)$. Eq. $(3)_2$ is satisfied if

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial \Omega_E}{\partial B}\right) = 0 \Leftrightarrow \frac{\partial \Omega_E}{\partial B} = c_o, \tag{70}$$

where c_o is a constant. In this case $(70)_2$ would be an in general nonlinear algebraic equation for B. From $(12)_1$ we obtain the expressions for $\boldsymbol{B}^{(o)}$ for the exterior vacuum space near the surfaces $r = r_i$ and $r = r_o$, which can be used to obtain $\boldsymbol{\tau}_M$ from (13). Considering the same boundary conditions for τ_{rr} mentioned in the previous section, we can obtain an expression similar to (69) to be used, for example, to find r_i for a given pressure P and constant c_o. For brevity we do not show details for those expressions.

5. An alternative formulation

In the above formulations, we use $\boldsymbol{B}_l$ as the independent magnetic variable and the dependent variable $\boldsymbol{H}_l$ is obtained from (8.2). However, treating $\boldsymbol{H}_l$ as the independent variable instead of $\boldsymbol{B}_l$ has certain advantages, especially in variational principal formulations and tackling some boundary value problems. Hence, it is convenient to write the energy function Ω in the form

$$\Omega^*(\boldsymbol{C}, \boldsymbol{H}_l) = \Omega(\boldsymbol{C}, \boldsymbol{B}_l) - \boldsymbol{H}_l \cdot \boldsymbol{B}_l, \tag{71}$$

where the above is obtained via the partial Legendre transformation (Dorfmann and Ogden 2004a,2004b, 2005) and it is assumed that for every $\boldsymbol{C}$ there is a one-to-one relationship between $\boldsymbol{H}_l$ and $\boldsymbol{B}_l$. The total stress for an incompressible solid takes the form

$$\boldsymbol{\tau} = 2\boldsymbol{F}\frac{\partial \Omega^*}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p^*\boldsymbol{I}, \tag{72}$$

where p^* is the associated Lagrange multiplier due to the incompressibility constraint. From (71) we then have

$$\boldsymbol{B}_l = -\frac{\partial \Omega^*}{\partial \boldsymbol{H}_l}. \tag{73}$$

For the formulation in this section we let

$$\boldsymbol{a} = \frac{1}{H}\boldsymbol{H}_l,\ H \neq 0, \tag{74}$$

where $H = |\boldsymbol{H}_l|$. In terms of the principal axis invariants, the energy function is of the form

$$\Omega_M = \Omega_H(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, H), \tag{75}$$

where ζ_i, $i = 1,2,3$ are calculated as in (15) using in this case $\boldsymbol{a}$ from (74). The rest of the formulations are similar to the formulations where $\boldsymbol{B}_l$ is treated as an independent variable; hence we will omit them in this section.

6. On the constitutive equation of an isotropic MS elastomer

If the energy function is written explicitly in terms of the classical in invariants, in view of (17) we can also expressed it explicitly in terms of principal axis invariants; hence, isotropic MS problems can be formulated using both classical and principal axis invariants. However, explicit expressions for principal axis invariants in terms of classical invariants are not straightforward, especially when two or three of the principal stretches have the same value (Shariff, 2008). Hence, if on the onset, we write the energy function in terms of the principal axis invariants, we can only formulate MS problems using principal axis formulations as given in this paper. In view of this, writing the energy function in terms of principal axis invariants is more general (this is evident in dealing with non MS purely isotropic materials (Ogden, 1997)) than writing in terms of classical invariants.

In the past, when a transversely isotropic material was modelled[7], due to an unclear physical meaning of most of the classical invariants and the complexity in including all the invariants, only a subset of the full invariants had usually been included in the strain energy function. For example, a strain energy of the form $W(I_1, I_4)$ for an incompressible solid has been commonly presented in the literature (see, for example, Dorfmann and Ogden, 2003; Dorfmann et al., 2004a). However, the authors are not sure whether this form can fully characterise a transversely isotropic material. Discussions on which classical invariants should or should not be included in the strain energy function is ongoing and the outcome is not clear. In addition to this, a strain energy function written in terms of the classical invariants is not experimentally attractive. For example, a simple isochoric deformation, such as a uniaxial stretch in the preferred direction, perturbs the invariants I_1, I_2, I_4 and I_5 and a pure dilatation deformation perturbs all the

[6] The constant c_1 is associated to the magnetic induction outside the body through $B_\theta^{(o)} = c_1/r$ for either the exterior $r \geq r_o$ or the interior vacuum space $r \leq r_i$.

[7] Here by transversely isotropic body we mean the magneto-active isotropic elastomer, which shows a behaviour similar to a transversely isotropic material considering only elastic deformations (no coupling with magnetic fields).

invariants. These deformations are not ideal in obtaining a specific form of the strain energy if the specific form is determined by doing rigorous tests, which hold four out of five invariants constant so that the dependence in the remaining invariant can be identified. However, using principal axis invariants, it is shown in Shariff (2008) that it is possible to conduct a test can vary a single invariant while keeping the remaining invariants fixed.

Because of their immediate physical meaning it is clear that we should include all the principal axis invariants in the energy function; for example, $I_5 = I_5(\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\zeta_3,B)$ itself depends on all the principal axis invariants. By writing the energy function in terms of the principal axis invariants directly we avoid its indirect principal axis representation via the restricted forms of classical invariants. The authors believe that these restricted forms maybe one of the reasons that there is no clear and general methodology on how to choose which classical invariants are suitable. Since, all the principal axis invariants in the energy function have to be included, future principal axis invariant workers will not waste their valuable time and energy discussing which invariants should or should not be included in the energy function as in the case of the classical invariants. Until recently, energy functions written directly in terms of principal axis invariants did not exist in the literature. Here, based on the recent work of Shariff (2008), a general incompressible energy function for an MS isotropic body could be written as

$$\Omega_M = \sum_{i=1}^{3} r(\lambda_i,\zeta_i,\beta) + \widehat{g}(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\beta) + \widehat{g}(\lambda_1,\lambda_3,\zeta_1,\zeta_3,\beta) + \widehat{g}(\lambda_2,\lambda_3,\zeta_2,\zeta_3,\beta), \tag{76}$$

where $\beta = B$ or H and ζ_i depends on $\boldsymbol{B}_l$ or $\boldsymbol{H}_l$. The function $\widehat{g}$ has the symmetry $\widehat{g}(x,y,\phi,\psi,B) = \widehat{g}(y,x,\psi,\phi,B)$ and $\widehat{\Omega}$ should be independent of ζ_i and ζ_j, $i \neq j$ when $\lambda_i = \lambda_j$, and independent of ζ_1, ζ_2, ζ_3, when $\lambda_1 = \lambda_2 = \lambda_3$. Below, as an example, we give a simple specific form for r and $\widehat{g}$ (for moderate strain values), i.e.

$$r(\lambda_i,\zeta_i,H) = \mu ln(\lambda_i)^2 + c_0\left|\frac{\lambda_i^2}{2} - 2\lambda_i + ln(\lambda_i) + 1.5\right| + c_1\frac{\zeta_i\nu_0(H)}{\lambda_i^2+\lambda_i} - \mu_0\frac{\zeta_i H^2}{2\lambda_i^2} + \mu_0\nu_0(H)e^{\frac{1}{H^2}}\frac{\zeta_i}{2\lambda_i^2} \tag{77}$$

and $\widehat{g} = 0$. For simplicity we let $\nu_0(H) = H^2$. We note that this is just an illustrative example, a more sophisticated specific form could easily be constructed if required. It is clear from the above that the energy Ω has a unique value if two or more of the principal stretches have the same value. We note that μ is the ground state shear modulus of the isotropic body and $\mu_0 = 1.2566 \times 10^{-3}$ kN/kA^2. All the other constants are ground state constants and hence we could easily put restrictions on their values for physically reasonable responses; however, we shall not do it here. Shariff (2013b) have used a form similar to (77) in biomechanics. In general, the constitutive equation (77) cannot be simply and explicitly expressed in terms of classical invariants and, hence, are not suitable for classical invariant formulations. Using the above simple form, we show in Fig. 1, that the function (77) compares well with the uniaxial experimental data of Bellan and Bossis (2002) (See Appendix B for uniaxial deformation theoretical results when $\beta = H_0$). The stress–strain behaviour depicted in Fig. 2 for different values of H is similar to that of Kankanala and Triantafyllidis (2004). Fig. 2 indicates that to maintain a zero strain under an external magnetic field, a tensile stress is required to overcome the attractive interparticle forces and for some range

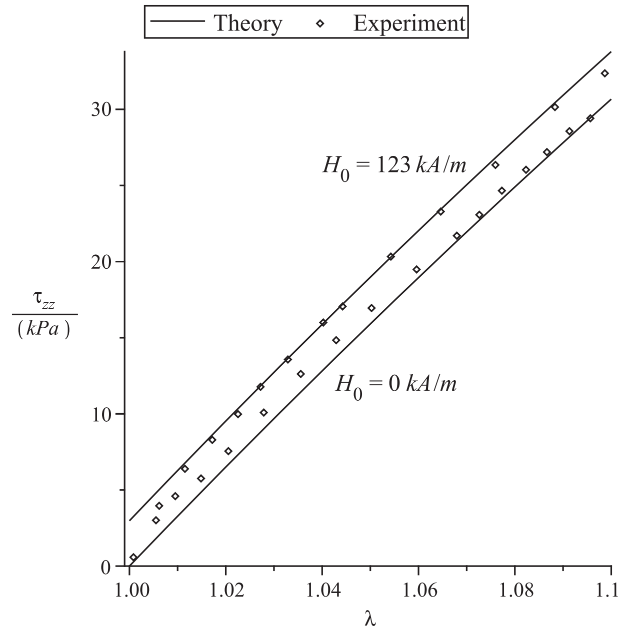

Fig. 1. Bellan and Bossis (2002) uniaxial experiment. $\mu = 110$, $c_0 = -100$, $c_1 = -.0011$.

of compressive strains the magnitude of the compressive traction increases with increasing applied magnetic field; indicating repulsive forces when the particle distance is considerably diminished. The tensile stress is given by

$$\tau_{ZZ} = \lambda s(\lambda) - \lambda^{-\frac{1}{2}}S\left(\lambda^{-\frac{1}{2}}\right) - c_1 H_0^2\frac{2\lambda^2+\lambda}{\left(\lambda^2+\lambda\right)^2} + \frac{\mu_0 H_0^2}{2\lambda^2}\left[1 - 2e^{-\left(\frac{1}{H_0^2}\right)}\right] \tag{78}$$

where

$$s(x) = 2\mu ln(x) + c_0(x-1)^2 . \tag{79}$$

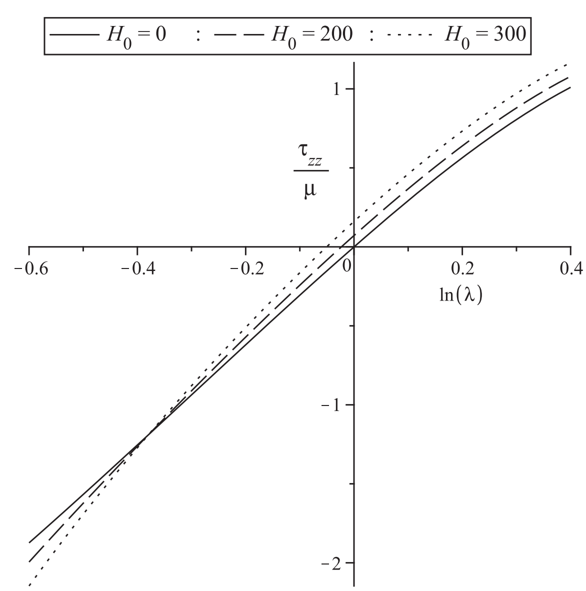

Fig. 2. Theoretical uniaxial stress strain curves for $H_0 = 0$, 200, 300 kA/m. Note that for $H_0 \neq 0$, a positive traction is needed to prevent magnetostriction at $\lambda = 1$. $\mu = 110$, $c_0 = -100$, $c_1 = -0.0011$.

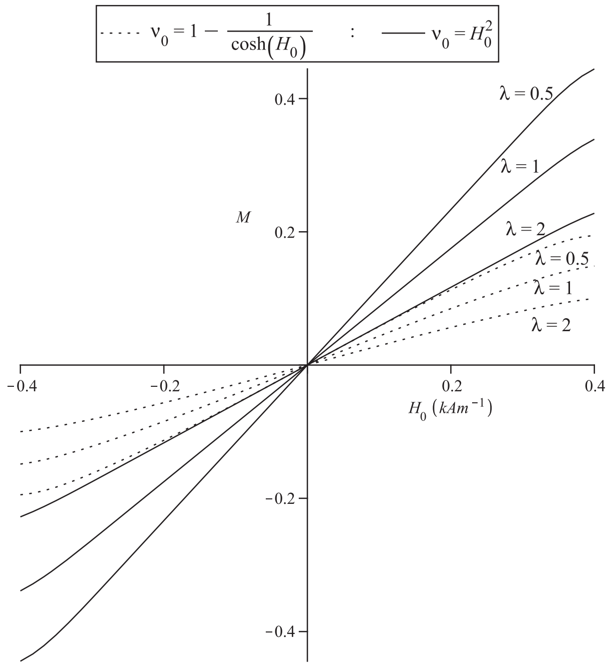

Fig. 3. Magnetization M vs. magnetic field H_0 for the cylindrical body for $\nu_0 = H_0^2$ and $\nu_0 = 1 - \cosh^{-1}(H_0)$. For extension ($\lambda > 1$) the particle distance increases thus lowering the specimen's magnetization compared to $\lambda = 1$. The opposite is true for compression ($\lambda < 1$) due to shorter distances among the magnetic particles. $\mu = 110$, $c_0 = -100$, $c_1 = -0.0011$.

It is evident from (78) that the influence of H_0 on the tensile stress τ_{zz} diminishes as λ increases. This is due to the average interparticle distance increases and the interparticle forces are weaker for the same imposed H_0, hence the influence of H_0 on the tensile stress τ_{zz} diminishes. In Fig. 3 we depict the behaviour of the magnetization $\boldsymbol{M}$ with respect to $\boldsymbol{H}_l$ for several uniaxial strain values and it shows that for extension ($\lambda > 1$) the particle distance increases thus lowering the specimen's magnetization compared to λ=1. The opposite is true for compression ($\lambda < 1$) due to shorter distances among the magnetic particles. For illustrative purposes the behaviour for $\nu_0 = H_0^2$ and $\nu_0 = 1 - \cosh^{-1}(H_0)$ is shown in Fig. 3. In view of the above illustrations, we note that a construction of more sophisticated specific constitutive equation is trivial and we will do this in the future when the appropriate experimental data is available.

7. Conclusions

In this work we have proposed a principal set of invariants in order to study problems in nonlinear magneto-elasticity. This new set of invariants which can characterise a more general MS elastomer is based on the recent work by Shariff (2008), where the case of purely elastic deformations was treated considering transversely isotropic bodies. The invariants used in this work can be more attractive than the standard invariants presented in the literature (see Eq. (17)), because a clearer physical meaning can be attached to each one of them, and because when solving boundary value problems, the different expressions for the stresses in terms of the deformation and the magnetic induction are simpler than when considering the standard theory, see, for example, Eqs. (32), (33) and (61), (62) for the problems of biaxial extension, simple shear and inflation and extension of a tube, and compare, for example, with the results presented in §3.1, 3.2 and 4 of Dorfmann and Ogden (2005).

In future works, the case of transversely isotropic MS bodies will be treated. Such materials are produced, for example, if a magnetic field or magnetic induction is applied on the matrix rubber during the curing process, when MS particles are being added to the matrix (Bica, 2012). Because of the magnetic induction field, the particles align in a preferred direction, and the resultant solid behaves as an orthotropic body, because it has already a preferred direction due to the chain of particles, plus the additional preferred direction caused by the external field. This complex problem, which requires the use of several invariants (see Bustamante, 2010) will be treated in a future work using as a basis the recent works of Shariff (2011, 2013a). Another problem to be treated following the same procedure to define invariant as presented here, will be the case of considering electro–elastic interactions, for a body which has two families of fibres when there is no electric field (Bustamante and Merodio, 2011); that problem could be of interests in biomechanics.

Appendix A. Relations with the standard set of invariants

In this section we explore the connections of (21), (22) and (26) with the expressions for the stresses and the magnetic field, considering the total energy function expressed in terms of the set of invariants (17).

Let us study first the case of the expression for the magnetic field. From (14), (15) we have

$$\zeta_i = \frac{(\boldsymbol{B}_l \cdot \boldsymbol{a})^2}{B^2}, \tag{A1}$$

therefore

$$\frac{\partial \zeta_i}{\partial \boldsymbol{B}_l} = \frac{2}{B}\left[(\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i - \zeta_i \boldsymbol{a}\right].. \tag{A2}$$

Now, considering the notation $\Omega = \widehat{\Omega}(B, \lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3)$, we have

$$\frac{\partial \Omega}{\partial \boldsymbol{B}_l} = \frac{\partial \widehat{\Omega}}{\partial B}\boldsymbol{a} + \frac{2}{B}\sum_{i=1}^{3}\frac{\partial \widehat{\Omega}}{\partial \zeta_i}\left[(\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i - \zeta_i \boldsymbol{a}\right]. \tag{A3}$$

For $\widehat{\Omega}(B, \lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) = \overline{\Omega}(I_1, I_2, I_3, I_4, I_5, I_6)$, where I_i, $i = 1,2,3,4,5,6$ are defined in (17). From (25) we have

$$\begin{aligned}\boldsymbol{H}_l = \frac{\partial \Omega}{\partial \boldsymbol{B}_l} &= \boldsymbol{a}\sum_{k=4}^{6}\frac{\partial \overline{\Omega}}{\partial I_k}\frac{\partial I_k}{\partial B} + \sum_{i=1}^{3}\left[\frac{\partial \overline{\Omega}}{\partial I_5}\frac{\partial I_5}{\partial \zeta_i} + \frac{\partial \overline{\Omega}}{\partial I_6}\frac{\partial I_6}{\partial \zeta_i}\right]\frac{\partial \zeta_i}{\partial \boldsymbol{B}_l},\\ &= 2\frac{\partial \overline{\Omega}}{\partial I_4}\boldsymbol{B}_l + 2B\frac{\partial \overline{\Omega}}{\partial I_5}\sum_{i=1}^{3}\lambda_i^2(\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i + 2B\frac{\partial \overline{\Omega}}{\partial I_6}\sum_{i=1}^{3}\lambda_i^4(\boldsymbol{a}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i,\\ &= 2\frac{\partial \overline{\Omega}}{\partial I_4}\boldsymbol{B}_l + 2B\frac{\partial \overline{\Omega}}{\partial I_5}\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{a} + 2B\frac{\partial \overline{\Omega}}{\partial I_6}\boldsymbol{C}^2\boldsymbol{a},\end{aligned} \tag{A4}$$

hence from $(4)_1$ we obtain

$$\boldsymbol{H} = \boldsymbol{F}^{-T}\boldsymbol{H}_l = 2\frac{\partial \overline{\Omega}}{\partial I_4}\boldsymbol{b}^{-1}\boldsymbol{B} + 2\frac{\partial \overline{\Omega}}{\partial I_5}\boldsymbol{B} + 2\frac{\partial \overline{\Omega}}{\partial I_6}\boldsymbol{b}\boldsymbol{B}, \tag{A5}$$

which can be compared with the results presented, for example, in Dorfmann et al. (2004b).

Let us consider the expressions for the total stress tensor. For a compressible body the Total Cauchy stress tensor is given by $(18)_1$

$$\boldsymbol{\tau} = \frac{2}{J}\boldsymbol{F}\frac{\partial\Omega}{\partial\boldsymbol{C}}\boldsymbol{F}^T. \tag{A6}$$

Using again the notation $\Omega_M = \widehat{\Omega}(B, \lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3)$, we have

$$\frac{\partial\Omega}{\partial\boldsymbol{C}} = \sum_{i=1}^{3}\frac{1}{2\lambda_i}\frac{\partial\widehat{\Omega}}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i + \sum_{i\neq j}^{3}\frac{\frac{\partial\widehat{\Omega}}{\partial\zeta_i}-\frac{\partial\widehat{\Omega}}{\partial\zeta_j}}{\left(\lambda_i^2-\lambda_j^2\right)}\left(\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j\right)\boldsymbol{e}_i\otimes\boldsymbol{e}_j. \tag{A7}$$

Consider the tensors $\boldsymbol{G}_i$, $i = 1, 2, 3, 4, 5$ defined as:

$$\boldsymbol{G}_1 = \frac{2\boldsymbol{F}}{J}\left(\sum_{i=1}\frac{1}{2\lambda_i}\frac{\partial I_1}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right)\boldsymbol{F}^T = \frac{2}{J}\boldsymbol{b}, \tag{A8}$$

$$\boldsymbol{G}_2 = \frac{2\boldsymbol{F}}{J}\left(\sum_{i=1}\frac{1}{2\lambda_i}\frac{\partial I_2}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right)\boldsymbol{F}^T = \frac{2}{J}\boldsymbol{F}\left(I_1\boldsymbol{I} - \sum_{i=1}^{3}\lambda_i^2\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right)\boldsymbol{F}^T = \frac{2}{J}\left(I_1\boldsymbol{b} - \boldsymbol{b}^2\right), \tag{A9}$$

$$\boldsymbol{G}_3 = \frac{2\boldsymbol{F}}{J}\left(\sum_{i=1}\frac{1}{2\lambda_i}\frac{\partial I_3}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i\right)\boldsymbol{F}^T = 2J\boldsymbol{I}, \tag{A10}$$

$$\boldsymbol{G}_4 = \frac{2\boldsymbol{F}}{J}\left(\sum_{i=1}\frac{1}{2\lambda_i}\frac{\partial I_4}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i + \sum_{i\neq j}^{3}\frac{\frac{\partial I_4}{\partial\zeta_i}-\frac{\partial I_4}{\partial\zeta_j}}{\left(\lambda_i^2-\lambda_j^2\right)}\left(\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j\right)\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right)\boldsymbol{F}^T, = \frac{2B^2}{J}\boldsymbol{F}^T\left(\sum_{i=1}^{3}\zeta_i\boldsymbol{e}_i\otimes\boldsymbol{e}_i + \sum_{i\neq j}^{3}\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right)\boldsymbol{F}^T = \frac{2}{J}\boldsymbol{B}\otimes\boldsymbol{B}, \tag{A11}$$

$$\boldsymbol{G}_5 = \frac{2\boldsymbol{F}}{J}\left(\sum_{i=1}\frac{1}{2\lambda_i}\frac{\partial I_5}{\partial\lambda_i}\boldsymbol{e}_i\otimes\boldsymbol{e}_i + \sum_{i\neq j}^{3}\frac{\frac{\partial I_5}{\partial\zeta_i}-\frac{\partial I_5}{\partial\zeta_j}}{\left(\lambda_i^2-\lambda_j^2\right)}\left(\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j\right)\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right)\boldsymbol{F}^T, = \frac{2B^2}{J}\boldsymbol{F}\left(\sum_{i=1}^{3}\lambda_i^2\zeta_i\boldsymbol{e}_i\otimes\boldsymbol{e}_i + \sum_{i\neq j}^{3}\left(\lambda_i^2+\lambda_j^2\right)\left(\boldsymbol{e}_i\cdot\boldsymbol{A}\boldsymbol{e}_j\right)\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right), = \frac{2}{J}(\boldsymbol{B}\otimes\boldsymbol{b}\boldsymbol{B} + \boldsymbol{b}\boldsymbol{B}\otimes\boldsymbol{B}). \tag{A12}$$

Consider again $\Omega_M = \widehat{\Omega}(B, \lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3)$ and the relations

$$\frac{\partial\widehat{\Omega}}{\partial\lambda_i} = \sum_{k=1,k\neq 4}^{6}\frac{\partial\overline{\Omega}}{\partial I_k}\frac{\partial I_k}{\partial\lambda_i}, \tag{A13}$$

$$\frac{\partial\widehat{\Omega}}{\partial\zeta_i} = \frac{\partial\overline{\Omega}}{\partial I_5}\frac{\partial I_5}{\partial\zeta_i} + \frac{\partial\overline{\Omega}}{\partial I_6}\frac{\partial I_6}{\partial\zeta_i}. \tag{A14}$$

In view of (A6)–(A14), in (A6) we finally obtain

$$\boldsymbol{\tau} = \frac{2}{J}\left[\frac{\partial\overline{\Omega}}{\partial I_1}\boldsymbol{b} + \frac{\partial\overline{\Omega}}{\partial I_2}\left(I_1\boldsymbol{b} - \boldsymbol{b}^2\right) + I_3\frac{\partial\overline{\Omega}}{\partial I_3}\boldsymbol{I} + \frac{\partial\overline{\Omega}}{\partial I_5}\boldsymbol{B}\otimes\boldsymbol{B} + \frac{\partial\overline{\Omega}}{\partial I_6}(\boldsymbol{B}\otimes\boldsymbol{b}\boldsymbol{B} + \boldsymbol{b}\boldsymbol{B}\otimes\boldsymbol{B})\right], \tag{A15}$$

which can be compared with the expressions presented in Dorfmann et al. (2004b).

Appendix B. Uniaxial Deformation using Ω_H

The homogeneous deformation of an incompressible circular cylindrical bar is described by

$$r = \lambda^{-\frac{1}{2}}R\,,\ \ \theta = \Theta\,,\ \ z = \lambda Z\,, \tag{B1}$$

where (R,Θ,Z) are cylindrical polar coordinates in the reference configuration and (r,θ,z) the corresponding coordinates in the deformed configuration and $0 < \lambda$ is the axial stretch. This type of deformation was used by Bellan and Bossis (2002) to obtain experimental results depicted in Section 5. In this case $\boldsymbol{F} \equiv (\lambda^{-1/2},\lambda^{-1/2},\lambda)$. Here we only consider $\boldsymbol{H}_l = H_0\boldsymbol{a} \equiv (0,0,H_0)^T$ and, hence $\boldsymbol{H} \equiv (0,0,\lambda^{-1}H_0)^T$, $\zeta_1 = \zeta_2 = 0$ and $\zeta_3 = 1$. In view of (73), (4.2) and (6), we have

$$\boldsymbol{B}_l = -\frac{\partial\Omega_H}{\partial H_0}\boldsymbol{a},\quad \boldsymbol{B} = -\lambda\frac{\partial\Omega_H}{\partial H_0}\boldsymbol{a},\quad \boldsymbol{M} = M\boldsymbol{a} = -\left(\frac{\lambda\frac{\partial\Omega_H}{\partial H_0}}{\nu_0} + \frac{H_0}{\lambda}\right)\boldsymbol{a}. \tag{B2}$$

For the special form of the constitutive equation given in (77), the axial stress is given by

$$\tau_{zz} = \lambda r'(\lambda, 1, H_0) - \lambda^{-\frac{1}{2}}r'\left(\lambda^{-\frac{1}{2}}, 0, H_0\right) - \frac{\mu_0 H_0^2}{2\lambda^2}\,. \tag{B3}$$

The derivation of the above equation takes into account the Maxwell stress

$$\tau_{M_{rr}} = \tau_{M_{\theta\theta}} = -\tau_{M_{zz}} = -\frac{\mu_0 H_0^2}{2\lambda^2} \tag{B4}$$

exterior to the cylinder.

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