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Feedback Arc Set Problem and NP-Hardness of Minimum Recurrent Configuration Problem of Chip-Firing Game on Directed Graphs*

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Abstract. In this paper we present further studies of recurrent configurations of chip-firing games on Eulerian directed graphs (simple digraphs), a class on the way from undirected graphs to general directed graphs. A computational problem that arises naturally from this model is to find the minimum number of chips of a recurrent configuration, which we call the *minimum recurrent configuration* (MINREC) problem. We point out a close relationship between MINREC and the minimum feedback arc set (MINFAS) problem on Eulerian directed graphs, and prove that both problems are NP-hard.

Keywords: chip-firing game, critical configuration, recurrent configuration, Eulerian digraph, feedback arc set, complexity, sandpile model

1. Introduction

A *feedback arc set* of a directed graph (digraph) G is a subset A of arcs of G such that removing A from G leaves an acyclic graph. The *minimum feedback arc set* (MINFAS) problem is a classical combinatorial optimization on graphs in which one tries to minimize $|A|$. This problem has a long history and its decision version was one of Karp's 21 NP-complete problems [19]. The problem is known to be still NP-hard for many smaller classes of digraphs such as tournaments, bipartite tournaments, and

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Eulerian multi-digraphs [7, 11, 15]. We will prove that it is also NP-hard on Eulerian digraphs, a class in-between classes of undirected graphs and digraphs, in which the in-degree and the out-degree of each vertex are equal.

The chip-firing game is a discrete dynamical system that has received a great attention in recent years, with many variants. The model is a kind of diffusion process on graphs that can be defined informally as follows. Each vertex of a graph has a number of chips and it can give one chip to each of its out-neighbors if it has as many chips as its out-degree. A distribution of chips on the vertices of the graph is called a *configuration*. The model has several equivalent definitions [1, 4, 5, 10]. In this paper we refer to the definition on digraphs given by Björner, Lovász, and Shor [5]. The most important property of chip-firing games is that if the game converges, it always converges to a unique stable configuration. This property leads to some research directions. A natural direction is the classification of all lattices generated by the converging games [20, 21]. Most recently, the authors of [26] gave a criterion that provides an algorithm for deciding that class of lattices. In this paper we pay attention to another important direction initiated in a paper of Biggs. The author defined a variant of chip-firing game on undirected graphs, i.e., the *Dollar game* [3], and studied some special configurations called *critical configurations*. A generalization to the case of digraphs was given in [10, 17] where the authors defined *recurrent configurations* and presented many properties that are similar to those of critical configurations on undirected graphs. Holroyd et al. in [17] also studied the chip-firing game on Eulerian digraphs and presented several typical properties that can also be considered as natural generalizations of the undirected case. In this paper we continue this work and present a connection between the MINREC problem and the MINFAS problem on Eulerian digraphs. As a corollary of this connection, the MINREC problem is NP-hard on Eulerian digraphs, therefore on general digraphs.

A typical property of recurrent configurations is that any stable configuration being component-wise greater than a recurrent configuration is also a recurrent configuration. If the set of minimal recurrent configurations is known, one knows the set of all recurrent configurations. Hence, it is worth studying properties of such recurrent configurations. It turns out from the study in [30] that we can associate a minimal recurrent configuration of an undirected graph G with an acyclic orientation of G. The acyclic orientations of G have the same number of arcs, namely $|E(G)|$, so do the total number of chips of minimal recurrent configurations. A direct consequence of this fact is that we can compute the minimum total number of chips of a recurrent configuration in polynomial time since we can compute easily a minimal recurrent configuration, in the undirected case. It is natural to ask whether this problem can be solved in polynomial time for the case of digraphs. We show in this paper that the problem is NP-hard for digraphs, in particular for the class of Eulerian digraphs. Our method for attacking this problem is described succinctly as follows. By giving a polynomial-time reduction from the MINFAS problem on general digraphs to the MINFAS problem on Eulerian digraphs we show that the MINFAS problem on Eulerian digraphs is also NP-hard. The MINFAS problem is obviously equivalent to the problem of finding the number of arcs of a maximum acyclic subgraph. We show that the set of acyclic subgraphs of an Eulerian digraph with exactly one fixed source s always contains a maximum acyclic subgraph. Moreover, the number of arcs of such a maximum acyclic subgraph can be computed directly knowing the number of chips of a minimum recurrent configuration with the sink s (via firing graphs), and vice versa. This implies the NP-hardness of the MINREC problem.

The paper is divided into two main sections. In the first section we recall the definition of recurrent configurations and give a connection between minimal recurrent configurations and maximal acyclic subgraphs of an Eulerian digraph via the Dhar's burning algorithm. This connection leads to the study of the maximum acyclic subgraphs on Eulerian digraphs that is done in the second section. Most importantly, we show that the set of acyclic subgraphs with exactly one fixed source always contains a maximum acyclic subgraph. By using this property we show that the MINREC problem is NP-hard at the end of this section. Note that we will directly manipulate sets of arcs of acyclic subgraphs, to which we conveniently assign the name *acyclic arc sets*.

2. Recurrent Configurations of Chip-Firing Game and Acyclic Arc Sets

A simple digraph is a digraph that has no loop and has no more than one arc from v to v' for any two distinct vertices v, v'. Throughout this paper a graph always means a simple digraph. All results in this paper can be generalized easily to the case of multi-graphs. Traditionally, the vertex set and the arc set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An *Eulerian graph* is a connected graph in which the in-degree and the out-degree of each vertex are equal. An undirected graph is considered as a graph in which for any edge linking u and v, we consider two arcs: one from u to v and another from v to u. With this convention a connected undirected graph is an Eulerian graph.

2.1. Chip-Firing Game on Graphs

Let $G = (V, E)$ be a graph. A vertex s is called a *global sink* if $deg_G^+(s) = 0$ and for any $v \in V$ there is a path from v to s (possibly a path of length 0). Clearly, if G has a global sink then it is unique.

A *configuration* c of G is a map from V to $\mathbb{N}$. The value $c(v)$ can be regarded as the number of chips stored at v. A vertex v of G is *active* if $c(v) \geq deg_G^+(v) \geq 1$. Configuration c is *stable* if c has no active vertex. *Firing* at v results in the map $c' \colon V \to \mathbb{Z}$ defined by

$$c'(w) = \begin{cases} c(w) - deg_G^+(w), & \text{if } w = v, \\ c(w) + 1, & \text{if } v \neq w \text{ and } (v, w) \in E, \\ c(w), & \text{otherwise.} \end{cases}$$

This firing is often denoted by $c \xrightarrow{v} c'$. Clearly, if v is active then c' is also a configuration of G. In this case the firing $c \xrightarrow{v} c'$ is called *legal*. If d is obtained from c by a sequence of legal firings (possibly a sequence of length 0), we write $c \xrightarrow{*} d$.

A game beginning with initial configuration c_0 and playing with legal firings is called *a chip-firing game*. Note that at each step of firing there are possibly more than one active vertex, therefore there are possibly more than one choice of legal firing. As a consequence, it may be a complicated problem if one wants to know the termination of the game. Hopefully, it is not the case for the chip-firing model since the termination has a good characterization.

Lemma 2.1. ([4]) *Let G be a graph and c an initial configuration. Then the game either plays forever or arrives at a unique stable configuration. Moreover, if G has a global sink, the game arrives at a stable configuration. We denote by c° this stable configuration.*

Fix a linear order $v_1 < v_2 < \cdots < v_n$ on V, where $n = |V|$. The *Laplacian matrix* Δ of G with respect to the order is given by

$$\Delta_{i,j} = \begin{cases} -d_G(v_i, v_j), & \text{if } i \neq j, \\ deg_G^+(v_i), & \text{if } i = j, \end{cases}$$

where

$$d_G(v_i, v_j) = \begin{cases} 1, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

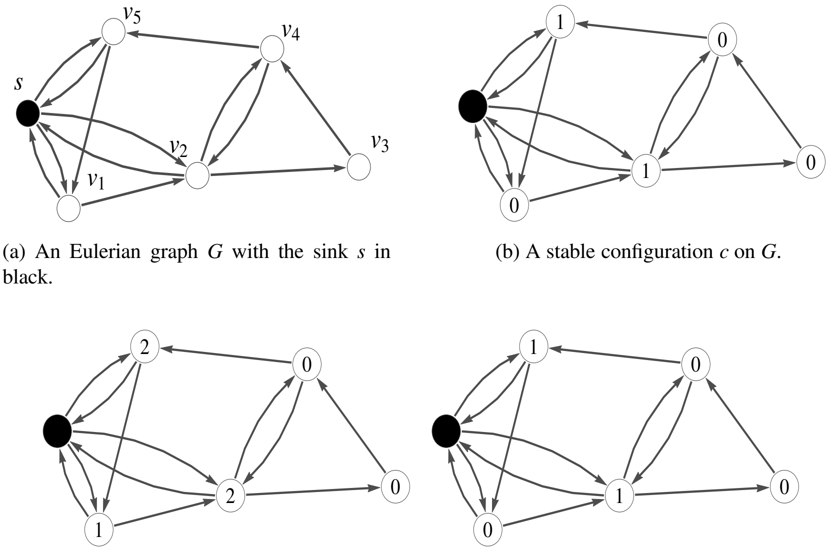
With the order a configuration can be represented by a vector of $\mathbb{Z}^n$, therefore can be regarded as an element of the group $(\mathbb{Z}^n, +)$. It follows from the firing rule that if $c \xrightarrow{*} d$ then there is $z \in \mathbb{Z}^n$ such that $c - d = z.\Delta$.

If the graph G is strongly connected, the game can play forever. However, the game converges to a unique configuration for any initial configuration if we disable a vertex s of G so that it cannot be fired no matter how many chips it has. The reason for the convergence of this variant is that when we disable a vertex s, the game is equivalent to the ordinary one that plays on the graph G in which all out-going arcs of s have been deleted, therefore plays on a graph with a global sink s. This observation leads to the notion of *recurrent configuration* due to Dhar and Biggs [3, 10], and also known under the name Dhar's burning algorithm.

2.2. Chip-Firing Game on Eulerian Graphs with a Sink and Recurrent Configurations

Let $G = (V, E)$ be an Eulerian graph with a distinguised vertex s called the *sink* of the game. A chip disappears when going to the sink. We define a configuration on G to be a map from $V \setminus \{s\}$ to $\mathbb{N}$. Although the sink does not hold chips, we define it to be *active* if all other vertices are not active. A configuration c is called *stable* if s is active in c. When s is active, firing s means the process of adding one chip to each out-neighbor of s. For a configuration c, let c° denote the stable configuration that is reachable from c by repeatedly firing active vertices distinct from s. It follows from the above observation that c° is unique. The configuration c° is called the *stabilization* of c. A configuration c is *recurrent* if after firing s and stabilizing, the game arrives c again. The set of all recurrent configurations is denoted by REC(G).

There is a naturally generalized definition of recurrent configuration on graphs with a global sink. In that definition a configuration is recurrent if it is stable and *accessible* [17]. The graph G can be made into a graph with a global sink s if we remove all out-going arcs of s in G. Then the two definitions are equivalent.



(a) An Eulerian graph G with the sink s in black.

(b) A stable configuration c on G.

(c) The configuration $c+\beta$.

(d) The configuration $(c+\beta)^{\circ}$.

Figure 1: Verifying the recurrence of a configuration.

Let β be the configuration given by $\beta(v) = 1$ if $(s, v) \in E$, and $\beta(v) = 0$ otherwise. Clearly, firing s results in the configuration $c+\beta$ (the sum is done vertex-wise). Thus a configuration c is recurrent if and only if $c = (c+\beta)^{\circ}$. Moreover, each vertex distinct from s is fired exactly once during the process of stabilizing $(c+\beta)$ until s is active [17]. See Figure 1 for an illustration of verifying the recurrence of a configuration. The configuration in Figure 1b is recurrent since it is the same as the configuration in Figure 1d.

We define an equivalence relation $\sim$ on the set of all configurations of G by $c_1 \sim c_2$ if $c_1 - c_2$ are in the Abelian group in $\left(\mathbb{Z}^{|V|-1}, +\right)$ generated by the rows of the matrix Δ', where Δ' is the Laplacian matrix of G in which the row and the colum corresponding to s have been deleted. Note that the above definition of recurrent configuration does not work on a general strongly connected graph since β may not be in the same equivalence class as the configuration $\mathbf{0}$ (a configuration does not have any chip), therefore $c \neq (c+\beta)^{\circ}$ for any configuration c. Speer presents the *script algorithm* as a generalization of the Dhar's burning algorithm to graphs with a global sink [33]. The following shows a basic algebraic property of recurrent configurations.

Lemma 2.2. ([17]) *The set of recurrent configurations* $\mathrm{REC(G)}$ *is an Abelian group with the addition defined by* $c \oplus c' := (c+c')^{\circ}$. *Moreover, each equivalence class according to* $\sim$ *contains exactly one recurrent configuration, and* $|\mathrm{REC(G)}|$ *is equal to the number of the equivalence classes.*

The number of equivalence classes is equivalent to the determinant of Δ'. By the Matrix-tree theorem the number of spanning trees rooted at s is equivalent to $\det(\Delta')$, therefore to $|\mathrm{REC(G)}|$ [35]. By using the burning algorithm Majumdar and Dhar gave a natural bijection between REC(G) and the spanning trees on undirected graphs [22]. Cori and Le Borgne also gave a remarkable bijection such that the difference between the number of chips of a recurrent configuration and the external activity of the associate tree is equal to the number of edges minus the degree of the sink [9]. The bijection gives a bijective proof for a result of López about the connection between the generating function of recurrent configurations and the Tutte polynomial [23]. For graphs Holroyd et al. presented a natural bijection by using the rotor router operation [17].

A configuration c is called a *parking function* if for any non-empty subset U of $V\backslash\{s\}$ there is $v \in U$ such that $c(v)$ is less than the number of arcs from v to the vertices outside of U. It is known that the number of parking functions with respect to s is equal to the number of spanning trees of G rooted at s [27]. For a bijective proof of this fact a family of bijections between parking functions and rooted spanning trees is presented in [8].

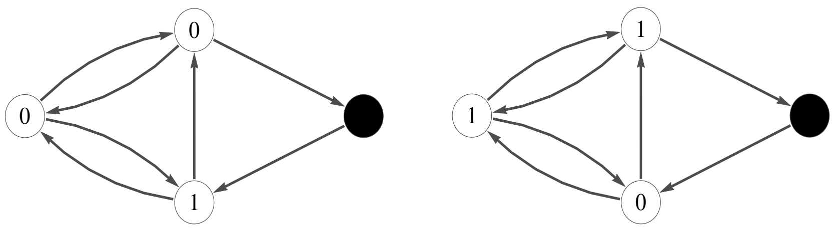
A configuration c is called *superstable* if for any non-empty subset A of $V\backslash\{s\}$ the vector $c - \sum_{v\in A} \Delta_v^{'}$ has a negative entry, where $\Delta_v^{'}$ denotes the row of $\Delta^{'}$ corresponding to the vertex v [17]. Let δ be the configuration given by $\delta(v) = deg_G^+(v) - 1$ for any $v \neq s$. It is well known that on undirected graphs a configuration c is recurrent if and only if $\delta - c$ is a parking function. Moreover, a configuration is superstable if and only if it is a parking function [10, 17]. A generalized definition of superstable configurations on a graph with a global sink can be found in [16, 24].

Eulerian graphs are exactly those connected graphs G in which a configuration does not change if we fire simultaneously all vertices of G (each vertex is fired exactly once). Equivalently, the kernel of homomorphism $x \mapsto x\Delta$ from $\left(\mathbb{Z}^{|V|}, +\right)$ to $\left(\mathbb{Z}^{|V|}, +\right)$ is generated by the vector $\mathbf{1}$ (a vector having 1 on all entries). This is a reason why the Dhar's burning algorithm is still valid for defining recurrent configurations on Eulerian digraphs, and therefore working with the chip-firing game on Eulerian graphs is easier than the one on general graphs (strongly connected graphs with a sink, or graphs with a global sink). There are many properties of the chip-firing game on undirected graphs which still hold on Eulerian graphs such as the sandpile group is independent of the choice of sink, recurrent configurations are dual to superstable configurations [17]. There are also many properties which hold for undirected graphs but do not hold for Eulerian graphs such as a minimal recurrent configuration is no longer minimum (see Figure 5), recurrent configurations are no longer dual (in the sense $\delta - c$) to parking functions in general (see Figure 2). So it is interesting to study properties of the chip-firing game on Eulerian graphs to know how those properties are related to the properties of the game on undirected graphs. In this paper we present some unusual properties which hold not only for undirected graphs but also hold for Eulerian graphs (Propositions 3.7 and 3.8). Those properties are clear from the undirected case. We recommend our recent work on the generating function of recurrent configurations in [25] for more suprising properties of the chip-firing game on Eulerian graphs.

Although the recurrent configurations may not be dual to the parking functions in an Eulerian graph G, they are dual to the parking functions of the dual graph $\overleftarrow{G}$ which is obtained from G by replacing each arc (v, v') of G by the reverse arc

(v', v). Since undirected graphs are exactly those Eulerian graphs G such that $G = \overleftarrow{G}$, this duality can be considered as a generalization of the duality between recurrent configurations and parking functions on undirected graphs. In Section 2.3 we will construct a bijection between the minimal recurrent configurations and the maximal acyclic arc sets with a unique source (sink for the game) of an Eulerian graph. That bijection is a generalization of the one presented in [2], between the maximal G-parking functions and the acyclic orientations with a unique source of an undirected graph. The following observation plays an important role in the construction of the bijection.

When the Dhar's burning algorithm is applied to a recurent configuration, it induces a linear order on V according to a legal firing sequence. This linear order naturally induces an acyclic subgraph of G. This observation gives a connection between recurrent configurations and acyclic subgraphs of G via the notion of *firing graph* [30].



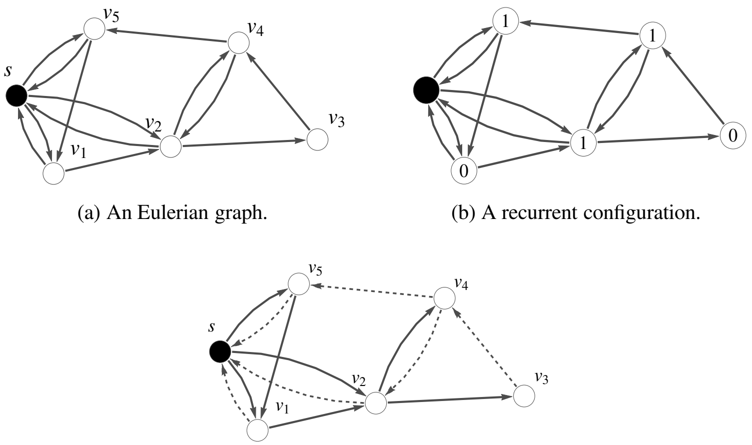
(a) A parking function c with respect to s (in black).

(b) Non-recurrent configuration $\delta - c$.

Figure 2

Definition 2.3. *Let c be a recurrent configuration and $c \xrightarrow{w_0=s} d_0 \xrightarrow{w_1} d_1 \xrightarrow{w_2} d_2 \xrightarrow{w_3} d_3 \rightarrow \cdots \xrightarrow{w_k} d_k$ a legal firing sequence of c such that $d_k = c$ and $w_i \neq s$ for any $1 \leq i \leq k$. This sequence of legal firings can be presented by $(w_0, w_1, w_2, \ldots, w_k)$ since d_i is completely defined by $w_0, w_1, w_2, \ldots, w_i$ for $i \geq 0$. Clearly, we have $k = |V| - 1$ and $\{w_1, w_2, \ldots, w_k\} = V \backslash \{s\}$. The graph $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = V$ and $\mathcal{E} = \{(w_i, w_j) \colon i < j \text{ and } (w_i, w_j) \in E\}$ is called a* firing graph *of c.*

Note that the vertex s is a unique vertex that has in-degree 0 in the firing graph $\mathcal{F}$, and for any vertex v distinct from s there is a path from s to v in $\mathcal{F}$. Figure 3 presents an Eulerian graph with the sink s in black. Figure 3b presents a recurrent configuration. Starting with the configuration c we can fire consecutively the vertices $s, v_5, v_1, v_2, v_4, v_3$ of V in this order to reach again c. With the legal firing sequence $(s, v_5, v_1, v_2, v_4, v_3)$ we have the firing graph that is presented by the undashed arcs in Figure 3c. Note that legal firing sequences of c are possibly not unique, so are firing graphs of c. In the next part we are going to study a kind of recurrent configurations that always have a unique firing graph.



(a) An Eulerian graph.

(b) A recurrent configuration.

(c) Firing graph corresponding to the sequence $(s, v_5, v_1, v_2, v_4, v_3)$.

Figure 3: An example of a firing graph.

2.3. Minimal Recurrent Configurations and Maximal Acyclic Arc Sets

Let $G = (V, E)$ be a graph. For a subset A of E let $G[A]$ denote the graph (V', E') with $V' = V$ and $E' = A$. A *feedback arc set* F of G is a subset of E such that removing the arcs in F from G leaves an acyclic graph. An *acyclic arc set* A of G is a subset of E such that the graph $G[A]$ is acyclic. Clearly, an acyclic arc set is the complement of a feedback arc set. A feedback arc set (acyclic arc set, respectively) is *minimum* (*maximum*, respectively) if it has minimum (maximum, respectively) number of arcs over all feedback arc sets (acyclic arc sets, respectively) of G. A feedback arc set A (acyclic arc set A, respectively) is *minimal* (*maximal*, respectively) if for any $e \in A$ ($e \in E \backslash A$, respectively) we have $A \backslash \{e\}$ ($A \cup \{e\}$, respectively) is not a feedback arc set (acyclic arc set, respectively). A vertex s is called a *source* of an acyclic arc set A if s has in-degree 0 in $G[A]$. Note that an isolated vertex (if existent) in $G[A]$ is a source. If A has a unique source s then $G[A]$ is connected and for any vertex v there is a path in $G[A]$ from s to v.

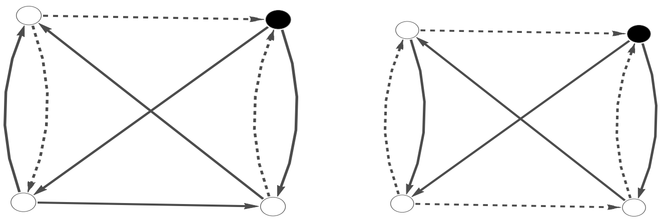
From now until the end of this subsection we work with the chip-firing game on an Eulerian graph $G = (V, E)$ with sink s. For two configurations c' and c we write $c' \leq c$ if $c'(v) \leq c(v)$ for every $v \in V \backslash \{s\}$. A recurrent configuration c is *minimal* if whenever $c' \neq c$ and $c' \leq c$, c' is not recurrent. When c has the minimum total number of chips over all recurrent configurations, we say that c is *minimum*. Let $\mathcal{M}$ denote the set of all minimal recurrent configurations of the game, and $\mathcal{A}$ denote the set of all maximal acyclic arc sets A of G such that s is a unique source of A.

If G is an undirected graph, the definition of maximal acylic arc set is equivalent to the definition of acyclic orientation. In this case $\mathcal{A}$ and $\mathcal{M}$ have the following nice properties:

- each acyclic arc set A in $\mathcal{A}$ (each recurrent configuration in $\mathcal{M}$, respectively) is a maximum acyclic arc set (a minimum recurrent configuration, respectively) and has $\frac{|E|}{2}$ arcs $\big($has $\frac{|E|}{2} - deg_G^+(s)$ chips, respectively$\big)$.

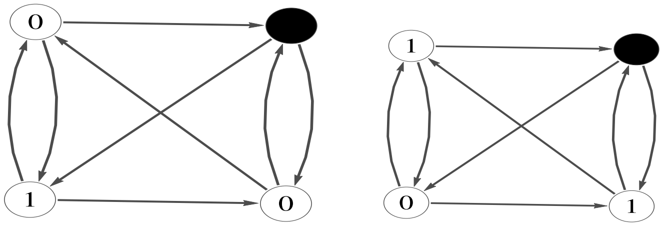
- $|\mathcal{A}| = |\mathcal{M}|$.

Figure 4 presents an Eulerian graph with two acyclic arc sets that are shown in Figures 4a and 4b. The first one in Figure 4a is a maximum acyclic arc set of 5 arcs. The second one is a maximal acyclic arc set of 4 arcs, therefore it is not maximum. Additionally, Figure 5a presents a minimum configuration of one chip. Figure 5b presents a minimal recurrent configuration of 2 chips, therefore it is not minimum. This implies that the first property does not hold for a general Eulerian graph.



(a) A maximum acyclic arc set. (b) A maximal acyclic arc set.

Figure 4



(a) A minimum recurrent configuration. (b) A minimal recurrent configuration.

Figure 5

The second property follows from the fact that the map from $\mathcal{M}$ to $\mathcal{A}$ given by $c \mapsto \mathcal{F}_c$ is well defined and bijective, where $\mathcal{F}_c$ is a firing graph of c. This bijection was given explicitly in [2,30]. However, the relation between the acyclic orientations and the minimum recurrent configurations of an undirected graph had been mentioned in the earlier work of Björner, Lovász, and Shor [5,12]. It is natural to ask whether this property holds for Eulerian graphs and whether the map is well defined and bijective on this class of graphs. Although the first property does not hold for Eulerian graphs, we show that, in this subsection, the second property still holds for Eulerian graphs. In particular, we show that the map $c \mapsto \mathcal{F}_c$ is well defined and bijective. The inverse map $\mathcal{F}_c \mapsto c$ sends a maximum acyclic arc set to a minimum recurrent configuration. This property will play an important role in proving the NP-hardness of the MINREC problem shown in the next section.

Lemma 2.4. *Let A be an acyclic arc set such that s is a unique source of A. Then the configuration c defined by $c(v) = deg^+_G(v) - deg^-_{G[A]}(v)$ for every $v \in V \backslash \{s\}$ is recurrent.*

Proof. Since $G[A]$ is acyclic, there is a linear order $v_0 < v_1 < v_2 \cdots < v_{|V|-1}$ on V such that if $(v_i, v_j) \in A$ then $i < j$. Clearly, $v_0 = s$. The proof is completed by showing that $\left(v_0, v_1, v_2, \ldots, v_{|V(G)|-1}\right)$ is a legal firing sequence of c. Since c is stable, s is active in c, therefore we can fire s in c. Now by induction, suppose that $(v_0, v_1, v_2, \ldots, v_j)$ is a legal firing sequence of c, where $j < |V(G)| - 1$. By firing consecutively the vertices $v_0, v_1, v_2, \ldots, v_j$ in this order we arrive at the configuration c'. It suffices to show that v_{j+1} is active in c'. It is clear that v_{j+1} receives $\sum\limits_{0 \leq i \leq j} d_G(v_i, v_{j+1})$ chips from its in-neighbors after all vertices $v_0, v_1, v_2, \ldots, v_j$ have been fired. Since $\sum_{0 \leq i \leq j} d_G(v_i, v_{j+1}) \geq deg^-_{G[A]}(v_{j+1})$, the number of chips stored at v_{j+1} in c' is not less than $deg^+_G(v_{j+1})$, therefore, v_{j+1} is active in c'. This completes the proof. ∎

From the definition of firing graph, a recurrent configuration may have many firing graphs. However, the following implies that the numbers of arcs of those firing graphs have a lower bound that depends on the recurrent configuration.

Lemma 2.5. *If c is a recurrent configuration then for every firing graph $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ of c, $\mathcal{E}$ is an acyclic arc set of G and s is a unique source of $\mathcal{E}$. Moreover, $\mathcal{F}$ is connected and for each $v \in V \backslash \{s\}$ we have $c(v) \geq deg^+_G(v) - deg^-_{\mathcal{F}}(v)$.*

Proof. It follows immediately from the definition of firing graph that s is a vertex of in-degree 0 in $\mathcal{F}$ and $\mathcal{E}$ is an acyclic arc set. We show that there is no other vertex of in-degree 0 in $\mathcal{F}$. Let $\left(v_0, v_1, v_2, \ldots, v_{|V|-1}\right)$ be a legal firing sequence of c that is used to construct $\mathcal{F}$. For each $i \in \{1, 2, \ldots, |V| - 1\}$ let c' denote the configuration obtained from c by firing consecutively the vertices $v_0, v_1, v_2, \ldots, v_{i-1}$. Since v_i is not active in c but active in c', v_i must receive some chips during this firing process. This implies that there is $j < i$ such that $(v_j, v_i) \in E$. It follows from the definition of firing graph that $(v_j, v_i) \in \mathcal{F}$, therefore $deg^-_{\mathcal{F}}(v_i) \geq 1$. Since $\mathcal{F}$ is acyclic and has exactly one vertex of in-degree 0, $\mathcal{F}$ is connected.

It remains to prove that for every $v \in V \backslash \{s\}$ we have $c(v) \geq deg^+_G(v) - deg^-_{\mathcal{F}}(v)$. For every $i \in \{1, 2, \ldots, |V| - 1\}$, vertex v_i receives $deg^-_{\mathcal{F}}(v_i)$ chips from its in-neighbors when all vertices $v_0, v_1, v_2, \ldots, v_{i-1}$ have been fired. At this point v_i is active, therefore $c(v_i) \geq deg^+_G(v_i) - deg^-_{\mathcal{F}}(v_i)$. ∎

When c is in $\mathcal{M}$, each firing graph of c has the same number of arcs. This is implied by the following lemma.

Lemma 2.6. *Let $c \in \mathcal{M}$ and $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ a firing graph of c. Then $c(v) = deg^+_G(v) - deg^-_{\mathcal{F}}(v)$ for every $v \in V \backslash \{s\}$ and $\mathcal{E} \in \mathcal{A}$. Moreover, the configuration c contains $|E| - deg^+_G(s) - |\mathcal{E}|$ chips.*

Proof. Let c' be the configuration defined by $c'(v) = deg^+_G(v) - deg^-_{\mathcal{F}}(v)$ for every $V \backslash \{s\}$. By Lemma 2.4, c' is a recurrent configuration. It follows from Lemma 2.5 that $c' \leq c$. Since c is minimal, we have $c' = c$, therefore, $c(v) = deg^+_G(v) - deg^-_{\mathcal{F}}(v)$ for every $v \in V \backslash \{s\}$.

To prove $\mathcal{E} \in \mathcal{A}$, we assume to the contrary that there is $A \in \mathcal{A}$ such that $\mathcal{E} \subsetneq A$ (from Lemma 2.5 we know that $\mathcal{E}$ is an acyclic arc set, hence it is not maximal). Let c'' be the configuration defined by $c''(v) = deg^+_G(v) - deg^-_{G[A]}(v)$ for every $v \in V \backslash \{s\}$. Let $(u, u') \in A \backslash \mathcal{E}$. Clearly $deg^-_{G[A]}(u') > deg^-_{\mathcal{F}}(u')$, therefore $c''(u') < c(u')$. It implies that $c'' \neq c$ and $c'' \leq c$, a contradiction to the fact that $c \in \mathcal{M}$.

The number of chips c contains is

$$\begin{aligned}\sum_{v \neq s} c(v) &= \sum_{v \neq s} \left(deg^+_G(v) - deg^-_{\mathcal{F}}(v)\right) \\ &= \sum_{v \in V} deg^+_G(v) - deg^+_G(s) - |\mathcal{E}| \\ &= |E| - deg^+_G(s) - |\mathcal{E}|.\end{aligned}$$

The second statement follows. ∎

For two non-repeated sequences $\mathfrak{f} = (v_0, v_1, v_2, \ldots, v_{|V|-1})$ and $\mathfrak{g} = (w_0, w_1, w_2, \ldots, w_{|V|-1})$ of the vertices in V with $v_0 = w_0 = s$, let $\texttt{pref}(\mathfrak{f}, \mathfrak{g})$ denote the maximum integer k such that for every $i \in \{0, 1, 2, \ldots, k\}$, we have $v_i = w_i$. The following implies that the map from $\mathcal{M}$ to $\mathcal{A}$ that sends c to $\mathcal{F}_c$ is well defined, where $\mathcal{F}_c$ denotes a firing graph of c.

Lemma 2.7. *For every $c \in \mathcal{M}$, c has exactly one firing graph.*

Proof. Let $\mathfrak{f}_1 = \left(v_0, v_1, v_2, \ldots, v_{|V|-1}\right)$ and $\mathfrak{f}_2 = \left(w_0, w_1, w_2, \ldots, w_{|V|-1}\right)$ be two different legal firing sequences of c. Let j denote $\texttt{pref}(\mathfrak{f}_1, \mathfrak{f}_2)$ and $\mathfrak{f}' = \left(v_0, v_1, v_2, \ldots, v_j, w_{j+1}, v_{j+1}, v'_{j+3}, v'_{j+4}, \ldots, v'_{|V|-1}\right)$ the sequence of vertices of G, where $\left(v'_{j+3}, v'_{j+4}, \ldots, v'_{|V|-1}\right)$ is the sequence $\left(v_{j+2}, \ldots, v_{|V|-1}\right)$ with w_{j+1} deleted. Clearly, $\mathfrak{f}'$ is also a legal firing sequence of c. Let $\mathcal{F}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{F}' = (\mathcal{V}', \mathcal{E}')$ denote the firing graphs of c with respect to $\mathfrak{f}_1$ and $\mathfrak{f}'$, respectively.

We claim that $\mathcal{F}_1 = \mathcal{F}'$. Lemma 2.6 implies that

$$|\mathcal{E}_1| = |\mathcal{E}'| = \sum_{v \in V \backslash \{s\}} deg^+_G(v) - \sum_{v \in V \backslash \{s\}} c(v).$$

Hence, it suffices to prove that $\mathcal{E}_1 \backslash \mathcal{E}' = \emptyset$. We assume to the contrary that $\mathcal{E}_1 \backslash \mathcal{E}' \neq \emptyset$. Let k denote the integer such that $w_{j+1} = v_k$. Note that $k > j+1$. It follows from the definition of firing graph that $\mathcal{E}_1 \backslash \mathcal{E}' = \left\{(v_i, v_k) \in E \colon j+1 \leq i \leq k-1\right\}$. Let $X = \left\{(v_i, v_k) \colon (v_i, v_k) \in \mathcal{F}'\right\}$ and $Y = \left\{(v_i, v_k) \colon (v_i, v_k) \in \mathcal{F}_1\right\}$. Since $\mathfrak{f}'$ can be viewed as $\mathfrak{f}_1$ in which v_k has been moved backward, we have $X \subseteq Y$. It follows from $\mathcal{E}_1 \backslash \mathcal{E}' \neq \emptyset$ that $X \subsetneq Y$, therefore, $deg^-_{\mathcal{F}'}(v_k) < deg^-_{\mathcal{F}_1}(v_k)$, a contradiction to the assertion of Lemma 2.6.

Let $\mathcal{F}_2$ denote the firing graph of c constructed by $\mathfrak{f}_2$. The proof is completed by showing that $\mathcal{F}_1 = \mathcal{F}_2$. Let $\delta = \left(\delta_0, \delta_1, \delta_2, \ldots, \delta_{|V|-1}\right)$ be a legal firing sequence of c such that the firing graph constructed by δ is $\mathcal{F}_1$ and $\texttt{pref}(\delta, \mathfrak{f}_2)$ is maximum. We are going to show that $\delta = \mathfrak{f}_2$. Let p denote $\texttt{pref}(\delta, \mathfrak{f}_2)$. If $\delta \neq \mathfrak{f}_2$ then $p < |V| - 1$. Let δ' denote the sequence $\left(\delta_0, \delta_1, \delta_2, \ldots, \delta_p, w_{p+1}, \delta_{p+1}, u_{p+3}, u_{p+4}, \ldots, u_{|V|-1}\right)$, where $\left(u_{p+3}, u_{p+4}, \ldots, u_{|V|-1}\right)$ is the sequence $\left(\delta_{p+2}, \delta_{p+3}, \ldots, \delta_{|V|-1}\right)$ with the vertex w_{p+1} deleted. The above claim implies that the firing graph of c constructed by δ' is the same as the one constructed by δ. It is clear that $\mathtt{pref}(\delta', \mathfrak{f}_2) > \mathtt{pref}(\delta, \mathfrak{f}_2)$, a contradiction to the maximum of $\mathtt{pref}(\delta, \mathfrak{f}_2)$ ∎

For two non-repeated sequences $\mathfrak{f} = \left(v_0, v_1, v_2, \ldots, v_{|V|-1}\right)$, $\mathfrak{g} = \left(w_0, w_1, w_2, \ldots, w_{|V|-1}\right)$ of vertices in V with $v_0 = w_0 = s$ we denote by $\mathtt{inter}(\mathfrak{f}, \mathfrak{g})$ the sequence $\left(v_0, v_1, v_2, \ldots, v_k, w_{k+1}, v_{k+1}, v'_{k+3}, v'_{k+4}, \ldots, v'_{|V|-1}\right)$, where $k = \mathtt{pref}(\mathfrak{f}, \mathfrak{g})$ and $\left(v'_{k+3}, v'_{k+4}, \ldots, v'_{|V|-1}\right)$ is the sequence $\left(v_{k+2}, v_{k+3}, \ldots, v_{|V|-1}\right)$ with the vertex w_{k+1} deleted. It is easy to see that $\mathtt{pref}(\mathfrak{f}, \mathfrak{g}) < \mathtt{pref}(\mathtt{inter}(\mathfrak{f}, \mathfrak{g}), \mathfrak{g})$. Note that if $\mathfrak{f}$ and $\mathfrak{g}$ are two legal firing sequences of a configuration c, $\mathtt{inter}(\mathfrak{f}, \mathfrak{g})$ is also a legal firing sequence of c. The following result is the converse of Lemma 2.6.

Lemma 2.8. *Let $A \in \mathcal{A}$ and $\mathcal{F}$ denote $G[A]$. Then the configuration c defined by $c(v) = deg^+_G(v) - deg^-_{\mathcal{F}}(v)$ for every $v \in V \backslash \{s\}$ is a minimal recurrent configuration.*

Proof. We assume to the contrary that c is not minimal. There is $c' \in \mathcal{M}$ such that $c' \neq c$ and $c' \leq c$. Let $\mathcal{F}'$ be the firing graph of c'. By Lemma 2.6 we have $E(\mathcal{F}') \in \mathcal{A}$ and $\mathcal{F}' \neq \mathcal{F}$.

Since A is acyclic, there is a non-repeated sequence $\mathfrak{f}_1 = (v_0, v_1, v_2, \ldots, v_{|V|-1})$ of vertices in V such that if $(v_i, v_j) \in A$ then $i < j$. Clearly, $\mathfrak{f}_1$ is a legal firing sequence of c. Similarly, there is a non-repeated sequence $\mathfrak{f}_2 = \left(w_0, w_1, w_2, \ldots, w_{|V|-1}\right)$ of vertices V such that if $(w_i, w_j) \in E(\mathcal{F}')$ then $i < j$. Clearly, $\mathfrak{f}_2$ is a legal firing sequence of c'. We define the sequence $\left\{\mathfrak{g}_i\right\}_{i \in \mathbb{N}}$ as follows

$$\mathfrak{g}_0 = \mathfrak{f}_1,$$

$$\mathfrak{g}_{i+1} = \mathtt{inter}(\mathfrak{g}_i, \mathfrak{f}_2), \quad i \geq 0.$$

Let p be the minimum integer such that $\mathfrak{g}_p = \mathfrak{f}_2$. Note that for every $i \geq p$, $\mathfrak{g}_i = \mathfrak{f}_2$. Since $\mathcal{F} \neq \mathcal{F}'$, there is a minimum integer $q < p$ such that the firing graph constructed by $\mathfrak{g}_q = \left(\delta_0, \delta_1, \delta_2, \ldots, \delta_{|V|-1}\right)$ is distinct from the firing graph constructed by $\mathfrak{g}_{q+1}$. Let $k = \mathtt{pref}(\mathfrak{g}_q, \mathfrak{f}_2)$ and l the integer such that $\delta_l = w_{k+1}$. The firing graphs constructed by $\mathfrak{g}_q$ and $\mathfrak{g}_{q+1}$ are denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

We claim that for every $k+1 \leq i \leq l-1$ we have $(\delta_i, \delta_l) \notin E$. For a contradiction we assume to the contrary. By a similar argument as in the proof of Lemma 2.7, the set of arcs of $\mathcal{G}_2$ with head δ_l is a subset of the set of arcs of $\mathcal{G}_1$ with head δ_l. The assumption implies that there is an arc $e \in E$ with head δ_l such that $e \in \mathcal{G}_1$ and $e \notin \mathcal{G}_2$, therefore $deg^-_{\mathcal{G}_2}(\delta_l) < deg^-_{\mathcal{G}_1}(\delta_l)$. Since $\mathtt{pref}(\mathfrak{g}_i, \mathfrak{f}_2) < \mathtt{pref}(\mathfrak{g}_{i+1}, \mathfrak{f}_2)$ for every $0 \leq i \leq p-1$, $deg^-_{\mathcal{G}_2}(\delta_l)$ is equal to the in-degree of δ_l in the firing graph constructed by $\mathfrak{g}_p = \mathfrak{f}_2$, namely, $\mathcal{F}'$. It follows that $deg^-_{\mathcal{F}}(\delta_l) = deg^-_{\mathcal{G}_1}(\delta_l) > deg^-_{\mathcal{G}_2}(\delta_l) = deg^-_{\mathcal{F}'}(\delta_l)$, therefore, $c(\delta_l) < c'(\delta_l)$, a contradiction to the fact that $c' \leq c$.

Since $E(\mathcal{G}_1) \backslash E(\mathcal{G}_2) = \left\{(\delta_i, \delta_l) \in E : k+1 \leq i \leq l-1\right\}$, it follows from the above claim that $E(\mathcal{G}_1) \backslash E(\mathcal{G}_2) = \emptyset$, therefore $E(\mathcal{G}_1) \subsetneq E(\mathcal{G}_2)$. The choice of q implies that $E(\mathcal{G}_1) = A$, a contradiction to the fact that A is a maximal acyclic arc set. ∎

The following is the main result of this subsection.

Theorem 2.9. *Let $\mathcal{F}_c$ denote the firing graph of c, the map from $\mathcal{M}$ to $\mathcal{A}$ defined by $c \mapsto \mathcal{F}_c$ is bijective.*

Proof. Lemmas 2.6 and 2.7 imply that the map is well defined and injective. Lemma 2.8 implies the surjectivity. ∎

3. Acyclic Arc Sets on Eulerian Graphs and NP-Hardness of Minimum Recurrent Configuration Problem

3.1. Acyclic Arc Sets on Eulerian Graphs

From now until Proposition 3.8 we work with an Eulerian graph $G = (V,E)$. We will show in this subsection that finding a maximum acyclic arc set on an Eulerian graph can be restricted to finding a maximum acyclic arc set with a unique given source. This gives a connection between the MINFAS problem and the MINREC problem on Eulerian graphs via Theorem 2.9. By using this connection we show that the MINREC problem is NP-hard in the next subsection.

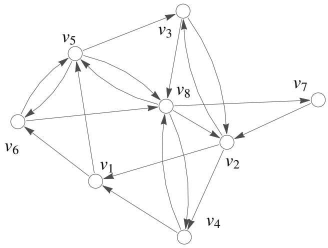
For two subsets A and B of V, we denote by $cut_G(A,B)$ the set $\{(u,v) \in E \colon u \in A \text{ and } v \in B\}$. We write $cut_G(A)$ for $cut_G(A,V\backslash A)$, and $cut_G^{-1}(A)$ for $cut_G(V\backslash A,A)$. The following appears stronger than the property $\forall v \in V, deg_G^-(v) = deg_G^+(v)$, but are actually equivalent.

Lemma 3.1. *For every $A \subseteq V$, we have $|cut_G(A)| = \left|cut_G^{-1}(A)\right|$.*

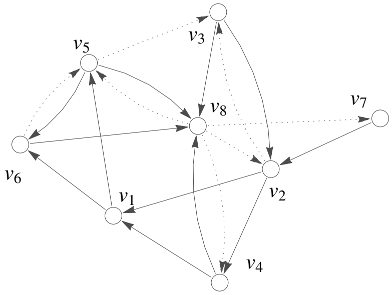
Proof. Let $X = \{(u,v) \in E \colon v \in A\}$, $Y = \{(u,v) \in E \colon u \in A\}$, $Z = \{(u,v) \in E \colon u \in A \text{ and } v \in A\}$. We have $X = cut_G^{-1}(A) \cup Z$ and $Y = cut_G(A) \cup Z$. Since $cut_G(A)$, $cut_G^{-1}(A)$, and Z are pairwise disjoint, $|X| = \left|cut_G^{-1}(A)\right| + |Z|$ and $|Y| = |cut_G(A)| + |Z|$. Since G is Eulerian, we have $0 = \sum\limits_{v \in A} \left(deg_G^-(v) - deg_G^+(v)\right) = |X| - |Y| = \left|cut_G^{-1}(A)\right| - |cut_G(A)|$. ∎

Definition 3.2. *Let A be an acyclic arc set and s a vertex of G. Let $r_G(A,v)$ denote the subset of all vertices of G that are reachable from s by a path in $G[A]$. The set $A \backslash cut_G^{-1}(r_G(A,s)) \cup cut_G(r_G(A,s))$ is called* cut-stretch *of A at s. We denote this set by $Cs_G(A,s)$.*

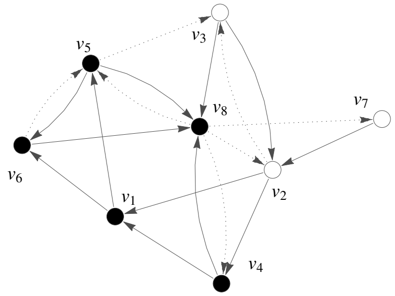
The idea of cut-stretch is to construct a new acyclic arc set, so that it does not contain less arcs than the old one. Moreover, the number of vertices, that are reachable from a fixed vertex, increases after performing the cut-stretch. For an intuitive illustration of this definition let us give here an example. Figure 6a shows an Eulerian graph with an acyclic arc set A shown in Figure 6b (plain arcs). If we want to compute the cut-stretch of A at v_4, we look at all vertices reachable from v_4 in $G[A]$. These vertices are the set $r_G(A,v_4)$ drawn in black in Figure 6c. The plain arcs in Figure 6d form the set $cut_G^{-1}(r_G(A,v_4))$: arcs of A going from the outside (the set $\{v_2, v_3, v_7\}$) to $r_G(A,v_4)$; and the other dotted arcs in this figure form the set $cut_G(r_G(A,v_4))$: arcs of G going from $r_G(A,v_4)$ to the outside. Remove the plain arcs in A from A and add the dotted arcs of Figure 6d, we obtain $Cs_G(A,v_4)$ shown in Figure 6e.



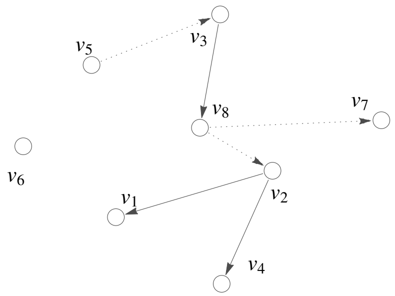
(a) An Eulerian graph.



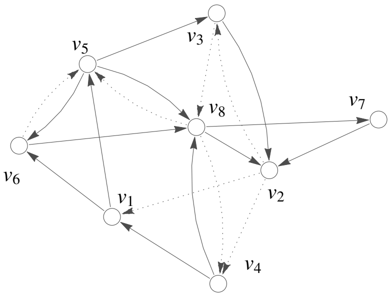
(b) An acyclic arc set A.



(c) v_4 is chosen and the set R of vertices reachable from v_4 in $G[A]$.



(d) the arcs of G going into R from the outside (plain arcs) and the arcs of G going from R to the outside (the dotted arcs).



(e) The cut-stretch $Cs_G(A, v_4)$.

Figure 6: An example of cut-stretch.

A simple observation from the above example is that a cut-stretch is still an acyclic arc set and its number of arcs is not less than the number of arcs of the old one. The following shows that this property holds not only for this example but also holds for the general case.

Lemma 3.3. *Let A be an acyclic arc set and s a vertex of G. Then $Cs_G(A,s)$ is also an acyclic arc set of G. Moreover, $|A| \leq |Cs_G(A,s)|$.*

Proof. By the definition of cut-stretch there is no arc in $Cs_G(A,s)$ from a vertex in $V\backslash r_G(A,s)$ to a vertex in $r_G(A,s)$. It implies that if $Cs_G(A,s)$ contains a cycle, the vertices in this cycle must be completely contained either in $r_G(A,s)$ or in $V\backslash r_G(A,s)$. In this case the arcs of the cycle are also the arcs of A, therefore the cycle is also a cycle of A, a contradiction to the acyclicity of A.

To prove $|A| \leq |Cs_G(A,s)|$, we observe that $A \cap cut_G(r_G(A,s)) = \emptyset$ $\big($from the maximality of $r_G(A,s)\big)$. From Lemma 3.1, we have $|Cs_G(A,s)| \geq |A| + |cut_G(r_G(A,s))| - \left|cut_G^{-1}(r_G(A,s))\right| = |A|$, which completes the proof. ∎

The following is the main result of this subsection.

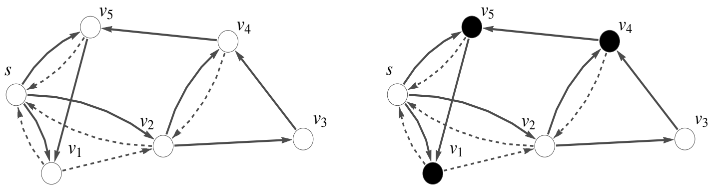
Theorem 3.4. *Let N be the maximum number of arcs of an acyclic arc set of G. For every vertex s of G there is an acyclic arc set of N arcs such that s is a unique source of this acyclic arc set.*

Proof. Let X be an acyclic set of G of N arcs. We construct a sequence $\left\{A_i\right\}_{i\in\mathbb{N}}$ as follows: $A_0 = X$ and $A_i = Cs_G(A_{i-1},s)$ for every $i \geq 1$. Lemma 3.3 and the maximum of N imply that $|A_i| = N$ for every $i \in \mathbb{N}$. If $r_G(A_k,s) = V$ for some k, then A_k is an acyclic set that has the required property since for any vertex $v \neq s$ of G there is a path in A_k from s to v. The proof is completed by showing that there always exists such a k.

Since a path from s in $G[A_i]$ is also a path from s in $G[A_{i+1}]$, we have $r_G(A_i,s) \subseteq r_G(A_{i+1},s)$. It suffices to show that if $r_G(A_i,s) \subsetneq V$ then $r_G(A_i,s) \subsetneq r_G(A_{i+1},s)$. Since $r_G(A_i,s) \subsetneq V$, there is an arc $e = (v_1, v_2)$ of G such that $v_1 \in r_G(A_i,s)$ and $v_2 \notin r_G(A_i,s)$. Since $e \in A_{i+1}$, there is a path in A_{i+1} from s to v_2 going through v_1. It implies that $v_2 \in r_G(A_{i+1},s)$, therefore $r_G(A_i,s) \subsetneq r_G(A_{i+1},s)$. ∎

Definition 3.5. *Let A be an acyclic arc set of G such that A has a unique source s. A vertex s' of G distinct from s is called* sourceable *in A if there is an arc of G whose head is s and whose tail is in $r_G(A,s')$.*

Figure 7a presents an Eulerian graph with an acyclic arc set (undashed arcs). This acyclic arc set has a unique source s. The black vertices in Figure 7b are all vertices that are reachable from v_4 by a path in the acyclic arc set. Since v_5 is one of these vertices and $(v_5, s) \in E$, v_4 is sourceable.



(a) An Eulerian graph. (b) Reachable vertices from v_4.

Figure 7: An example of sourceability.

We call such a vertex s' sourceable because the idea is to use the arc from s' to s to construct an acyclic arc set where it becomes a source. The fact that this is done by the cut-stretch at s' is stated in the following lemma.

Lemma 3.6. *Let A be an acyclic arc set of G having exactly one source s. If s' is sourceable in A then $Cs_G(A,s')$ has exactly one source s' and s is sourceable in $Cs_G(A,s')$. Moreover, $A \subseteq Cs_G(Cs_G(A,s'),s)$.*

Proof. Let X denote $r_G(A,s')$ and $Y = V \backslash X$. Since A has exactly one source s, for any $v \in V$ there is a path in A from s to v, therefore from s to s'. The acyclicity of $G[A]$ implies that $s \in Y$.

Clearly, s' is a source of $Cs_G(A,s')$. To prove that $Cs_G(A,s')$ has a unique source, it suffices to show that for any $v \in V$ there is a path in $Cs_G(A,s')$ from s' to v. It is trivial if $v \in X$. We consider the case $v \in Y$. Let (v', s) be an arc of G such that $v' \in X$. Such an arc exists because of the assumption of the lemma. By the definition of cut-stretch we have $(v', s) \in Cs_G(A,s')$. Let P_1 and P_2 be paths in A from s' to v' and from s to v, respectively. It follows from the definition of cut-stretch that $Cs_G(A,s')$ contains P_1. Since $v \in Y$, the path P_2 goes through only the vertices in Y. Therefore, $Cs_G(A,s')$ also contains P_2. Hence, the path $P_1 \cup \{(v', s)\} \cup P_2$ is a path in $Cs_G(A,s')$ from s' to v.

Let P_3 be a path in A from s to s'. The acyclicity of $G[A]$ implies that P_3 goes through only the vertices in $Y \cup \{s'\}$. Therefore there is an arc (v'', s') such that $v'' \in Y$. Clearly, we have $r_G(Cs_G(A,s'),s) = Y$. By the definition of sourceability we have s is sourceable in $Cs_G(A,s')$.

It remains to show that $A \subseteq Cs_G(Cs_G(A,s'),s)$. This follows immediately from the fact that $Cs_G(Cs_G(A,s'),s) = A \cup cut_G(Y,X)$. ∎

For each $s \in V$, let χ_s denote the number of maximum acyclic arc sets of G with exactly one source s. It is well known that for an undirected graph G, $T_G(1,0)$ counts the number of acyclic orientations with a unique fixed source, therefore counts χ_s, where $T_G(x,y)$ is the Tutte polynomial of G [14]. This implies that if G is an undirected graph, χ_s is independent of the choice of s. The following is a generalization of this fact to Eulerian graphs.

Proposition 3.7. *For any two vertices s_1, s_2 of G we have $\chi_{s_1} = \chi_{s_2}$.*

Proof. We claim that if $(v', v) \in E(G)$ then $\chi_v \leq \chi_{v'}$. Let $\mathcal{A}_1$ denote the set of maximum acyclic arc sets of G having exactly one source v, and $\mathcal{A}_2$ the set of maximum acyclic arc sets having exactly one source v'. Since $(v', v) \in E(G)$, v' is sourceable in every acyclic arc set in $\mathcal{A}_1$. It follows from Theorem 3.4 and Lemma 3.6 that the map $\theta\colon \mathcal{A}_1 \to \mathcal{A}_2$, defined by $A \to Cs_G(A,v')$, is well defined. Let A be arbitrary in $\mathcal{A}_1$. It follows from Lemma 3.6 that $A \subseteq Cs_G(Cs_G(A,v'),v)$. Since A is maximum, we have $A = Cs_G(Cs_G(A,v'),v)$. This implies that θ is injective. Therefore, $|\mathcal{A}_1| \leq |\mathcal{A}_2|$, equivalently $\chi_v \leq \chi_{v'}$.

The claim implies that for any two vertices v' and v of G such that there is a path in G from v' to v, we have $\chi_v \leq \chi_{v'}$. Since G is strongly connected, there is a path in G from s_1 to s_2 and a path in G from s_2 to s_1. Hence, $\chi_{s_1} = \chi_{s_2}$. ∎

Note that in an undirected graph a maximal acyclic arc set is also a maximum acyclic arc set (and vice versa). This fact no longer holds for Eulerian graphs. The assertion in Proposition 3.7 is not correct if we replace the maximum acyclic arc sets by the maximal acyclic arc sets. By using the relation in Theorem 2.9 we have a similar result for recurrent configurations.

Proposition 3.8. *The number of minimum recurrent configurations is independent of the choice of sink.*

Proof. Let s denote the sink of the game. Theorem 2.9 and Lemma 2.6 imply that the map $c \mapsto \mathcal{F}_c$ induces a bijective map from the minimum recurrent configurations to the maximum acyclic arc sets of G with exactly one source s, therefore their cardinalities are equal. It follows, from Proposition 3.7, that the number of maximum acyclic arc sets of G with exactly one source is independent of the choice of source, so is the number of minimum recurrent configurations. ∎

Proposition 3.8 states that the number of minimum recurrent configurations is characteristic of the graph itself. We recall the definition of the MINFAS problem:

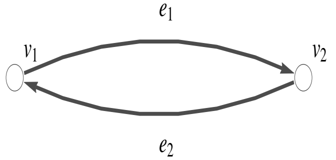
MINFAS Problem
Input: A graph G
Output: Minimum number of arcs of a feedback arc set of G

When the problem is restricted to Eulerian graphs, we call it *EMINFAS* problem for short. Although the EMINFAS problem was known to be NP-hard for its multigraph version [11], it is worth studying the computational complexity of the EMINFAS problem since most variants of the MINFAS problem are restrictions of the class of graphs (simple) (see [18]). In the following we describe briefly the reduction presented in [11], and why it cannot be simply extended to the case of graphs (simple).

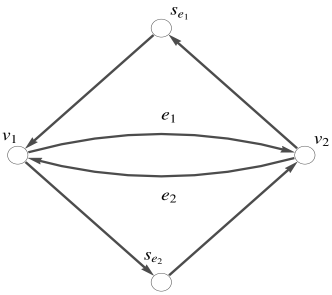
For a graph $G = (V, E)$ we create a new vertex s. Let $G' = (V \cup \{s\}, E)$. For each arc (v, v') of G we add two arcs (s, v) and (v', s) to G', thus obtaining an Eulerian digraph G'. Let k denote the number of arcs of a minimum feedback arc set of G. Then the number of arcs of a minimum feedback arc set of G' is $k + |E|$. Note that if G has two arcs whose heads are the same, say v, then the graph G' is not simple since it has at least two arcs with tail v and head s. As a consequence the reduction does not seem to be applicable to the case of simple graphs. As suggested by Flier in private correspondence, a natural way to avoid multi-graphs is to create a new vertex s_e for each arc e of G and add two arcs (s_e, v) and (v', s_e) to G', where v, v' are tail and head of e, respectively. However, with this construction the number of arcs of a minimum feedback arc set of G' is no longer equal to $k + |E|$, and it is not easy to understand how to patch the reduction, as shown by an example in Figure 8.

By using Theorem 3.4 and a stronger construction we show that the EMINFAS is NP-hard. We work with a general graph $G = (V, E)$, and construct an Eulerian graph G' so that an optimum value of the MINFAS problem on G implies an optimum value of the EMINFAS problem on G'. The graph $G' = (V', E')$ used in the reduction of MINFAS to EMINFAS is constructed as follows.

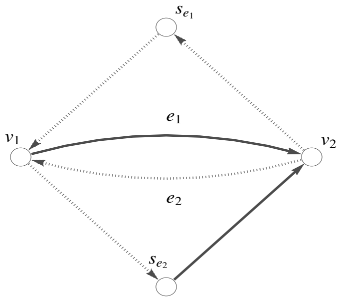
The basic idea to construct an Eulerian graph G' from G would be to create a new vertex and add arcs from this new vertex to any vertex that has more out-degree than in-degree, and arcs from vertices which have in-degree greater than out-degree to the new vertex. To avoid multi-graphs, we furthermore add for each of those arcs a new vertex in between, which has in-degree and out-degree 1. More precisely, the vertices of G are denoted by $v_1, v_2, \ldots, v_n$ for some n. If G is already an Eulerian graph then



(a) A graph G.



(b) Eulerian graph G'.



(c) A minimum feedback arc set (undotted arcs) of size 2.

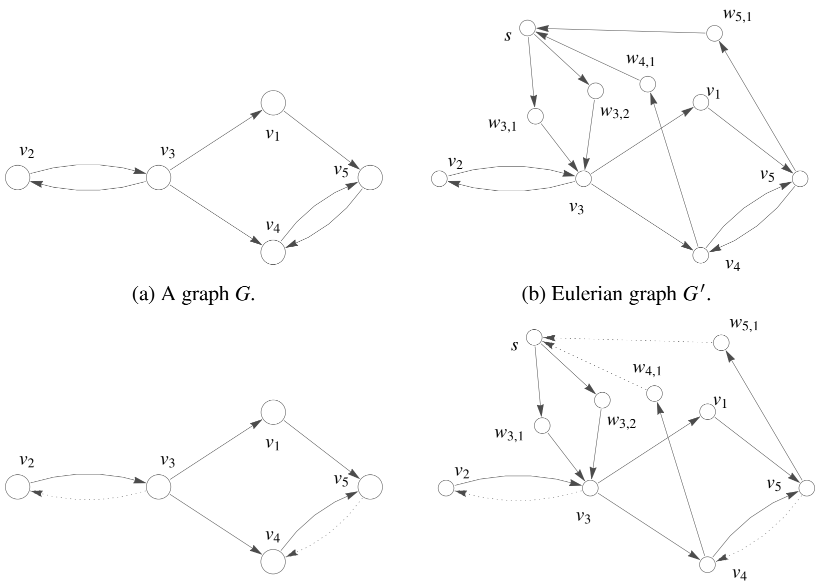
Figure 8

$G' := G$. Otherwise let G' be a copy of G. We add to G' a new vertex s. For each vertex v_i such that $deg^-_G(v_i) < deg^+_G(v_i)$ we add p_i new vertices $w_{i,1}, w_{i,2}, \ldots, w_{i,p_i}$ to G', and for each $j \in \{1, 2, \ldots, p_i\}$ we add two arcs $(s, w_{i,j})$ and $(w_{i,j}, v_i)$ to G', where $p_i = deg^+_G(v_i) - deg^-_G(v_i)$. For each vertex v_i such that $deg^+_G(v_i) < deg^-_G(v_i)$ we add q_i new vertices $w_{i,1}, w_{i,2}, \ldots, w_{i,q_i}$ to G', and for each $j \in \{1, 2, \ldots, q_i\}$ we add two arcs $(w_{i,j}, s)$ and $(v_i, w_{i,j})$ to G', where $q_i = deg^-_G(v_i) - deg^+_G(v_i)$. Formally, the vertex set and the arc set of G' are defined by

$$V' := \{s\} \cup V \cup \bigcup_{1 \le i \le n} \{w_{i,j} \colon 1 \le j \le |deg^-_G(v_i) - deg^+_G(v_i)|\},$$

$$\begin{aligned} E' := E \;&\cup \bigcup_{deg^-_G(v_i) < deg^+_G(v_i)} \{(s, w_{i,j}) colon 1 \le j \le deg^+_G(v_i) - deg^-_G(v_i)\} \\ &\cup \bigcup_{deg^-_G(v_i) < deg^+_G(v_i)} \{(w_{i,j}, v_i) \colon 1 \le j \le deg^+_G(v_i) - deg^-_G(v_i)\} \\ &\cup \bigcup_{deg^+_G(v_i) < deg^-_G(v_i)} \{(w_{i,j}, s) \colon 1 \le j \le deg^-_G(v_i) - deg^+_G(v_i)\} \end{aligned}$$

$$\cup \bigcup_{deg_G^+(v_i)<deg_G^-(v_i)} \left\{(v_i, w_{i,j})\colon 1 \le j \le deg_G^-(v_i) - deg_G^+(v_i)\right\}.$$



(a) A graph G.

(b) Eulerian graph G'.

(c) An acyclic arc set of G of maximum cardinality.

(d) An acyclic arc set of G'.

Figure 9: Maximum acyclic arc sets.

Figure 9 shows an example of G (Figure 9a) and the corresponding Eulerian graph G' (Figure 9b). Figure 9c shows an acyclic arc set of G of maximum cardinality. In order to construct an acyclic arc set of G', we add the arcs $(s, w_{i,j})$, $(w_{i,j}, v_i)$ (all the arcs created to offset vertices having out-degree greater than in-degree in G) and $(v_i, w_{i,j})$ (half of the arcs created to offset vertices having in-degree greater than out-degree in G) to this set, which indeed results in an acyclic arc set of G' of maximum cardinality. The following shows that we can always obtain an acyclic arc set of G' of maximum cardinality with this construction.

Lemma 3.9. *Let r be the maximum number of arcs of an acyclic arc set of G, and*

$$d = \sum_{deg_G^-(v_i) < deg_G^+(v_i)} \left(deg_G^+(v_i) - deg_G^-(v_i)\right).$$

The maximum number of arcs of an acyclic arc set of G' is $3d + r$.

Proof. The lemma clearly holds if G is an Eulerian graph, in which case $G' = G$. We assume to the contrary. Note that $4d$ arcs and $2d+1$ vertices are added to G in order to construct G'. Let r' be the maximum number of arcs of an acyclic arc set of G'.

First, we show that $3d + r \le r'$. Let A be an acyclic arc set of G of r arcs. Let $A' = A \cup \left\{(s, w_{i,j})\colon (s, w_{i,j}) \in E'\right\} \cup \left\{(w_{i,j}, v_i)\colon (w_{i,j}, v_i) \in E'\right\} \cup \left\{(v_i, w_{i,j})\colon (v_i, w_{i,j}) \in\right.$

$E'\}$. Since A is an acyclic arc set of G and A' contains no arc $(w_{i,j}, s)$ of E', A' is an acyclic arc set of G'. The sets $\{(s, w_{i,j})\colon (s, w_{i,j}) \in E'\}$, $\{(w_{i,j}, v_i)\colon (w_{i,j}, v_i) \in E'\}$, and $\{(v_i, w_{i,j})\colon (v_i, w_{i,j}) \in E'\}$ are pairwise-disjoint, and each of them has exactly d arcs, therefore we have constructed an acyclic arc set A' of size $|A'| = 3d + r$. It implies that $3d + r \leq r'$.

It remains to show that $r' \leq 3d + r$. Let B be an acyclic arc set of G' of r' arcs. By Theorem 3.4 there is an acyclic arc set B' of G' of r' arcs such that B' contains no arc $(w_{i,j}, s)$ of E'. The set B' must contain all arcs e of G' of the form $(s, w_{i,j})$, $(w_{i,j}, v_i)$ or $(v_i, w_{i,j})$ since if otherwise, $B' \cup \{e\}$ is an acyclic arc set of G' containing $r' + 1$ arcs. Let A'' denote $B' \backslash (\{(s, w_{i,j})\colon (s, w_{i,j}) \in E'\} \cup \{(w_{i,j}, v_i)\colon (w_{i,j}, v_i) \in E'\} \cup \{(v_i, w_{i,j})\colon (v_i, w_{i,j}) \in E'\})$. The set A'' is an acyclic arc set of G, therefore $|A''| \leq r$. It implies $r' = |B'| = 3d + |A''| \leq 3d + r$. ∎

A direct consequence of Lemma 3.9 is a NP-hardness proof for the EMINFAS problem.

Theorem 3.10. *The EMINFAS problem is NP-hard.*

Proof. Given a general graph G, the Eulerian graph G' can be constructed in polynomial time. Let b be the minimum number of arcs of a feedback arc set of G', that is, the solution of EMINFAS on G'. Clearly $|E'| - b$ is the maximum number of arcs of an acyclic arc set of G'. By Lemma 3.9, the maximum number of arcs of an acyclic arc set of G is $|E'| - b - 3d$, where d is defined as in Lemma 3.9 and is computable in polynomial time. Thus the minimum number of arcs of a feedback arc set of G is $|E| - (|E'| - b - 3d) = b + 3d + |E| - |E'|$. This implies a polynomial-time reduction from the MINFAS problem to the EMINFAS problem. The MINFAS problem is NP-hard, so is the EMINFAS problem. ∎

3.2. NP-Hardness of Minimum Recurrent Configuration Problem

In this subsection we study the computational complexity of the following problem:

> **MINREC problem**
>
> **Input:** A graph G with a global sink
> **Output:** Minimum total number of chips of a recurrent configuration of G

If the input graphs are restricted to undirected graphs G with a sink s, the problem can be solved in polynomial time since a minimum recurrent configuration has $\frac{|E(G)|}{2} - deg^+_G(s)$ chips. Nevertheless, we show that the problem is NP-hard when the input graphs are restricted to Eulerian graphs.

> **EMINREC problem**
>
> **Input:** An Eulerian graph G with a sink s
> **Output:** Minimum total number of chips of a recurrent configuration of G

Theorem 3.11. *The EMINREC problem is NP-hard, so is the MINREC problem.*

Proof. Let G be an Eulerian graph with sink s. Let k be the maximum number of arcs of a feedback arc set of G and let k' be the minimum number of chips of a recurrent configuration of G. Since the EMINFAS problem is NP-hard, the proof is completed by showing that $k+k' = \sum_{v\in V\setminus\{s\}} deg^+_G(v)$.

By Theorem 3.4, there is an acyclic arc set A of G such that $|A| = k$ and s is a unique vertex of in-degree 0 in $G[A]$. Lemma 2.4 implies that the configuration c defined by $c(v) = deg^+_G(v) - deg^-_{G[A]}(v)$ for every $v \in V\setminus\{s\}$ is recurrent. Clearly, $k + \sum_{v\in V\setminus\{s\}} c(v) = \sum_{v\in V\setminus\{s\}} deg^+_G(v)$ and $k+k' \leq \sum_{v\in V\setminus\{s\}} deg^+_G(v)$ since G is Eulerian.

It remains to prove that $k+k' \geq \sum_{v\in V\setminus\{s\}} deg^+_G(v)$. Let $\bar{c}$ be a recurrent configuration such that $\sum_{v\in V\setminus\{s\}} \bar{c}(v) = k'$. Let $\mathcal{F}$ be a firing graph of $\bar{c}$. Lemma 2.5 implies that $\bar{c}(v) \geq deg^+_G(v) - deg^-_{\mathcal{F}}(v)$ for every $v \in V\setminus\{s\}$, therefore $k+k' \geq \sum_{v\in V\setminus\{s\}} \bar{c}(v) + |E(\mathcal{F})| \geq \sum_{v\in V\setminus\{s\}} deg^+_G(v)$. ∎

Note that it follows directly from [34] that the EMINFAS problem restricted to planar Eulerian graphs is solvable in polynomial time, so is the EMINREC problem. This class of graphs is pretty big since it contains planar undirected graphs.

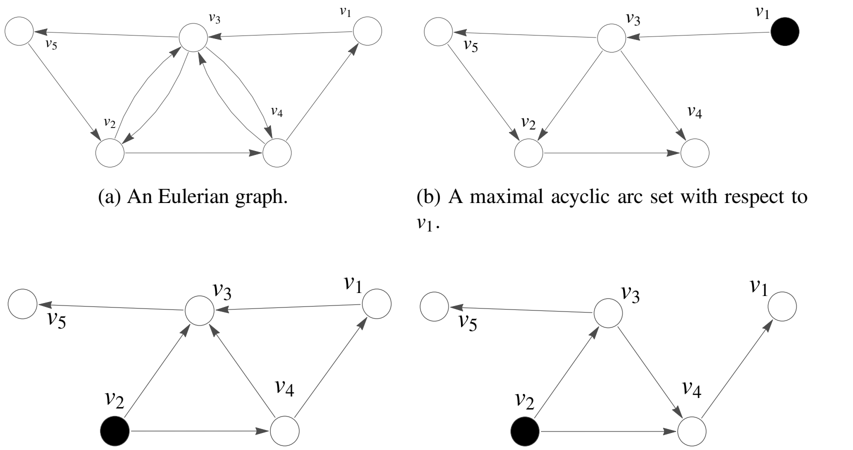
4. Conclusion and Perspectives

In this paper we pointed out a close relation between the MINFAS problem and the MINREC problem. The important consequence of this relation is the NP-hardness of the MINREC problem. It would be interesting to investigate classes of graphs that are situated strictly between the class of undirected graphs and the class of Eulerian graphs, for which the MINFAS and MINREC problems are solvable in polynomial time. We discuss here about such a class.

It follows from Theorem 3.4 that to compute the maximum number of arcs of an acyclic arc set of an Eulerian graph, we can restrict to the acyclic arc sets that satisfy the condition in Theorem 3.4. With different choices of s we have different maximal acyclic arc sets. One would prefer to choose a vertex s such that all maximal acyclic arc sets have the same number of arcs since a maximal acyclic arc set can be computed quickly, therefore a maximum acyclic arc set. Figure 10a shows an Eulerian graph. If v_1 is chosen, we have exactly one maximal acyclic arc set that is shown in Figure 10b. If v_2 is chosen, we have exactly two maximal acyclic arc sets with different sizes. Thus one computes easily a maximum acyclic arc set if v_1 is chosen.

Note that there are many Eulerian graphs in each of which there is no vertex s that satisfies this good property. By an experimental observation we see that the class of Eulerian graphs, for which at least one vertex s has the property, is rather large. However, a characterization for this class of graphs, on which the MINFAS problem is polynomial, is still unknown. In addition, the observation also provides a heuristic algorithm for the EMINFAS problem. It is interesting to investigate the properties of this algorithm.

For an undirected graph the minimal recurrent configurations coincide with the minimum recurrent configurations, and has an one-to-one correspondence to the acyclic orientations with a fixed source. This is the reason why the number of minimum (minimal) recurrent configurations is independent of the choice of sink. For



(a) An Eulerian graph.

(b) A maximal acyclic arc set with respect to v_1.

(c) Maximal acyclic arc sets with respect to v_2.

Figure 10: Maximal acyclic arc sets with different choices of s.

a general Eulerian graph, minimum recurrent configurations may no longer coincide with minimal recurrent configurations. Moreover, the number of minimal recurrent configurations depends on the choice of sink. Nevertheless, we point out in this paper (Proposition 3.8) that the number of minimum recurrent configurations is independent of the choice of sink. This property is one of the most interesting and promising results in this paper. Note that this property has a stronger version given in [25], however with a different and complicated proof technique. It would be interesting to investigate whether the technique we present in this paper can be applied to solve the stronger result in [25]. The authors of [25] also conjecture a generalization of this property to strongly connected graphs. Those problems are open and remain to be investigated.

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