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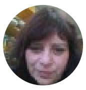

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# FINITE MASS SOLUTIONS FOR A NONLOCAL INHOMOGENEOUS DISPERSAL EQUATION


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Abstract. In this paper we study the asymptotic behavior of the following nonlocal inhomogeneous dispersal equation

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}dy - u(x,t) \qquad x\in\mathbb{R},\ t>0,$$

where $J$ is an even, smooth, probability density, and $g$, which accounts for a dispersal distance, is continuous and positive. We prove that if $g(|y|)\sim a|y|$ as $|y|\to\pm\infty$ for some $0<a<1$, there exists a unique (up to normalization) positive stationary solution, which is in $L^1(\mathbb{R})$. On the other hand, if $g(|y|)\sim |y|^p$, with $p>2$ there are no positive stationary solutions. We also establish the asymptotic behavior of the solutions of the evolution problem in both cases.


## 1. Introduction

Integro-differential equations have been used in population ecology to model the long range dispersal of species. A simple model for nonlocal dispersal is the following convolution equation in the real line:

$$u_t(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy - u(x,t) \quad x\in\mathbb{R},\ t\in\mathbb{R}, \tag{1.1}$$

here $u(x,t)$ accounts for the density of a population at site $x$ and time $t$, and $J$ is an even probability density. In this model, the rate at which individuals of site $y$ move to site $x$, given by $J(x-y)$, only depends on the distance between sites $x$ and $y$, thus dispersal is homogeneous in space. In the last years, there have been several works that consider the effect of a heterogenous environment adding a growth term to equation (1.1) which is space dependent, as in [17, 18, 12, 6, 9, 8].

Another way in which the heterogeneity of the environment can affect the distribution of a species is through space-dependent dispersal strategies. In [2] the dispersal equation

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}dy - u(x,t) \quad x\in\mathbb{R},\ t\in\mathbb{R}, \tag{1.2}$$

is proposed. The function $g(y)$ accounts for the *dispersal distance* which depends on the departing point. In this equation, the rate at which individuals arrive at site $x$ is

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy,$$

while the rate at which individuals leave site $x$ is

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g(x)}\,dy = u(x,t).$$

In this model the dispersal is not homogeneous and under suitable time scaling, when $J$ approximates Dirac's delta, this equation resembles $u_t = \Delta(g^2u)$ (see [5, 13]).

The objective of this paper is to address a basic issue for equation (1.2), which is to study the asymptotic behavior of its solutions when $u(\cdot,0)\in L^1(\mathbb{R})$, trying to understand the role played by the function $g$.

From now on we will assume that the following hold:

(J1) $J$ is an even, positive, Hölder continuous function, $\int_{\mathbb{R}} J(x)dx = 1$ and $\operatorname{supp} \mathrm{J} = [-1,1]$.

(g1) $g$ is a positive continuous function.

Basic properties for (1.2) are the global existence of solutions when $u(\cdot,0)\in L^1(\mathbb{R})$, the mass preserving property (i.e. $\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u(x,0)dx$ for all $t$), and infinite speed of propagation, that is if $u(\cdot,0)\ge 0$ a.e. and $u(\cdot,0)\ne 0$, then $u(\cdot,t)>0$ a.e. for $t>0$. We refer to [2] for details.

The dispersal distance function $g$ affects the location of the sites that can be reached from a certain departure point, as well as the location of the sites that can reach a given point. Indeed, note that since $\operatorname{supp}\mathrm{J} = [-1,1]$ then the individuals departing from a site $x$ will move to sites satisfying $|y-x|<g(x)$. Also, observe that

$$J\left(\frac{x-y}{g(y)}\right)\ne 0 \ \text{ if and only if } \ |x-y|<g(y); \tag{1.3}$$

hence, the only points $y$ that contribute to the the integral term in (1.2) are those where $g(y) > |x-y|$. We denote this set $D_x = \{y \,/\, |x-y|<g(y)\}$ which accounts for the *domain of dependence* of $x$. In the case where $0<\kappa_1<g(y)<\kappa_2$ for all $y$ it is easy to check that

$$(x-\kappa_1, x+\kappa_1)\subset D_x \subset (x-\kappa_2, x+\kappa_2),$$

which means that the domain of dependence consists roughly in an interval of fixed size around the point. As we will discuss below, in this case the asymptotic behavior of the finite mass solutions of (1.2) is basically the same as when $g$ is a constant.

Observe that if $g$ becomes unbounded as $y\to\infty$ (or $y\to-\infty$) then as $x\to\infty$ the set $D_x$ will contain points $y$ such that $|y-x|\to\infty$, that is, points whose distance from $x$ become unbounded will be in the domain of dependence of $x$. Moreover, if $\frac{g(y)}{|y|}>a>1$ for large $|y|$, then for all $x$ the set $D_x$ is unbounded. Thus, the dispersal term in equation (1.2) can behave quite different from the one in (1.1). We will see that in these cases the behavior of the solutions does actually differ from the case when $g$ is constant.

The asymptotic behavior of the solutions of (1.2) for $g=1$ is already understood; in this case all solutions with finite mass converge to 0 locally in $\mathbb{R}$ as $t\to\infty$ (see [1] for precise estimates).

The same happens when $g$ is a continuous function bounded above and below, as was shown in [2]. A key point to prove this last result was the use of the following energy functional defined in [16]:

$$E(u) = \int_{\mathbb{R}} \frac{u^2}{p} dx, \tag{1.4}$$

where $p$ is a positive steady state solution of (1.2), i.e. it satisfies

$$p(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} dy \quad x \in \mathbb{R}. \tag{1.5}$$

It can be shown that $E'$ is given by

$$E'(t) = -\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} \left(\frac{u(x,t)}{p(x)} - \frac{u(y,t)}{p(y)}\right)^2 dx\, dy, \tag{1.6}$$

thus, if $\lim_{t\to\infty} u(\cdot,t) \to v$ locally, then $\frac{v}{p}$ should be a constant. In the case that $g$ is bounded above and below, it is shown in [2] that any positive solution $p$ of (1.5) is bounded above and below, hence $v \equiv 0$. If $p$ happens to be in $L^1(\mathbb{R})$ then we should have that $v = \frac{\langle u(\cdot,0)\rangle}{\langle p\rangle} p$, where $\langle\cdot\rangle$ denotes the average of a function, and $\lim_{t\to\infty} u(\cdot,t) \to v$ in $L^1(\mathbb{R})$. In this case, we see that the pure dispersal equation will always produce an inhomogeneous asymptotic distribution, when the initial condition is nontrivial.

In the present paper, we will provide examples that show how the behavior of $g$ near $\pm\infty$ affects the existence and profiles of the positive solutions of (1.5). This type of dependence was hinted in [2]. Indeed, it was proved that if $g$ is bounded above, then (1.5) has a positive solution; and if $g$ is bounded away from zero at $\pm\infty$ then the positive solution of (1.5) (which is unique up to a multiplicative constant) is bounded above and below, while if $\lim_{|y|\to\infty} g(y) = 0$ any positive solution is unbounded. As the behavior of $g$ near $\pm\infty$ becomes unbounded we have that $L^1(\mathbb{R})$ solutions do appear. Actually, we will show:

**Theorem 1.1.** *Suppose that*

$$0 < \liminf_{|y|\to\infty} \frac{g(y)}{|y|} \le \limsup_{|y|\to\infty} \frac{g(y)}{|y|} < 1.$$

*Then* (1.5) *has a unique positive solution* $p$*, up to a multiplicative constant, and it belongs to* $L^1(\mathbb{R})$*. In this case, all solutions of* (1.2) *with initial value* $u(\cdot,0) \in L^1(\mathbb{R})$ *converge in* $L^1(\mathbb{R})$ *to* $\lambda p$ *where* $\lambda = \frac{\langle u(\cdot,0)\rangle}{\langle p\rangle}$.

It turns out that, when $g$ grows faster than linearly at $\pm\infty$, the stationary problem (1.5) does no longer have a positive solution. More precisely, we will prove the following result:

**Theorem 1.2.** *Suppose that, for some* $p > 2$,

$$0 < \liminf_{|y|\to\infty} \frac{g(y)}{|y|^p} \le \limsup_{|y|\to\infty} \frac{g(y)}{|y|^p} < \infty.$$

*Then* (1.5) *does not have any positive solution. In this case, all solutions of* (1.2) *with initial value* $u(\cdot,0) \in L^1(\mathbb{R})$ *converge to 0 in* $L^1_{\text{loc}}(\mathbb{R})$.

The last theorem can be generalized to the case $p > 1$, but we are only dealing here with $p > 2$ for simplicity.

These two results shed some light on the relationship between the behavior of $g$ at $\pm\infty$ and the existence and profile of the positive solutions of (1.5). Finally, we should mention that other types of results relating the behavior of $g$ and the profile of the positive solutions of (1.5) are obtained in [3], when the problem is considered in a bounded domain and $g$ vanishes near the boundary. For instance, if for all $y$ we have that $0 < g(y) \le a\,\mathrm{dist}(y,\partial I)$, where $I$ is a bounded interval and $a < 1$, then the positive solutions of (1.5) become unbounded on $\partial I$.

The paper is organized as follows: in Section §2 we will establish some preliminary results regarding the energy $E$ defined above and the convergence of solutions of (1.2). In Section §3 we will prove Theorem 1.1, using an approximation procedure, and Section §4 will be devoted to the proof of Theorem 1.2.

## 2. Preliminary results

In this section we will prove some preliminary results needed to characterize the asymptotic behavior of the solutions of the initial value problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}dy - u(x,t) \quad x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in L^1(\mathbb{R}). \end{cases} \tag{2.7}$$

We define the operator $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ by

$$Tv(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{v(y)}{g(y)}dy,$$

which is continuous. Problem (2.7) above can be written as $u_t = Tu - u$, with $u(\cdot,0) = u_0$, and it can be shown with a simple application of semigroup theory that for $u_0 \in L^1(\mathbb{R})$ there exists a unique solution $u \in C^1(\mathbb{R}, L^1(\mathbb{R}))$ of (2.7), which in addition satisfies

$$\|u(\cdot,t)\|_{L^1} \le \|u_0\|_{L^1}. \tag{2.8}$$

As was mentioned above, we will study the behavior of solutions of (2.7) using the energy functional

$$E(u) = \int_{\mathbb{R}} \frac{u^2}{p}dx,$$

where $p$ is a positive supersolution of (1.2), that is, it satisfies

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}dy - p(x) \le 0 \quad x \in \mathbb{R}. \tag{2.9}$$

We will prove the following result:

**Proposition 1.** *Suppose that (J1) and (g1) hold, and that for all $R > 0$ there exists $\rho(R) > 0$ such that $D_x \subset B(0,\rho(R))$ for all $|x| \le R$. Moreover, assume that there exists a continuous $p > 0$ which satisfies* (2.9)*. If*

- (i) *$p \notin L^1(\mathbb{R})$, then, as $t \to \infty$, all solutions of* (2.7) *converge to 0 in $L^1_{\rm loc}(\mathbb{R})$.*
- (ii) *$p \in L^1(\mathbb{R})$, then, as $t \to \infty$, all solutions $u$ of* (2.7) *converge in $L^1(\mathbb{R})$ to $\lambda p$, where $\lambda = \int_{\mathbb{R}} u_0 dx / \int_{\mathbb{R}} p dx$.*

The proof of this proposition is based on showing that, for a class of initial conditions $u_0$, $E(u)$ is well defined for all $t$, it is decreasing as a function of $t$, and $E'(u) \to 0$ as $t \to \infty$.

Suppose that there exists a continuous $p > 0$ satisfying (2.9). As was done in [3], we define the Banach space $X_p = \{u \in C(\mathbb{R}) \mid \|u\|_p < \infty\}$, where

$$\|u\|_{X_p} = \sup_{x\in\mathbb{R}} \frac{|u(x)|}{p(x)}.$$

It can be proved (see Theorems 7 and 8 of [3]) that if the initial condition $u_0 \in X_p$ then there exists a unique solution $u \in C^1(\mathbb{R}, X_p)$ of (2.7), moreover

$$\|u(\cdot,t)\|_{X_p} \le \|u_0\|_{X_p} \quad \text{for all } t > 0. \tag{2.10}$$

Notice that for $u_0 \in L^1(\mathbb{R}) \cap X_p$ we have from (2.10) that

$$\frac{|u(x,t)|^2}{p(x)} \leqslant \|u(\cdot,t)\|_{X_p}|u(x,t)| \le \|u_0\|_{X_p}|u(x,t)|,$$

thus for all $t > 0$, $E(t) := E(u)(t)$ is well defined and, by (2.8), it satisfies

$$E(u) \le \|u_0\|_{X_p}\|u_0\|_{L^1}.$$

**Proof of Proposition 1**. The proof of (i) follows the ideas of Lemmas 10 and 11 in [3]. We provide the details for completeness. Taking $u_0 \in X_p \cap L^1(\mathbb{R})$, the function $E(t)$ is well defined. Observe that by (2.8) we obtain $\|u_t\|_{L^1} = \|J * u\|_{L^1} + \|u\|_{L^1} = 2\|u\|_{L^1} \leq 2\|u_0\|_{L^1}$, and by (2.9) and (2.10) we have

$$|u_t| \leq \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{|u(y,t)|}{p(y)}dy + \frac{|u(x,t)|}{p(x)}p(x) \leq \|u_0\|_{X_p}p(x) + \|u_0\|_{X_p}p(x),$$

whence $\|u_t\|_{X_p} \leqslant 2\|u_0\|_{X_p}$. Also, differentiating (2.7) we easily get $\|u_{tt}\|_{L^1} \leq 4\|u_0\|_{L^1}$ and $\|u_{tt}\|_{X_p} \leqslant 4\|u_0\|_{X_p}$. Then, $E \in C^2(0,+\infty)$ and $E' = 2\int_{\mathbb{R}} \frac{uu_t}{p}\,dx$, $E'' = 2\int_{\Omega}(\frac{u_t^2}{p} + \frac{uu_{tt}}{p})dx$, from where we obtain that

$$|E''| \leq 2(\|u_t\|_{X_p}\|u_t\|_{L^1} + \|u\|_{X_p}\|u_t\|_{L^1}) \leq 12\|u_0\|_{X_p}\|u_0\|_{L^1},$$

and $E''(t)$ is bounded. The derivative of $E$ is given by (1.6), thus $E$ is decreasing and since $E''$ is bounded, we obtain that $E' \to 0$ as $t \to \infty$.

Now we claim that, for any sequence $t_n \to \infty$, $u(\cdot, t_n)$ converges uniformly in compact sets. Indeed, we see that if $x, z \in B(0,R)$, then $Tu(x,t) \leq \|J\|_\infty \sup_{y\in B(0,\rho(R))} \frac{1}{g(y)}\|u_0\|_{L^1(\mathbb{R})}$ and

(2.11)

$$\begin{aligned}|Tu(x,t) - Tu(z,t)| &\leq \int_{\mathbb{R}} \left| J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right)\right| \frac{|u(y,t)|}{g(y)}dy \\ &\leq C_J|x-z|^\alpha \int_{B(0,\rho(R))} \frac{|u(y,t)|}{g(y)^{1+\alpha}}dy \leq C_J \sup_{y\in B(0,\rho(R))} \frac{1}{g(y)^{1+\alpha}}|x-z|^\alpha \|u\|_{L^1},\end{aligned}$$

where $0 < \alpha < 1$ and $C_J$ are the Hölder exponent and constant of $J$, respectively. Therefore, the family $\{Tu(\cdot,t)\}_{t>0}$ is locally bounded and equicontinuous. On the other hand, multiplying (2.7) by $e^s$ and integrating in $[0,t]$ we obtain that

$$u(x,t) = e^{-t}u_0(x) + \int_0^t e^{s-t}Tu(x,s)dx.$$

Thus, except possibly for a subsequence, the sequence $\{u(\cdot,t_n)\}_{n=1}^\infty$ converges uniformly on compact sets of $\mathbb{R}$ as $n \to \infty$ to a function $u$. Using (1.6) we must have $u = \lambda p$, where $\lambda$ is a constant. In addition, Fatou's lemma implies $\lambda p \in L^1(\mathbb{R})$, so when $p \notin L^1(\mathbb{R})$ we have $\lambda = 0$, and we conclude that $u(\cdot,t) \to 0$ locally uniformly. If $p \in L^1(\mathbb{R})$ then by dominated convergence we have $\int_{\mathbb{R}} u(\cdot,t_n)dx \to \lambda\int_{\mathbb{R}} p dx$ and so $\lambda = \int_{\mathbb{R}} u_0 dx / \int_{\mathbb{R}} p dx$ and the convergence is in $L^1(\mathbb{R})$.

Finally, to conclude the general result we proceed by a density argument using that $C_0^\infty(\mathbb{R}) \subset X_p$ hence $X_p$ is dense in $L^1(\mathbb{R})$ and inequality (2.8). □

**Remark 1.** It is interesting to note that if $p \in L^1(\mathbb{R})$ is continuous and positive, and satisfies (2.9), then it is a stationary solution of (2.7), that is, it satisfies (1.5). Indeed, integrating (2.9) in $\mathbb{R}$ and using Fubini's theorem, we obtain

$$\int_{\mathbb{R}} p(x)dx \geq \int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}dydx = \int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}dx\, p(y)dy = \int_{\mathbb{R}} p(y)dy.$$

Whence we deduce that $p$ has to satisfy (1.5).

## 3. Existence of positive steady states in $L^1(\mathbb{R})$

In this section we will prove Theorem 1.1, which establishes the existence of an integrable positive solution of (1.5). Throughout this section we will assume that

(g2) there exist $M > 0$ and $0 < a < 1$ such that $g(x) \leq a|x|$ for all $|x| \geq M$.

We observe that under this assumption we have that for all $x$

$$D_x \cap \{y \,/\, |y| \geq M\} \subset \left\{y\,:\, \frac{1}{1+a}|x| \leq |y| \leq \frac{1}{1-a}|x|\right\}. \tag{3.12}$$

It follows in particular that, for all $R > 0$ there exists $\rho(R)$ such that $D_x \subset B(0, \rho(R))$ for all $|x| \le R$.

The proof of Theorem 1.1 will rely on the use of the following function, defined in [2]:

$$W_p(x) = \int_0^\infty \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw.$$

It is proved in [2] that, provided that $W_p$ is well defined, $p$ is a positive solution of (1.5) if and only if $W_p$ is a constant. Observe that if $\frac{w}{g(s)} < 1$ with $|s - x| < w$ we need that $s \in D_x$. Therefore, by (g2) $w$ has to belong to a bounded set, which generally depends on $x$. Moreover, it can be easily checked that if $|x| < R$, with $R > 0$ fixed, there exists $M_R > 0$ such that

$$W_p(x) = \int_0^{M_R} \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\, dz\, ds\, dw.$$

To obtain solutions of (1.5) we will proceed as in [2]: we will construct first solutions of a related problem in a bounded interval, and then, passing to the limit, we will construct solutions in $\mathbb{R}$.

For $N > 0$, we consider the following finite interval problem

$$\int_{-N}^{N} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} dy - \int_{-N}^{N} J\left(\frac{x-y}{g(x)}\right) \frac{p(x)}{g(x)} dy = 0 \quad x \in [-N, N], \tag{3.13}$$

which can be written as

$$T_N p(x) = a_N(x) p(x) \quad x \in [-N, N],$$

where

$$T_N v = \int_{-N}^{N} J\left(\frac{x-y}{g(y)}\right) \frac{v(y)}{g(y)} dy, \text{ and } a_N(x) = \int_{-N}^{N} J\left(\frac{x-y}{g(x)}\right) \frac{1}{g(x)} dy.$$

It follows by (J1) and (g1) that $T_N : C([-N, N]) \to C([-N, N])$ is compact and strongly positive, and the function $a_N$ is bounded above and below. Thus, by Krein-Rutman theorem there exists $\lambda > 0$ and a positive function $p \in C([-N, N])$ such that

$$\int_{-N}^{N} J\left(\frac{x-y}{g(y)}\right) \frac{p(y)}{g(y)} dy = \lambda \int_{-N}^{N} J\left(\frac{x-y}{g(x)}\right) \frac{p(x)}{g(x)} dy \quad x \in [-N, N].$$

Integrating in $[-N, N]$ and using Fubini's theorem we see that $\lambda = 1$, and therefore $p$ is a positive solution of (3.13). This solution is unique up to multiplicative constants, so that we can select it such that $p(0) = 1$. Let us denote by $p_N$ the solution so chosen.

We observe that for every $R > 0$, there exists $N_0$ such that for all $N > N_0$ any solution $p_N$ of (3.13) satisfies

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right) \frac{p_N(y)}{g(y)} dy - p_N(x) = 0 \quad x \in [-R, R]. \tag{3.14}$$

**Proposition 2.** *Assume that (g1), (g2) and (J1) are satisfied. Then there exists a positive continuous solution of* (1.5)*.*

**Proof.** Fix $R > 0$ and let $N_0$ be defined as in (3.14). By Theorem 1.3 in [7] there exist $C_R^2 > C_R^1 > 0$ such that for all $|x| < \rho(R)$ we have $C_R^1 p_N(0) < p_N(x) < C_R^2 p_N(0)$. Since $p_N(0) = 1$ it follows that $p_N$ is uniformly bounded above and below in $|x| < \rho(R)$.

If $x, z \in B(0, R)$ and $N > N_0$ then

$$\begin{aligned} |p_N(x) - p_N(z)| &\le \int_{B(0,\rho(R))} \left| J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right) \right| \frac{p_N(y)}{g(y)} dy \\ &\le C_J C_R^2 |x - z|^\alpha \int_{B(0,\rho(R))} \frac{1}{g(y)^{1+\alpha}} dy \le \kappa_R |x - z|^\alpha, \end{aligned}$$

so that $\{p_N\}_{N>0}$ is equicontinuous in $B(0,R)$. Therefore, by means of Arzelá-Ascoli's theorem and with a diagonal procedure, $\{p_N\}$ converges locally uniformly to a positive continuous function $p$. Passing to the limit in (3.14), we see that $p$ satisfies (1.5). □

**Proof of Theorem 1.1.** We will prove that if $p$ is any positive solution of (1.5), then $p \in L^1(\mathbb{R})$. Once we have shown this, uniqueness is be deduced with the help of Proposition 1.

First of all, we normalize $p$ such that $W_p \equiv 1$. By our hypotheses, there exist $c \in (0,1)$ and $M > 0$ such that $g(x) > c|x|$ for $|x| > M$. Then, for $x > M$:

$$1 = W_p(x) \geq \int_{\frac{\delta}{2}x}^{\delta x} \int_x^{x+\frac{\delta}{2}x} p(s) \int_{\frac{\delta x}{g(s)}}^1 J(z)\,dz\,ds\,dw, \tag{3.15}$$

where $\delta > 0$ will be chosen suitably. Since $g(s) > c|s|$ in the interval $[x, x+\frac{\delta}{2}x]$, we have

$$\int_{\frac{\delta x}{g(s)}}^1 J(z)\,dz \geq \int_{\frac{\delta x}{cx}}^1 J(z)\,dz = \int_{\frac{\delta}{c}}^1 J(z)\,dz \geq \frac{1}{4} > 0$$

provided that $\delta$ is chosen small. Then from (3.15) we obtain,

$$\int_x^{x+\frac{\delta}{2}x} p(s)\,ds \leq \frac{8}{\delta x}.$$

If we define $x_0 = x$, $x_{n+1} = (1+\frac{\delta}{2})x_n$, $n \in \mathbb{N}$ we obtain recursively

$$\int_{x_n}^{x_{n+1}} p(s)\,ds \leq \frac{8}{\delta x_n} = \frac{8}{\delta x}\frac{1}{(1+\frac{\delta}{2})^n},$$

for every $n \in \mathbb{N}$. Adding up in $n$ yields

$$\int_x^\infty p(s)\,ds \leq \frac{16}{\delta^2 x}.$$

It can be shown analogously that, for $x < -M$, $\int_{-\infty}^x p(s)\,ds \leq \frac{16}{\delta^2|x|}$; therefore $p \in L^1(\mathbb{R})$, as was to be shown. □

## 4. Nonexistence of positive steady states

In this section we will prove Theorem 1.2; hence from now on we assume

(g2*) There exist $p > 2$, $M, C_1, C_2 > 0$ such that $C_1|x|^p \leq g(x) \leq C_2|x|^p$ for all $|x| > M$.

We will show that under this hypothesis (1.5) does not have any positive solution, and therefore all solutions of (2.7) converge to 0 in $L^1_{\text{loc}}(\mathbb{R})$.

Note that in this situation for all $x \in \mathbb{R}$ the set $D_x = \{y \,/\, |x-y| \leq g(y)\}$ is unbounded. We observe that by taking $M_x$ suitably large, we have that

$$J\left(\frac{x-y}{g(y)}\right) \geq \frac{1}{2}$$

for $|y| > M_x$. Thus, every positive solution $p$ should verify

$$p(x) \geq \frac{1}{2}\int_{|y|\geq M_x} \frac{p(y)}{g(y)}dy.$$

Hence, to give meaning to the concept of solution of (1.5) we should require $\frac{p}{g} \in L^1(\mathbb{R})$.

**Lemma 4.1.** *Suppose that there exists a nonnegative solution $p$ of (1.5). Then $p$ is continuous and belongs to $L^2(\mathbb{R})$.*

**Proof.** Due to our hypotheses, the function $g$ is bounded from below. Then

$$|p(x)-p(z)| \le C_J \int_{\mathbb{R}} \left|\frac{x-z}{g(y)}\right|^{\alpha} \frac{p(y)}{g(y)} dy \le C_1 |x-y|^{\alpha} \int_{\mathbb{R}} \frac{p(y)}{g(y)} dy,$$

where as before $\alpha, C_J$ are the Hölder exponent and constant of $J$, respectively, and $C_1 = C_J / \inf_{\mathbb{R}} g$. Thus $p$ is continuous.

Let us show that $p$ is also bounded. To prove this, notice that by taking $M$ large enough there exists $C > 0$ such that $D_x \subset \{y \,/\, |y| \ge C|x|^{1/p}\}$ for all $|x| > M$; hence

$$p(x) \le \|J\|_\infty \int_{|y| \ge C|x|^{1/p}} \frac{p(y)}{g(y)} dy,$$

and in particular $p(x) \to 0$ as $|x| \to \infty$; thus $p \in L^\infty$. Finally, for $|x| > M$

$$p(x) \le \|J\|_\infty \|p\|_\infty \int_{|y| \ge C|x|^{1/p}} \frac{1}{g(y)} dy \le C_0 |x|^{\frac{1-p}{p}},$$

for a suitable constant $C_0 > 0$. Since $p > 2$, we get that $p \in L^2(\mathbb{R})$. $\square$

**Proof of Theorem 1.2.** We observe that $\frac{1}{g} \in L^1(\mathbb{R})$, and then

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)^2 \frac{1}{g(y)^2} dy\, dx = \int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)^2 \frac{1}{g(y)^2} dx\, dy = \int_{\mathbb{R}} J(z)^2 dz \int_{\mathbb{R}} \frac{1}{g(y)} dy < \infty,$$

thus $T$ is a Hilbert-Schmidt operator. Hence $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $T^* : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are compact.

If there exists a solution $p$ of (1.5), we deduce by Lemma 4.1 that $p \in L^2(\mathbb{R})$, and then by Krein-Rutman theorem $\mu = \mathrm{spr}(T) \ge 1$, is an eigenvalue associated to positive eigenfunctions for $T$ and $T^*$ in $L^2(\mathbb{R})$. Thus, there exists $q \in L^2(\mathbb{R})$ such that $q \ge 0$ a.e. and

$$\int_{\mathbb{R}} J\left(\frac{x-y}{g(x)}\right) \frac{q(y)}{g(x)} dy = \mu q(x), \quad \text{a.e.} \tag{4.16}$$

Let us prove that $q$ is continuous. Observe that the integration in (4.16) is performed in the set $O_x := B(x, g(x))$, the ball of center $x$ and radius $g(x)$. Moreover if we choose and arbitrary compact set $K$ and $x, z \in K$, then $O_x \cup O_z \subset \bar{K}$ for some compact set $\bar{K}$, so that

$$\begin{aligned}
\mu|q(x)g(x) - q(z)g(z)| &\le \int_{O_x \cup O_z} \left| J\left(\frac{x-y}{g(x)}\right) - J\left(\frac{z-y}{g(z)}\right)\right| q(y) dy \\
&\le \left( \int_{\bar{K}} \left| J\left(\frac{x-y}{g(x)}\right) - J\left(\frac{z-y}{g(z)}\right)\right|^2 dy \right)^{1/2} \|q\|_{L^2} \\
&\le C_J \left( \int_{\bar{K}} \left| \frac{x-y}{g(x)} - \frac{z-y}{g(z)} \right|^{2\alpha} dy \right)^{1/2} \|q\|_{L^2} \\
&\le C_J |\bar{K}| \frac{1}{g(x)^\alpha g(z)^\alpha} \left( M|g(z)-g(x)| + |xg(z) - zg(x)| \right)^\alpha \|q\|_{L^2},
\end{aligned}$$

where $C_J$, $\alpha$ are the Hölder constant and exponent of $J$, respectively, and $M = \sup_{\bar{K}} |y|$. Hence $qg$ is continuous, and so is $q$.

On the other hand, $q(x) \to 0$ as $|x| \to +\infty$. Actually, we can even obtain an estimate of the decay of $q$. More precisely, it follows from (4.16) that:

$$\mu^2 q(x)^2 \le \int_{\mathbb{R}} J\left(\frac{x-y}{g(x)}\right)^2 \frac{1}{g(x)^2} dy \int_{\mathbb{R}} q^2(y) dy \le \frac{1}{g(x)} \int_{\mathbb{R}} J^2(z) dz \int_{\mathbb{R}} q^2(x) dx, \tag{4.17}$$

so that $q \le C \frac{1}{|x|^{p/2}}$ for large $|x|$ and some $C > 0$.

Since $q$ is continuous and $q(x) \to 0$ as $|x| \to \infty$, it is clear that the maximum of $q$ is achieved at some point $x_0$, and then by (4.16) evaluated at $x_0$:

$$\mu \max q = \int_{\mathbb{R}} J\left(\frac{x_0 - y}{g(x_0)}\right) \frac{q(y)}{g(x_0)} dy \leq \max q.$$

Since $q$ is nontrivial, we deduce $\mu < 1$, which is a contradiction. Thus no positive solutions $p$ of (1.5) may exist.

In spite of the nonexistence result above, there exists a continuous supersolution $p$ of (1.5). According to Remark 1 , we have $p \notin L^1(\mathbb{R})$, so that by 1 we obtain that any solution of (2.7) converges to 0 in $L^1_{\mathrm{loc}}(\mathbb{R})$.

To conclude the proof, it only remains to construct the supersolution. Take $h \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ with $h > 0$ in $\mathbb{R}$. It is easy to check (cf. the proof of Theorem 1.2) that the operator $T : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is compact and $I - T$ is invertible. Then the unique solution of $(I - T)p = h$ is given by the series $p = \sum_{n=0}^{\infty} T^n h \geq 0$ which converges in $L^1(\mathbb{R})$. Clearly $p \geq Tp$, hence $p$ satisfies (2.9). It can also be proved that $p$ is continuous; indeed, since $p = Tp + h$ and

$$|Tp(x) - Tp(z)| \leq C_J|x - z|^\alpha \int_{\mathbb{R}} \frac{p(y)}{g(y)^{1+\alpha}} dy \leq C_J|x - z|^\alpha \|g^{-(1+\alpha)}\|_{L^2} \|p\|_{L^2}.$$

the continuity of $p$ follows by that of $h$. This concludes the proof. □

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