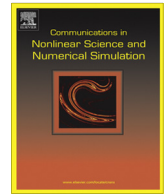




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# Comments on “Fractional order Lyapunov stability theorem and its applications in synchronization of complex dynamical networks”



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## ABSTRACT

This letter shows an incorrect application of the chain rule for fractional order derivatives reported in paper (Chen et al., 2014). Due to this misleading application, the proof of Theorem 2 and Theorem 5 in Chen et al., (2014) are incorrect. However, the mentioned Theorem 2 is a straightforward conclusion from results already available in literature (Jarad et al., 2013; Matignon 1996), and consequently there is no need to prove it, as it is stated in this letter. In the same way, although the proof of Theorem 5 in Chen et al. (2014) is not valid, Theorem 5 is indeed true, and a recommendation as to how to prove it is made to the authors. Besides, this letter shows that the proposed Theorem 1 in Chen et al., (2014) is also a straightforward conclusion from well known results available in literature (Jarad et al., 2013; Slotine and Li, 1999), so no demonstration is needed for this result neither.

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## 1. Introduction

Recently, a paper has been published [1], in which the authors present two theorems; one regarding the fractional extension of Lyapunov direct method (Theorem 1) and the other one regarding the stability of linear time invariant fractional order systems (Theorem 2). Theorem 1 and Theorem 2, however, do not need demonstrations, since they are straightforward conclusions from other results already available in literature [2–4].

Nevertheless, the proof given of Theorem 2 in [1] contains an incorrect use of the chain rule for fractional derivatives. Moreover, the proof of Theorem 5 presented in the paper [1] is also incorrect, since the authors make a misleading application of the chain rule as well.

This letter gives an explanation of these topics. Section 2 justify why the Theorem 1 in [1] is a straightforward conclusion from results already available in literature [2,4]. Section 3 deals with the erroneous application of the chain rule for fractional derivatives in the proof of Theorem 2 and Theorem 5 in [1], and also includes a counterexample. Finally, Section 4 justify why the Theorem 2 is a straightforward conclusion from results already present in literature [2,3].

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### 2. Theorem regarding fractional extension of Lyapunov direct method

Theorem 1 in [1] relates with the asymptotic stability of fractional order systems. The authors make first a reference to Theorem A.1 [2]. Based on this reference, the authors introduce then Theorem 1 in [1].

However, Theorem 1 in [1] is just another form to write Theorem A.1 [2], given the relation between class-K functions and positive definite functions, which is show in the following Lemma 1, taken from [4].

**Lemma 1** [4]. A time varying scalar function  $V(x, t) \in \mathbb{R}$  with  $x \in \mathbb{R}^n$  is positive definite if and only if there exists a class-K function  $\alpha$  such that  $V(0, t) = 0, \forall t \geq t_0$  and

$$V(x, t) \geq \alpha(\|x\|), \quad \forall t \geq t_0 \tag{1}$$

Using the same notation as in [1] and based on Lemma 1, if  $\psi$  exists in Theorem A.1 [2], then  ${}_a^C D_t^q V(x)$  is negative definite, that is  ${}_a^C D_t^q V(x) < 0$  for all  $t \geq t_0$  and  ${}_a^C D_t^q V(x) = 0$  if and only if  $x = 0$ . Thus, Theorem 1 in [1] is just another way to write Theorem A.1 [2].

### 3. Application of the chain rule for fractional derivatives

The incorrect application of the chain rule for fractional derivatives is one of the most frequent mistakes when using fractional operators. In the integer order case, it is true that

$$\frac{d}{dt} [f(t)g(t)] = f(t) \frac{d}{dt} [g(t)] + g(t) \frac{d}{dt} [f(t)] \tag{2}$$

However, in the fractional order case, the chain rule differs from the integer order case, as it is stated by [5].

**Property 1** (Leibniz rule for fractional differentiation [5]). If  $f(t)$  and  $g(t)$  along with all its derivatives are continuous in  $[a, t]$ , then the Leibniz rule for fractional differentiation takes the form

$${}_a^C D_t^q [f(t)g(t)] = \sum_{k=0}^{\infty} \binom{q}{k} f^{(k)}(t) {}_a^C D_t^{q-k} g(t) \tag{3}$$

where  $q \in (0, 1]$  and  ${}_a^C D_t^q$  corresponds to the Caputo fractional derivative [6].

As can be seen from Property 1, taking the fractional derivative of the product of two functions implies having an infinite sum, which includes fractional order and integer order derivatives of the functions.

Although Property 1 is available in literature, many authors use the chain rule in the fractional order case as it is in the integer order case, stating for example that

$${}_a^C D_t^q [f(t)g(t)] = f(t) {}_a^C D_t^q [g(t)] + g(t) {}_a^C D_t^q [f(t)] \tag{4}$$

which is in general not true.

This is precisely the mistake that the authors made in the paper by [1]. In the proof of Theorem 2, the authors state that

$${}_a^C D_t^q [x^H P x] = {}_a^C D_t^q [x^H] P x + x^H P {}_a^C D_t^q [x] \tag{5}$$

where  $x \in \mathbb{R}^n, x^H$  is the conjugate transpose (Hermit) of  $x$  and  $P \in \mathbb{R}^{n \times n}$  is a positive definite matrix such that  $P^H = P$ . However, as was stated before, expression (5) is not true. To make the statement more clear, let us show a counterexample.

#### 3.1. Counterexample

Let  $x(t) = [t^\beta \ t^\gamma]^T$ , with  $\beta, \gamma > 1$ . Let  $P = I_{2 \times 2}$  and  $t_0 = 0$ . According to the authors, the result should be

$${}_0^C D_t^q [x^T(t) P x(t)] = {}_0^C D_t^q [x^T(t)] P x(t) + x^T(t) P {}_0^C D_t^q [x(t)] = [{}_0^C D_t^q t^\beta \quad {}_0^C D_t^q t^\gamma] \begin{bmatrix} t^\beta \\ t^\gamma \end{bmatrix} + [t^\beta \ t^\gamma] \begin{bmatrix} {}_0^C D_t^q t^\beta \\ {}_0^C D_t^q t^\gamma \end{bmatrix} = 2 [t^\beta \ t^\gamma] \begin{bmatrix} {}_0^C D_t^q t^\beta \\ {}_0^C D_t^q t^\gamma \end{bmatrix} \tag{6}$$

According to [7], for  $q \in (0, 1], \beta > 1$

$${}_0^C D_t^q t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - q)} t^{\beta - q} \tag{7}$$

Using (7) in (6) it follows that

$${}_0^C D_t^q [x^T(t) P x(t)] = 2 [t^\beta \ t^\gamma] \begin{bmatrix} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - q)} t^{\beta - q} \\ \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - q)} t^{\gamma - q} \end{bmatrix} = 2 \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - q)} t^{2\beta - q} + 2 \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - q)} t^{2\gamma - q} \tag{8}$$

Let us now look for the result solving directly the expression  ${}^C_0D_t^q [x^T(t)Px(t)]$ .

$${}^C_0D_t^q [x^T(t)Px(t)] = {}^C_0D_t^q \left[ \begin{matrix} t^\beta & t^\gamma \\ t^\beta & t^\gamma \end{matrix} \right] = {}^C_0D_t^q [t^{2\beta} + t^{2\gamma}] \tag{9}$$

and using (7) in (9) it follows that

$${}^C_0D_t^q [x^T(t)Px(t)] = \frac{\Gamma(2\beta + 1)}{\Gamma(2\beta + 1 - q)} t^{2\beta - q} + \frac{\Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1 - q)} t^{2\gamma - q} \tag{10}$$

As can be seen, the result (8) according to the authors of [1] is different from the result (10) that follows directly from solving  $x^T Px$  and applying the fractional derivative. This proves the incorrect use of the chain rule for fractional derivatives that the authors made.

According to what has been exposed in this section, the proof of Theorem 2 given in the paper [1] is not valid.

Unfortunately, the same mistake applying the chain rule for fractional derivatives is made in the proof of Theorem 5 in the paper [1], so this result is also analytically not valid, even when the numerical simulations show that the control actually works.

Nevertheless, Theorem 5 is indeed true, and in order to give a valid proof for it, we suggest to use Lemma 4 in [8].

#### 4. Theorem regarding the stability of linear time invariant fractional order systems

Besides the proof of Theorem 2 in [1] is incorrect, as it was stated in the previous section, we would like to point out that Theorem 2 in [1] has no contribution, since it is a direct consequence of results available in literature [2,3]. Let us first make an introduction to the Theorem 2, as it is stated in [1].

First, the authors reference a lemma in [9], which actually is not in this reference. Nevertheless, the result is already available in [3], and it is stated in this note as Theorem A.2. As can be seen from Theorem A.2, it is related to the stability analysis of linear time invariant fractional order systems.

Based on the Theorem A.2 [3], the authors state Theorem 2 in [1].

However, again in this case, there is no need to proof Theorem 2 in [1], since it is a direct consequence of Theorem A.2 [3] and Theorem A.1 [2].

Theorem A.1 [2] is a sufficient condition for asymptotic stability, and Theorem A.2 [3] is a necessary and sufficient condition. So, obviously, if system (A.2) satisfies Theorem A.1 [2], it will satisfy Theorem A.2 [3] as well.

#### 5. Conclusions

Based on the analysis performed in this letter, some conclusions can be drawn.

- Theorem 1 in [1] is just another way to write Theorem A.1 [2].
- There is no need to prove Theorem 2 in [1], since it is a trivial consequence of Theorem A.2 [3] and Theorem A.1 [2].
- An erroneous application of the chain rule is made in the proof of Theorem 2 and Theorem 5 in [1], which is clarified in this letter. This incorrect application of the chain rule implies that the proof of Theorem 5 in [1] is not valid. However, Theorem 5 is indeed true, and it can be proved using Lemma 4 in [8].

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#### Appendix A. Referenced results

This appendix contains the theorems that are referenced in this note.

**Theorem A.1 [2].** *Let the fractional order system*

$${}^C_0D_t^q x = f(x) \tag{A.1}$$

where  $f(x) \in \mathbb{R}^n$  is a smooth nonlinear vector field,  $x \in \mathbb{R}^n$  is the state vector of the system,  $q \in (0, 1]$ .

If there exists a positive definite Lyapunov function  $V(x)$  such that  ${}^C_0D_t^q V(x) \leq -\psi(V(x))$ , for all  $t \geq t_0$  where  $\psi$  is a class-K function, then the trivial solution of system (A.1) is asymptotically stable.

Furthermore, a function  $\psi(r)$  is said to belong to class-K if and only if  $\psi \in [[0, \rho), \mathbb{R}_+]$ , where  $\rho$  is a positive real number,  $\psi(0)=0$  and  $\psi(r)$  is strictly monotonically increasing.

**Theorem A.2** [3]. Let a linear time invariant fractional order system be given by

$${}^C D_t^q x = Ax \quad (\text{A.2})$$

where  $q \in (0, 1]$  and  $x \in \mathbb{R}^n$ .

System (A.2) is asymptotically stable if and only if  $|\arg(\lambda_i)| > q\pi/2$  is satisfied for all eigenvalues  $\lambda_i$  of matrix  $A$ . Furthermore, this system is stable if and only if  $|\arg(\lambda_i)| \geq q\pi/2$  is satisfied for all eigenvalues  $\lambda_i$  of matrix  $A$  and those critical eigenvalues that satisfy the condition  $|\arg(\lambda_i)| = q\pi/2$  have geometric multiplicity one.

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