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Vortex-type solutions to a magnetic nonlinear Choquard equation

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Abstract. We consider the stationary nonlinear magnetic Choquard equation

$$(-\mathrm{i}\nabla + A(x))^2 u + W(x)u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3, \alpha \in (0, N), p \in \left[2, \frac{2N-\alpha}{N-2}\right), A : \mathbb{R}^N \to \mathbb{R}^N$ is a magnetic potential and $W : \mathbb{R}^N \to \mathbb{R}$ is a bounded electric potential. For a given group Γ of linear isometries of $\mathbb{R}^N$, we assume that $A(gx) = gA(x)$ and $W(gx) = W(x)$ for all $g \in \Gamma, x \in \mathbb{R}^N$. Under some assumptions on the decay of A and W at infinity, we establish the existence of solutions to this problem which satisfy

$$u(\gamma x) = \phi(\gamma)u(x) \quad \text{for all } \gamma \in \Gamma,\, x \in \mathbb{R}^N,$$

where $\phi : \Gamma \to \mathbb{S}^1$ is a given continuous group homomorphism into the unit complex numbers.

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1. Introduction

We consider the problem

$$\begin{cases} (-\mathrm{i}\nabla + A(x))^2 u + (V_\infty + V(x))u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \nabla u + \mathrm{i}\, A(x)u \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases} \tag{1.1}$$

where $N \geq 3, \alpha \in (0, N), p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right), A : \mathbb{R}^N \to \mathbb{R}^N$ is a $\mathcal{C}^1$-vector potential and $V_\infty + V$ is a scalar potential which satisfies

$$V \in \mathcal{C}^0(\mathbb{R}^N),\ V_\infty \in (0, \infty),\ \inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0,\ \lim_{|x|\to\infty} V(x) = 0. \tag{V_0}$$

Here, $*$ denotes the convolution operator and i is the imaginary unit.

This equation arises in various physical contexts, especially in the case where $A = 0, V = 0, N = 3, \alpha = 1$, and $p = 2$. Depending on the context, it is called the stationary nonlinear Choquard equation or the nonlinear Schrödinger–Newton equation. As the Choquard equation, it comes from an approximation to the Hartree–Fock theory of a one-component plasma and describes an electron trapped in its own hole (we refer to Lieb and Lieb–Simon's papers [9,11] in the 1970s for a wide discussion on the relevance of Choquard equation and Hartree–Fock equations to physics). As the nonlinear Schrödinger–Newton

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equation, it was proposed by Penrose in a model where quantum state reduction is considered as a phenomenon that occurs due to some gravitational influence [21–23].

Because of its relevance to physics, the existence of solutions to Eq. (1.1) has been extensively investigated in the context of $H^1(\mathbb{R}^3)$ in the nonmagnetic case $A = 0$. In particular, when $V = 0, \alpha = 1$, and $p = 2$, there is a well-known result due to Lieb [9] which asserts that (1.1) possess a unique minimizer, up to translations. A result concerning the existence of infinitely many radially symmetric solutions was obtained by Lions [12]. We also refer to [15,16,25] and the references therein for interesting existence results of Schrödinger–Newton equations.

Families of semiclassical solutions to problem (1.1) for $N = 3, \alpha = 1$, and $p = 2$ have been obtained in [19,24,26] when $A = 0$ and in [3,5,6] when $A \neq 0$. The question of the existence of semiclassical solutions in arbitrary dimensions $N \in \mathbb{N}$ for $\alpha \in (0, N)$ and an appropriate range of exponents $p \geq 2$ has been recently studied by Moroz and Van Schaftingen [18] when $A = 0$. Further results for related problems may be found in [1,4,14] and the references therein.

Recently, Cingolani, Clapp, and Secchi considered the stationary nonlinear magnetic Choquard equation (1.1) for the case in which $|A|^2 + V$ tends to its limit at infinity exponentially from below at an appropriate speed of convergence. Under symmetry assumptions on the data and the additional condition

$$\left[2, \frac{2N}{N-2}\right] \cap \left(p, \frac{pN}{N-\alpha}\right) \cap \left(\frac{(2p-2)N}{N+2-\alpha}, \frac{(2p-1)N}{N+2-\alpha}\right] \cap \left[\frac{(2p-1)N}{2N-\alpha}, \infty\right) \neq \emptyset, \tag{1.2}$$

they proved, in [3], the existence of a complex-valued solution to this problem which exhibits a vortex-type behavior. The main goal of this paper is to allow potentials A and V such that $|A|^2 + V$ approaches to its limit at infinity exponentially from above.

For the local nonlinear Schrödinger equation

$$-\Delta u + (V_\infty + V(x))u = |u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N),$$

which corresponds to the local version of (1.1) when $A = 0$, Bahri and Lions solved the question of the existence, for potentials that approach to its limit from above, without any symmetry assumption. Unfortunately, some of the facts that they used are not available in the nonlocal setting. For example, the uniqueness of positive solutions to (1.1) when $A = 0$ and $V = 0$ is a difficult problem that has only been solved in the case $N = 3, \alpha = 1$, and $p = 2$ [13].

The existence of infinitely many symmetric solutions of (1.1) is known in the case where A and V are compatible with the action of some group G of linear isometries of $\mathbb{R}^N$ and every nontrivial G-orbit in $\mathbb{R}^N$ is infinite [3]. When the data have only finite symmetries and $A = 0$, it was shown in [8] that there exist a positive and multiple sign-changing solutions to (1.1). Our purpose is to obtain vortex-type solutions to the magnetic problem above when both A and V have finite symmetries given by the action of a closed subgroup Γ of the group $O(N)$ of linear isometries of $\mathbb{R}^N$.

More precisely, we assume that A and V satisfy

$$A(gx) = gA(x) \quad \text{and} \quad V(gx) = V(x) \quad \text{for all } g \in \Gamma \text{ and } x \in \mathbb{R}^N. \tag{1.3}$$

We consider a continuous group homomorphism $\phi : \Gamma \to \mathbb{S}^1$ into the unit complex numbers $\mathbb{S}^1$ and we look for solutions that satisfy

$$u(gx) = \phi(g)u(x) \qquad \text{for all } g \in \Gamma \text{ and } x \in \mathbb{R}^N. \tag{1.4}$$

We denote by $\Gamma_x := \{g \in \Gamma : gx = x\}$ the isotropy group of x, by $\Gamma x := \{gx : g \in \Gamma\}$ the Γ-orbit of x and by $\#\Gamma x$ its cardinality. Let

$$\ell(\Gamma) := \min\left\{\#\Gamma x : x \in \mathbb{R}^N \smallsetminus \{0\}\right\}.$$

Recall that a function is called Γ-invariant if it is constant on each Γ-orbit in its domain. Note that if u satisfies (1.4), then the absolute value $|u|$ of u is Γ-invariant, i.e.,

$$|u(gx)| = |u(x)| \qquad \text{for all } g \in \Gamma \text{ and } x \in \mathbb{R}^N.$$

Moreover, the phase of $u(gx)$ is that of $u(x)$ multiplied by $\phi(g)$. If $\phi \equiv 1$ is the trivial homomorphism, then (1.4) simply says that u is Γ-invariant.

Observe that it may happen that every function satisfying (1.4) is trivial. Indeed, if $\Gamma = O(N)$ and $\phi(g)$ is the determinant of g, then for each $x \in \mathbb{R}^N$, we may choose a $g_x \in O(N)$ with $g_x x = x$ and $\phi(g_x) = -1$. If u satisfies (1.4), then $u(x) = u(g_x x) = \phi(g_x)u(x) = -u(x)$. Thus, $u \equiv 0$. To avoid this situation, we will restrict our attention to those $x \in \mathbb{R}^N$ such that $\Gamma_x \subset \ker\phi$. Set

$$\Sigma^\phi := \left\{x \in \mathbb{R}^N : |x| = 1,\, \#\Gamma x = \ell(\Gamma),\, \Gamma_x \subset \ker\phi\right\}.$$

Note that Σ^ϕ is Γ-invariant, i.e., $\Gamma x \subset \Sigma^\phi$ for every $x \in \Sigma^\phi$.

We are going to consider only the case $\ell(\Gamma) < \infty$ because, as we mentioned before, if all Γ-orbits of $\mathbb{R}^N \smallsetminus \{0\}$ are infinite and $\Sigma^\phi \neq \emptyset$, it was already shown in [3, Theorem 1.1] that (1.1) has infinitely many solutions satisfying (1.4).

Let $z \in \Sigma^\phi$. If there exists $\alpha \in \Gamma$ such that $\alpha z \neq z$ and $\mathrm{Re}(\phi(\alpha)) > 0$, we define

$$\mu^\phi(\Gamma z) := \min\left\{|gz - hz| : g, h \in \Gamma,\ gz \neq hz, \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) > 0\right\}.$$

Otherwise, we set $\mu^\phi(\Gamma z) = 2$.

We denote by c_∞ the energy of a ground state of the problem

$$\begin{cases} -\Delta u + V_\infty u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.5}$$

We shall look for solutions with small energy, i.e., which satisfy

$$\frac{p-1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\mathrm{d}x\,\mathrm{d}y < \ell(\Gamma)\,c_\infty. \tag{1.6}$$

In what follows, we assume that V satisfies (V_0). We shall prove the following result.

Theorem 1.1. *Let $p = 2$. Let $\phi : \Gamma \to \mathbb{S}^1$ be a homomorphism. If A and V satisfy (1.3) and the following hold:*
(Z_0) There exist $z \in \Sigma^\phi$ and $a_0 > 1$ such that

$$|gz - hz| \geq a_0\,\mu^\phi(\Gamma z) \qquad \textit{for all } g, h \in \Gamma \textit{ with } Re\big(\phi(g)\overline{\phi(h)}\big) < 0,$$

(AV) There exist $c_0 > 0$ and $\kappa > \mu^\phi(\Gamma z)\sqrt{V_\infty}$ such that

$$|A(x)|,\ \big||A(x)|^2 + V(x)\big| \leq c_0 e^{-\kappa|x|} \qquad \textit{for all } x \in \mathbb{R}^N,$$

then (1.1) has at least one vortex-type solution u, which satisfies (1.4) and (1.6).

Let us look at an example. Fix $k \in \mathbb{N}, k \geq 2$ and let Γ_k be the cyclic group of order k generated by $\xi := e^{\mathrm{i}\frac{2\pi}{k}}$. If N is even, let us consider the action of Γ_k on $\mathbb{R}^N \equiv \mathbb{C}^{N/2}$ given by complex multiplication on each complex coordinate. Assume that A and V satisfy (1.3) for this action. For example, the magnetic potential $A(z_1, \ldots, z_{N/2}) = (\mathrm{i}z_1, \ldots, \mathrm{i}z_{N/2})$ associated with the constant magnetic field $B = \mathrm{curl}A$ has this property for every k. For each $m \in \mathbb{N}, m \geq 1$, consider the homomorphism $\phi_m : \Gamma_k \to \mathbb{S}^1$ given by $\phi_m(\xi) = \xi^m$. Observe that the kernel of ϕ_m is the group Γ_t being $t = \gcd(k, m)$. Since Γ_k acts freely on $\mathbb{R}^N \smallsetminus \{0\}, \Sigma^\phi = \mathbb{S}^{N-1}$. We assert that assumption (Z_0) is satisfied for any $z \in \mathbb{S}^{N-1}$ provided $k > 4m$. Indeed, if $k > 4m$, then for $n = 0, 1, \ldots, k-1, \mathrm{Re}\big(\phi(\xi^{n+1})\overline{\phi(\xi^n)}\big) = \cos\frac{2\pi m}{k} > 0$ and so $\mu^\phi(\Gamma z) := |e^{\mathrm{i}\frac{2\pi}{k}} - 1| = 2\sin\frac{\pi}{k}$. Now, if $s, n \in \mathbb{N}$ are such that $0 \leq s < n \leq k-1$ and $\mathrm{Re}\big(\phi(\xi^n)\overline{\phi(\xi^s)}\big) = \cos\frac{2\pi m}{k}(n-s) < 0$, from $0 < \frac{2\pi m}{k} < \frac{\pi}{2}$, one has that $1 < n - s < k - 1$ and thus

$$|\xi^n z - \xi^s z| = |e^{\mathrm{i}\frac{2\pi}{k}(n-s)} - 1| = 2\sin\frac{\pi}{k}(n-s) > \mu^\phi(\Gamma z).$$

This proves the assertion. Hence, if, additionally, A and V satisfy (AV), Theorem 1.1 yields at least one solution to problem (1.1) satisfying (1.4) and (1.6).

Observe that Theorem 1.1 deals with the case in which $|A|^2 + V$ is nontrivial and takes nonnegative values. To our knowledge, this is the first existence result for this kind of potentials in the magnetic setting. The problem of existence without symmetries is open and seems to be nowhere studied in the literature.

Note also that the speed of convergence of $|A|^2 + V$ depends on the distance between the elements of a certain orbit of $\mathbb{R}^N$. Weaker conditions on the decay of the potentials require stronger conditions on the symmetries.

The argument we are going to apply to prove Theorem 1.1 follows the same pattern of that used in [8] to produce multiple solutions.

It would be interesting to establish an analogous result to Theorem 1.1 for $p > 2$. Unfortunately, in the case when $2 < p \leq 4$, the approach used in [8] to obtain the asymptotic estimates does not work, while when $p > 4$, the inequality $p < \frac{2N-\alpha}{N-2}$ holds only for $N = 3$ and $\alpha \in (0, 2)$; however, condition (Z_0) cannot be realized in dimension $N = 3$.

On the other hand, we remark that the techniques used in [8] can be applied to obtain Γ-invariant solutions to the magnetic problem (1.1). In fact, if $\phi \equiv 1$, defining $\mu_\Gamma := \inf_{z\in\Sigma^\phi} \mu^\phi(\Gamma z)$, we observe that [8, Theorem 1.3] can be extended to the magnetic setting in the following way.

Theorem 1.2. *Let $p \geq 2$ and $\ell(\Gamma) \geq 3$. If A and V satisfy (1.3) and the following holds*

(AV_1), *there exist $c_0 > 0$ and $\kappa > \mu_\Gamma\sqrt{V_\infty}$ such that*

$$\left||A(x)|^2 + V(x)\right| \leq c_0 e^{-\kappa|x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

then (1.1) has at least one solution u which is Γ-invariant and satisfies (1.6).

To prove this theorem, we just follow the same lines of the proof of [8, Theorem 1.3] taking into account Lemma 4.3 below.

As it is remarked in [8], when N is even, there are many groups satisfying the symmetry assumption in Theorem 1.2. Particularly, the group Γ_k in the above example satisfies $\ell(\Gamma) = k$. In contrast, when N is odd, not many groups satisfy $\ell(\Gamma) \geq 3$. For example, if $N = 3$, the only subgroups of $O(3)$ which satisfy this condition are the rotation groups of the icosahedron, octahedron, and tetrahedron, I, O and T, and the groups $I \times \mathbb{Z}_2^c, O \times \mathbb{Z}_2^c, T \times \mathbb{Z}_2^c$ and O^- described in [2, Appendix A].

Finally, we remark that it is possible to remove assumption (1.2) from the statement of Cingolani, Clapp, and Secchi's result [3, Theorem 1.2.] thanks to a recent analysis on the qualitative properties and decay asymptotics of the ground states of (1.5) given by Moroz and Van Schaftingen [17].

The outline of this paper is the following. In Sect. 2, we discuss the variational setting for problem (1.1). In Sect. 3, we collect some asymptotic estimates given in [8] to control the energy of the interaction between positive minimizers of (1.5) and we derive some others that are required to estimate the energy of the interaction between positive minimizers and the absolute value of the magnetic potential. Finally, Sect. 4 is devoted to a careful estimate of the energy of a suitable test function. This, combined with a result given by Cingolani et al. [3, Proposition 3.1], which establishes a lower bound for the lack of compactness of the variational functional associated with (1.1) in the appropriate symmetric subspaces of $H^1(\mathbb{R}^N)$, enables us to conclude that the infimum of the variational functional is attained on the symmetric Nehari manifold and so we get a vortex-type solution of problem (1.1).

2. The variational framework

From now on, we shall assume without loss of generality that $V_\infty = 1$. Let $\nabla_A u := \nabla u + \mathrm{i}Au$, and consider the real Hilbert space

$$H^1_A(\mathbb{R}^N, \mathbb{C}) := \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}^N)\right\}$$

with the scalar product

$$\langle u,v\rangle_{A,V} := \operatorname{Re}\int\limits_{\mathbb{R}^N} \left(\nabla_A u\cdot\overline{\nabla_A v} + (1+V(x))\, u\bar{v}\right). \tag{2.1}$$

Assumption (V_0) guarantees that the induced norm

$$\|u\|_{A,V} := \left(\int\limits_{\mathbb{R}^N} \left(|\nabla_A u|^2 + (1+V(x))\,|u|^2\right)\right)^{1/2}$$

is equivalent to the usual one, defined by taking $V\equiv 0$ [10, Definition 7.20]. If $A\equiv 0\equiv V$, we write $\langle u,v\rangle$ and $\|u\|$ instead of $\langle u,v\rangle_{0,0}$ and $\|u\|_{0,0}$.

We define

$$\mathbb{D}(u) := \int\limits_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}\mathrm{d}x\,\mathrm{d}y$$

and set $r := \frac{2N}{2N-\alpha}$. As $p\in(\frac{2N-\alpha}{N},\frac{2N-\alpha}{N-2})$, one has that $pr\in(2,\frac{2N}{N-2})$. Thus, the Hardy–Littlewood–Sobolev inequality [10, Theorem 4.3] implies the existence of a positive constant $\bar{C}=\bar{C}(\alpha,N)$ such that

$$\mathbb{D}(u)\le \bar{C}|u|_{pr}^{2p} \qquad \text{for all } \ u\in H_A^1(\mathbb{R}^N,\mathbb{C}),$$

where $|u|_q := \left(\int_{\mathbb{R}^N}|u|^q\right)^{1/q}$ is the norm in $L^q(\mathbb{R}^N)$. This proves that $\mathbb{D}$ is well defined.

We shall assume from now on that $p\in\left[2,\frac{2N-\alpha}{N-2}\right)$. Thus, the energy functional $J_{A,V}: H_A^1(\mathbb{R}^N,\mathbb{C})\to\mathbb{R}$ associated with problem (1.1), defined by

$$J_{A,V}(u) := \frac{1}{2}\|u\|_{A,V}^2 - \frac{1}{2p}\mathbb{D}(u)$$

is of class $\mathcal{C}^2$. Its derivative is given by

$$J'_{A,V}(u)v = \langle u,v\rangle_{A,V} - \operatorname{Re}\int\limits_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u\bar{v}.$$

Hence, the solutions to problem (1.1) are the critical points of $J_{A,V}$.

The homomorphism $\phi:\Gamma\to\mathbb{S}^1$ induces an orthogonal action of Γ on $H_A^1(\mathbb{R}^N,\mathbb{C})$ as follows: for $\gamma\in\Gamma$ and $u\in H_A^1(\mathbb{R}^N,\mathbb{C})$, we define $\gamma u\in H_A^1(\mathbb{R}^N,\mathbb{C})$ by

$$(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x).$$

Since the functional $J_{A,V}$ is Γ-invariant, the principle of symmetric criticality [20,27] guarantees that the critical points of the restriction of $J_{A,V}$ to the fixed point space of this action, namely

$$\begin{aligned} H_A^1(\mathbb{R}^N,\mathbb{C})^\phi :&= \left\{u\in H_A^1(\mathbb{R}^N,\mathbb{C}) : \gamma u = u \ \ \forall\gamma\in\Gamma\right\}\\ &= \left\{u\in H_A^1(\mathbb{R}^N,\mathbb{C}) : u(\gamma x)=\phi(\gamma)u(x) \ \ \forall\gamma\in\Gamma,\,\forall x\in\mathbb{R}^N\right\}, \end{aligned}$$

are the solutions to problem (1.1) that satisfy (1.4). The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}_{A,V}^\phi := \left\{u\in H_A^1(\mathbb{R}^N,\mathbb{C})^\phi : u\neq 0,\ \|u\|_{A,V}^2 = \mathbb{D}(u)\right\},$$

which is of class $\mathcal{C}^2$ and radially diffeomorphic to the unit sphere in $H_A^1(\mathbb{R}^N,\mathbb{C})^\phi$.

Now, the radial projection $\pi: H_A^1(\mathbb{R}^N,\mathbb{C})^\phi\smallsetminus\{0\}\longrightarrow\mathcal{N}_{A,V}^\phi$ onto the Nehari manifold is given by

$$\pi(u) := \left(\frac{\|u\|_{A,V}^2}{\mathbb{D}(u)}\right)^{\frac{1}{2(p-1)}} u$$

and so, for every $u \in H^1_A(\mathbb{R}^N,\mathbb{C})^\phi \smallsetminus \{0\}$, one has that

$$J_{A,V}(\pi(u)) = \frac{p-1}{2p}\left(\frac{\|u\|^2_{A,V}}{\mathbb{D}(u)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}. \tag{2.2}$$

We set

$$c^\phi_{A,V} := \inf_{\mathcal{N}^\phi_{A,V}} J_{A,V}.$$

For the limit problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{2.3}$$

we write $J_\infty, \mathcal{N}_\infty$, and c_∞ instead of $J_{0,0}, \mathcal{N}^\phi_{0,0}$ and $c^\phi_{0,0}$ with $\phi \equiv 1$.

It is known that c_∞ is attained at a positive function $\omega \in H^1(\mathbb{R}^N)$ (see for example [17, Theorem 3]). The behavior of the ground states to problem (2.3) was recently described in [3,17].

We denote by $\nabla J_{A,V}$ the gradient of $J_{A,V}$ with respect to the scalar product (2.1). We shall say that $J_{A,V} : H^1_A(\mathbb{R}^N,\mathbb{C})^\phi \to \mathbb{R}$ *satisfies condition* $(PS)^\phi_c$ if every sequence (u_n) such that

$$u_n \in H^1_A(\mathbb{R}^N,\mathbb{C})^\phi, \qquad J_{A,V}(u_n) \to c, \qquad \nabla J_{A,V}(u_n) \to 0,$$

contains a convergent subsequence in $H^1_A(\mathbb{R}^N,\mathbb{C})$.

Proposition 2.1. *$J_{A,V} : H^1_A(\mathbb{R}^N,\mathbb{C})^\phi \to \mathbb{R}$ satisfies condition $(PS)^\phi_c$ for all*

$$c < \ell(\Gamma)\, c_\infty.$$

Proof. See Proposition 3.1 in [3] with $G = \Gamma$ and $\tau = \phi$. □

We write $\nabla_{\mathcal{N}} J_{A,V}(u)$ for the orthogonal projection of $\nabla J_{A,V}(u)$ onto the tangent space $T_u\mathcal{N}^\phi_{A,V}$ to the Nehari manifold $\mathcal{N}^\phi_{A,V}$ at the point $u \in \mathcal{N}^\phi_{A,V}$. We say that $J_{A,V}$ *satisfies condition* $(PS)^\phi_c$ *on* $\mathcal{N}^\phi_{A,V}$ if every sequence (u_n) such that

$$u_n \in \mathcal{N}^\phi_{A,V}, \qquad J_{A,V}(u_n) \to c, \qquad \nabla_{\mathcal{N}} J_{A,V}(u_n) \to 0,$$

has a convergent subsequence in $H^1_A(\mathbb{R}^N,\mathbb{C})$.

Corollary 2.2. *$J_{A,V}$ satisfies condition $(PS)^\phi_c$ on $\mathcal{N}^\phi_{A,V}$ for all*

$$c < \ell(\Gamma) c_\infty.$$

Proof. The proof is similar to that of Corollary 3.2 in [7]. □

3. Preliminary asymptotic estimates

In what follows, ω will denote a positive ground state of problem (2.3) which is radially symmetric with respect to the origin. The existence, qualitative properties, and decay asymptotics of ω have been recently studied in [3,17]. In particular, it is known that $\omega \in L^1(\mathbb{R}^N) \cap \mathcal{C}^\infty(\mathbb{R}^N)$ and that ω is monotone decreasing in the radial direction with respect to the origin. Moreover, from [17, Theorem 4], it can be deduced that ω has the following asymptotic behavior:

Lemma 3.1.

$$\lim_{|x|\to\infty} \omega(x)|x|^{\frac{N-1}{2}} e^{a|x|} = \begin{cases} \infty & \text{if } a > 1, \\ 0 & \text{if } a \in (0,1). \end{cases}$$

Proof. See Lemma 3.2 in [8]. ☐

For $\zeta \in \mathbb{R}^N$, we define

$$w_\zeta(x) := w(x-\zeta) \qquad I(\zeta) := \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega^p\right) \omega^{p-1}\omega_\zeta. \tag{3.1}$$

In the next proposition, we collect some asymptotic estimates that were proved in [8] and that are going to be useful to prove Theorem 1.1.

Proposition 3.2. *Set $b := \frac{N-1}{2}$. The following hold*

(1) For each $a \in (0,1)$,

$$\lim_{|\zeta|\to\infty} \int_{\mathbb{R}^N} \omega\, \omega_\zeta |\zeta|^b e^{a|\zeta|} = 0 \tag{3.2}$$

$$\lim_{|\zeta|\to\infty} I(\zeta)\, |\zeta|^b\, e^{a|\zeta|} = 0. \tag{3.3}$$

(2) For every $a > 1$, there exists a positive constant k_a such that

$$I(\zeta)|\zeta|^b e^{a|\zeta|} \geq k_a \qquad \text{for all } |\zeta| \geq 1. \tag{3.4}$$

Lemma 3.3. *Let $M \in (0,2)$ and $z, z' \in \mathbb{R}^N$ with $|z| = |z'| = 1$. If $|A(x)|, \big||A(x)|^2 + V(x)\big| \leq ce^{-k|x|}$ for all $x \in \mathbb{R}^N$ with $c > 0$ and $k > M$, then*

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} \big||A(x)|^2 + V(x)\big|\, \omega_{Rz}\, \omega_{Rz'}\, \mathrm{d}x\, R^{\frac{N-1}{2}} e^{MR} = 0 \tag{3.5}$$

and

$$\lim_{R\to\infty} \int_{\mathbb{R}^N} |A(x)|\, \omega_{Rz}\, |\nabla \omega_{Rz'}|\, \mathrm{d}x\, R^{\frac{N-1}{2}} e^{MR} = 0. \tag{3.6}$$

Proof. Fix $\nu \in (0,1)$ such that $M < 2\nu < k$. Lemma 3.1 insures the existence of a positive constant C_ν such that

$$\omega(x) \leq C_\nu e^{-\nu|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^N} \big||A(x)|^2 + V(x)\big|\, \omega_{Rz}\, \omega_{Rz'}\, \mathrm{d}x &\leq C \int_{\mathbb{R}^N} e^{-k|x|} e^{-\nu|x-Rz|} e^{-\nu|x-Rz'|}\, \mathrm{d}x \\
&= C \int_{\mathbb{R}^N} e^{-\nu(|x|+|x-Rz|)} e^{-\nu(|x|+|x-Rz'|)} e^{-(k-2\nu)|x|}\, \mathrm{d}x \\
&\leq Ce^{-2\nu R} \int_{\mathbb{R}^N} e^{-(k-2\nu)|x|}\, \mathrm{d}x \\
&= Ce^{-2\nu R},
\end{aligned}$$

where C denotes different positive constants depending only on ν.

Consequently,

$$0 \leq \int_{\mathbb{R}^N} \big||A(x)|^2 + V(x)\big|\, \omega_{Rz}\, \omega_{Rz'}\, \mathrm{d}x\, R^{\frac{N-1}{2}} e^{MR} \leq CR^{\frac{N-1}{2}} e^{-(2\nu-M)R}.$$

This implies (3.5).

On the other hand, by Lemma 3.1 and Proposition A.2 in [3], we have that, for each $\nu \in (0,1)$, there exists a constant $C_\nu > 0$ such that

$$|\nabla \omega(x)| \leq C_\nu e^{-\nu|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

This, combined with the decay assumption on $|A|$, allows us to argue as above to prove (3.6). □

4. The existence of a vortex-type solution

Let $\phi : \Gamma \to \mathbb{S}^1$ be a continuous group homomorphism. Let $\omega \in H^1(\mathbb{R}^N)$ be a positive ground state of problem (2.3) which is radially symmetric about the origin. Thus, for $z \in \Sigma^\phi$ and $R > 0$, the function

$$\sigma_{Rz} := \sum_{gz \in \Gamma z} \phi(g)\omega_{Rgz}, \qquad \text{where} \quad \omega_\zeta(x) := \omega(x - \zeta),$$

is well defined and satisfies (1.4).

We shall prove the following result.

Proposition 4.1. *Let $p = 2$. If A and V satisfy (AV) and (Z_0) holds, then there exist $C_0, R_0 > 0$ and $\beta > 1$ such that*

$$\frac{\|\sigma_{Rz}\|^2_{A,V}}{\mathbb{D}(\sigma_{Rz})^{\frac{1}{p}}} \leq \left(\ell(\Gamma)\, \|\omega\|^2\right)^{\frac{p-1}{p}} - C_0 e^{-\beta R} \quad \textit{for any}\, R \geq R_0. \tag{4.1}$$

Consequently, $c^\phi_{A,V} < \ell(\Gamma)\, c_\infty$.

To prove this proposition, we require some additional asymptotic estimates which will be derived from the results in the previous section.

Observe that, since ω is a solution of problem (2.3), for any $z, z' \in \mathbb{R}^N$, one has that $J'_\infty(\omega_z)\omega_{z'} = 0$, which is equivalent to

$$\int_{\mathbb{R}^N} [\nabla\omega_z \cdot \nabla\omega_{z'} + \omega_z\omega_{z'}] = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega_z^p\right) \omega_z^{p-1}\omega_{z'}.$$

Performing a change of variable in the right-hand side of this inequality, one can express it as

$$\langle \omega_z, \omega_{z'} \rangle = I(z' - z) \qquad \text{for all} \quad z, z' \in \mathbb{R}^N, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $H^1_A(\mathbb{R}^N, \mathbb{C})$ with $A \equiv 0$ and I is the function defined in (3.1). We denote by $Fz := \{(gz, hz) \in \Gamma z \times \Gamma z : gz \neq hz\}$ and define

$$\varepsilon_{Rz} := \sum_{\substack{(gz,hz) \in Fz \\ \mathrm{Re}\left(\phi(g)\overline{\phi(h)}\right) > 0}} \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) I(Rgz - Rhz),$$

$$\widehat{\varepsilon}_{Rz} := - \sum_{\substack{(gz,hz) \in Fz \\ \mathrm{Re}\left(\phi(g)\overline{\phi(h)}\right) < 0}} \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) I(Rgz - Rhz) \quad \text{if } \phi \not\equiv 1,$$

and $\widehat{\varepsilon}_{Rz} := 0$ if $\phi \equiv 1$. Let $z \in \Sigma^\phi$ be as in condition (Z_0). We choose $g_z, h_z \in \Gamma$ such that

$$|g_z z - h_z z| = \mu^\phi(\Gamma z) := \min\{|gz - hz| : g, h \in \Gamma,\ gz \neq hz,\, \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) > 0\}$$

and set

$$\xi_z := g_z z - h_z z.$$

The proof of the following lemma is similar to that of Lemma 5.5 in [8]. However, we include it here for the sake of completeness.

Lemma 4.2. *If* (Z_0) *holds, then*

$$\widehat{\varepsilon}_{Rz} = o\left(\varepsilon_{Rz}\right).$$

Proof. Let $a_0 > 1$ be as in condition (Z_0). Choosing $\widehat{a} \in (0, 1)$ such that $a := \widehat{a}a_0 > 1$, we obtain that $a\,|\xi_z| = a\mu^{\phi}(\Gamma z) \leq \widehat{a}\,|gz - hz|$ for any $g, h \in \Gamma$ with $gz \neq hz$ and $\mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big) < 0$. From (3.4), it follows that there exists a constant $k_a > 0$ such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \qquad \text{if} \;\; R \geq \mu^{\phi}(\Gamma z)^{-1},$$

where $b := \frac{N-1}{2}$. So, setting $C := k_a^{-1}$, we get

$$\begin{aligned}\frac{I(Rgz - Rhz)}{I(R\xi_z)} &\leq \frac{I(Rgz - Rhz)\,|Rgz - Rhz|^b\, e^{\widehat{a}|Rgz - Rhz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C I(Rgz - Rhz)\,|Rgz - Rhz|^b\, e^{\widehat{a}|Rgz-Rhz|} \qquad \text{if} \;\; R \geq \mu^{\phi}(\Gamma z)^{-1}.\end{aligned}$$

Let $\varepsilon > 0$. By (3.3), there exists $S > 0$ such that $I(\zeta)\,|\zeta|^b\, e^{\widehat{a}|\zeta|} < \varepsilon$ if $|\zeta| > S$. Since $\widehat{a}\,|Rgz - Rhz| \geq Ra\mu^{\phi}(\Gamma z) > 0$, we may take $R_0 := \max\{\frac{\widehat{a}S}{a\mu^{\phi}(\Gamma z)}, \mu^{\phi}(\Gamma z)^{-1}\}$ to conclude that

$$0 \leq \frac{\widehat{\varepsilon}_{Rz}}{\varepsilon_{Rz}} \leq \sum_{\substack{gz \neq hz \in \Gamma z \\ \mathrm{Re}\left(\phi(g)\overline{\phi(h)}\right) < 0}} \frac{I(Rgz - Rhz)}{I(R\xi_z)} \leq \ell(\Gamma)^2 C\varepsilon \qquad \text{if } R \geq R_0.$$

□

Lemma 4.3. *If* A *and* V *satisfy* (AV) *and* (Z_0) *holds, then*

$$\sum_{(gz,hz)\in\Gamma z\times\Gamma z} \int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz} = o(\varepsilon_{Rz}).$$

Proof. Assumption (Z_0) guarantees that $\mu^{\phi}(\Gamma z) < 2$. Let $\kappa > \mu^{\phi}(\Gamma z)$ be as in assumption (AV) (recall that $V_\infty = 1$ is assumed). We fix $a > 1$ such that $M := a\mu^{\phi}(\Gamma z) < \min\{2, \kappa\}$. By (3.4), there exists a positive constant k_a such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \qquad \text{if} \;\; R \geq \mu^{\phi}(\Gamma z)^{-1},$$

where $b := \frac{N-1}{2}$. Since $MR = aR\mu^{\phi}(\Gamma z) = a|R\xi_z|$, we have that

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz}}{\varepsilon_{Rz}} &\leq C \sum_{gz\in\Gamma z} \frac{\int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz}}{I(R\xi_z)} \\ &\leq C \sum_{gz\in\Gamma z} \frac{\int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz} R^b e^{MR}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C \sum_{gz\in\Gamma z} \int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz} R^b e^{MR},\end{aligned}$$

if $R \geq \mu^{\phi}(\Gamma z)^{-1}$. Here, C denotes distinct positive constants. Taking (3.5) into account, we get

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} \left||A|^2 + V\right| \omega_{Rgz}\omega_{Rhz}}{\varepsilon_{Rz}} = 0$$

as claimed. □

Lemma 4.4. *If* A *and* V *satisfy* (AV) *and* (Z_0) *holds, then*

$$\sum_{(gz,hz)\in\Gamma z\times\Gamma z} Im\big(\phi(g)\overline{\phi(h)}\big)\; Im\left(\int_{\mathbb{R}^N} \nabla_A\omega_{Rgz} \cdot \overline{\nabla_A\omega_{Rhz}}\right) = o(\varepsilon_{Rz}).$$

Proof. First note that

$$\mathrm{Im}\left(\int_{\mathbb{R}^N} \nabla_A\omega_{Rgz}\cdot\overline{\nabla_A\omega_{Rhz}}\right)=\int_{\mathbb{R}^N}\omega_{Rgz}\,A\cdot\nabla\omega_{Rhz}-\int_{\mathbb{R}^N}\omega_{Rhz}\,A\cdot\nabla\omega_{Rgz}$$

and so, it is enough to prove that

$$\left|\int_{\mathbb{R}^N}\omega_{Rgz}\,A\cdot\nabla\omega_{Rhz}\right|=o(\varepsilon_{Rz}).$$

Since

$$\left|\int_{\mathbb{R}^N}\omega_{Rgz}\,A\cdot\nabla\omega_{Rhz}\right|\leq\int_{\mathbb{R}^N}|A|\,\omega_{Rgz}\,|\nabla\omega_{Rhz}|$$

and (3.6) holds, we can argue as in the proof of Lemma 4.3 to get the conclusion. □

Lemma 4.5. *Let $g,h,\eta,\gamma\in\Gamma$ be such that $Re\big(\phi(\eta)\overline{\phi(\gamma)}\big)<0$. If (Z_0) holds, then*

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*\big(\omega_{Rgz}\,\omega_{Rhz}\big)\right)\omega_{R\eta z}\,\omega_{R\gamma z}=o(\varepsilon_{Rz}).$$

Proof. Since $\frac{1}{|x|^\alpha}*\big(\omega_{Rgz}\,\omega_{Rhz}\big)\in L^\infty(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*\Big(\omega_{Rgz}\,\omega_{Rhz}\Big)\right)\omega_{R\eta z}\,\omega_{R\gamma z}\leq C\int_{\mathbb{R}^N}\omega_{R\eta z}\,\omega_{R\gamma z}=C\int_{\mathbb{R}^N}\omega\,\omega_{R\eta z-R\gamma z}.$$

From (3.2), it follows that, for $\epsilon>0$ given,

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\alpha}*\Big(\omega_{Rgz}\,\omega_{Rhz}\Big)\right)\omega_{R\eta z}\,\omega_{R\gamma z}|R\eta z-R\gamma z|e^{\hat{a}|R\eta z-R\gamma z|}<\epsilon$$

provided $R>\mu^\phi(\Gamma z)^{-1}$ and $\hat{a}\in(0,1)$. Therefore, if $\mathrm{Re}\big(\phi(\eta)\overline{\phi(\gamma)}\big)<0$, we may argue as in the proof of Lemma 4.2 to get the conclusion. □

Finally, we need the following result.

Lemma 4.6. *Let $\psi:(0,\infty)\to\mathbb{R}$ be the function given by*

$$\psi(t):=\frac{a+t+o(t)}{(a+bt+o(t))^\beta},$$

where $a>0,\beta\in(0,1)$ and $b\beta>1$. Then, there exist constants $C_0,t_0>0$ such that

$$\psi(t)\leq a^{1-\beta}-C_0t\quad\text{for all } t\in(0,t_0).$$

Proof. See Lemma 5.9 in [8]. □

Lemma 4.7. *If A and V satisfy (AV) and (Z_0) holds, then*

$$\|\sigma_{Rz}\|^2_{A,V}\leq\ell(\Gamma)\,\|\omega\|^2+\varepsilon_{Rz}+o(\varepsilon_{Rz}).$$

Proof. From

$$\langle \phi(g)\omega_{Rgz}, \phi(h)\omega_{Rhz}\rangle_{A,V} = \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\langle \omega_{Rgz}, \omega_{Rhz}\rangle_{A,V} - \mathrm{Im}\big(\phi(g)\overline{\phi(h)}\big)\ \mathrm{Im}\int\limits_{\mathbb{R}^N} \nabla_A\omega_{Rgz}\cdot\overline{\nabla_A\omega_{Rhz}}$$

and

$$\langle \omega_{Rgz}, \omega_{Rhz}\rangle_{A,V} = \langle \omega_{Rgz}, \omega_{Rhz}\rangle + \int\limits_{\mathbb{R}^N} \left(|A|^2 + V(x)\right)\omega_{Rgz}\omega_{Rhz},$$

we obtain that

$$\begin{aligned}
\|\sigma_{Rz}\|_{A,V}^2 &= \left\langle \sum_{gz\in\Gamma z}\phi(g)\omega_{Rgz}, \sum_{hz\in\Gamma z}\phi(h)\omega_{Rhz}\right\rangle_{A,V}\\
&= \sum_{(gz,hz)\in\Gamma z\times\Gamma z} \langle \phi(g)\omega_{Rgz}, \phi(h)\omega_{Rhz}\rangle_{A,V}\\
&= \sum_{(gz,hz)\in\Gamma z\times\Gamma z} \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\langle \omega_{Rgz}, \omega_{Rhz}\rangle\\
&\quad + \sum_{(gz,hz)\in\Gamma z\times\Gamma z} \mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\int\limits_{\mathbb{R}^N}\left(|A|^2+V(x)\right)\omega_{Rgz}\omega_{Rhz}\\
&\quad - \sum_{(gz,hz)\in\Gamma z\times\Gamma z} \mathrm{Im}\big(\phi(g)\overline{\phi(h)}\big)\ \mathrm{Im}\int\limits_{\mathbb{R}^N}\nabla_A\omega_{Rgz}\cdot\overline{\nabla_A\omega_{Rhz}}.
\end{aligned}$$

Taking into account Lemmas 4.2, 4.3, and 4.4, we deduce that

$$\|\sigma_{Rz}\|_{A,V}^2 \le \ell(\Gamma)\,\|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz}).$$

□

Lemma 4.8. *Let $p = 2$. If (Z_0) holds, then*

$$\mathbb{D}(\sigma_{Rz}) \ge \ell(\Gamma)\,\mathbb{D}(\omega) + 4\,\varepsilon_{Rz} + o(\varepsilon_{Rz}).$$

Proof. Let z be as in assumption (Z_0). Recall that if $z_1, \ldots, z_n \in \mathbb{C}$, then

$$\left|\sum_{j=1}^n z_j\right|^2 = \sum_{j=1}^n |z_j|^2 + 2\sum_{k=1}^{n-1}\sum_{j=k+1}^n \mathrm{Re}\left(z_k\overline{z_j}\right).$$

Hence,

$$\left|\sum_{gz\in\Gamma z}\phi(g)\omega_{Rgz}\right|^2 = \sum_{gz\in\Gamma z}\omega_{Rgz}^2 + \sum_{(gz,hz)\in Fz}\mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\omega_{Rgz}\,\omega_{Rhz}.$$

Observe that

$$\begin{aligned}
\left|\sum_{gz\in\Gamma z}\phi(g)\omega_{Rgz}(x)\right|^2 \left|\sum_{gz\in\Gamma z}\phi(g)\omega_{Rgz}(y)\right|^2 &\ge \sum_{gz\in\Gamma z}\omega_{Rgz}^2(x)\,\omega_{Rgz}^2(y)\\
&\quad +2\sum_{(gz,hz)\in Fz}\mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\,\omega_{Rgz}^2(x)\,\omega_{Rgz}(y)\,\omega_{Rhz}(y)\\
&\quad +2\sum_{(gz,hz)\in Fz}\mathrm{Re}\big(\phi(g)\overline{\phi(h)}\big)\,\omega_{Rgz}^2(y)\,\omega_{Rgz}(x)\,\omega_{Rhz}(x)
\end{aligned}$$

$$-C_1 \sum_{\substack{\mathrm{Re}(\phi(g)\overline{\phi(h)})>0 \\ \mathrm{Re}(\phi(\eta)\overline{\phi(\gamma)})<0}} \omega_{Rgz}(x)\,\omega_{Rhz}(x)\,\omega_{R\eta z}(y)\,\omega_{R\gamma z}(y)$$
$$-C_2 \sum_{\substack{\mathrm{Re}(\phi(g)\overline{\phi(h)})<0 \\ \mathrm{Re}(\phi(\eta)\overline{\phi(\gamma)})>0}} \omega_{Rgz}(x)\,\omega_{Rhz}(x)\,\omega_{R\eta z}(y)\,\omega_{R\gamma z}(y),$$

where C_1, C_2 are positive constants. Therefore, using Lemmas 4.2 and 4.5, we conclude that

$$\mathbb{D}(\sigma_{Rz}) \geq \ell(\Gamma)\,\mathbb{D}(\omega) + 4\,\varepsilon_{Rz} + o(\varepsilon_{Rz}).$$

□

Proof of Proposition 4.1. Lemmas 4.7 and 4.8 yield

$$\frac{\|\sigma_{Rz}\|^2_{A,V}}{\mathbb{D}(\sigma_{Rz})^{\frac{1}{2}}} \leq \frac{\ell(\Gamma)\,\|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz})}{(\ell(\Gamma)\mathbb{D}(\omega) + 4\,\varepsilon_{Rz} + o(\varepsilon_{Rz}))^{\frac{1}{2}}}.$$

Consequently, since $\|\omega\|^2 = \mathbb{D}(\omega)$ and $\varepsilon_{Rz} \to 0$ as $R \to \infty$, the assumptions of Lemma 4.6 are satisfied and so there exist $c_1, R_1 > 0$ such that

$$\frac{\|\sigma_{Rz}\|^2_{A,V}}{\mathbb{D}(\sigma_{Rz})^{\frac{1}{2}}} \leq \left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{1}{2}} - c_1\varepsilon_{Rz}$$

for $R \geq R_1$. Using (3.4), we conclude that there exist $C_0, R_0 > 0$ and $\beta > 1$ such that

$$\frac{\|\sigma_{Rz}\|^2_{A,V}}{\mathbb{D}(\sigma_{Rz})^{\frac{1}{2}}} \leq \left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{1}{2}} - C_0 e^{-\beta R} \quad \text{for any} \quad R \geq R_0,$$

which is inequality (4.1). Finally, since $\pi\sigma_{Rz} \in \mathcal{N}^{\phi}_{A,V}$ and (2.2) implies

$$J_{A,V}(\pi(\sigma_{Rz})) = \frac{1}{4}\left(\frac{\|\sigma_{Rz}\|^2_{A,V}}{\mathbb{D}(\sigma_{Rz})^{\frac{1}{2}}}\right)^2 < \frac{1}{4}\,\ell(\Gamma)\,\|\omega\|^2 = \ell(\Gamma)\,c_\infty,$$

one has that $c^{\phi}_{A,V} < \ell(\Gamma)\,c_\infty$. □

Proof of Theorem 1.1. Proposition 4.1 guarantees that

$$c^{\phi}_{A,V} := \inf_{\mathcal{N}^{\phi}_{A,V}} J_{A,V} < \ell(\Gamma)c_\infty.$$

Corollary 2.2 implies that this infimum is attained. □

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