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FULL LENGTH PAPER

Equilibrium routing under uncertainty

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Abstract We provide a brief introduction to the basic models used to describe traffic on congested networks, both in urban transport and telecommunications. We discuss traffic equilibrium models, covering atomic and non-atomic routing games, with emphasis on situations where the travel times are subject to random fluctuations. We use convex optimization to present the models in a unified framework that stresses the common underlying structures. As a prototypical example of traffic equilibrium with elastic demands, we discuss some models for routing and congestion control in telecommunications. We also describe a class of stochastic dynamics that model the adaptive behavior of agents and which provides a plausible micro-foundation for the equilibrium. Finally we present some recent ideas on how risk-averse behavior might be incorporated in the equilibrium models.

Keywords Routing games · Network congestion · Repeated games · Adaptive dynamics · Stochastic travel times · Risk aversion

Mathematics Subject Classification 90B18 · 90B20 · 68M12 · 91A13 · 91A20 · 91A35

1 Introduction

Congestion is a recurring phenomenon in large urban areas as well as in telecommunication networks, especially during peak hours when transport demands approach the saturation capacities of the links. Traffic equilibrium models provide a static

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description of how the flows circulate in such congested networks. Equilibrium is usually described as a steady state that emerges from the adaptive behavior of selfish agents who strive to minimize their own travel time while competing for the limited capacity of the routes. These models can be used to forecast flows in order to evaluate network performance under different scenarios of demand as well as under alternative traffic management policies.

The literature on traffic congestion and equilibrium models is extensive, and there exist excellent surveys both for urban traffic [15,45,46,85,104] as well as telecommunications [27,58]. In this paper we offer a personal view on these topics based on some of our recent work, emphasizing models that take into account the stochastic nature of travel times and how this feature affects the traffic flows. Our main goals, which roughly correspond to the three main sections of the paper, are:

(a) to provide a brief introduction to the basic atomic and non-atomic traffic equilibrium models,
(b) to describe a plausible model of adaptive dynamics which provides a micro-foundation for steady state traffic equilibrium models,
(c) to present some preliminary ideas on how risk-averse behavior of the agents may be incorporated to the traffic equilibrium models.

We use convex optimization to develop a unifying framework that stresses the common structures of the different equilibrium models, and highlights the connections between urban traffic and telecommunications. Thus, we just consider equilibrium models that admit an equivalent convex programming formulation, restricting to link travel times that depend only on the flow over that particular link and disregarding the case of non-separable costs that lead to models described by general variational inequalities.

The emphasis of the paper is methodological and presupposes a working knowledge in mathematical programming and operations research, including some familiarity with convex duality and basic concepts of stochastic processes. The results are presented without proofs, though we provide an informal discussion of the basic techniques and ideas involved in the analysis. We hope that the presentation provides enough clues so that an interested reader may either work out the details on his own or refer to the original sources.

Along the paper we provide additional references for further reading. We apologize for any involuntary omissions, and we refer to the existing surveys for more comprehensive accounts of the huge literature in the fields of traffic equilibrium and telecommunications, including other relevant topics not covered in this paper such as transit networks, dynamic equilibria, or the study of the price of anarchy and the efficiency of equilibria.

Summary of the paper

A more detailed summary of the topics covered in the paper is as follows. Section 2 focuses on the non-atomic traffic equilibrium models that describe a large population of players each of which has a negligible effect on the congestion. The agents behave rationally by selecting shortest paths according to the prevailing traffic conditions, while congestion is modeled by travel times that increase with the total flow on each route. Equilibrium is defined as a consistent traffic pattern in which the routes that are actually used are the shortest ones according to the induced congestion.

We begin by briefly recalling the classical deterministic model of Wardrop [122] for a homogeneous population of agents that share the same perceptions of travel times, and then we review the concept of stochastic user equilibrium (SUE) which introduces heterogeneity by combining the idea of stochastic assignment introduced by Dial [37] with an equilibrium model as described in Daganzo and Sheffi [34]. For a more complete account of these classical models and relevant historical notes the reader is referred to the books by Ben-Akiva and Lermann [15] and Patriksson [104], as well as the surveys by Marcotte and Patriksson [85] and Florian and Hearn [45,46]. In large networks, the assumption that drivers are able to compare all possible routes becomes less and less plausible and route-based models become computationally intractable. Methods to circumvent path enumeration were developed in [2,3,14,78,84]. These ideas evolved into the notion of Markovian traffic equilibrium (MTE) [10] which is a special case of SUE with an extra additive structure of route travel times. In this model, described in Sect. 2.3, route choice is conceived as a stochastic dynamic programming process in which the route is no longer chosen at the origin but it is the outcome of a sequential process of arc selection at every intermediate node. In this setting, driver movements are governed by a Markov chain and the network flows are the corresponding invariant measures. Beckmann et al. [12] realized that Wardrop equilibrium could be characterized as an optimal solution of an equivalent convex minimization problem.[1] A similar characterization for SUE was later established by Fisk [44], while in the 1980's the results were supplemented by Daganzo [33] and Fukushima [54] who investigated the corresponding dual formulations. An analog primal-dual characterization holds for MTE. These variational formulations are very similar and provide a common framework that connects and unifies the different models.

In Sect. 2.5 we turn to the case of endogenous demands, a relevant feature in telecommunications where the traffic rate of each source is controlled by a congestion sensitive protocol. The situation is similar to models of urban traffic with elastic demands, though this connection has not been fully exploited. As a matter of fact, the standard transmission protocols in telecommunications use single path connections and the current research in multipath routing could benefit from combining the ideas of urban traffic and telecommunications. One possible approach to multipath routing that exploits the ideas of Wardrop equilibrium was presented in [74]. In Sect. 2.5 we describe an alternative approach proposed in [29] which combines a congestion control protocol with a Markovian multipath routing scheme based on MTE.

In Sect. 3 we move to a discrete setting by considering the routing game introduced by Rosenthal [110] in which drivers are considered as individual atomic players. In analyzing this model, Rosenthal discovered a potential function that allowed him to establish the existence of a Nash equilibrium in pure strategies. This discrete potential, which can be seen as a discrete analog of Beckmann et al.'s, provides a bridge to connect the continuous and discrete models. In this framework of atomic routing games, we discuss a simple stochastic process that provides a plausible model for the

[1] We thank Michael Patriksson for pointing out the earlier paper [105] which already formulated an equivalent optimization problem for equilibrium.

adaptive behavior of drivers and which provides a micro-foundation for equilibrium: equilibrium is not assumed from the outset but it is derived from basic assumptions on the behavior of individual drivers. We analyze the conditions under which these dynamics converge, and we show that the limit is indeed a Nash equilibrium for a perturbed routing game.

In the final Sect. 4 we present some preliminary ideas on how the standard theories of choice under risk may apply in the context of risk-averse routing. As a matter of fact, while all the previous models are based on the assumption that players are risk-neutral and evaluate routes by their expected travel times, a rather natural question is how traffic equilibrium might change in the presence of risk-averse players that are concerned with travel time reliability.

Standing assumptions

Throughout the paper we consider a fixed traffic network consisting of a directed graph $G = (N, A)$ together with a set of traffic demands $g_k > 0$ indexed by $k \in K$, each one from a given origin $o_k \in N$ to a corresponding destination $d_k \in N$. Without loss of generality we assume that the pairs (o_k, d_k) are all distinct. We also suppose that the set R_k of simple paths (i.e. paths without cycles) connecting o_k to d_k is nonempty, and we denote by R their union. The time required to traverse an arc $a \in A$ is modeled as a random variable $\tilde{t}_a$ whose expected value $t_a = s_a(w_a)$ is a non-negative and increasing continuous function of the arc load w_a. For simplicity, and unless otherwise stated, we assume that all random variables have continuous distributions and that the functions $s_a : [0, \infty) \to [0, \infty)$ are strictly increasing.

2 Network equilibrium with a continuum of players

2.1 Wardrop equilibrium

Traffic equilibrium models aim to describe how the demands g_k flow through the network under the effects of congestion. In his seminal paper, Wardrop [122] introduced a deterministic model for a continuous population of rational agents that travel along shortest paths. In this model, the demands g_k are split into non-negative path-flows $h_r \geq 0$ so that $g_k = \sum_{r \in R_k} h_r$. These path-flows induce the arc-loads $w_a = \sum_{r \ni a} h_r$ which determine the link travel times $t_a = s_a(w_a)$ and corresponding path travel times $c_r = \sum_{a \in r} t_a$. Denoting by H the polytope of all pairs (h, w) satisfying the previous flow conservation constraints, a *Wardrop equilibrium* is any $(h, w) \in H$ that uses shortest paths only. Formally,

$$(\forall k \in K)\,(\forall r \in R_k)\;\; h_r > 0 \Rightarrow c_r = \tau_k, \tag{1}$$

where $\tau_k = \min_{r \in R_k} c_r$ is the minimum time for the origin-destination pair k.

Wardrop equilibrium can be stated in a number of equivalent forms as a variational inequality or a fixed point of a suitable map (see e.g. [1,32] and the more recent survey [85]). However, the most powerful characterization was discovered by Beckmann et al. [12] who realized that (1) are the first order optimality conditions for the convex program

$$\text{(P-WE)} \quad \min_{(h,w)\in H} \; S(w)$$

where $S(w) = \sum_{a\in A} \int_0^{w_a} s_a(z)dz$. Indeed, a pair $(w, h) \in H$ is optimal iff it satisfies the first order optimality condition

$$\sum_{a\in A} s_a(w_a)(w'_a - w_a) \geq 0 \quad \forall\, (w', h') \in H.$$

Replacing the expressions $w'_a = \sum_{r\ni a} h'_r$ and $w_a = \sum_{r\ni a} h_r$ into this equation, and rearranging the summation, this may be equivalently stated as

$$\sum_{k\in K} \sum_{r\in R_k} c_r(h'_r - h_r) \geq 0 \quad \forall\, (w', h') \in H$$

and then the flow conservation constraints satisfied by h' and h imply that this is equivalent to the fact that for each $k \in K$ and each route $r \in R_k$ the total expected time c_r must be minimal whenever $h_r > 0$. These are precisely Wardrop's conditions.

Using convex duality (see [109]) it turns out that the variables (t, τ) are dual optimal solutions and correspond to Lagrange multipliers for the constraints that define H. Specifically, inverting $t_a = s_a(w_a)$ gives $w_a = s_a^{-1}(t_a)$ which can be written as $w = \nabla S^*(t)$ where $S^*(t) = \sum_{a\in A} \int_{t_a^0}^{t_a} s_a^{-1}(z)\, dz$ with $t_a^0 = s_a(0)$ is the so-called Fenchel conjugate of S. It turns out that (t, τ) is an optimal solution for the dual problem

$$\min_{(t,\tau)\in T} \; S^*(t) - \sum_{k\in K} g_k \tau_k,$$

where T is defined by the constraints $\tau_k \leq \sum_{a\in r} t_a$ for all $k \in K$ and $r \in R_k$.

For any fixed t, the optimal τ_k is $\bar{\tau}_k(t) = \min_{r\in R_k} \sum_{a\in r} t_a$ which gives the minimum delay from o_k to s_k. These polyhedral concave functions can be efficiently computed by any shortest path algorithm and allow to write the dual as an unconstrained convex problem in the variables $t = (t_a)_{a\in A}$, namely

$$\text{(D-WE)} \quad \min_t \; S^*(t) - \sum_{k\in K} g_k \bar{\tau}_k(t).$$

The optimal solutions for the primal and dual are related as $t_a = s_a(w_a)$. Since the feasible set in (P-WE) is compact it follows that Wardrop equilibria always exist. Moreover, since $s_a(\cdot)$ is strictly increasing, the function S is strictly convex and the vector of equilibrium arc loads w is unique. Since S^* is also strictly convex, $t = \nabla S(w)$ is the unique optimal solution of (D-WE).

Note that the primal objective function $S(w)$ does not depend on h. Thus, although the model is stated using path flows and requires path enumeration, (P-WE) admits an equivalent arc-flow formulation by expressing $w_a = \sum_{k\in K} v_a^k$ where each $v^k = (v_a^k)_{a\in A}$ satisfies flow conservation for the pair $k \in K$. Since travel times are non-negative, flows along cycles are excluded in any optimal solution and then v^k may be decomposed into path-flows h_r. Any such decomposition gives an equilibrium and every equilibrium is of this form.

2.2 Stochastic user equilibrium

Wardrop equilibrium assumes implicitly that all users perceive the same path delays. This was signaled as a possible explanation for the relatively poor agreement between the predictions of the model and the flows observed in real networks. Actually, travel times have some inherent variability so that perceptions are likely to differ depending on the past experience of each driver. This called for extending the concept of equilibrium to a stochastic setting in which the travel times are modeled as random variables, representing the variability of the perceptions of travel times across an heterogeneous population.

The idea was formalized by looking at route selection as a discrete choice based on random utility theory. Route travel times are modeled as $\tilde{c}_r = c_r + \epsilon_r$ where $c_r = \sum_{a \in r} t_a$ as before and ϵ_r is a continuous random variable with $\mathbb{E}[\epsilon_r] = 0$ that accounts for the variability in the perceptions of travel times across the population. The equilibrium conditions (1) are replaced by a stochastic assignment where each demand g_k splits among the paths $r \in R_k$ according to the probability that this path is optimal, that is

$$(\forall r \in R_k) \;\; h_r = g_k \, \mathbb{P}\left(\tilde{c}_r \leq \tilde{c}_p \; \forall p \in R_k\right), \tag{2}$$

with $t_a = s_a(w_a)$ and $w_a = \sum_{r \ni a} h_r$. Such a pair (h, w) is called a SUE. Note that (2) implies that $(h, w) \in H$.

As for Wardrop equilibrium, SUE admits a variational characterization. This follows from a basic fact in discrete choice theory (see e.g. [15]): if $j \in J$ represents a finite set of alternatives with random costs $\tilde{x}_j = x_j + \epsilon_j$ where $x_j \in \mathbb{R}$ and ϵ_j is a random variable with $\mathbb{E}(\epsilon_j) = 0$, the probability that any given $j \in J$ attains the minimum cost is given by $\mathbb{P}(\tilde{x}_j \leq \tilde{x}_i \; \forall i \in J) = \frac{\partial \varphi}{\partial x_j}(x)$ where $\varphi(x)$ is the smooth concave *expected utility* function

$$\varphi(x) = \mathbb{E}\left[\min_{j \in J} \left\{x_j + \epsilon_j\right\}\right].$$

Thus, the probability in (2) of choosing path r can be expressed as the partial derivatives $\frac{\partial \varphi_k}{\partial c_r}$ of the *expected travel time* functions

$$\varphi_k((c_r)_{r \in R_k}) = \mathbb{E}\left[\min_{r \in R_k} \{c_r + \epsilon_r\}\right].$$

Hence, by considering the composite concave functions

$$\tau_k(t) = \varphi_k\left(\left(\sum\nolimits_{a\in r} t_a\right)_{r\in R_k}\right),$$

a direct application of the chain rule gives

$$s_a^{-1}(t_a) = w_a = \sum_{r\ni a} h_r = \sum_{k\in K} g_k \frac{\partial \tau_k}{\partial t_a}(t),$$

which shows that t must be an optimal solution of the strictly convex program

$$\text{(D-SUE)} \quad \min_t \ S^*(t) - \sum_{k\in K} g_k \tau_k(t).$$

By Fenchel's duality, if ψ_k denotes the conjugate of the convex function $-\varphi_k$, we get a corresponding primal problem that characterizes the SUE flows

$$\text{(P-SUE)} \quad \min_{(h,w)\in H} \ S(w) + \sum_{k\in K} g_k\, \psi_k\left(\left(\frac{h_r}{g_k}\right)_{r\in R_k}\right).$$

These are the stochastic analogs of the characterizations of Wardrop equilibrium, to which they reduce in the deterministic case when the random variables ϵ_r are concentrated at 0. The objective function in (D-SUE) has bounded level sets and is strictly convex so there is a unique optimal t, from which we get unique equilibrium arc-loads $w_a = s_a^{-1}(t_a)$ which give the solution of (P-SUE).

Examples Dial [37] considered a Logit model with route travel times $\tilde{c}_r$ given by independent Gumbel random variables, which yields the stochastic assignment

$$h_r = g_k \frac{\exp(-\beta_k c_r)}{\sum_{p\in R_k} \exp(-\beta_k c_p)}.$$

The parameter β_k controls the repartition of flows: for $\beta_k \sim 0$ every path gets an approximately equal share of the flow, while for β_k large the flow concentrates on the shorter paths. Logit assignment was used by Fisk [44] to develop a SUE model for which (P-SUE) and (D-SUE) take the explicit forms

$$\text{(P-SUE)} \quad \min_{(w,h)\in H} S(w) + \sum_{k\in K} \frac{1}{\beta_k} \sum_{r\in R_k} h_r \ln\left(\frac{h_r}{g_k}\right),$$

$$\text{(D-SUE)} \quad \min_t \ S^*(t) + \sum_{k\in K} \frac{g_k}{\beta_k} \ln\left(\sum_{r\in R_k} e^{-\beta c_r}\right).$$

While these require path enumeration, when $\beta_k \equiv \beta$ is constant across the network, an equivalent formulation in the space of arc flows was given in [3,14,78,84]

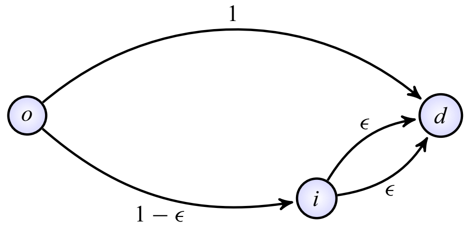


Fig. 1 Path choice v/s arc choice on a small network

by exploiting a Markovian property of the Logit assignment. Note that a constant β_k conveys the assumption that the travel time variance is constant and does not scale with distance, which seems somewhat restrictive. In a different direction, since independence of route travel times is an unlikely assumption when dealing with overlapping routes, Daganzo and Sheffi [34] proposed an alternative model based on a Probit stochastic assignment in which $\epsilon = (\epsilon_r)_{r\in R}$ has Normal distribution $\mathcal{N}(0, \Sigma)$. In this case there is no close form expression for the probabilities in (2), which must be estimated using Monte Carlo simulation. Note that the noises ϵ_r in both Logit and Probit models are supported on $\mathbb{R}$ so that $\tilde{c}_r$ takes negative values with positive probability and flow is assigned to every route no matter how large its expected travel time is. This does not occur if the random terms ϵ_r have bounded support. Dual formulations of the type (D-SUE) were first considered by Fukushima [54], Daganzo [33] and Miyagi [88].

2.3 Markovian traffic equilibrium

On large networks one may argue that agents are unable to compare exponentially many paths, whereas the task is much simpler if route choice is conceived as a sequential process of arc choices. In this view, the path is not fixed a priori but is built along the way: drivers move towards their destination by sequentially choosing the next arc at each node visited. Such dynamics are naturally modeled as Markov processes that determine how flows are distributed across the network. As an informal motivation for the basic idea, consider the simple network in Fig. 1 with 3 routes from o to d, all of which have expected travel time $c_r = 1$. A route-based Logit model would assign one third of the demand to each path, whereas a recursive scheme based on arc-choices would lead to a repartition close to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Indeed, at the origin o there are only two options (upper and lower), both offering the same expected travel time to the destination so one might expect that each should get approximately half of the demand. The half taking the lower arc faces a second choice at node i where it splits again giving roughly $\frac{1}{4}$ on both lower routes. Since these lower routes have a significant overlap and are highly correlated, an arc-based approach seems more appropriate.

This idea is captured by the notion of MTE which was introduced in [10] and further extended in [28,29]. The description below differs slightly as it is presented as a particular case of SUE in which the travel time of each route $r \in R_k$ is given by a

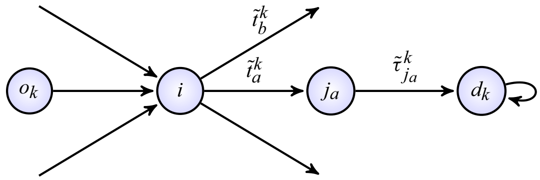


Fig. 2 Variables for dynamic programming equations

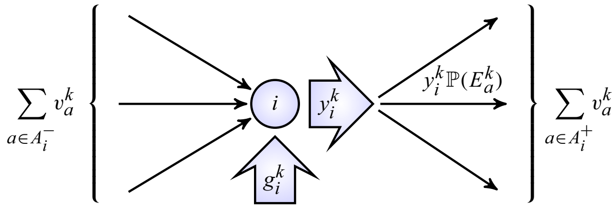


Fig. 3 Flow distribution diagram

sum $\tilde{c}_r = \sum_{a\in r} \tilde{t}_a^k$ of random arc-delays $\tilde{t}_a^k = t_a + \epsilon_a^k$ with $\mathbb{E}[\epsilon_a^k] = 0$.[2] This additive structure has implicit the fact that flows are governed by some underlying Markov chains. Indeed, let $\tilde{\tau}_i^k$ be the random variable giving the minimum travel time among all paths from i to d_k, and denote $\tilde{z}_a^k = \tilde{t}_a^k + \tilde{\tau}_{j_a}^k$ the cost-to-go for destination d_k when taking arc a (see Fig. 2).

Bellman's equations give $\tilde{\tau}_i^k = \min_{a\in A_i^+} \tilde{z}_a^k$ which is interpreted by the fact that a driver of type k traveling on a shortest path and reaching i, chooses the outgoing link $a \in A_i^+$ with smallest $\tilde{z}_a^k$ and moves to the next node j_a where the process repeats. Thus, denoting E_a^k the event $\{\tilde{z}_a^k \leq \tilde{z}_b^k \ \forall b \in A_i^+\}$, the demand $k \in K$ moves across the network following a Markov chain with transition probabilities $P_{ij}^k = \mathbb{P}(E_a^k)$ for $ij = a, i \neq d_k$, whereas the destination d_k is an absorbing state.

The expected flows correspond to the invariant measures of these Markov chains, so that the *route-based* assignment (2) is replaced by an *arc-based* recursive scheme in which the total inflow y_i^k that enters each node $i \neq d_k$ is distributed among the outgoing links $a \in A_i^+$ according to $v_a^k = y_i^k \, \mathbb{P}(E_a^k)$. The inflow is given by the sum $y_i^k = g_i^k + \sum_{a\in A_i^-} v_a^k$ where the first term describes the demand with $g_i^k = 0$ for $i \neq o_k$ and equal to the demand g_k for $i = o_k$, while the second summation represents the flows that enter node i through the incoming links $a \in A_i^-$ (see Fig. 3).

The model admits a concise description using expected utility maps. Namely, let us write $\tilde{z}_a^k = z_a^k + \nu_a^k$ as a sum of its expected value z_a^k plus a random term ν_a^k with $\mathbb{E}[\nu_a^k] = 0$. Assuming that the distribution of ν_a^k is not affected by a change in the expected value z_a^k and denoting

[2] The ϵ_a^k's are allowed to depend on k, which may capture differences in travel time perceptions for different classes of agents. However, they are not required to be independent and may be taken the same for all k, allowing even for correlations among arcs.

$$\varphi_i^k(z^k) \triangleq \mathbb{E}\left[\min_{a\in A_i^+}\left\{z_a^k + \nu_a^k\right\}\right]$$

we may express the transition probabilities as $\mathbb{P}(E_a^k) = \frac{\partial \varphi_i^k}{\partial z_a^k}(z^k)$ and the equilibrium flows y_i^k, v_a^k are characterized by the set of linear equations

$$\begin{cases} y_i^k = g_i^k + \sum_{a\in A_i^-} v_a^k & \forall i \neq d_k, \\ v_a^k = y_i^k\, \frac{\partial \varphi_i^k}{\partial z_a^k}(z^k) & \forall a \in A_i^+. \end{cases} \tag{3}$$

Taking expectation in Bellman's equations, the expected values $\tau_i^k = \mathbb{E}[\tilde{\tau}_i^k]$ of the travel times are characterized by the nonlinear system.[3]

$$\begin{cases} z_a^k = t_a + \tau_{j_a}^k & \forall a \in A, \\ \tau_i^k = \varphi_i^k(z^k) & \forall i \in N. \end{cases} \tag{4}$$

A MTE is any solution of (3)–(4) with $t_a = s_a(w_a)$ and $w_a = \sum_{k\in K} v_a^k$. In words, the link flows w_a determine expected link delays $t_a = s_a(w_a)$ which in turn determine the expected travel times τ_i^k as well as the expected flows v_a^k. An equilibrium is then a fixed point in which the total induced flows $\sum_{k\in K} v_a^k$ coincide with the original link flows w_a.

Note that the maps φ_i^k convey all the information required to state the model. In the sequel we assume that the model is formulated directly by prescribing these maps. It is worth noting that while the φ_i^k's are determined by the random variables ν_a^k, which in turn depend on the noise ϵ_a^k, they can also be characterized (see [10]) as the maps $\varphi : \mathbb{R}^d \to \mathbb{R}$ that are smooth, concave, component-wise nondecreasing, with

(a) $\varphi(x_1 + c, \ldots, x_d + c) = \varphi(x_1, \ldots, x_d) + c$ for every constant c
(b) $\varphi(x) \to x_i$ when $x_j \to \infty$ for all $j \neq i$
(c) for x_i fixed, $\frac{\partial \varphi}{\partial x_i}$ is a cumulative distribution on the remaining variables.

The formulas (3) and (4) avoid path enumeration and facilitate the computation of equilibria. In particular they give the minimum expected time function $\tau_k(t) = \tau_{o_k}^k$ from o_k to d_k appearing in (D-SUE). These equations uniquely define the flows v_a^k, y_i^k and the times τ_i^k, z_a^k as implicit functions of the link travel times t_a, and are defined over the domain $\mathcal{D}$ composed by all the vectors $t = (t_a)_{a\in A}$ for which

$$\exists\ \hat{\tau} = (\hat{\tau}_i^k)_{i\in N, k\in K} \text{ with } \hat{\tau}_i^k \leq \varphi_i^k\left(\left(t_a + \hat{\tau}_{j_a}^k\right)_{a\in A}\right) \text{ for all } i \neq d_k, k \in K. \tag{5}$$

[3] Since $\tau_{d_k}^k \equiv 0$ we adopt the convention $\varphi_{d_k}^k(\cdot) \equiv 0$. Note that φ_i^k is a smooth and concave lower approximation of the polyhedral function $\bar{\varphi}_i^k(z^k) = \min_{a\in A_i^+} z_a^k$, so that (4) is a smoothed version of the standard shortest path equations $\bar{\tau}_i^k = \min_{a\in A_i^+} t_a + \bar{\tau}_{j_a}^k$.

Note that the latter is obviously a necessary condition for the existence of solutions for (4). Since the maps φ_i^k are continuous, concave and component-wise nondecreasing, the domain $\mathcal{D}$ is closed and convex and for each $t \in \mathcal{D}$ we have $t' \in \mathcal{D}$ for all $t' \geq t$. As in Sect. 2.1, we denote $\bar{\tau}_i^k(t)$ the minimum time from i to d_k with deterministic link costs t_a.

Proposition 1 ([10,28,29])

(a) *The Eq.* (4) *uniquely define* $z_a^k = z_a^k(t)$ *and* $\tau_i^k = \tau_i^k(t)$ *as implicit functions for* $t \in \mathcal{D}$. *These maps are concave, component-wise nondecreasing, and we have* $\tau_i^k(t) \leq \bar{\tau}_i^k(t)$. *The maps* $\tau_i^k(t)$ *can be computed as the limit of the nonincreasing sequence generated from* $\tau^{k,0} = \bar{\tau}^k$ *by the iteration*

$$\tau_i^{k,n+1} = \varphi_i^k\left(\left(t_a + \tau_{j_a}^{k,n}\right)_{a\in A}\right) \quad n = 0, 1, 2, \ldots$$

(b) *Let* $t \in \mathcal{D}$ *and let* $z_a^k = z_a^k(t)$ *the unique solution of* (4). *Then the equations* (3) *uniquely define* $y_i^k = y_i^k(t)$ *and* $v_a^k = v_a^k(t)$ *as implicit functions of* $t \in \mathcal{D}$.

(c) *The previous implicit maps are smooth at each* $t \in \operatorname{int}(\mathcal{D})$, *namely, at every* t *for which there exists* $\hat{\tau} = (\hat{\tau}_i^k)_{i\in N, k\in K}$ *satisfying* (5) *with strict inequalities.*

These implicit maps allow to rewrite the conditions for MTE as $s_a^{-1}(t_a) = \sum_{k\in K} v_a^k(t)$. Moreover, an implicit differentiation of (4) gives $v_a^k(t) = g_k \frac{\partial \tau_k}{\partial t_a}(t)$ where $\tau_k(t) = \tau_{o_k}^k(t)$ which turn the MTE equations into $s_a^{-1}(t_a) = \sum_{k\in K} g_k \frac{\partial \tau_k}{\partial t_a}$. These are precisely the optimality conditions for the strictly convex program (D-SUE), which yields the following characterization for the MTE.

Theorem 1 ([10,28,29]) *Let* $t^0 = (t_a^0)_{a\in A}$ *denote the vector of uncongested travel times* $t_a^0 = s_a(0)$ *and assume the Slater-type constraint qualification* $t^0 \in \operatorname{int}(\mathcal{D})$. *Then the following strictly convex program has a unique optimal solution*

$$(\mathrm{D-MTE}) \quad \min_{t\in\mathcal{D}} \; S^*(t) - \sum_{k\in K} g_k \tau_k(t).$$

The optimal solution t *belongs to int*$(\mathcal{D})$ *and gives a unique MTE* $w_a = s_a^{-1}(t_a)$ *which decomposes into per-destination flows* $w_a = \sum_{k\in K} v_a^k(t)$. *These equilibrium flows can be characterized as the optimal solution of the strictly convex* primal *problem*

$$(\mathrm{P-MTE}) \quad \min_{(w,(v^k)_{k\in K})\in V} \; S(w) + \sum_{k\in K}\sum_{i\in N} \chi_i^k\left(\left(v_a^k\right)_{a\in A_i^+}, \sum\nolimits_{b\in A_i^+} v_b^k\right),$$

where $\chi_i^k(v, y) = y\,(-\varphi_i^k)^*(-\frac{v}{y})$ *and the feasible set* V *is defined by the constraints* $w_a = \sum_{k\in K} v_a^k$ *with each* v^k *satisfying flow conservation for the demand* $k \in K$.

Although (D-MTE) includes the constraint $t \in \mathcal{D}$, it is essentially unconstrained since the optimum is attained in the interior. Observe also that, in contrast with the deterministic case, for MTE not only the aggregated link flows w but also the per-destination link flows v_a^k are uniquely determined.

Example A Markovian Logit model may be defined by considering the maps

$$\varphi_i^k(z^k) = -\frac{1}{\beta_i^k} \ln\left(\sum\nolimits_{a\in A_i^+} e^{-\beta_i^k z_a^k}\right).$$

This extends the case $\beta_i^k \equiv \beta$ analyzed by Akamatsu [2]. The corresponding dual functionals are in this case

$$\chi_i^k(v, y) = \frac{1}{\beta_i^k}\left[\sum\nolimits_{a\in A_i^+} v_a \ln v_a - y \ln y\right].$$

2.4 Extensions

Non-decreasing travel times: The equilibrium models WE, SUE and MTE, can be extended to nondecreasing functions $s_a(\cdot)$. In this case the primal objective function $S(w)$ is still convex though not strictly convex, and the equilibrium flows w are no longer unique. However, the dual functional $S^*(t)$ remains strictly convex so that the equilibrium times t are unique. Any link flow w_a with $t_a = s_a(w_a)$ constitute equilibrium flows. For details we refer to [10].

Arc capacities and saturation: When arcs have a maximal capacity one must deal with increasing continuous functions $s_a : [0, c_a) \to [0, \infty)$ such that $s_a(w_a) \to \infty$ as w_a approaches the capacity c_a. To ensure feasibility the capacities must be large enough to support the demands. Theorem 1 for MTE remains valid as long as there exist feasible flows $(\hat{w}, \hat{v}) \in V$ satisfying $\hat{w}_a < c_a$ for all $a \in A$. The proof of this fact is based on general results in convex analysis. For details we refer to [10,28,29].

General random variables: The SUE and MTE models can be adapted to deal with random variables ϵ_a^k that may have atoms. In this case the maps $\tau^k(\cdot)$ are still concave but no longer smooth. The characterizations as well as the results on existence and uniqueness remain true by replacing the derivatives of these functions with their sub-differentials. In particular, when ϵ_a^k are Dirac masses at 0, we recover the concept of Wardrop equilibrium.

Mixed choice models: MTE allows different discrete choice models at every node. This can be exploited to model more complex decision processes such as the simultaneous choice of transportation *mode* and *route*. It suffices to consider a multi-modal network that combines the subgraphs of the basic modes (car, bus, metro, walk, etc.) which are connected among them as well as with origins and destinations through *transfer arcs*. At origins one may adopt a distribution rule based on a Logit or Probit model, while at other nodes (e.g. the metro subgraph) one may use a deterministic rule.

2.5 Traffic equilibrium with elastic demands

So far we considered a set of fixed traffic demands g_k. This is inadequate when demands are endogenous and depend on congestion. As a relevant case let us consider packet-switched communication networks where the transmission rate g_k of a source is controlled by a congestion sensitive protocol such as TCP. Namely, each source dynamically adjusts its transmission rate based on a feedback signal of the aggregate congestion along the route to its destination, so that the higher the congestion the smaller the rate. The predominant TCP protocols in use are Tahoe and Reno which use *packet losses* to measure congestion, and Vegas which is based on *queueing delays*. We refer to [27,81] for a detailed description of current protocols and their models, including other related techniques such as random early drop (RED) and random exponential marking (REM) which anticipate congestion and react before packet losses actually occur.

The routing of traffic in packet-switched networks is performed by routers in a decentralized manner using routing tables that determine the next hop for each destination. These routing tables are updated periodically by an asynchronous distributed shortest path iteration that finds optimal paths according to some metric such as hop count, delay, or bandwidth. Updates occur on a much slower time scale compared with TCP rate control, so that in first approximation one may assume that all the traffic for any given source is routed along a single and fixed path. This ensures that packets arrive to the destination in the same order as they were sent, an important feature that allows to detect when a packet was lost and is not simply delayed. We refer again to [27] for a more detailed description of routing protocols.

The interaction of many sources performing a decentralized congestion control based on feedback signals that are subject to estimation errors and communication delays, gives rise to very complex dynamics. Assuming that the system stabilizes, the steady state can be characterized as an optimal solution of a network utility maximization (NUM) problem [72,80,127]. The TCP mechanism may then be viewed as a decentralized algorithm that seeks to optimize an aggregate utility function. To describe NUM we assume that each source routes its flow along a single path $r_k \in R_k$, so that the arc-loads are given by $w_a = \sum_{k \ni a} g_k$ where the summation is over all the sources k whose route r_k contains the link a. These loads induce link congestion measures $\lambda_a = \rho_a(w_a)$ where $\rho_a : [0, c_a) \to [0, \infty)$ is continuous and strictly increasing with maximal link capacity c_a. Then, each source k observes the aggregate congestion along its path $\mu_k = \sum_{a \in r_k} \lambda_a$ and adjusts its rate as $g_k = f_k(\mu_k)$ where $f_k : (0, \infty) \to (0, \infty)$ is a continuous and strictly decreasing function. These equations may be written as

$$f_k^{-1}(g_k) = \mu_k = \sum_{a \in r_k} \lambda_a = \sum_{a \in r_k} \rho_a(w_a) = \sum_{a \in r_k} \rho_a \left(\sum\nolimits_{s \ni a} g_s \right),$$

so that denoting $P_a(\cdot)$ a primitive of $\rho_a(\cdot)$ it follows that the rates g_k can be characterized as the unique optimal solution of the strictly convex program.[4]

[4] Note that the common usage in telecommunications is to state the model as the *maximization* of a network utility function. Here we follow the convention in traffic equilibrium by stating the model in the form of a convex *minimization* problem. This choice also facilitates the use of the convex duality theory.

$$\text{(P-NUM)} \quad \min_{g\in\mathbb{R}^K} \sum_{a\in A} P_a\left(\sum\nolimits_{s\ni a} g_s\right) - \sum_{k\in K}\int_0^{g_k} f_k^{-1}(z)dz.$$

Alternatively, the equations may also be stated in terms of the link congestion variables as

$$\rho_a^{-1}(\lambda_a) = w_a = \sum_{k\ni a} g_k = \sum_{k\ni a} f_k(\mu_k) = \sum_{k\ni a} f_k\left(\sum\nolimits_{b\in r_k}\lambda_b\right),$$

which are the optimality conditions for the strictly convex dual program

$$\text{(D-NUM)} \quad \min_{\lambda\in\mathbb{R}^A} \sum_{a\in A}\int_0^{\lambda_a} \rho_a^{-1}(y)\,dy - \sum_{k\in K} F_k\left(\sum\nolimits_{b\in r_k}\lambda_b\right),$$

where $F_k(\cdot)$ is a primitive of $f_k(\cdot)$. This setting has been used to model different protocols each one characterized by specific maps f_k and ρ_a (see [39,57,72,76,81,101]).

Examples A model for Vegas might take $\lambda_a = \rho_a(w_a) = \frac{w_a}{c_a(c_a-w_a)}$ which are the queuing delays for links considered as M/M/1 queues with service rates c_a. The rate functions can be modeled as $g_k = f_k(\mu_k) = \alpha_k D_k/\mu_k$ where α_k is a parameter and D_k denotes the uncongested round trip time between o_k an d_k (see [80,82]). For protocols based on packet losses (Reno, Tahoe, RED, REM), if $p_a = \psi_a(w_a)$ denotes the probability that a packet is lost on link a, then the probability of traversing the path successfully is $\prod_{a\in r_k}(1-p_a)$. This product may be transformed into an additive congestion measure by taking logarithms and considering $\lambda_a = -\ln(1-p_a)$ which leads to $\rho_a(w_a) = -\ln(1-\psi_a(w_a))$.

Although NUM is restricted to single path routing, the idea of multipath routing protocols that seek to increase throughput by exploiting the available transmission capacity on a set of alternative paths has been considered since the seminal papers by Gallager [55] and Kelly et al. [72]. For general surveys and discussions of the challenges involved in multipath routing we refer to Gojmerac [58], He and Rexford [65] and Lee and Choi [79]. A few multipath techniques are available in today's Internet (e.g. MPLS tunnels [119], *overlay TCP* protocol [60], or the multipath TCP (MPTCP) developed by the IETF working group, the main Internet standardization body (see http://www.multipath-tcp.org/). Two important papers that provide support to the IETF initiative are Kelly and Voice [73] and Key et al. [74].

An additional advantage of multipath routing is its ability to redirect flows in case of link failures which improves the reliability of the communications. Moreover it also contributes to stability. Indeed, when route choice is based on metrics that are affected by congestion, such as queueing delays or latencies, routing and rate control become mutually inter-dependent and equilibrium must consider both aspects jointly: routing affects the rate control through the induced congestion signals, while rate control induces flows that determine in turn which routes are optimal.

A simple way to capture the interactions between routing and congestion control is to combine the NUM model for rate control with a stochastic routing such as MTE.

Note that congestion metrics are subject to estimation errors and random effects, which fits naturally into the framework of stochastic equilibrium. The idea mixes a NUM rate control with a decentralized routing scheme in which routers do not use routing tables but rather split the flow over its outgoing links taking into account the time to destination variables $\tilde{z}_a^k$. These random variables may incorporate for instance the current queue length of the outgoing links. Instead of using hop count, queuing delays, or bandwidth, we evaluate routes using the total delays $t_a = s_a(w_a)$ so that packets are routed along paths with smaller travel times. The rationale is that the earlier each packet is delivered, the higher the transmission rate. At the same time, this contributes to ensure that packets reach the destination in the same order as they are sent, reducing the conflicts with the duplicate *ack* mechanism for detecting packet losses.

The model, as described in [29], assumes that packets are routed according to an MTE strategy characterized by a family of maps φ_i^k, while sources adjust their rates as a function $g_k = f_k(\mu_k)$ of the total queueing delay $\mu_k = \tau_k(t) - \tau_k^0$ where $\tau_k(t)$ is the end-to-end expected delay defined by (4) and $\bar{\tau}_k^0$ is the cost of a shorter path considering the uncongested travel times t_a^0. Informally, the source rates g_k induce flows v_a^k and total link loads w_a. These loads determine link expected delays $t_a = s_a(w_a)$ that yield end-to-end expected optimal delays $\tau_k(t)$ and corresponding queueing delays μ_k. At equilibrium, these queueing delays must be consistent with the original rates $g_k = f_k(\mu_k)$. More precisely, a pair (g, w) is a *Markovian NUM equilibrium* (MNUM) iff $t_a = s_a(w_a)$ with $w_a = \sum_{k\in K} v_a^k(t)$ where $v^k(t)$ are the implicit functions defined by (3) with $g_k = f_k(\mu_k)$ and $\mu_k = \tau_k(t) - \tau_k^0$. These equations can be written in condensed form as

$$s_a^{-1}(t_a) = \sum_{k\in K} v_a^k(t),$$

which turn out to be the optimality conditions for the strictly convex program

$$\text{(D-MNUM)} \quad \min_{t\in\mathcal{D}} \; S^*(t) - \sum_{k\in K} F_k(q^k(t)),$$

where as before $F_k(\cdot)$ denotes a primitive of $f_k(\cdot)$. Assuming that $t^0 \in \mathcal{D}$ the problem (D-MNUM) is coercive so that it has a unique optimal solution and therefore there exists a unique MNUM equilibrium (see [29]). This model can be seen as an extension of the MTE model in which the second term of the objective function incorporates the effect of elastic demands.

For a comparison of MNUM with other multipath routing protocols, particularly Paganini [102] and Paganini and Mallada [103] who consider a similar idea in which routers split the traffic among the outgoing links using dynamically adjusted ratios, we refer to [29]. That paper also describes how MNUM can be used to design a packet-level protocol, although the convergence of these packet-level dynamics towards the steady state described by MNUM is not yet understood. Promising results along this line can be found in [71,121].

3 Routing equilibrium with atomic players

3.1 Atomic routing games

An alternative to Wardrop's model in which drivers are considered as individual players in a game was introduced by Rosenthal [110]. In this framework each $k \in K$ corresponds to a single player that has to route a unit demand $g_k = 1$ from o_k to d_k. The flow cannot be split among routes so that the strategy for player k consists in choosing a single path $r_k \in R_k$. Given a strategy profile $p = (r_k)_{k \in K}$ giving the choices of every player, the load of an arc $a \in A$ is simply $w_a(p) = |\{k \in K : a \in r_k\}|$ which determines the time $t_a = s_a(w_a(p))$ and then the total delay experienced by player k is given by

$$u_k(p) = \sum_{a \in r_k} s_a\left(w_a(p)\right).$$

These delays and strategy sets define a finite game: a *Nash equilibrium* is a strategy profile $p = (r_k)_{k \in K}$ such that for each player k the route r_k yields the minimum delay among all paths in R_k, given the strategies $r_{-k} = (r_i)_{i \neq k}$ chosen by the other players, namely, $u_k(p) \leq u_k(s_k, r_{-k})$ for each $s_k \in R_k$.

The potential $\sum_{a \in A} \int_0^{w_a} s_a(z)dz$ that characterizes Wardrop's equilibrium has the following discrete analog discovered by Rosenthal [110]

$$V(p) = \sum_{a \in A} \sum_{i=1}^{w_a(p)} s_a(i). \tag{6}$$

The crucial observation is that, given the strategies r_{-k} of his opponents, player k can evaluate the difference between two paths $r_k, s_k \in R_k$ as

$$u_k(r_k, r_{-k}) - u_k(s_k, r_{-k}) = V(r_k, r_{-k}) - V(s_k, r_{-k}).$$

This equality, which can be readily checked by direct computation, implies that the minimizers of $u_k(\cdot, r_{-k})$ coincide with those of $V(\cdot, r_{-k})$. It follows that Nash equilibria are precisely the profiles p that minimize $V(\cdot)$ with respect to each coordinate r_k. In particular, a Nash equilibrium in pure strategies can be found by solving the discrete optimization problem

$$(\mathrm{P-N}) \quad \min_p \left\{ V(p) : p \in \prod\nolimits_{k \in K} R_k \right\}.$$

The existence of the potential function V has relevant consequences. It implies for instance that an iteration in which players update cyclically their route r_k as a best response to the current choices of his opponents will monotonically decrease the potential $V(\cdot)$ and then, since the set of strategy profiles is finite, after finitely many steps the process must stop at a fixed point which is precisely a Nash equilibrium. In particular this proves the existence of equilibria.

3.2 Adaptive dynamics in traffic games

The models discussed so far assume implicitly the existence of an adaptive mechanism by which traffic flows stabilize on a steady state equilibrium. Empirical evidence of some form of adaptive behavior has been reported in [9,38,69,87,117,123], though the steady states observed differ from the standard equilibria. Also, continuous time dynamics describing plausible adaptive mechanisms that converge to Wardrop equilibrium have been studied in [52,114,118], while a class of finite-lag adjustment procedures was considered in [24,25,35,64]. All these dynamics are of an aggregate nature and are not directly tied to the behavior of individual drivers, so that the informational and strategic aspects are ignored. The latter features are taken into account in the study of learning and adaptation in repeated games (see e.g. Fudenberg and Levine [53] and Young [128]). The most prominent procedure is *fictitious play* which assumes that at every stage players choose a best reply to the empirical distribution of past moves by their opponents [23,108]. A variant called *smooth fictitious play* for games with perturbed payoffs and reminiscent of Logit random choice was studied in [53,67]. However, these models assume that players are able to observe the moves of their opponents which might be very stringent for games involving many players. A milder assumption requires players to observe only the payoffs at every stage, including those that would have been obtained if a different move had been played. Procedures such as *no-regret* [61,62], *exponential weight* [51], *reinforcement* [7,13,21,41,63], and *calibration* [50], deal with limited information contexts in which players adjust their behavior based on rough statistics of past performance. A general overview of learning dynamics in games with application in routing can be found in Sandholm [115]. Also in the context of repeated congestion games, and specifically for single origin-destination routing games with bounded rationality and partial monitoring, Scarsini and Tomala [116] studies the efficiency of a class of adaptive strategies that can be implemented by finite automata.

An alternative approach is considered in this section. We study the emergence of steady states from some specific dynamics that describe the adaptive behavior of drivers [22,30], with the aim of providing a micro-foundation for equilibrium. The approach proceeds bottom-up: a discrete time stochastic model for individual behavior gives rise to an associated deterministic dynamic which leads to an equilibrium of a particular limit game. Equilibrium is not postulated a priori but it is rather derived from basic assumptions on player behavior. Specifically, each player has a prior estimate of the average payoff of each route and selects a path based on this rough information. The payoff of the chosen route is then observed and is used to update the perception for that particular move. This procedure is repeated day after day, generating a discrete time stochastic process that progressively reveals to each player the congestion on all the routes. Under suitable conditions the dynamics converge to a steady state which turns out to be a Nash equilibrium for a specific limit game. The idea is similar to reinforcement but the resulting dynamics are structurally different.

We describe the dynamics in the framework of routing games, though the basic idea apply to any finite game (see [30]). We denote $\Delta = \prod_{k\in K} \Delta_k$ where Δ_k is the set of mixed strategies or probability distributions over R_k. The game is played repeatedly. At every stage, each driver k selects a route $r_k \in R_k$ at random using a mixed strategy

$\pi_k = (\pi_{kr})_{r\in R_k} \in \Delta_k$. The vector $\pi = (\pi_k)_{k\in K} \in \Delta$ induces a product distribution on the set of strategy profiles $p = (r_k)_{k\in K}$, which in turn determine random arc loads $w_a(p)$ and route travel times $u_k(p)$ as in the previous section. At the end of the stage each player observes the payoff obtained for his chosen route. We suppose that the mixed strategy $\pi_k = \sigma_k(x_k)$ derives from a discrete choice model that depends on a vector $x_k = (x_{kr})_{r\in R_k}$ where x_{kr} is an estimate of the travel time of route r that player k has built from his prior observations. Namely, $\sigma_k(x_k) = \nabla\varphi_k(x_k)$ where $\varphi_k(x_k) = \mathbb{E}[\min_{r\in R_k}\{x_{kr} + \varepsilon_{kr}\}]$ is a smooth and concave map defined by the continuous random variables ε_{kr}. The map $\sigma_k : \mathbb{R}^{R_k} \to \Delta_k$ is continuous and we assume that $\sigma_{kr}(x_k) > 0$ for all $r \in R_k$.

The perceptions evolve along discrete stages $n = 0, 1, 2, \ldots$ as follows. The only information available to player k after stage n is the travel time $u_k(p^n)$ for the route r_k^n that was chosen on that day. Based on this minimal piece of information, player k updates the perception for that route keeping the others unchanged, namely

$$x_{kr}^{n+1} = \begin{cases} (1-\gamma_n)\, x_{kr}^n + \gamma_n u_k(p^n) & \text{if } r = r_k^n, \\ x_{kr}^n & \text{otherwise}, \end{cases}$$

where $\gamma_n \in (0, 1)$ is a fixed sequence of averaging factors with $\sum_n \gamma_n = \infty$ and $\sum_n \gamma_n^2 < \infty$. In contrast with other dynamics such as fictitious play or reinforcement, which give rise to evolution dynamics in the space of mixed strategies or correlated strategies (see e.g. [50]), the above dynamics describe the evolution of perceptions. The iteration may be written in vector form

$$x^{n+1} - x^n = \gamma_n[w^n - x^n], \tag{7}$$

with $w_{kr}^n = u_k(p^n)$ for $r = r_k^n$ and $w_{kr}^n = x_{kr}^n$ otherwise. The distribution of the random vector w^n is determined by the current x^n, so that (7) yields a stochastic process for the evolution of perceptions. It is interpreted as a process in which drivers probe the different routes to *learn* their payoffs, and adapt their behavior according to the accumulated information. The information gathered at every stage is very limited—only the travel time of the specific route chosen on that day—but it conveys implicit information on the behavior of the rest of the players. A basic question is whether this procedure can lead players to coordinate on a steady state.

Dividing (7) by the small parameter γ_n the iteration may be interpreted as a finite difference scheme for a related differential equation, except that the right hand side is a random field. Using the techniques from stochastic algorithms (see e.g. Kushner and Yin [77] as well as [16–19]), the long term behavior of the discrete time random process (7) is related to the asymptotics as $t \to \infty$ of the continuous-time deterministic *averaged* dynamics

$$\dot{x} = \mathbb{E}_x(w) - x \tag{8}$$

where $w = (w_{kr})_{k\in K, r\in R}$ is a random vector whose distribution is determined as above by the mixed strategies $\pi_k = \sigma_k(x_k)$. The expectation here is with respect to the distribution induced by the mixed strategies $\pi = \Sigma(x)$, where $\Sigma : \mathbb{B} \to \Delta$ is the map $\Sigma(x) = (\sigma_k(x_k))_{k\in K}$ with $\mathbb{B} = \prod_{k\in K} \mathbb{R}^{R_k}$ the space of perceptions.

In order to give a more explicit description of (8), let us consider the functions $U_k(\pi) = \mathbb{E}_\pi[u_k(p)]$ and define the map $F : \Delta \to \mathbb{B}$ by setting $F_{kr}(\pi) = U_k(\delta_r, \pi_{-k})$ with δ_r the Dirac mass. This is simply the expected travel time for player k when he/she chooses route $r \in R_k$ while the other players use mixed strategies $\pi_{-k} = (\pi_j)_{j\neq k}$. By conditioning on player k's move we have

$$\mathbb{E}_x[w_{kr}] = \pi_{kr} F_{kr}(\pi) + (1 - \pi_{kr})\, x_{kr},$$

so that considering the composed map $C : \mathbb{B} \to \mathbb{B}$ given by $C(x) = F(\Sigma(x))$, (8) may be expressed as

$$(\forall k \in K)\,(\forall r \in R_k) \quad \dot{x}_{kr} = \sigma_{kr}(x_k)\,[C_{kr}(x) - x_{kr}]. \tag{9}$$

These evolution equations are interpreted as a process in which driver k's estimate x_{kr} dynamically tracks the expected value $C_{kr}(x)$ of the travel time of route r, which is determined by the other player's behavior $(\pi_j)_{j\neq k}$. We stress that (9) is not postulated as a mechanism of adaptive behavior, but it is just an auxiliary tool for analyzing (7).

3.2.1 Convergence of the dynamics: rest points and perturbed game

From general results on stochastic approximation we know that the attractors of (9) capture all limit points of the stochastic process (7). For our current purposes it suffices to mention that if (9) has a unique rest point $\bar{x}$ which is a global attractor for the dynamics, then x^n converges to $\bar{x}$ almost surely (see e.g. [16, Corollary 5.4]). It is worth noting that the convergence of the state variables $x^n \to \bar{x}$ entails the convergence of the corresponding mixed strategies $\pi_k^n = \sigma_k(x_k^n)$ and therefore of the behavior of players.

Clearly, the rest points of (9) are the fixed points of C. When C is a contraction the fixed point is unique and it is a global attractor. More precisely,

Theorem 2 ([30]) *If $C : \mathbb{B} \to \mathbb{B}$ is a $\|\cdot\|_\infty$-contraction then its unique fixed point $\bar{x} \in \mathbb{B}$ is a global attractor for the adaptive dynamics (9) and the sequence x^n generated by the learning process (7) converges almost surely towards $\bar{x}$.*

In general, since C is continuous with bounded range we may use Brouwer's theorem to deduce that rest points exist, though they may not be unique and are not necessarily attractors. These rest points can be characterized in terms of an associated perturbed routing game. Namely, let $\mathcal{F}$ be the set of solutions of $x = C(x)$ and $\mathcal{E}$ the corresponding set of equilibrium probabilities $\pi = \Sigma(x)$. The fixed point equation can be written as a coupled system $x = F(\pi)$ and $\pi = \Sigma(x)$, which shows that the map $x \mapsto \Sigma(x)$ is a bijection from $\mathcal{F}$ to $\mathcal{E}$ with inverse $\pi \mapsto F(\pi)$. Moreover, the equilibrium probabilities $\pi \in \mathcal{E}$ are characterized by the equation $\pi = \Sigma(F(\pi))$, namely $\pi_k = \nabla\varphi_k(F_k(\pi))$ for all $k \in K$. The latter are the optimality conditions saying that each π_k minimizes the function

$$G_k(\pi_k, \pi_{\text{-}k}) = \langle \pi_k, F_k(\pi) \rangle + \psi_k(-\pi_k),$$

with $\psi_k = (-\varphi_k)^*$. Recall that $F_k(\pi)$ depends only on $\pi_{\text{-}k}$. Hence π is a Nash equilibrium for an underlying game which is a perturbation of the original routing game with payoffs $\langle \pi_k, F_k(\pi) \rangle$.

Proposition 2 ([30]) *$\mathcal{E}$ is the set of Nash equilibria for the perturbed game $\mathcal{G}$ with strategy sets Δ_k and payoffs G_k.*

This result is somewhat surprising since the adaptive dynamics (7) assumed very little in terms of rationality of the drivers. They need not even be aware that they are playing a game, yet their long term behavior can be described as a fully rational equilibrium for an underlying game $\mathcal{G}$.

Example To illustrate how the previous results can be used, let us consider the case where $\Sigma(x)$ is given by a Logit discrete choice

$$\sigma_{kr}(x_k) = \frac{\exp(-\beta_k x_{kr})}{\sum_{p \in R_k} \exp\left(-\beta_k x_{kp}\right)}. \tag{10}$$

In this case the perturbed game in Proposition 2 is defined by the payoffs

$$G_k(\pi) = \langle \pi_k, F_k(\pi) \rangle + \frac{1}{\beta_k} \sum\nolimits_{r \in R_k} \pi_{kr} \left[\ln \pi_{kr} - 1\right],$$

and the points $\pi \in \mathcal{E}$ correspond to the so-called *quantal response equilibria* studied in [87]. Note that in the limit when $\beta_k \to \infty$ the model becomes deterministic and we get back to Rosenthal's model. The conditions under which C is a $\|\cdot\|_\infty$-contraction can be worked out explicitly. To this end we denote $\omega = \max_{k \in K} \sum_{j \neq k} \beta_j$ and $\theta = \max_{r \in R} \sum_{a \in r} \delta_a$ where δ_a measures the increment in the congestion of link a as a result of an additional user, namely

$$\delta_a = \max_{u=2,\ldots,|K|} s_a(u) - s_a(u-1).$$

The parameter θ allows to estimate the impact over a player's payoff when another player changes her move. It turns out that C is $\|\cdot\|_\infty$-Lipschitz with constant $L = 2\omega\theta$. Thus, if $2\omega\theta < 1$ we conclude that C is a $\|\cdot\|_\infty$-contraction and (7) converges almost surely towards the unique rest point $\bar{x}$. The corresponding probabilities $\bar{\pi} = \Sigma(\bar{x})$ are the unique quantal response equilibrium.

3.3 Parallel link networks: a potential function

Let us now consider a more specific situation in which player choices are described by the Logit rule (10), and the network consists of a single origin destination pair connected by a set of parallel links as shown in Fig. 4. In this case we have $o_k = o$ and $d_k = d$ for all $k \in K$, and also $R_k \equiv A = \{a_1, \ldots, a_m\}$.

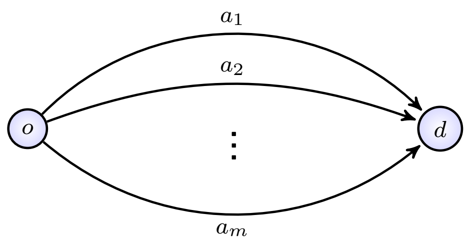


Fig. 4 Parallel link network

In this setting, the vector payoff map $F(\cdot)$ can be expressed as the gradient of a potential function which is inspired from Rosenthal [110]. Namely, consider the map $H : [0, 1]^{K \times A} \to \mathbb{R}$ defined by

$$H(\pi) = \mathbb{E}_{\pi}^{B}\left[\sum\nolimits_{a \in A} \sum\nolimits_{i=1}^{\tilde{w}_a} s_a(i)\right],$$

where $\tilde{w}_a = \sum_{k \in K} X_{ka}$ is a sum of independent Bernoullis with $\mathbb{P}(X_{ka} = 1) = \pi_{ka}$. Denoting $\tilde{w}_a^{-k} = \sum_{i \neq k} X_{ia}$ we have for all $\pi \in \Delta$

$$F_{ka}(\pi) = \mathbb{E}_{\pi}^{B}\left[s_a(\tilde{w}_a^{-k} + 1)\right] = \frac{\partial H}{\partial \pi_{ka}}(\pi). \tag{11}$$

Hence $F(\pi) = \nabla H(\pi)$, which yields an alternative characterization of equilibria as stationary points of the perturbed potential function

$$\Psi(\pi) = H(\pi) + \sum\nolimits_{k \in K} \frac{1}{\beta_k} \sum\nolimits_{a \in A} \pi_{ka} \left[\ln(\pi_{ka}) - 1\right].$$

and its associated Lagrangian

$$\mathcal{L}(\pi, \mu) = \Psi(\pi) + \sum\nolimits_{k \in K} \mu_k \left[\sum\nolimits_{a \in A} \pi_{ka} - 1\right].$$

Theorem 3 ([30]) *For each $x \in \mathbb{B}$ and $\pi = \Sigma(x)$ the following are equivalent:*

(a) *$x \in \mathcal{F}$,*
(b) *π is a Nash equilibrium of the perturbed game $\mathcal{G}$,*
(c) *there exist multipliers $\mu_k \geq 0$ such that $\nabla_{\pi} \mathcal{L}(\pi, \mu) = 0$,*
(d) *π is a critical point of Ψ, i.e. $\nabla \Psi(\pi)$ is orthogonal to the tangent set $T_{\Delta}(\pi)$.*

The potential function H can be used to improve the convergence results for the learning process (7). To this end we note that $F_{ka}(\pi)$ is a symmetric polynomial in the variables $(\pi_{ja})_{j \neq k}$ only, and does not depend on the probabilities $\pi_{ja'}$ for $a' \neq a$. Hence the second derivatives of H are all zero except for $\frac{\partial^2 H}{\partial \pi_{ka} \partial \pi_{ja}}$ for $a \in A$ and $k \neq j$. Using (11) one can prove that the latter belong to the interval $[0, \theta]$, from which it follows that C is $\|\cdot\|_\infty$-Lipschitz with constant $L = \frac{1}{2}\omega\theta$. This improves by a factor four the estimate in the example at the end of the previous section, namely

Theorem 4 ([30]) *If $\omega\theta < 2$ then C is a $\|\cdot\|_\infty$-contraction. Its unique fixed point $\bar{x}$ is a global attractor of* (9) *and the process (7) converges almost surely to $\bar{x}$.*

To interpret this result we note that the coefficients β_k are inversely proportional to the standard deviation of the random terms in the Logit choice model. Thus, the condition $\omega\theta < 2$ requires either a weak congestion effect (small θ) or a sufficiently large noise (small ω). At lower noise levels (large β_k) player behavior becomes increasingly deterministic and multiple equilibria coexist just as in Rosenthal's game [110]. The case of small noise was studied by Duffy and Hopkins [38] in the context of a market entry game,[5] by considering two alternative dynamics: proportional reinforcement and hypothetical reinforcement.

The parameter ω involves sums of β_k's so the condition becomes more and more stringent as the number of players increases. The following results show that uniqueness of the rest point still holds under the much weaker conditions in which ω is replaced by $\beta = \max_{k\in K} \beta_k$. As a matter of fact, these conditions ensure that the perturbed potential Ψ is strongly convex so that the equilibrium is its unique minimizer.

Proposition 3 ([30]) *If $\beta\theta < 1$ then Ψ is strongly convex with parameter $(\frac{1}{\beta} - \theta)$. Also, if $\beta\theta < 2$ and the arc travel times are linear, then Ψ is quadratic and strongly convex with parameter $(\frac{2}{\beta} - \theta)$ over the set Δ. In both cases Ψ attains its minimum at a unique point $\bar{\pi} \in \Delta$. This point is the only Nash equilibrium of the game $\mathcal{G}$ while $\bar{x} = F(\bar{\pi})$ is the corresponding unique rest point of the adaptive dynamics* (9).

In the symmetric case where all players have the same Logit parameter $\beta_k \equiv \beta$, one might expect that rest points are also symmetric with all players sharing the same perceptions: $\bar{x}_k = \bar{x}_j$ for all $k, j \in K$. This is indeed the case when $\beta\theta$ is small, but beyond a certain threshold there is a multiplicity of rest points all of which except for one are non-symmetric.

Theorem 5 ([30]) *For identical players the adaptive dynamics (9) has exactly one symmetric rest point $\hat{x} = (\hat{y}, \ldots, \hat{y})$. When $\beta\theta < 2$ this is the unique rest point and is a local attractor for (9). As a consequence, the game $\mathcal{G}$ has a unique symmetric Nash equilibrium which is the only equilibrium when $\beta\theta < 2$.*

The assumption $\beta\theta < 2$ above is much weaker than the condition in Theorem 4 which for identical players becomes $\beta\theta < \frac{2}{N-1}$. However, since $\beta\theta < 2$ already guarantees a unique rest point $\hat{x}$, one may hope that it remains a global attractor under this weaker condition. Although numerical experiments confirm this conjecture, we have only been able to prove that $\hat{x}$ is a *local* attractor. Unfortunately this does not allow to conclude the almost sure convergence of the learning process (7). The numerical simulations also show that convergence to an equilibrium still holds for $\beta\theta > 2$, but there is a bifurcation value beyond which the symmetric equilibrium becomes unstable (in the sense of dynamical systems) and the dynamics converge towards one

[5] This corresponds to the case of 2 parallel links with linear costs and identical players.

of the multiple non-symmetric equilibria. The structure of this bifurcation seems quite intricate and is not well understood.

3.4 State dependent adaptive dynamics

The dynamics (7) use an exogenous sequence of averaging factors γ_n which depend only on the stage n and not on the history of the process. A meaningful variant studied by Bravo [22] considers a state dependent factor $\gamma_n = 1/(1+\theta^n_{kr^n_k})$ where θ^n_{kr} counts the number of times that each route $r \in R_k$ has been observed up to stage n. This yields again a Markov process which can be analyzed using stochastic approximation. Under very mild conditions, Bravo [22] shows that this process converges to an equilibrium with positive probability, while almost sure convergence holds under the same assumptions as in the state independent case (see Propositions 4.2 and 5.1 in [22]). Additionaly, [22, Theorem 5.1 and Proposition 5.2] show that the state-dependent dynamics exhibit a much faster convergence rate. To provide some intuition, we note that the continuous expected dynamics associated with the state-dependent process are given by the following coupled dynamics, for each $k \in K$ and $r \in R_k$

$$\begin{cases} \dot{x}_{kr} = \frac{\sigma_{kr}(x_k)}{\theta_{kr}(t)}\,[C_{kr}(x) - x_{kr}]\,, \\ \dot{\theta}_{kr} = \sigma_{kr}(x_k) - \theta_{kr}, \end{cases}$$

whose rest points are the same as for (9). The second equation implies that θ_{kr} tends to $\sigma_{kr}(x_k)$ very fast, so that in the first equation we have $\frac{\sigma_{kr}(x_k)}{\theta_{kr}(t)} \sim 1$. In contrast, the multiplicative factor in (9) is $\sigma_{kr}(x_k)$ which slows down the convergence for routes with large travel times and small probabilities $\sigma_{kr}(x_k)$.

4 Risk-aversion and traffic equilibrium

The models discussed in the previous sections evaluate routes by their expected travel times. While this might be appropriate for a risk neutral agent, a risk-averse driver will be more concerned with travel time reliability.

Risk-sensitive route choice is a relatively young research area. General discussions on risk evaluation in route choice can be found in Bates et al. [11], Noland [97], Hollander [68]. Also, the literature review in Nikolova and Stier-Moses [96] describes a number of alternative approaches that have been recently proposed to model risk in the context of network traffic models. In Sect. 4.4 we briefly discuss some of these risk-averse route choice models.

A basic question here is to understand the mechanisms of route choice under risk and how risk-aversion may affect congestion and equilibrium. While this calls for modeling the actual behavior of agents, it can also be approached from a normative angle by asking which properties characterize a rational route choice under risk. General theories of choice under risk provide a natural framework to address these questions. For a recent and complete account of these theories we refer to Föllmer and Schied [49].

In what follows we present an overview of the general approaches that have been proposed to measure risk, and how these measures and postulates are interpreted in the context of risk-averse routing. Within these classical frameworks in the theory of choice, we show that the so-called *entropic risk measures* emerge as the only ones that satisfy *additive consistency*, an intuitive axiom for route choice. This axiom is relevant since it allows to formulate risk averse equilibrium in the standard frameworks of Wardop, SUE, and Rosenthal.

4.1 Risk valuations: axioms and examples

In the sequel we consider a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A *risk valuation* is a map $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ which associates to each bounded random variable X a scalar risk value $\rho(X)$ satisfying the two following postulates

NORMALIZATION: $\rho(0) = 0$,
MONOTONICITY: if $X \le Y$ almost surely then $\rho(X) \le \rho(Y)$.

Such a map induces a preference relation:[6] $X \preceq Y$ iff $\rho(X) \le \rho(Y)$. Against common usage, in our context where random variables represent travel times, we read this as "*X is preferred to Y*". Normalization is not restrictive as one can always replace $\rho(X)$ by $\rho(X) - \rho(0)$, while monotonicity has a clear intuitive meaning: routes with smaller travel times are preferred.

A particular class of risk valuations, the so-called *risk measures* introduced in the area of finance by Artzner et al. ([8]), assume as a third postulate

TRANSLATION INVARIANCE: $\rho(X + m) = \rho(X) + m$ for all $m \in \mathbb{R}$.

Among these, Artzner [8] distinguishes the subclass of *coherent risk measures* which are moreover sub-additivite and positively homogeneous.

Under the normalization axiom, traslation invariance is equivalent to requiring simultaneously

NORMALIZATION ON CONSTANTS: $\rho(m) = m$ for all $m \in \mathbb{R}$.
TRANSLATION CONSISTENCY: $\rho(X) \le \rho(Y) \Rightarrow \rho(X + m) \le \rho(Y + m)$.

Normalization on constants is a mild requirement: it suffices to have $m \mapsto \rho(m)$ strictly increasing and continuous, since then this function has an inverse σ and we may substitute ρ by $\sigma \circ \rho$. Translation consistency is a plausible condition stating that the preference between X and Y is not altered when we add a constant. While this postulate has been contested in finance (risk attitudes may change after receiving a heritage), it seems very likely for route choice. As a matter of fact we will consider the stronger axiom

ADDITIVE CONSISTENCY: if $\rho(X) \le \rho(Y)$ then $\rho(X + Z) \le \rho(Y + Z)$ for all Z independent of (X, Y).

It is an easy exercise to check that under translation consistency and normalization on constants, this is equivalent to

[6] Not all preference relations can be represented in this form, though this is not too restrictive (see e.g. [47, Proposition 2.13]).

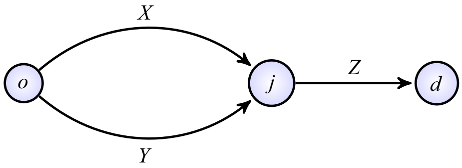


Fig. 5 The additive consistency axiom for route choice

ADDITIVITY: if $\rho(X+Y) = \rho(X) + \rho(Y)$ for all independent variables X, Y.

The meaning of additive consistency is illustrated in Fig. 5. Suppose that the preferred choice to go from o to j is the upper link. Then, if we extend our trip to d we should still prefer the upper link: since the arc (j, d) is compulsory and one must inevitably incur in the cost Z, the decision at o should not depend on Z. This seems all the more plausible since, due to independence, even if one observes Z this reveals no information that could affect the choice between X and Y. A change of preference to the lower arc would appear as paradoxical!

4.1.1 Examples

The mean-stdev maps $\rho_\gamma^{std}(X) = \mathbb{E}(X) + \gamma\sigma(X)$ quantify risk by the sum of the mean plus a multiple of the standard deviation, or some other variability index [86,98,99]. These maps satisfy normalization and translation invariance, but monotonicity fails: take $X \sim U[0,1]$ a uniform variable and $Y = (1+X)/2$ so that $X \leq Y$ almost surely, yet for γ large we have $\rho_\gamma^{std}(Y) < \rho_\gamma^{std}(X)$. They also lack additive consistency: for independent normals $X \sim N(11,1)$, $Y \sim N(10,5)$, $Z \sim N(10,2)$ and $\gamma = 1$ we have $\rho_\gamma^{std}(X) < \rho_\gamma^{std}(Y)$, but $\rho_\gamma^{std}(X+Z) > \rho_\gamma^{std}(Y+Z)$. Truncating these normals we get an example in $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Similar examples can be built for each $\gamma > 0$.

A popular risk measure is *Value-at-Risk* defined as the p-percentile

$$\mathrm{VaR}_p(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(X \leq m) \geq 1 - p\}.$$

This map is positively homogeneous but not convex nor sub-additive (see [8]). The best known coherent risk measure is *Average Value-at-Risk*, introduced in [8] and defined for a level $p \in (0,1)$ by

$$\mathrm{AVaR}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)dq.$$

For continuous variables it coincides with the *Tail Conditional Expectation* $\mathbb{E}(X|X \geq \mathrm{VaR}_p(X))$. When restricted to normal random variables both VaR_p and AVaR_p coincide with ρ_γ^{std} for appropriate corresponding constants γ. In particular they do not satisfy additive consistency either.

A family of convex (but not coherent) risk measures are the entropic measures defined as (cf. [47–49,113])

$$\rho_{\beta}^{ent}(X) = \frac{1}{\beta}\ln\left(\mathbb{E}\left(e^{\beta X}\right)\right).$$

These measures are clearly additive and thus satisfy additive consistency. They can be derived from additive premium principles [56], as well as from expected utilities with constant absolute risk aversion CARA ([6,106]). The case $\beta > 0$ characterizes risk-averse behavior while $\beta < 0$ corresponds to a risk-prone agent. The limit $\beta \to 0$ gives $\rho_0^{ent}(X) = \mathbb{E}(X)$ which reflects risk neutrality.[7]

4.2 Additive consistency and risk-averse equilibrium

As noted above, a risk valuation which is normalized on constants and additive consistent, is automatically additive. This has useful consequences for risk-averse route choice and equilibrium. Namely, suppose that the random travel times $\tilde{t}_a$ are independent across arcs $a \in A$. Let $\tilde{c}_r = \sum_{a \in r} \tilde{t}_a$ denote the route travel times, and consider the problem of finding a risk-minimizing path

$$\min_{r \in R_k} \rho(\tilde{c}_r). \tag{12}$$

When ρ is additive the objective function separates as $\rho(\tilde{c}_r) = \sum_{a \in r} \rho(\tilde{t}_a)$ and (12) reduces to a standard shortest path problem with arc lengths $d_a = \rho(\tilde{t}_a)$. This can be efficiently solved using standard algorithms.

Consider next a risk-averse equilibrium model with traffic demands $g_k \geq 0$. As in Sect. 2.1 the demands $g_k = \sum_{r \in R_k} h_r$ are decomposed into path-flows $h_r \geq 0$, which induce arc loads $w_a = \sum_{r \ni a} h_r$. Suppose that the distribution $\tilde{t}_a \sim \mathcal{F}_a(w_a)$ depends on the total link flow w_a. We may then define a *risk-averse equilibrium* as a path-flow vector h which uses only risk-minimizing paths, namely, for each OD pair $k \in \mathcal{K}$ and every path $r \in R_k$ we must have

$$h_r > 0 \Rightarrow \rho(\tilde{c}_r) = \min_{p \in R_k} \rho(\tilde{c}_p).$$

If ρ is additive and the function $s_a(w_a) \triangleq \rho(\tilde{t}_a)$ increases with w_a, the situation falls into the standard framework of Wardrop equilibrium.

Similarly, in an atomic setting where each player $k \in K$ selects a path $r \in R_k$ that minimizes the risk $\rho(\tilde{c}_r)$ and the distribution $\tilde{t}_a \sim \mathcal{F}_a(w_a)$ depends on the number of players on the link, denoting $s_a(w_a) = \rho(\tilde{t}_a)$ we get a congestion game of the type treated in Sect. 3.1.

The situation seems much more difficult when the link travel times $\tilde{t}_a$ are not independent. A possible approach is to use *dynamic risk measures* which are specifically designed to deal with time-inconsistency in multi-stage decision processes [111–113]. However, in a routing context the notion of stage is unclear and may even depend on

[7] The limit cases $\beta \to \pm\infty$ can also be considered as extreme attitudes toward risk with $\rho_{\infty}^{ent}(X) =$ ess sup X and $\rho_{-\infty}^{ent}(X) =$ ess inf X. In this paper we only consider finite β's.

the representation of the network. A short discussion of this issue is presented in the final section of [31].

The models should also be changed if drivers are not homogeneous with respect to their valuation of risk. In the next section we show that additive consistency limits the choice to entropic risk measures. Thus, some homogeneity among users might be expected, though they can still differ in their risk aversion index β. This case leads to an equilibrium model with multiple user classes. One might still get the existence of equilibria, but a simple variational characterization such as (P-WE) or the existence of a potential function like (6) seems unlikely.

4.3 Additive consistency for classical theories of choice

The roots of the theories of choice can be traced back to Bernoulli [20] who argued that preferences among random variables could be expressed in terms of an expected utility index $\mathbb{E}(c(X))$ with $c : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous. Such expected utility preferences hold an intuitive appeal for route choice. Imagine for instance a fire truck rushing to an emergency so that reaching the destination quickly is critical since the damage by fire increases nonlinearly with time. A route with small expected time but affected by events of high congestion might be too risky, and a longer but more reliable route could be a better choice. Expected utility captures the nonlinear relation between "time" and "cost", and minimizing expected cost is a natural goal. A convex $c(\cdot)$ characterizes a risk-averse agent who always prefers the expected value $\mathbb{E}(X)$ over the uncertain outcome X.

Cearly, expected utility preferences may also be expressed using the so-called *certainty equivalent*

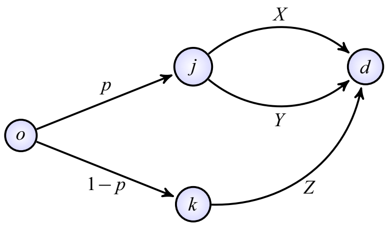
$$\rho_c(X) = c^{-1}\left(\mathbb{E}\left(c(X)\right)\right).$$

This map is normalized on constants and monotone so that it is a risk valuation, though in general it is not a risk measure since translation invariance may fail. Axiomatic characterizations of the preferences that can be represented in this form were given by Kolmogorov [75], Nagumo [89], de Finetti [42], and von Neumann and Morgenstern [90]. For a thorough discussion on expected utilities we refer to [36,43,49]. In particular, a risk valuation ρ is of the form ρ_c, with c unique up to a positive affine transformation, if and only if it satisfies

- Law invariance: $F_X = F_Y \Rightarrow \rho(X) = \rho(Y)$,
- Strict monotonicity: if $X < Y$ almost surely then $\rho(X) < \rho(Y)$,
- Weak continuity: if $X_n \overset{\mathcal{D}}{\to} X$ in distribution then $\rho(X_n) \to \rho(X)$,
- Independence: if $\rho(X) \leq \rho(Y)$ then $\rho(\mathcal{L}(p; X; Z)) \leq \rho(\mathcal{L}(p; Y; Z))$.

Here $\mathcal{L}(p; X; Z)$ denotes the lottery with distribution $pF_X(x) + (1-p)F_Z(x)$ for all $x \in \mathbb{R}$. To interpret this condition, imagine a driver who has two options X, Y to travel from j to d of which he prefers X (see Fig. 6). Suppose now that he is actually at a point o on the other side of a river, and to reach the intermediate node j he must first cross a bridge which is open with probability p. Otherwise he must take a

Fig. 6 The independence axiom



long detour Z. Thus, our driver faces a choice between the lotteries $\mathcal{L}(p; X; Z)$ and $\mathcal{L}(p; Y; Z)$ and the independence axiom requires that the first should be preferred. Although this independence axiom seems quite plausible for route choice, it has not been without critics. The paradoxes of Allais [4] and Ellsberg [40] show specific situations in which the axiom is violated and agents do not behave consistently with the predictions of this theory. Further empirical evidence by Kahneman and Tversky [70] motivated alternative independence axioms that lead to theories of choice based on other representations. These include the dual theory of choice [126] and the rank-dependent utilities [26, 107, 120].

Namely, while expected utility introduces risk-aversion by magnifying the effects of bad outcomes through a nonlinear transformation of the cost $c(X)$, Yaari's dual theory of choice [126] uses the idea that a risk-averse agent tends to overstate the probability of bad outcomes. An agent is then characterized by a continuous nondecreasing *distortion map* $h : [0, 1] \to [0, 1]$ with $h(0) = 0$ and $h(1) = 1$, so that the probability $\mathbb{P}(X > x)$ is distorted as $\theta_X^h(x) = h(\mathbb{P}(X > x))$. Risk-aversion corresponds to $h(x) \geq x$ for all $x \in [0, 1]$, while a risk-prone agent satisfies the reverse inequality. The function $\theta_X^h(\cdot)$ is a decumulative distribution so we may find a random variable X^h such that $\mathbb{P}(X^h > x) = \theta_X^h(x)$. The agent's preferences are then described by the map[8]

$$\rho^h(X) = \mathbb{E}(X^h) = \int_{-\infty}^{0} \left[\theta_X^h(x) - 1\right] dx + \int_0^{\infty} \theta_X^h(x)dx.$$

This is clearly a law invariant risk measure which is also positively homogeneous and normalized on constants. It is called a *distortion risk measure*. Axiomatic characterizations of the preferences that can be represented in this form are given in [36, 126].

The two previous ideas for modeling risk-aversion are complementary and can be combined by considering preference functionals of the form

$$\rho_c^h(X) = c^{-1}\left(\mathbb{E}\left(c(X^h)\right)\right) = c^{-1}\left(\mathbb{E}\left(c(X)^h\right)\right),$$

where c is a utility function and h is a distortion map. These maps are normalized on constants but they need not be translation invariant nor additive consistent. For $h(x) = x$ we recover expected utilities, while $c(x) = x$ gives the distortion risk measures. The

[8] As a technical remark, we note that in order to properly define a random variable X^h from its distribution, we need $(\Omega, \mathcal{F}, \mathbb{P})$ to be a *standard non-atomic probability space*. However the explicit integral formula for $\rho^h(X)$ does not require this.

functionals ρ_c^h are called *rank-dependent expected utilities* and have been considered by several authors including Quiggin [107], Wakker [120], and Chateauneuf [26], who provide axiomatic characterizations of the preferences that can be represented in this form.

It turns out among these standard theories of choice, the only preferences that satisfy additive consistency are the *entropic risk measures*.

Theorem 6 ([31]) *A rank dependent expected utility ρ_c^h is translation invariant iff $c(\cdot)$ is an exponential function or the identity. Moreover, the only measures of form ρ_c^h which satisfy additive consistency are the entropic risk measures: h is the identity and c is either an exponential function or the identity.*

This result drastically narrows down the options for modeling route choice under risk, unless we give up additive consistency. Note however that beyond the standard theories there exist other additive consistent maps. In particular this holds for mixtures of entropic risk measures. Namely, for each probability distribution $F : \mathbb{R} \to [0, 1]$ the following gives an additive risk measure[9]

$$\rho_F^{ent}(X) = \int_{\mathbb{R}} \rho_\beta^{ent}(X)\, dF(\beta).$$

Theorem 6 was proved in Luan [83] assuming c and h twice differentiable and increasing, with h concave and c convex. An alternative proof was given in Goovaerts et al. ([59]) assuming in addition that c has a McLaurin expansion. Previous results for ρ_c and ρ^h, always under additional concavity and/or smoothness assumptions, were presented in Gerber [56] and Heilpern [66]. The proof without any *a priori* regularity assumptions is found in Cominetti and Torrico [31]. Note however that regularity of $c(\cdot)$ as well as additivity of ρ_c follow as a consequence.

4.4 Recent work on risk-averse routing

The use of mean-stdev risk maps ρ_γ^{std} (see Sect. 4.1.1) in the context of risk-averse routing were investigated by Nikolova and Stier-Moses [96], including the case when both the expected value and the variance of travel times depend on traffic intensity. Algorithms to compute optimal routes for the mean-stdev objective were studied by Nikolova [94] and Nikolova et al. [95], giving an exact algorithm with sub-exponential complexity as well as a PTAS. A similar model in which risk-aversion is measured by expected value plus a variability index was considered in Ordoñez and Stier-Moses [100] together with other alternative models based on robust optimization and α-percentiles. Percentile equilibria in route choice were also investigated by Nie [92]. A different approach to risk-averse path choice considers user preferences based on the on-time arrival probability. This was studied by Nie and Wu [93], addressing the question of whether or not route optimality is inherited by subpaths. An algorithm for this objective function was also given by Nikolova et al. [95]. A related approach by

[9] A simple entropic risk measure corresponds to the case of a Dirac distribution F.

Nie et al. [91,124] reconsiders the route choice question under stochastic dominance constraints. Some further recent papers are [5,125,129].

Except for the approach based on stochastic dominance, all the other involve risk maps that do not satisfy additive consistency. While this raises a concern whether these models describe a fully rational behavior in route choice, it does not invalidate them. From a practical viewpoint, all these models may plausibly describe the behavior of some agents and there is no firm empirical evidence to assert that drivers make consistent choices. Furthermore, Theorem 6 deals with preferences defined over all of $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ which might be asking too much as one could argue that drivers make choices in a much narrower subset, e.g. a set of uniformly bounded nonnegative variables. In summary, although additive consistency is a natural axiom for route choice, there is still work to be done before one can tell which is the most appropriate model for route choice under risk.

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