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Fluctuation bounds for entropy production estimators in Gibbs measures

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Abstract
We give bounds for the fluctuations of estimators of the mean entropy production in Gibbsian sources. These bounds are valid for every n, where n denotes the size-length of the sample. We consider two estimators which are based on waiting and hitting times.

Keywords: entropy production, Gibbs measures, return times, concentration inequalities, non-equilibrium processes

1. Introduction

Entropy and entropy production are very important quantities in the theory of the out-of-equilibrium processes. Since most of the phenomena occurring in nature are irreversible, there exists a great interest in creating a mathematical formulation concerning these phenomena.

There is a huge amount of literature about entropy and entropy production in non-equilibrium statistical mechanics. We refer the interested reader to [14] and the references therein, for a review. Here we will focus on a mathematical framework that has been built in order to formalize the concept of entropy production in deterministic and stochastic dynamics within the set-up of Gibbs measures [10–12]. Specifically in [10], the entropy production was introduced as a function that quantifies the degree of irreversibility of the system, by using one single trajectory of the system. In this formalism the non-equilibrium systems are formalized as a Gibbs measure of space-time with a non-symmetric part with respect to time inversion, which is associated with the entropy production. Thus the mean entropy production is defined as the relative entropy density between the forward and backward processes. That quantity is equal to zero if and only if the process is reversible. In [12] the entropy production is introduced for different classes of systems from stochastic and deterministic dynamics, using the Gibbs formalism as an unifying framework.

From the practical point of view, if one is provided with a sample of some processes, one would like to know from a single typical trajectory if the process under study is irreversible or not. In case it is, then to give an estimate of the 'degree of irreversibility' or the rate of entropy production, is an important issue. Let us remark that a typical and important case of study are DNA sequences considered as generated by a stochastic process. We refer the reader to [13], where DNA sequences are suggested to be out-of-equilibrium structures.

In [6] Chazottes and Redig gave tests for irreversibility for a Gibbsian source. They introduced two estimators of the entropy production. Those estimators are based on the hitting and waiting times of finite sequences and use one single or two typical trajectories of the systems respectively. They showed the convergence properties of those estimators and proved its asymptotic normality and gave large deviations estimates under the assumption of a Gibbsian measure associated to potentials with summable variations.

In this work we study the fluctuation properties of those two estimators of the entropy production defined in [6]. Our main contribution is that we give bounds which are valid for every n, where n stands for the sample's length. This type of results possess practical importance since they give *a priori* bounds depending on the size of the given sample. And thus they are complementary to asymptotic results such as the central limit theorem or large deviations. We restrict ourselves to the case where the Gibbs measure is associated to a Hölder continuous potential. Following the strategy in [6], we use an exponential law approximation of the hitting time proved by Abadi [1], and a mixture of exponential laws for the return time proved by Abadi and Vergne [2]. We also make use of the concentration phenomenon in Gibbs measures to obtain our bounds. Is in this step where the Hölder assumption becomes important.

The outline of this paper is as follows: section 2 concerns the set-up. We recall generalities about Gibbs measures and entropy production. We briefly introduce concentration inequalities in Gibbs measures. We prove a concentration bound for the entropy production at a finite step to its limit. Section 3 is about entropy production estimation. There we introduce the estimators of interest. We recall their convergence properties. Finally in section 4 we state and prove our bounds.

2. Setting and generalities

2.1. Gibbs measures

Complete details for this section can be found in [4]. We consider the set $\Omega = A^{\mathbb{Z}}$ of all bi-infinite sequences made by symbols in a finite set (alphabet) A. Let $\sigma : \Omega \to \Omega$ be the shift-map, defined as follows, for every $\omega \in \Omega$ and for every $n \in \mathbb{Z}$, $(\sigma^n \omega)_k = \omega_{k+n}$. The block x_1^n denotes the sequence $x_1 \cdots x_n$, where $x_i \in A$ for $i = 1, \ldots, n$. We denote by $[x_1^n]$ the cylinder set defined as follows, $[x_1^n] := \{\omega \in \Omega : \omega_i = x_i \;\text{ for } i = 1, \ldots, n\}$.

Let $\phi : \Omega \to \mathbb{R}$ be a Hölder continuous function. Consider the probability measure μ_ϕ on Ω to be the Gibbs measure associated to ϕ. That is, μ_ϕ is the unique σ-invariant probability measure for which there exists some constants $C := C(\phi) > 1$ and $P := P(\phi) > 0$ such that for any $x_1^n \in A^n$ one has the estimate

$$C^{-1} \leqslant \frac{\mu_\phi\left(\left[x_1^n\right]\right)}{\exp\left(-nP + \sum_{j=0}^{n-1} \phi\left(\sigma^j \omega\right)\right)} \leqslant C,$$

for all $\omega \in [x_1^n]$ and $n \geqslant 1$. The constant P is the topological pressure of ϕ. The function ϕ is called 'potential' in the jargon of the thermodynamic formalism, but it is physically interpreted as a local energy function. For the sake of convenience we will consider from now on functions ϕ depending only on future coordinates. It is indeed known [9] that there exists a function $\varphi(\omega) := \varphi(\omega_1\omega_2, \dots)$ which is 'physically equivalent' to ϕ. This means that φ is also Hölder continuous and gives the same Gibbs measure than ϕ. This trick is used in [6], because in the one-sided shift-space one is able to use the transfer operator techniques, which in our case are necessary while using the concentration inequalities. Moreover, we assume that $P(\varphi) = 0$. This assumption is harmless since one can always replace φ by the physically equivalent potential $\varphi - P(\varphi)$. Therefore we have that there exists a constant $K > 0$ such that for every $x_1^n \in A^n$, and all $\omega \in [x_1^n]$, one has that the uniform estimate

$$K^{-1} \leqslant \frac{\mu_\varphi\left(\left[x_1^n\right]\right)}{\exp\left(\sum_{i=0}^{n-1}\varphi\left(\sigma^i\omega\right)\right)} \leqslant K, \tag{1}$$

holds. The Gibbs measure μ_φ is also an equilibrium state for φ. That is, it satisfies the so-called variational principle

$$\sup\left\{h(\nu) + \int\varphi \mathrm{d}\nu: \text{ for } \nu\ \sigma\text{-invariant}\right\} = h\left(\mu_\varphi\right) + \int\phi \mathrm{d}\mu_\varphi = P(\varphi) = 0.$$

Clearly one has that the formula $h\left(\mu_\varphi\right) = -\int\varphi \mathrm{d}\mu_\varphi$ holds.

Next, for any block x_1^n, its corresponding time-reversed block is denoted by $x_n^1 := x_n x_{n-1} \cdots x_1$. One can formalize the time-reversed process by introducing an involution θ, i.e. $\theta^2 = \mathrm{Id}$; defining the time-reversal operation on $A^{\mathbb{Z}}$. That allows us to define $\phi^r := \phi \circ \theta$ and then to consider the potential φ^r which is the physically equivalent potential to ϕ^r depending on future coordinates, just as we did above with φ. Consequently the measure μ_{φ^r} is the measure associated to φ^r. The analogous Gibbs property (1) holds for the measure μ_{φ^r} with the corresponding modifications.

2.2. Entropy production

For any $\omega \in \Omega$, the entropy production for the measure μ_φ up to time n is given by

$$\dot{\mathbf{S}}_n(\omega) := \log \frac{\mu_\varphi\left(\left[\omega_1^n\right]\right)}{\mu_\varphi\left(\left[\omega_n^1\right]\right)} = \log \frac{\mu_\varphi\left(\left[\omega_1^n\right]\right)}{\mu_{\varphi^r}\left(\left[\omega_1^n\right]\right)}.$$

This is a random variable that quantifies the 'irreversibility' of the process up to time n using one single orbit.

Let us remind the reader of some definitions concerning the relative entropy. The n-block relative entropy of a probability distribution ν with respect to μ is given by

$$H_n(\nu\,|\mu) := \sum_{x_1^n \in A^n} \nu\left(\left[x_1^n\right]\right) \log \frac{\nu\left(\left[x_1^n\right]\right)}{\mu\left(\left[x_1^n\right]\right)},$$

so the relative entropy density between ν and μ is given by the limit

$$h(\nu\,|\mu) = \lim_{n\to\infty} \frac{H_n(\nu\,|\mu)}{n}.$$

It is known [7] that the relative entropy density of an invariant measure ν with respect to a Gibbs measure μ_ϕ satisfies the following formula:

$$h\left(\nu|\mu_\phi\right) = P(\phi) - \int\phi \mathrm{d}\nu + h(\nu),$$

where $h(\nu)$ is the entropy density of ν. Moreover $h(\nu|\mu_\phi) \geqslant 0$, with equality if and only if ν is an equilibrium state for ϕ. Consider our measure μ_φ, then from the definition of the entropy production at time n, the Gibbs property and the ergodic theorem one has μ_φ-a.s. that

$$\lim_{n\to\infty} \frac{\dot{\mathbf{S}}_n(\omega)}{n} = h(\mu_\varphi|\mu_{\varphi^r}) =: \mathscr{H}(\mu_\varphi). \tag{2}$$

This quantity is called the *mean entropy production* and is zero if and only if the process is reversible. Observe that from the definition of the entropy production $\dot{\mathbf{S}}_n$ and the Gibbs property, there exists a constant $\widetilde{K} > 0$ such that

$$-\widetilde{K} \leqslant \dot{\mathbf{S}}_n - \sum_{i=0}^{n-1} (\varphi - \varphi^r) \circ \sigma^i \leqslant \widetilde{K}$$

for all $n \geqslant 0$. From Birkhoff's ergodic theorem and (2) one obtains

$$\mathscr{H}(\mu_\varphi) = \int (\varphi - \varphi^r) \mathrm{d}\mu_\varphi. \tag{3}$$

2.3. Concentration bounds

We use concentration inequalities as a tool to get fluctuation bounds for our estimators. Concentration inequalities are by now a well-known and powerful tool in applied probability. They have been extensively used in many other others fields (see [3] and references there in).

In the setting of Gibbs measures an exponential concentration inequality holds for a class of general Lipschitz observables of n variables [8, theorem 1]. That is, there exists a constant $D > 0$ such that, for any integer $n \geqslant 1$ and any separately Lipschitz function K of n variables one has

$$\int \mathrm{e}^{K(x, \ldots, \sigma^{n-1}x) - \int K(y, \ldots, \sigma^{n-1}y) \mathrm{d}\mu_\varphi(y)} \mathrm{d}\mu_\varphi(x) \leqslant \mathrm{e}^{D \sum_{i=0}^{n-1} \mathrm{Lip}_i^2(\mathrm{K})}.$$

Separately Lipschitz means that the function K is Lipschitz on each coordinate, and Lip_i is the corresponding Lipschitz constant at the ith coordinate. This inequality is based on martingale methods and the spectral gap of the transfer operator associated to the shift map, acting on the space of Lipschitz functions. For the proof and further details, see [8].

For our purposes, a useful consequence of the exponential concentration inequality is the following.

Proposition 2.1. *([5]) Let μ_φ be a Gibbs measure on the full-shift (Ω, σ). Then there exists a constant $B > 0$ such that for any Lipschitz function $f : \Omega \to \mathbb{R}$, every $t > 0$ and for every $n \geqslant 1$,*

$$\mu_\varphi \left\{ \omega \in \Omega : \left| \frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i \omega) - \int f \mathrm{d}\mu_\varphi \right| \geqslant t \right\} \leqslant 2\mathrm{e}^{-Bnt^2}.$$

This inequality says that the ergodic sum of f concentrates sharply around its average with respect to μ_φ. This is a consequence of the exponential concentration inequality for Gibbs measures above and the Markov inequality, which gives us a Gaussian inequality of the deviation of K about its expected value. Applying that result to the K equal to the Birkhoff sum of f, one is able to obtain the desired inequality. Further details also can be found in [5, section 3].

Next, our first result is the following proposition.

Proposition 2.2. *Let μ_φ be a Gibbs measure on the full-shift (Ω, σ). There exist constants $B > 0$ and $t' > 0$ such that for all $t \geqslant t'$ and for every $n \geqslant 1$,*

$$\mu_\varphi\left\{\omega \in \Omega\colon \left|\frac{1}{n}\dot{\mathbf{S}}_n(\omega) - \mathscr{H}(\mu_\varphi)\right| > t\right\} \leqslant 4\mathrm{e}^{-Bnt^2}.$$

This inequality gives a control for the speed of convergence of $\frac{1}{n}\dot{\mathbf{S}}_n$ to the mean entropy production of μ_φ.

Proof. From the definition of $\dot{\mathbf{S}}_n$ and the formula (3), for all $\omega \in \Omega$ one can write

$$\left|\frac{1}{n}\dot{\mathbf{S}}_n(\omega) - \mathscr{H}(\mu_\varphi)\right| = \left|\frac{1}{n}\log\mu_\varphi\left(\left[\omega_1^n\right]\right) - \frac{1}{n}\log\mu_\varphi\left(\left[\omega_n^1\right]\right) - \int(\varphi - \varphi^r)\mathrm{d}\mu_\varphi\right|.$$

Let $\epsilon > 0$. Using a union bound one may write

$$\mu_\varphi\left\{\left|\frac{1}{n}\dot{\mathbf{S}}_n(\omega) - \mathscr{H}(\mu_\varphi)\right| > \epsilon\right\} \leqslant \mu_\varphi\left\{\left|\frac{1}{n}\log\mu_\varphi\left(\left[\omega_1^n\right]\right) - \int\varphi\mathrm{d}\mu_\varphi\right| > \frac{\epsilon}{2}\right\}$$
$$+ \mu_\varphi\left\{\left|\frac{1}{n}\log\mu_\varphi\left(\left[\omega_n^1\right]\right) - \int\varphi^r\mathrm{d}\mu_\varphi\right| > \frac{\epsilon}{2}\right\}.$$

For the moment let us focus on the first part of the right-hand side. Taking advantage of the Gibbs property (1) and proposition 2.1 we obtain that

$$\mu_\varphi\left\{\left|\frac{1}{n}\log\mu_\varphi\left(\left[\omega_1^n\right]\right) - \int\varphi\mathrm{d}\mu_\varphi\right| > \frac{\epsilon}{2}\right\}$$
$$\leqslant \mu_\varphi\left\{\left|\frac{1}{n}\sum_{j=0}^{n-1}\varphi(\sigma^j\omega) - \int\phi\mathrm{d}\mu_\varphi\right| > \frac{\epsilon}{2} - \frac{1}{n}\log K\right\} \leqslant 2\mathrm{e}^{-Bn\epsilon^2},$$

for every $\epsilon > \frac{2}{n}\log K$. Since μ_{φ^r} is the Gibbs measure associated to φ^r, and the ergodic average of φ^r converges μ_φ-a.e. to $\int\varphi^r\mathrm{d}\mu_\varphi$, then we proceed analogously for μ_{φ^r}. Putting all together yields

$$\mu_\varphi\left\{\left|\frac{1}{n}\dot{\mathbf{S}}_n(\omega) - \mathscr{H}(\mu_\varphi)\right| > t\right\} \leqslant 4\mathrm{e}^{-Bnt^2},$$

for all $t \geqslant t' := 2\max\{2\log K, 2\log K_r\}$, where K, K_r are the constants appearing in the Gibbs estimate for the measures μ_φ and μ_{φ^r} respectively. □

3. Estimation of the entropy production

We introduce two estimators of the entropy production. For complete details see [6]. This estimators are based on the hitting and return times. Let $\omega \in \Omega$. The hitting time of a given cylinder set $[x_1^n]$ in ω is defined as follows,

$$\tau_{x_1^n}(\omega) := \inf\left\{k \geqslant 1 : \sigma^k\omega \in \left[x_1^n\right]\right\}.$$

That is the first time one 'observes' the block x_1^n in ω. For clarity and convenience we will adopt the following notations,

$$\tau_n := \tau_{\omega_1^n}(\omega) \quad \text{and} \quad \tau_n^r := \tau_{\omega_n^1}(\omega).$$

Observe that τ_n is nothing but the first return time but τ_n^r is a hitting time. Let us introduce the single trajectory estimator $\dot{\mathcal{S}}_n^\tau$ of the entropy production (referred to as the hitting-time estimator)

$$\dot{\mathcal{S}}_n^\tau(\omega) := \log \frac{\tau_n^r(\omega)}{\tau_n(\omega)}.$$

It turns out that typically $\tau_n^r \gg \tau_n$ if the process is not reversible [6]. Thus, the hitting-time estimator of the entropy production is typically positive.

Next, let ω and ω' be two independent and typical trajectories given by the measure μ_φ. The first time it appears the first n symbols of ω into ω' is defined as the waiting time which is denoted by $w(\omega, \omega')$. As before we introduce some additional notations:

$$w_n(\omega, \omega') := \tau_{\omega_1^n}(\omega') \quad \text{and} \quad w_n^r(\omega, \omega') := \tau_{\omega_n^1}(\omega').$$

The waiting-time estimator $\dot{\mathcal{S}}_n^{\mathrm{w}}$ of the entropy production is defined by

$$\dot{\mathcal{S}}_n^{\mathrm{w}}(\omega, \omega') := \log \frac{w_n^r(\omega, \omega')}{w_n(\omega, \omega')}.$$

We briefly recall some of facts about the convergence properties of the estimators. Assuming φ to be of summable variations (a more general condition than ours), then there exist a positive constant C such that

$$-C \log n \leqslant \dot{\mathcal{S}}_n^\tau - \dot{\mathbf{S}}_n \leqslant C \log n \quad \text{and} \quad -C \log n \leqslant \dot{\mathcal{S}}_n^{\mathrm{w}} - \dot{\mathbf{S}}_n \leqslant C \log n,$$

eventually μ_φ-a.s. for the first inequality and eventually $\mu_\varphi \times \mu_\varphi$-a.s. for the second one [6, theorem 1]. Moreover, these estimators are almost surely consistent [6, corollary 1], i.e.

$$\lim_{n\to\infty} \frac{1}{n} \dot{\mathcal{S}}_n^\tau = \mathscr{H}(\mu_\varphi) \quad \mu_\varphi\text{-a.s. and} \quad \lim_{n\to\infty} \frac{1}{n} \dot{\mathcal{S}}_n^{\mathrm{w}} = \mathscr{H}(\mu_\varphi) \quad \mu_\varphi \times \mu_\varphi\text{-a.s.}$$

If one assumes φ to be Hölder continuous, the previous results allow us to obtain central limit asymptotics for these estimators [6, theorem 2]. And under the less restrictive condition of summable variations a large deviation description is obtained [6, theorem 3]. We refer the reader for precise statements and proofs to [6].

4. Fluctuation bounds for the mean-entropy-production estimation

In this section we state and prove the main results of this paper. They are bounds for the fluctuations of the estimators of the entropy production. As we already mentioned, they are valid for every sample length n. And so, they bring practical information for the speed of convergence of the estimators under study.

Our first theorem gives bounds for the waiting-time estimator.

Theorem 4.1. *Let μ_φ be the Gibbs measure associated with the normalized Hölder continuous potential φ. Let μ_{φ^r} be its corresponding time-reversed measure. Then for the empirical estimator $\dot{\mathcal{S}}_n^{\mathrm{w}}$ of the mean entropy production, there exist constants $C, B > 0$ and $t^* > 0$ such that for all $t \geqslant t^*$ and all $n \geqslant 1$,*

$$\mu_\varphi \times \mu_\varphi \left\{ \left| \frac{1}{n} \dot{\mathcal{S}}_n^{\mathrm{w}}(\omega, \omega') - \mathscr{H}(\mu_\varphi) \right| > t \right\} \leqslant C\mathrm{e}^{-nt/4} + 4\mathrm{e}^{-Bnt^2}.$$

Remark 4.1. Hölder continuity is a sufficient condition, and it is assumed for the sake of simplicity. Although it could be enhanced provided that μ_φ satisfies at the same time the exponential concentration inequality and the exponential law for the hitting time, which are the main ingredients for the proof. This is expected to be true for measures associated to potentials satisfying $\sum_{k\geqslant 1} k \cdot \mathrm{var}_k(\varphi) < \infty$. Here $\mathrm{var}(\varphi)$ denotes the modulus of continuity of φ.

A weaker result is given for the hitting-time estimator, this is because it is based on the return time statistic which may present problems for short return times.

Theorem 4.2. *Consider the same assumptions of the previous theorem. Then for the empirical estimator $\dot{S}_n^\tau$ of the mean entropy production, there exist positive constants C, B, c_1, c_2, c_3 and $t^* > 0$ such that for all $t \geqslant t^*$ and all $n > 1$,*

$$\mu_\varphi \left\{ \mathscr{H}(\mu_\varphi) - \frac{1}{n}\dot{S}_n^\tau(\omega) > t \right\} \leqslant C\left(\mathrm{e}^{-Bnt^2} + \mathrm{e}^{-c_1 nt} + \mathrm{e}^{-c_2 n} + \mathrm{e}^{-c_3 \mathrm{e}^{nt/3}} \right).$$

Proof of theorem 4.1. The proof for the waiting-time estimator uses the exponential law for the hitting time. Let us first consider the positive side of the bound on the waiting-time estimator. Clearly

$$\mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n}\dot{S}_n^{\mathrm{w}}(\omega, \omega') - \mathscr{H} > t \right\} = \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log \frac{w_n^r(\omega, \omega')}{w_n(\omega, \omega')} - \mathscr{H} > t \right\}$$

$$= \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log\left(w_n^r \mu_\varphi\left(\left[\omega_n^1 \right] \right) \right) - \frac{1}{n} \log \mu_\varphi\left(\left[\omega_n^1 \right] \right) \right.$$
$$\left. - \frac{1}{n} \log\left(w_n \mu_\varphi\left(\left[\omega_1^n \right] \right) \right) + \frac{1}{n} \log \mu_\varphi\left(\left[\omega_1^n \right] \right) - \mathscr{H} > t \right\}$$

$$\leqslant \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log\left(w_n^r \mu_\varphi\left(\left[\omega_n^1 \right] \right) \right) - \frac{1}{n} \log\left(w_n \mu_\varphi\left(\left[\omega_1^n \right] \right) \right) > \frac{t}{2} \right\}$$
$$+ \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log \mu_\varphi\left(\left[\omega_1^n \right] \right) - \frac{1}{n} \log \mu_\varphi\left(\left[\omega_n^1 \right] \right) - \mathscr{H} > \frac{t}{2} \right\}.$$

For convenience we introduce the following quantities,

$$A_1 := \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log\left(w_n^r \mu_\varphi\left(\left[\omega_n^1 \right] \right) \right) - \frac{1}{n} \log\left(w_n \mu_\varphi\left(\left[\omega_1^n \right] \right) \right) > \frac{t}{2} \right\},$$

$$A_2 := \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log \mu_\varphi\left(\left[\omega_1^n \right] \right) - \frac{1}{n} \log \mu_\varphi\left(\left[\omega_n^1 \right] - \mathscr{H}(\mu_\varphi) > \frac{t}{2} \right\}.$$

We will provide a bound for each quantity. For A_2, observe that proposition 2.2 gives us automatically the following upper bound $A_2 \leqslant 2\mathrm{e}^{-Bnt^2}$, for some constant $B > 0$ and for any $t > t' := \max\{8 \log K, 8 \log K_r\}$.

It remains to find an upper bound for A_1. We clearly have that

$$A_1 \leqslant \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log\left(w_n^r(\omega, \omega') \mu_\varphi\left(\left[\omega_n^1\right]\right)\right) > \frac{t}{4} \right\}$$
$$+ \mu_\varphi \times \mu_\varphi \left\{ -\frac{1}{n} \log\left(w_n(\omega, \omega') \mu_\varphi\left(\left[\omega_1^n\right]\right)\right) > \frac{t}{4} \right\} =: D_1 + D_2. \quad (4)$$

Next, we use a simplification of two very general results due to Abadi [1, theorem 1 and lemma 9], which are valid for ψ-mixing processes. Since Gibbs measures with Hölder pontentials are exponentially ψ-mixing, in our setting the conditions of that theorem are immediately satisfied. We state them as lemmata.

Lemma 4.1. *([1]) There exist strictly positive constants* $C_1, c_1, \lambda_1, \lambda_2$*, with* $\lambda_1 < \lambda_2$*, such that for every* $n \in \mathbb{N}$*, every string* $a_1^n \in A^n$*, there exists* $\lambda(a_1^n) \in [\lambda_1, \lambda_2]$ *such that*

$$\left| \mu_\varphi \left\{ \omega\colon \tau_{a_1^n}(\omega) > \frac{u}{\lambda\left(a_1^n\right)\mu_\varphi\left(\left[a_1^n\right]\right)} \right\} - \mathrm{e}^{-u} \right| \leqslant C_1 \mathrm{e}^{-c_1 u}$$

for every $u > 0$.

We also need the following lemma (lemma 9 in [1]).

Lemma 4.2. *([1]) For any* $v > 0$ *and for any* $a_1^n \in A^n$ *such that* $v\mu([a_1^n]) \leqslant 1/2$*, one has*

$$\lambda_1 \leqslant -\frac{\log \mu_\varphi \left\{ \tau_{\left[a_1^n\right]} > v \right\}}{v\mu_\varphi\left(\left[a_1^n\right]\right)} \leqslant \lambda_2,$$

where λ_1, λ_2 *are the constants appearing in lemma 4.1.*

For the proofs we refer the reader to [1]. Now we are able to find an upper bound for A_1 in inequality (4) above. Write

$$D_1 := \mu_\varphi \times \mu_\varphi \left\{ \frac{1}{n} \log\left(w_n^r(\omega, \omega') \mu_\varphi\left(\left[\omega_n^1\right]\right)\right) > \frac{t}{4} \right\}$$
$$= \sum_{\omega_n^1} \mu_\varphi\left(\left[\omega_n^1\right]\right) \cdot \mu_\varphi \left\{ \tau_{\omega_n^1}(\omega') \mu_\varphi\left(\left[\omega_n^1\right]\right) > \mathrm{e}^{\frac{nt}{4}} \right\}.$$

At this point we simply apply lemma 4.1, with $u = \mathrm{e}^{nt/4}$. Then, summing up over all blocks we obtain

$$D_1 \leqslant C' \mathrm{e}^{-c' \mathrm{e}^{nt/4}},$$

for some adequate constants $C', c' > 0$.

For D_2 we proceed similarly. Write

$$D_2 := \mu_\varphi \times \mu_\varphi \left\{ -\frac{1}{n} \log\left(\mathrm{w}_n(\omega, \omega') \mu_\varphi\left(\left[\omega_1^n\right]\right)\right) > \frac{t}{4} \right\}$$
$$= \sum_{\omega_1^n} \mu_\varphi\left(\left[\omega_1^n\right]\right) \cdot \mu_\varphi \left\{ \tau_{\omega_1^n}(\omega') \mu_\varphi\left(\left[\omega_1^n\right]\right) < \mathrm{e}^{-\frac{nt}{4}} \right\}.$$

Observe that lemma 4.2 implies that

$$\mu_\varphi\left\{\tau_{\omega_1^n}(\omega')\cdot\mu_\varphi\left(\left[\omega_1^n\right]\right)<v\right\}\leqslant 1-\mathrm{e}^{-v\lambda_2}\leqslant\lambda_2 v,$$

provided that $v\mu([\omega_1^n])\leqslant 1/2$. Let us take $v=\mathrm{e}^{-nt/4}$, for which yields

$$\begin{aligned}D_2&=\sum_{\omega_1^n}\mu_\varphi\left(\left[\omega_1^n\right]\right)\mu_\varphi\left\{\tau_{\omega_1^n}(\omega')\mu_\varphi\left(\left[\omega_1^n\right]\right)<\mathrm{e}^{-nt/4}\right\}\\&\leqslant\lambda_2\ \mathrm{e}^{-nt/4}.\end{aligned}$$

This is true if $\mathrm{e}^{-nt/4}\mu_\varphi([\omega_1^n])\leqslant 1/2$, which is actually satisfied for any $n\geqslant 1$ if $t\geqslant t'':=4\log 2$.

Observe that the bound for D_1 is extremely smaller than that for D_2 for every choice of t and n. Thus one can absorb the contribution of D_1 into D_2 by changing the constant. Let us write $A_1\leqslant C''\ \mathrm{e}^{-nt/4}$. Adding up the bound for A_2, we obtain that

$$\mu_\varphi\times\mu_\varphi\left\{\frac{1}{n}\dot{\mathcal{S}}_n^{\mathrm{w}}(\omega,\omega')-\mathscr{H}>t\right\}\leqslant C''\mathrm{e}^{-nt/4}+2\mathrm{e}^{Bnt^2},$$

for all $n\geqslant 1$ and all $t>t^*:=\max\{t',t''\}$.

There still remains the bound for the negative side. To this end, one follows the same idea as before. Let

$$\mu_\varphi\times\mu_\varphi\left\{\frac{1}{n}\dot{\mathcal{S}}_n^{\mathrm{w}}(\omega,\omega')-\mathscr{H}>t\right\}\leqslant A_1'+A_2',$$

where

$$A_1':=\mu_\varphi\times\mu_\varphi\left\{-\frac{1}{n}\log\left(w_n^r\mu_\varphi\left(\left[\omega_n^1\right]\right)\right)+\frac{1}{n}\log\left(w_n\mu_\varphi\left(\left[\omega_1^n\right]\right)\right)>\frac{t}{2}\right\},$$

$$A_2':=\mu_\varphi\times\mu_\varphi\left\{-\frac{1}{n}\log\mu_\varphi\left(\left[\omega_1^n\right]\right)+\frac{1}{n}\log\mu_\varphi\left(\left[\omega_n^1\right]\right)+\mathscr{H}\left(\mu_\varphi\right)>\frac{t}{2}\right\}.$$

As before, for A_2' we use our proposition 2.2. For A_1' we have that

$$\begin{aligned}A_1'&\leqslant\mu_\varphi\times\mu_\varphi\left\{-\frac{1}{n}\log\left(w_n^r(\omega,\omega')\mu_\varphi\left(\left[\omega_n^1\right]\right)\right)>\frac{t}{4}\right\}\\&\quad+\mu_\varphi\times\mu_\varphi\left\{\frac{1}{n}\log\left(w_n(\omega,\omega')\mu_\varphi\left(\left[\omega_1^n\right]\right)\right)>\frac{t}{4}\right\}=:D_1'+D_2'.\end{aligned}$$

We apply lemmata 4.1 and 4.2. Since the calculation is completely analogous one obtains

$$\mu_\varphi\times\mu_\varphi\left\{\left|\dot{\mathcal{S}}_n^{\mathrm{w}}(\omega,\omega')-\mathscr{H}\left(\mu_\varphi\right)\right|>t\right\}\leqslant C\mathrm{e}^{-nt/4}+4\mathrm{e}^{-Bnt^2},$$

for some positive constants C, B and t^* and for all $n\geqslant 1$ and for all $t>t^*$. This is the desired inequality. □

Proof of theorem 4.2. The proof is very similar than the previous one. The only difference concerns the exponential law for the return time since the hitting-time estimator is defined using a return time. Let us write

$$\mu_\varphi\left\{\mathscr{H}(\mu_\varphi)-\frac{1}{n}\dot{S}_n^\tau(\omega)>t\right\}$$
$$=\mu_\varphi\left\{\mathscr{H}(\mu_\varphi)-\frac{1}{n}\log\left(\tau_n^r(\omega)\mu_\varphi\left(\left[\omega_n^1\right]\right)\right)+\frac{1}{n}\log\mu_\varphi\left(\left[\omega_n^1\right]\right)\right.$$
$$\left.+\frac{1}{n}\log\left(\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)\right)-\frac{1}{n}\log\mu_\varphi\left(\left[\omega_1^n\right]\right)>t\right\}$$
$$\leqslant\mu_\varphi\left\{\mathscr{H}(\mu_\varphi)+\frac{1}{n}\log\mu_\varphi\left[\omega_n^1\right]-\frac{1}{n}\log\mu_\varphi\left[\omega_1^n\right]>\frac{t}{3}\right\}$$
$$+\mu_\varphi\left\{-\frac{1}{n}\log\left(\tau_n^r(\omega)\mu_\varphi\left(\left[\omega_n^1\right]\right)\right)>\frac{t}{3}\right\}$$
$$+\mu_\varphi\left\{\frac{1}{n}\log\left(\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)\right)>\frac{t}{3}\right\}.$$

The first and second parts above were actually estimated previously. For the third part, i.e. $\mu_\varphi\left\{\frac{1}{n}\log\left(\tau_n(\omega)\mu_\varphi([\omega_1^n])\right)>\frac{t}{3}\right\}$, we will make use of another general result due to Abadi and Vergne [2, theorem 4.1], which is valid as well for ψ-mixing processes.

In order to find an estimation for that remaining part we need first to introduce some additional quantities. For $k\leqslant n$ let

$$\mathcal{P}_k(n):=\left\{\left[\omega_1^n\right]:\min\left\{j\in 1,\ldots,n:\left[\omega_1^n\right]\bigcap\sigma^j\left[\omega_1^n\right]\neq\varnothing\right\}=k\right\}.$$

This is the set of cylinders of n symbols with 'internal periodicity'. We introduce the following lemma ([2, theorem 4.1]).

Lemma 4.3. *[2] There exist strictly positive constants c, c', C' such that for any $n\in\mathbb{N}$, any $k\in\{1,\ldots,n\}$, any cylinder $[a_1^n]\in\mathcal{P}_k(n)$, one has for all $t\geqslant k$ that*

$$\left|\mu_\varphi\left\{\omega:\ \tau_n(\omega)\mu_\varphi\left[a_1^n\right]>t\,\middle|\,\left[a_1^n\right]\right\}-\zeta\left(a_1^n\right)e^{-\zeta\left(a_1^n\right)t}\right|\leqslant C'e^{-cn}\,e^{-c't},$$

where $\zeta(a_1^n)$ is such that $|\zeta(a_1^n)-\lambda(a_1^n)|\leqslant De^{-c_1n}$, for some $D>0$. And $\lambda(a_1^n)$ is the quantity introduced in lemma 4.1.

Again, for the proof and details of this lemma we refer the reader to [2]. Using the previous lemma we are able to write

$$\mu_\varphi\left\{\frac{1}{n}\log\left(\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)\right)>\frac{t}{3}\right\}=\mu_\varphi\left\{\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)>e^{nt/3}\right\}$$
$$\mu_\varphi\left\{\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)>e^{nt/3}\right\}\leqslant C'e^{-cn}\,e^{-c'e^{nt/3}}+\zeta\left(\omega_1^n\right)e^{-\zeta\left(\omega_1^n\right)e^{nt/3}}.$$

Since $|\zeta(\omega_1^n)-\lambda(\omega_1^n)|\leqslant De^{-cn}$, we have that $\frac{\lambda_1}{2}\leqslant\lambda_1-De^{-cn}\leqslant\zeta(\omega_1^n)\leqslant De^{-cn}+\lambda_2$, so that

$$\mu_\varphi\left\{\tau_n(\omega)\mu_\varphi\left(\left[\omega_1^n\right]\right)>e^{nt/3}\right\}\leqslant C'e^{-cn}\,e^{-c'e^{nt/3}}+\left(De^{-cn}+\lambda_2\right)e^{-\frac{\lambda_1}{2}e^{nt/3}}.$$

Putting all this together, we obtain

$$\mu_\varphi\left\{\mathscr{H}(\mu_\varphi)-\frac{1}{n}\dot{S}_n^\tau(\omega)>t\right\}\leqslant 2\mathrm{e}^{-Bnt^2}+C_1\mathrm{e}^{-c_1nt}$$
$$+(C'+D)\mathrm{e}^{-cn}\exp\left(-\tilde{c}\mathrm{e}^{nt/3}\right)+\lambda_2\exp\left(-\frac{\lambda_1}{2}\mathrm{e}^{nt/3}\right),$$

where $\tilde{c}=\min\{c',\lambda_1/2\}$. Let $C_2:=(C'+D)\mathrm{e}^{-\tilde{c}}$ and $c_3:=\lambda_1/2$, then finally one can define a new constant $C:=\max\{2, C_1, C_2, \lambda_2\}$. And that finishes the proof. □

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