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Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials

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ABSTRACT

We study the boundary behavior of positive functions u satisfying (E) $-\Delta u-\frac{\kappa}{d^2(x)}u+g(u)=0$ in a bounded domain Ω of $\mathbb{R}^N$, where $0<\kappa\leq\frac{1}{4}$, g is a continuous nondecreasing function and $d(.)$ is the distance function to $\partial\Omega$. We first construct the Martin kernel associated to the linear operator $\mathcal{L}_\kappa=-\Delta-\frac{\kappa}{d^2(x)}$ and give a general condition for solving equation (E) with any Radon measure μ for boundary data. When $g(u)=|u|^{q-1}u$ we show the existence of a critical exponent $q_c=q_c(N,\kappa)>1$ with the following properties: when $0<q<q_c$ any measure is eligible for solving (E) with μ for boundary data; if $q\geq q_c$, a necessary and sufficient condition is expressed in terms of the absolute continuity of μ with respect to some Besov capacity. The same capacity characterizes the removable compact boundary sets. At end any positive solution (F) $-\Delta u-\frac{\kappa}{d^2(x)}u+|u|^{q-1}u=0$ with $q>1$ admits a boundary trace which is a positive outer regular Borel measure. When $1<q<q_c$ we prove that to any positive outer regular Borel measure we can associate a positive solutions of (F) with this boundary trace.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $d(x)=\operatorname{dist}(x,\Omega^c)$. In this article we study several aspects of the nonlinear boundary value associated to the equation

$$-\Delta u-\frac{\kappa}{d^2(x)}u+|u|^{p-1}u=0\quad\text{in }\Omega,\tag{1.1}$$

where $p>1$. The study of the boundary trace of solutions of (1.1) is a natural framework for a general study of several nonlinear problems where the nonlinearity, the geometric properties of the domain and the coefficient κ interact. On this point of view, the case $\kappa=0$ has been thoroughly treated by Marcus and Véron (e.g. [23,24,26,25] and the synthesis presented in [31] and [27]). The associated linear Schrödinger operator

$$u\mapsto\mathcal{L}_\kappa u:=-\Delta u-\frac{\kappa}{d^2(x)}u\tag{1.2}$$

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plays an important role in functional analysis because of the particular singularity of the potential $V(x) := -\frac{\kappa}{d^2(x)}$. The case $\kappa < 0$ and more generally of nonnegative potentials has been studied by Ancona [2] who has shown the existence of a Martin kernel which allows a general representation formula of nonnegative solutions of

$$\mathcal{L}_\kappa u = 0 \quad \text{in } \Omega. \tag{1.3}$$

When $\kappa < \frac{1}{4}$, Ancona proved that $\mathcal{L}_\kappa$ is *weakly coercive* in $H^1_0(\Omega)$. Thus any positive solution u of (1.3) admits a representation under the form

$$u(x) = \int_{\partial\Omega} K_{\mathcal{L}_\kappa}(x,\xi)d\mu(\xi) \quad \text{in } \Omega, \tag{1.4}$$

see [2, Remark p. 523]. Furthermore the kernel $K_{\mathcal{L}_\kappa}(x,\xi)$ with pole at ξ is unique up to a multiplication [2, Th 3]. When $\kappa = \frac{1}{4}$, then $\mathcal{L}_\kappa$ is no longer weakly coercive in $H^1_0(\Omega)$ and Ancona's results cannot be applied.
Ancona's representation theorem turned out to be the key ingredient of the full classification of positive solutions of

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.5}$$

which was obtained by Marcus [20]. In a more general setting, Véron and Yarur [32] constructed a capacitary theory associated to the linear equation

$$\mathcal{L}_V u := -\Delta u + V(x)u = 0 \quad \text{in } \Omega, \tag{1.6}$$

where the potential V is nonnegative and singular near $\partial\Omega$. When $V(x) := -\frac{\kappa}{d^2(x)}$ with $\kappa > 0$, V is called a Hardy potential. There is a critical value $\kappa = \frac{1}{4}$. If $\kappa > \frac{1}{4}$, no positive solution of (1.3) exists. When $0 < \kappa \le \frac{1}{4}$, there exist positive solutions, and the geometry of the domain plays a fundamental role in the study of the mere linear equation (1.3). We define the constant c_Ω by

$$c_\Omega = \inf_{v \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega \frac{v^2}{d^2(x)}dx}. \tag{1.7}$$

It is known that $0 < c_\Omega \le \frac{1}{4}$, and if Ω is convex then $c_\Omega = \frac{1}{4}$ (see [21] and [6] for related results). When $0 < \kappa \le \frac{1}{4}$, *which is always assumed in the sequel* and $-\Delta d \ge 0$ in the sense of distributions, it is possible to define the first eigenvalue λ_κ of the operator $\mathcal{L}_\kappa$. If we define the two fundamental exponents α_+ and α_- by

$$\alpha_+ = 1 + \sqrt{1-4\kappa} \quad \text{and} \quad \alpha_- = 1 - \sqrt{1-4\kappa}, \tag{1.8}$$

then the first eigenvalue is achieved by an eigenfunction ϕ_κ which satisfies $\phi_\kappa(x) \approx d^{\frac{\alpha_+}{2}}(x)$ as $d(x) \to 0$. Similarly, the Green kernel $G_{\mathcal{L}_\kappa}$ associated to $\mathcal{L}_\kappa$ inherits this type of boundary behavior since there holds

$$\frac{1}{C_\kappa}\min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha_+}{2}}(x)d^{\frac{\alpha_+}{2}}(y)}{|x-y|^{N+\alpha_+-2}}\right\} \le G_{\mathcal{L}_\kappa}(x,y) \le C_\kappa \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha_+}{2}}(x)d^{\frac{\alpha_+}{2}}(y)}{|x-y|^{N+\alpha_+-2}}\right\}. \tag{1.9}$$

We show that $\mathcal{L}_\kappa$ satisfies the maximum principle in the sense that if $u \in H^1_{loc} \cap C(\Omega)$ is a subsolution, i.e. $\mathcal{L}_\kappa u \le 0$, such that

$$\begin{aligned} &\text{(i)} \quad \limsup_{x\to y} \frac{u(x)}{d^{\alpha_-}(x)} \le 0 && \text{if } 0 < \kappa < \frac{1}{4}, \\ &\text{(ii)} \quad \limsup_{x\to y} \frac{u(x)}{\sqrt{d(x)}|\ln d(x)|} \le 0 && \text{if } \kappa = \frac{1}{4}, \end{aligned} \tag{1.10}$$

for all $y \in \partial\Omega$, then $u \le 0$. This result has to be compared with the result on the existence of positive sub-harmonic functions in Ω given in [3, Theorem 2. 3] which is associated to the maximum principle in neighborhood of $\partial\Omega$ stated in [3, Lemma 2.4].

If $\xi \in \partial\Omega$ and $r > 0$, we set $\Delta_r(\xi) = \partial\Omega \cap B_r(\xi)$. We prove that a positive solution of $\mathcal{L}_\kappa u = 0$ which vanishes on a part of the boundary in the sense that

$$\begin{aligned} &\text{(i)} \quad \lim_{x\to y} \frac{u(x)}{d^{\alpha_-}(x)} = 0 && \forall y \in \Delta_r(\xi) \text{ if } 0 < \kappa < \frac{1}{4}, \\ &\text{(ii)} \quad \lim_{x\to y} \frac{u(x)}{\sqrt{d(x)}|\ln d(x)|} = 0 && \forall y \in \Delta_r(\xi) \text{ if } \kappa = \frac{1}{4}, \end{aligned} \tag{1.11}$$

satisfies

$$\frac{u(x)}{\phi_\kappa(x)} \le C_1 \frac{u(y)}{\phi_\kappa(y)} \quad \forall x, y \in \Delta_{\frac{r}{2}}(\xi), \tag{1.12}$$

for some $C_1 = C_1(\Omega,\kappa) > 0$.

For any $h \in C(\partial\Omega)$ we construct the unique solution $v := v_h$ of the Dirichlet problem

$$\begin{aligned} \mathcal{L}_\kappa v &= 0 \quad \text{in } \Omega \\ v &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{1.13}$$

noting that the boundary data h is achieved in the sense that

$$\lim_{x\to y} \frac{u(x)}{d^{\alpha_-}(x)} = h(y) \quad \text{if } 0 < \kappa < \frac{1}{4} \quad \text{and} \quad \lim_{x\to y} \frac{u(x)}{\sqrt{d(x)}|\ln d(x)|} = h(y) \quad \text{if } \kappa = \frac{1}{4}.$$

Using this construction and estimates (1.10) we show the existence of the $\mathcal{L}_\kappa$-measure, which is a bounded Borel measure ω^x with the property that for any $h \in C(\partial\Omega)$, the above function v_h satisfies

$$v_h(x) = \int_{\partial\Omega} h(y)d\omega^x(y). \tag{1.14}$$

Because of Harnack inequality, the measures ω^x and ω^z are mutually absolutely continuous for $x, z \in \Omega$, and for any $x \in \Omega$ we can define the Radon–Nikodym derivative

$$K(x, y) := \frac{d\omega^x}{d\omega^{x_0}}(y) \quad \text{for } \omega^{x_0}\text{-almost } y \in \partial\Omega. \tag{1.15}$$

There exists $r_0 := r_0(\Omega)$ such that for any $x \in \Omega$ verifying $d(x) \le r_0$, there exists a unique $\xi = \xi_x \in \partial\Omega$ with the property that $d(x) = |x - \xi_x|$. If we denote by Ω'_{r_0} the set of $x \in \Omega$ such that $0 < d(x) < r_0$, the mapping Π from $\overline{\Omega}'_{r_0}$ to $[0, r_0] \times \partial\Omega$ defined by $\Pi(x) = (d(x), \xi_x)$ is a C^1 diffeomorphism. If $\xi \in \partial\Omega$ and $0 \le r \le r_0$, we set $x_r(\xi) = \Pi^{-1}(r, \xi)$. Let W be defined in Ω by

$$W(x) = \begin{cases} d^{\frac{\alpha_-}{2}}(x) & \text{if } \kappa < \frac{1}{4}, \\ \sqrt{d(x)}|\ln d(x)| & \text{if } \kappa = \frac{1}{4}. \end{cases} \tag{1.16}$$

We prove that the $\mathcal{L}_\kappa$-harmonic measure can be equivalently defined by

$$\omega^x(E) = \inf\left\{\psi : \psi \in C_+(\Omega),\ \mathcal{L}_\kappa\text{-superharmonic in } \Omega \text{ and s.t. } \liminf_{x\to E} \frac{\psi(x)}{W(x)} \ge 1\right\}, \tag{1.17}$$

on compact sets $E \subset \partial\Omega$ and then extended by regularity to Borel subsets of $\partial\Omega$.

The $\mathcal{L}_\kappa$-harmonic measure is connected to the Green kernel of $\mathcal{L}_\kappa$ by the following estimates:

Theorem A. *There exists $C_3 := C_3(\Omega) > 0$ such that for any $r \in (0, r_0]$ and $\xi \in \partial\Omega$, there holds*

$$\begin{aligned} \frac{1}{C_3} r^{N+\frac{\alpha_-}{2}-2} G_{\mathcal{L}_\kappa}(x_r(\xi), x) &\le \omega^x(\Delta_r(\xi)) \\ &\le C_3 r^{N+\frac{\alpha_-}{2}-2} G_{\mathcal{L}_\kappa}(x_r(\xi), x) \quad \forall x \in \Omega \setminus B_{4r(\xi)}, \end{aligned} \tag{1.18}$$

if $0 < \kappa < \frac{1}{4}$, *and*

$$\begin{aligned} \frac{1}{C_3} r^{N-2+\frac{1}{2}} |\ln d(x)| G_{\mathcal{L}_{\frac{1}{4}}}(x_r(\xi), x) &\le \omega^x(\Delta_r(\xi)) \\ &\le C_3 r^{N-2+\frac{1}{2}} |\ln d(x)| G_{\mathcal{L}_{\frac{1}{4}}}(x_r(\xi), x) \quad \forall x \in \Omega \setminus B_{4r(\xi)}, \end{aligned} \tag{1.19}$$

when $\kappa = \frac{1}{4}$.

As a consequence ω^x has the doubling property. The previous estimates allow to construct a kernel function of $\mathcal{L}_\kappa$ in Ω, prove its uniqueness up to an homothety. When normalized, the kernel function denoted by $K_{\mathcal{L}_\kappa}$ is the Martin kernel, defined by

$$K_{\mathcal{L}_\kappa}(x, \xi) = \lim_{x\to\xi} \frac{G_{\mathcal{L}_\kappa}(x, y)}{G_{\mathcal{L}_\kappa}(x, x_0)} \quad \forall \xi \in \partial\Omega, \tag{1.20}$$

for some $x_0 \in \Omega$. Thank to this kernel we can represent any positive $\mathcal{L}_\kappa$-harmonic function u by mean of a Poisson type formula which endows the form

$$u(x) = \int_{\partial\Omega} K_{\mathcal{L}_\kappa}(x, \xi)d\mu(\xi) \tag{1.21}$$

for some unique positive Radon measure μ on $\partial\Omega$. The measure μ is called the boundary trace of u. Furthermore $K_{\mathcal{L}_\kappa}$ satisfies the following two-side estimates:

Theorem B. *There exists $C_3 := C_3(\Omega,\kappa) > 0$ such that for any $(x,\xi) \in \Omega \times \partial\Omega$ there holds*

$$\frac{1}{C_3}\frac{d^{\frac{\alpha_+}{2}}}{|x-\xi|^{N+\alpha_+-2}} \leq K_{\mathcal{L}_\kappa}(x,\xi) \leq C_3\frac{d^{\frac{\alpha_+}{2}}}{|x-\xi|^{N+\alpha_+-2}}. \tag{1.22}$$

In Sections 3–5 and Appendix of this paper we develop the study of the semilinear equation (E) and emphasize the particular case of Eq. (1.1). With the help of the previous estimates we adapt the approach developed in [16] and generalized in [31] to prove the existence of weak solutions to the nonlinear boundary value problem

$$\begin{aligned} -\Delta u - \frac{\kappa}{d^2(x)}u + g(u) &= \nu \qquad \text{in } \Omega \\ u &= \mu \qquad \text{in } \partial\Omega, \end{aligned} \tag{1.23}$$

where g is a continuous nondecreasing function such that $g(0) \geq 0$ and ν and μ are Radon measures on Ω and $\partial\Omega$ respectively. We define the class $\mathbf{X}_\kappa(\Omega)$ of test functions by

$$\mathbf{X}_\kappa(\Omega) = \left\{\eta \in L^2(\Omega) \text{ s.t. } \nabla\left(d^{-\frac{\alpha_+}{2}}\eta\right) \in L^2_{\phi_k}(\Omega) \text{ and } \phi_\kappa^{-1}\mathcal{L}_\kappa\eta \in L^\infty(\Omega)\right\}, \tag{1.24}$$

and we prove

Theorem C. *Assume g satisfies*

$$\int_1^\infty (g(s)+|g(-s)|)s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}ds < \infty. \tag{1.25}$$

Then for any Radon measures ν on Ω and such that $\int_\Omega \phi_\kappa d|\mu| < \infty$ and μ on $\partial\Omega$ there exists a unique $u \in L^1_{\phi_\kappa}(\Omega)$ such that $g(u) \in L^1_{\phi_\kappa}(\Omega)$ which satisfies

$$\int_\Omega (u\mathcal{L}_\kappa\eta + g(u)\eta)\,dx = \int_\Omega \left(\eta d\nu + \mathbb{K}_{\mathcal{L}_\kappa}[\mu]\mathcal{L}_\kappa\eta dx\right) \quad \forall \eta \in \mathbf{X}_\kappa(\Omega). \tag{1.26}$$

When $g(r) = |r|^{q-1}r$ the critical value is $q_c = \frac{N+\frac{\alpha_+}{2}}{N+\frac{\alpha_+}{2}-2}$ and (1.25) is satisfied for $0 \leq q < q_c$ (the subcritical range). In this range of values of q, existence and uniqueness of a solution to

$$\begin{aligned} -\Delta u - \frac{\kappa}{d^2(x)}u + |u|^{q-1}u &= 0 \qquad \text{in } \Omega \\ u &= \mu \qquad \text{in } \partial\Omega, \end{aligned} \tag{1.27}$$

has been recently obtained by Marcus and Nguyen [22]. However, when $q \geq q_c$ not all the Radon measures are eligible for solving problem (1.27).

We prove the following result in the statement of which $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$ denotes the Besov capacity associated to the Besov space $B^{2-\frac{2+\alpha_+}{2q'},q'}(\mathbb{R}^{N-1})$.

Theorem D. *Assume $q \geq q_c$ and μ is a positive Radon measure on $\partial\Omega$. Then problem (1.27) admits a weak solution if and only if μ vanishes on Borel sets $E \subset \partial\Omega$ such that $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(E) = 0$.*

Note that a special case of this result is proved in [22] when $\mu = \delta_a$ for a boundary point and $q \geq q_c$. In that case δ_a does not vanish on $\{a\}$ although $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(\{a\}) = 0$.

This capacity plays a fundamental for characterizing the removable compact boundary sets which can only exist in the *supercritical range $q \geq q_c$*.

Theorem E. *Assume $q \geq q_c$ and $K \subset \partial\Omega$ is compact. Then any function $u \in C(\overline{\Omega}\setminus K)$ which satisfies*

$$\begin{aligned} -\Delta u - \frac{\kappa}{d^2(x)}u + |u|^{q-1}u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega\setminus K, \end{aligned} \tag{1.28}$$

is identically zero if and only if $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(K) = 0$.

The proof of Theorems D and E is delicate and based upon the use of the *optimal lifting operator* which has been introduced in [23] and the kernels estimates of [26, Appendix].

We show that any positive solution u of (1.1) admits a boundary trace in the class of outer regular positive Borel measures, not necessarily locally bounded, and more precisely we prove that the following dichotomy holds:

Theorem F. *Let u be a positive solution of (1.1) in Ω and $a\in\partial\Omega$. Then*

(i) *either for any $\epsilon>0$*

$$\lim_{\delta\to0}\int_{\Sigma_\delta\cap B_\epsilon(a)} u d\omega^{x_0}_{\Omega'_\delta}=\infty, \tag{1.29}$$

where $\Omega'_\delta=\{x\in\Omega: d(x)>\delta\}$, $\Sigma_\delta=\partial\Omega'_\delta$ and $\omega^{x_0}_{\Omega'_\delta}$ is the harmonic measure in Ω'_δ,

(ii) *or there exist $\epsilon_0>0$ and a positive Radon measure λ on $\partial\Omega\cap B_{\epsilon_0}(a)$ such that for any $Z\in C(\overline{\Omega})$ with support in $\Omega\cup(\partial\Omega\cap B_{\epsilon_0}(a))$, there holds*

$$\lim_{\delta\to0}\int_{\Sigma_\delta\cap B_\epsilon(a)} Zu d\omega^{x_0}_{\Omega'_\delta}=\int_{\partial\Omega\cap B_\epsilon(a)} Z d\lambda. \tag{1.30}$$

The set of points $a\in\partial\Omega$ such that (i) (resp. (ii)) holds is closed (resp. relatively open) and denoted by $\mathcal{S}_u$ (resp $\mathcal{R}_u$). There exists a unique Radon measure μ_u on $\mathcal{R}_u$ such that, for any $Z\in C(\overline{\Omega})$ with support in $\Omega\cup\mathcal{R}_u$ there holds

$$\lim_{\delta\to0}\int_{\Sigma_\delta} Zu d\omega^{x_0}_{\Omega'_\delta}=\int_{\mathcal{R}_u} Z d\mu_u. \tag{1.31}$$

The couple $(\mathcal{S}_u,\mu_u)$ is called *the boundary trace of* u and denoted by $Tr_{\partial\Omega}(u)$. A notion of normalized boundary trace of positive *moderate solutions* of (1.1), i.e. solutions such that $u\in L^q(\phi_\kappa)$, is developed in [22]. It is proved therein that there exists a boundary trace $\mu\approx(\{\emptyset\},\mu_u)$, and that the corresponding representation of u via the Martin and Green kernels holds.

If $1<q<q_c$ we denote by $u_{k\delta_a}$ positive solution of (1.1) with $\mu=k\delta_a$ for some $a\in\partial\Omega$ and $k\geq0$. Then there exists $\lim_{k\to\infty}u_{k\delta_a}=u_{\infty,a}$ and we prove the following:

Theorem G. *Assume $1<q<q_c$ and $a\in\partial\Omega$. If u is a positive solution of (1.1) such that $a\in\mathcal{S}_u$, then $u\geq u_{\infty,a}$.*

In order to go further in the study of boundary singularities, we construct separable solutions of (1.1) in $\mathbb{R}^N_+=\{x=(x',x_N): x_N>0\}=\{(r,\sigma)\in\mathbb{R}_+\times S^{N-1}_+\}$ which vanish on $\partial\mathbb{R}^N_+\setminus\{0\}$ under the form $u(r,\sigma)=r^{-\frac{2}{q-1}}\omega(\sigma)$, where $r>0$, $\sigma\in S^{N-1}_+$. They are solutions of

$$\begin{aligned} -\Delta_{S^{N-1}}\omega-\ell_{q,N}\omega-\frac{\kappa}{\mathbf{e}_N\cdot\sigma}\omega+|\omega|^{q-1}\omega=0 &\quad \text{in } S^{N-1}_+\\ \omega=0 &\quad \text{in } \partial S^{N-1}_+, \end{aligned} \tag{1.32}$$

where $\Delta_{S^{N-1}}$ is the Laplace–Beltrami operator, $\mathbf{e}_N$ the unit vector pointing toward the North pole and $\ell_{q,N}$ is a positive constant. We prove that if $1<q<q_c$, then problem (1.32) admits a unique positive solution ω_κ while no such solution exists if $q\geq q_c$. To this phenomenon is associated a result of classification of the positive solutions of (1.1) in Ω which vanishes of $\partial\Omega\setminus\{0\}$ (here we assume that $0\in\partial\Omega$ and that the tangent hyperplane to $\partial\Omega$ at 0 is $\{x: x\cdot\mathbf{e}_N=0\}$, and that there exists $r_0>0$ such that $B_{r_0}(r_0\mathbf{e}_N)\subset\Omega$, $B_{r_0}(r_0\mathbf{e}_N)\subset\{x: x\cdot\mathbf{e}_N\geq0\}$ and $d(r_0\mathbf{e}_N)=|r_0\mathbf{e}_N|=r_0$).

Theorem H. *Assume $1<q<q_c$ and let $u\in C(\overline{\Omega}\setminus\{a\})$ be a solution of (1.1) in Ω which vanishes of $\partial\Omega\setminus\{a\}$. Then*

(i) *either $u=u_{\infty,a}$ and*

$$\lim_{r\to0} r^{\frac{2}{q-1}}u(r,.)=\omega_\kappa \tag{1.33}$$

locally uniformly in S^{N-1}_+,

(ii) *or there exists $k\geq0$ such that $u=u_{k\delta_a}$ and*

$$u(x)=kK_{\mathcal{L}_\kappa}(x,a)(1+o(1)) \quad \text{as } x\to0. \tag{1.34}$$

If $1<q<q_c$ we prove that to any couple (F,μ) where F is a closed subset of $\partial\Omega$ and μ a positive Radon measure on $R=\partial\Omega\setminus F$, we can associate a positive solution u of (1.1) in Ω with the property that $Tr_{\partial\Omega}(u)=(F,\mu)$. The construction is based upon the existence of barrier functions which allow to prove local a priori estimate that is satisfied by any positive solution with boundary trace $(F,0)$. The delicate proof of the existence of these barrier is presented in Appendix A.1. A priori estimates which follow from the barrier method are presented in Appendix A.2. In Appendix A.3 we develop some regularity results based upon Moser's iterative scheme adapted to the framework of the Hardy operator.

The results presented here are announced in [15].

2. The linear operator $\mathcal{L}_\kappa = -\Delta - \frac{\kappa}{d^2(x)}$

Throughout this article c_j ($j = 1, 2, \ldots$) denote positive constants the value of which may change from one occurrence to another. The notation κ is reserved to the value of the coefficient of the Hardy potential.

2.1. Classical results on Hardy's inequality and the operator $\mathcal{L}_\kappa$

We first recall some known results concerning Hardy's inequalities and the associated eigenvalue problem (see [11,14]).
1—The constant c_Ω defined in (1.7) has value in $(0, \frac{1}{4}]$. If Ω is convex or if the function d is super-harmonic then $c_\Omega = \frac{1}{4}$. Moreover this equality is verified if and only if there exists no minimizer to the problem (1.7) [21]. For any $\kappa \in (0, \frac{1}{4}]$ there exists

$$\inf\left\{\int_\Omega \left(|\nabla u|^2 - \frac{\kappa}{d^2}u^2\right)dx : \int_\Omega u^2 dx = 1\right\} = \lambda_\kappa > -\infty. \tag{2.1}$$

Furthermore $\lambda_\kappa > 0$ if $\kappa < c_\Omega$ or if $k \leq \frac{1}{4}$ and d is a superharmonic function in Ω. (See [7].)
2—If $0 < \kappa < \frac{1}{4}$ the minimizer ϕ_κ of (2.1) belongs to the space $H_0^1(\Omega)$ and it satisfies

$$\phi_\kappa \approx d^{\frac{\alpha_+}{2}}(x), \tag{2.2}$$

where α_+ (as well as α_-) are defined by (1.8).
3—If $\kappa = \frac{1}{4}$, there is no minimizer in $H_0^1(\Omega)$, but there exists a non-negative function $\phi_{\frac{1}{4}} \in H^1_{loc}(\Omega)$ such that

$$\phi_{\frac{1}{4}} \approx d^{\frac{1}{2}}(x), \tag{2.3}$$

and it solves

$$-\Delta u - \frac{1}{4d^2}u = \lambda_k u \quad \text{in } \Omega$$

in the sense of distributions. In addition, the function $\psi_{\frac{1}{4}} = d^{-\frac{1}{2}}\phi_{\frac{1}{4}}$ belongs to $H_0^1(\Omega; d(x)dx)$.
4—Let $H_0^1(\Omega, d^\alpha(x)dx)$ denote the closure of $C_0^\infty(\Omega)$ functions under the norm

$$\|u\|^2_{H_0^1(\Omega, d^\alpha(x)dx)} = \int_\Omega |\nabla u|^2 d^\alpha(x)dx + \int_\Omega |u|^2 d^\alpha(x)dx. \tag{2.4}$$

If $\alpha \geq 1$ there holds [14, Th. 2.11]

$$H_0^1(\Omega, d^\alpha(x)dx) = H^1(\Omega, d^\alpha(x)dx) \quad \forall \alpha \geq 1. \tag{2.5}$$

5—Let $0 < \kappa \leq c_\Omega$. Let $\mathbf{H}_\kappa(\Omega)$ be the subset of functions of $H^1_{loc}(\Omega)$ satisfying

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{\kappa}{d^2}\phi^2\right)dx < \infty. \tag{2.6}$$

Then the mapping

$$\phi \mapsto \left(\int_\Omega \left(|\nabla\phi|^2 - \frac{\kappa}{d^2}\phi^2\right)dx\right)^{\frac{1}{2}} \tag{2.7}$$

is a norm on $\mathbf{H}_\kappa(\Omega)$. The closure $\mathbf{W}_\kappa(\Omega)$ of $C_0^\infty(\Omega)$ into $\mathbf{H}_\kappa(\Omega)$ satisfies

$$\mathbf{W}_\kappa(\Omega) = H_0^1(\Omega) \quad \forall\, 0 < \kappa < c_\Omega \quad \text{and} \quad \mathbf{W}_{\frac{1}{4}}(\Omega) \subset W_0^{1,q}(\Omega) \quad \text{if } \lambda_{\frac{1}{4}} > 0\, \forall 1 \leq q < 2, \tag{2.8}$$

see [4, Th B]. As a consequence $\mathbf{W}_\kappa(\Omega)$ is compactly embedded into $L^r(\Omega)$ for any $r \in [1, 2^*)$.
6—Let $\alpha > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. There exists $c^* > 0$ depending on $\mathrm{diam}(\Omega)$, N and α such that for any $v \in C_0^\infty(\Omega)$

$$\left(\int_\Omega |v|^{\frac{2(N+\alpha)}{N+\alpha-2}} d^\alpha dx\right)^{\frac{N+\alpha-2}{N+\alpha}} \leq c^* \int_\Omega |\nabla v|^2 d^\alpha dx. \tag{2.9}$$

For a proof see [14, Th. 2.9].

The boundary behavior of the first eigenfunction yields a two-side similar estimate of the Green kernel for Schrödinger operators with a general Hardy type potentials [14, Corollary 1.9].

Proposition 2.1. *Consider the operator $E := -\Delta - V$, in Ω where $V = V_1 + V_2$, with*

$$|V_1| \leq \frac{1}{4d^2(x)} \quad \text{and} \quad V_2 \in L^p(\Omega), \quad p > \frac{N}{2}.$$

We also assume that

$$0 < \lambda_1 := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 dx - Vu^2\right) dx}{\int_\Omega u^2 dx},$$

and that to λ_1 is associated a positive eigenfunction ϕ_1. If, for some $\alpha \geq 1$ and $C_1, C_2 > 0$, there holds

$$c_1 d^{\frac{\alpha}{2}}(x) \leq \phi_1(x) \leq c_2 d^{\frac{\alpha}{2}}(x) \quad \forall x \in \Omega,$$

then the Green kernel G_E^Ω associated to E in Ω satisfies

$$G_E^\Omega(x, y) \approx c_3 \min\left(\frac{1}{|x-y|^{N-2}}, \frac{d^{\frac{\alpha}{2}}(x) d^{\frac{\alpha}{2}}(y)}{|x-y|^{N+\alpha-2}}\right). \tag{2.10}$$

Next we define the sets Ω_δ, Ω'_δ and Σ_δ by

$$\Omega_\delta = \{x \in \Omega : d(x) < \delta\}, \qquad \Omega'_\delta = \{x \in \Omega : d(x) > \delta\} \quad \text{and} \quad \Sigma_\delta = \{x \in \Omega : d(x) = \delta\}. \tag{2.11}$$

Definition 2.2. Let $G \subset \Omega$ be open and let $H_c^1(G)$ denote the subspace of $H^1(G)$ of functions with compact support in G. A function $h \in W_{loc}^{1,1}(G)$ is $\mathcal{L}_\kappa$-harmonic in G if

$$\int_G \nabla h \cdot \nabla \psi dx - \kappa \int_\Omega \frac{1}{d^2(x)} h\psi dx = 0 \quad \forall \psi \in H_c^1(G).$$

A function $\underline{h} \in H_{loc}^1(G) \cap C(G)$ is $\mathcal{L}_\kappa$-subharmonic in G if

$$\int_G \nabla \underline{h} \cdot \nabla \psi dx - \kappa \int_\Omega \frac{1}{d^2(x)} h\psi dx \leq 0 \quad \forall \psi \in H_c^1(G),\ \psi \geq 0.$$

We say that $\underline{h}$ is a local $\mathcal{L}_\kappa$-subharmonic function if there exists $\delta > 0$ such that $\underline{h} \in H_{loc}^1(\Omega_\delta) \cap C(\Omega_\delta)$ is $\mathcal{L}_\kappa$-subharmonic in Ω_δ. Similarly, (local) $\mathcal{L}_\kappa$-superharmonics $\overline{h}$ are defined with "$\geq$" in the above inequality.

Note that $\mathcal{L}_\kappa$-harmonic functions are C^2 in G by standard elliptic equations regularity theory. The Phragmen–Lindelöf principle yields the following alternative [3, Theorem 2.6].

Proposition 2.3. *Let $\kappa \leq \frac{1}{4}$. If $\underline{h}$ is a local $\mathcal{L}_\kappa$-subharmonic function, then the following alternative holds:*

(i) *either for every local positive $\mathcal{L}_\kappa$-superharmonic function $\overline{h}$*

$$\limsup_{d(x)\to 0} \frac{\underline{h}(x)}{\overline{h}(x)} > 0, \tag{2.12}$$

(ii) *or for every local positive $\mathcal{L}_\kappa$-superharmonic function $\overline{h}$*

$$\limsup_{d(x)\to 0} \frac{\underline{h}(x)}{\overline{h}(x)} < \infty. \tag{2.13}$$

Definition 2.4. If a local $\mathcal{L}_\kappa$-subharmonic function $\underline{h}$ satisfies (i) (resp. (ii)) it is called a large $\mathcal{L}_\kappa$-subharmonic (resp. a small $\mathcal{L}_\kappa$-subharmonic).

The next statement is [3, Theorem 2.9].

Proposition 2.5. *Let $\underline{h}$ be a small local $\mathcal{L}_\kappa$-subharmonic of $\mathcal{L}_\kappa$.*

(i) *If $\kappa < \frac{1}{4}$, then the following alternative holds:*

$$\text{either} \ \limsup_{x\to\partial\Omega} \frac{\underline{h}(x)}{(d(x))^{\frac{\alpha_-}{2}}} > 0 \quad \text{or} \quad \limsup_{x\to\partial\Omega} \frac{\underline{h}(x)}{(d(x))^{\frac{\alpha_+}{2}}} < \infty.$$

(ii) *If* $\kappa = \frac{1}{4}$, *then the following alternative holds:*

$$\text{either } \limsup_{x\to\partial\Omega} \frac{h(x)}{(d(x))^{\frac{1}{2}}\log\left(\frac{1}{d}\right)} > 0 \quad \text{or} \quad \limsup_{x\to\partial\Omega} \frac{h(x)}{(d(x))^{\frac{1}{2}}} < \infty.$$

Definition 2.6. Let $f_0 \in L^2_{loc}(\Omega)$. We say that a function $u \in H^1_{loc}(\Omega)$ is a solution of

$$\mathcal{L}_\kappa u = f_0 \quad \text{in } \Omega, \tag{2.14}$$

if there holds

$$\int_\Omega \nabla u \cdot \nabla \psi dx - \kappa \int_\Omega \frac{1}{d^2(x)} u\psi dx = \int_\Omega f_0 \psi dx \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.15}$$

2.2. Preliminaries

In this part we study some regularity properties of solutions of linear equations involving $\mathcal{L}_\kappa$.

Lemma 2.7. (i) *If* $\alpha > 1$ *and* $d^{-\frac{\alpha}{2}} u \in H^1(\Omega, d^\alpha(x)dx)$, *then* $u \in H^1_0(\Omega)$.
(ii) *If* $\alpha = 1$ *and* $d^{-\frac{1}{2}} u \in H^1(\Omega, d(x)dx)$, *then* $u \in W_0^{1,p}(\Omega)$, $\forall p < 2$.

Proof. There exists $\beta_0 > 0$ such that $d \in C^2(\overline{\Omega_{\beta_0}})$ and set $u = d^{\frac{\alpha}{2}} v$. In the two cases (i)–(ii), our assumptions imply

$$u \in L^2(\Omega) \quad \text{and} \quad \nabla u - \frac{\alpha}{2} u d^{-1} \nabla d \in L^2(\Omega). \tag{2.16}$$

(i) Since $v \in H^1(\Omega, d^\alpha(x)dx)$, by (2.5) there exists a sequence $v_n \in C_0^\infty(\Omega)$ such that $v_n \to v$ in $H^1(\Omega, d^\alpha(x)dx)$. Set $u_n = d^\alpha v_n$. Let $0 < \beta \le \frac{\beta_0}{2}$ and ψ_β be a cut of function such that $\psi_\beta = 0$ in Ω'_β and $\psi_\beta = 1$ in $\Omega_{\frac{\beta}{2}}$. Then $u_n = d^{\frac{\alpha}{2}}(\psi_\beta v_n + (1-\psi_\beta)v_n)$. Thus it is enough to prove that $\widetilde{u}_n = d^{\frac{\alpha}{2}} \psi_\beta v_n$ remains bounded in $H^1(\Omega)$ independently of n. Set $w_n = \psi_\beta v_n$, then

$$\int_\Omega |\nabla \widetilde{u}_n|^2 dx = \int_{\Omega_\beta} |\nabla w_n|^2 dx \le c_4 \left(\int_{\Omega_\beta} d^\alpha |\nabla w_n|^2 dx + \int_{\Omega_\beta} d^{\alpha-2} w_n^2 dx \right).$$

Note that $\alpha - 2 > -1$. Now

$$\int_{\Omega_\beta} d^{\alpha-2} w_n^2 dx = \frac{1}{\alpha-1} \int_{\Omega_\beta} w_n^2 \operatorname{div}(d^{\alpha-1}\nabla d) dx - \frac{1}{\alpha-1} \int_{\Omega_\beta} d^{\alpha-1}(\Delta d) w_n^2 dx.$$

Now since $|\Delta d(x)| < c_5$, $\forall x \in \Omega_{\beta_0}$, we have

$$\left| \frac{1}{\alpha-1} \int_{\Omega_\beta} d^{\alpha-1}(\Delta d) w_n^2 dx \right| \le \frac{c_5 \beta_0^{\alpha-1}}{\alpha-1} \int_{\Omega_\beta} w_n^2 dx.$$

Also

$$\begin{aligned} \left| \int_{\Omega_\beta} w_n^2 \operatorname{div}(d^{\alpha-1}\nabla d) dx \right| &= 2 \left| \int_{\Omega_\beta} w_n d^{\frac{\alpha}{2}} d^{\frac{\alpha}{2}-1} \nabla d \cdot \nabla w_n dx \right| \\ &\le c_6 \int_{\Omega_\beta} d^\alpha |\nabla w_n|^2 dx + \delta \int_{\Omega_\beta} d^{\alpha-2} w_n^2 dx, \end{aligned}$$

where $c_6 = c_6(\delta) > 0$. The result follows in this case, if we choose δ small enough and then let $n \to \infty$.
(ii) By the same calculations we have

$$\int_\Omega d^{-\frac{p}{2}} |w_n|^p dx \le c_7 \int_{\Omega_\beta} d^{\frac{p}{2}} |\nabla w_n|^p dx \le c_7 \left(\int_\Omega d(x) dx \right)^{\frac{p}{2}} \int_{\Omega_\beta} d |\nabla w_n|^2 dx. \quad \square$$

In the following statement we prove regularity up to the boundary for the function $\frac{u}{\phi_\kappa}$.

Proposition 2.8. *Let $f_0 \in L^2(\Omega)$. Then there exists a unique $u \in H^1_{loc}(\Omega)$ such that $\phi_\kappa^{-1}u \in H^1(\Omega, d^{\alpha_+}(x)dx)$, satisfying (2.14). Furthermore, if $f_1 := \frac{f_0}{\phi_\kappa} \in L^q(\Omega, \phi_\kappa^2 dx)$, $q > \frac{N+\alpha_+}{2}$, then there exists $0 < \beta < 1$ such that*

$$\sup_{x,y\in\Omega,\, x\neq y} |x-y|^{-\beta} \left| \frac{u(x)}{\phi_\kappa(x)} - \frac{u(y)}{\phi_\kappa(y)} \right| < c_8 \|f_1\|_{L^q(\Omega,\phi_\kappa^2 dx)}. \tag{2.17}$$

Proof. If there exists a solution u, then $\psi = \frac{u}{\phi_\kappa}$ satisfies

$$-\phi_\kappa^{-2}\operatorname{div}(\phi_\kappa^2 \nabla\psi) + \lambda_\kappa \psi = \phi_\kappa^{-1} f_0, \tag{2.18}$$

and we recall that $\phi_\kappa(x) \approx d^{\frac{\alpha_+}{2}}(x)$. We endow the space $H^1(\Omega, \phi_\kappa^2 dx)$ with the inner product

$$\langle a, b\rangle = \int_\Omega (\nabla a \cdot \nabla b + \lambda_\kappa ab)\, \phi_\kappa^2 dx.$$

By a solution ψ of (2.18) we mean that $\psi \in H^1_0(\Omega, \phi_\kappa^2 dx)$ satisfies

$$\langle \nabla\psi, \nabla\zeta\rangle = \int_\Omega \nabla\psi \cdot \nabla\zeta\, \phi_\kappa^2 dx + \lambda_\kappa \int_\Omega \psi\zeta\phi_\kappa^2 dx = \int_\Omega f_0 \zeta \phi_\kappa dx \quad \forall \zeta \in H^1_0(\Omega, \phi_\kappa^2 dx). \tag{2.19}$$

By Riesz's representation theorem we derive the existence and uniqueness of the solution in this space. Since $H^1(\Omega, \phi_\kappa^2 dx) = H^1_0(\Omega, \phi_\kappa^2 dx)$ by [14, Th 2.11], any weak solution u of (2.14) such that $\phi_\kappa^{-1}u \in H^1(\Omega, \phi_\kappa^2 dx)$ is obtained by the above method.

Finally if $f_0 \in L^q(\Omega, \phi_\kappa^2 dx)$, where $q > \frac{N+\alpha_+}{2}$, thanks to (2.9) we can prove the estimate

$$\|\psi\|_{L^\infty(\Omega)} \le c_8 \|f_0\|_{L^q(\Omega,\phi_\kappa^2 dx)}, \tag{2.20}$$

where $c_8 = c_8(\Omega, \kappa, q)$. Then we can apply the Moser iteration (see Appendix A.3) to derive the Hölder regularity up to the boundary. □

In the next results we make more precise the rate of convergence of a solution of (2.14) to its boundary value.

Proposition 2.9. *Assume $\kappa < \frac{1}{4}$. If $f_0 \in L^2(\Omega)$ and $h \in H^1(\Omega)$ there exists a unique weak solution u of (2.14) belonging to $H^1_{loc}(\Omega)$ and such that $d^{-\frac{\alpha_+}{2}}(u - d^{\frac{\alpha_-}{2}}h) \in H^1(\Omega,\ d^{\alpha_+}(x)dx)$. Furthermore, if $f_1 := \frac{f_0}{\phi_\kappa} \in L^q(\Omega, \phi_\kappa^2 dx)$, $q > \frac{n+\alpha}{2}$ and $h \in C^2(\overline{\Omega})$, then there exists $0 < \beta < 1$ with the property that*

$$\lim_{x\in\Omega,\, x\to y\in\partial\Omega} \frac{u(x)}{(d(x))^{\frac{\alpha_-}{2}}} = h(y) \quad \forall y \in \partial\Omega,$$

uniformly with respect to y,

$$\left\| \frac{u}{d^{\frac{\alpha_-}{2}}} \right\|_{L^\infty(\Omega)} \le c_9 \left(\|h\|_{C^2(\overline{\Omega})} + \|f_1\|_{L^q(\Omega,\phi_\kappa^2 dx)} \right),$$

and

$$\sup_{x,y\in\Omega,\, x\neq y} |x-y|^{-\beta} \left| \frac{u(x)}{(d(x))^{\frac{\alpha_-}{2}}} - \frac{u(y)}{(d(y))^{\frac{\alpha_-}{2}}} \right| \le c_{10} \left(\|h\|_{C^2(\overline{\Omega})} + \|f_1\|_{L^q(\Omega,\phi_\kappa^2 dx)} \right), \tag{2.21}$$

with c_9 and c_{10} depending on Ω, N, q, and κ.

Remark. By Lemma 2.7 we already know that $u - d^{\frac{\alpha_-}{2}}h \in H^1_0(\Omega)$.

Proof. Let $\beta \le \beta_0$ and $\eta \in C^2(\Omega)$ be a function such that $\eta = d^{\frac{\alpha_-}{2}}(x)$ in Ω_β and $\eta(x) > c > 0$, if $x \in \Omega'_\beta$. We set $u = \phi_\kappa v + \eta h$. Then v is a weak solution of

$$-\frac{\operatorname{div}(\phi_\kappa^2 \nabla v)}{\phi_\kappa^2} + \lambda_\kappa v = \frac{1}{\phi_\kappa}\left(f_0 + \left(\Delta\eta + \kappa\frac{\eta}{d^2}\right) h + 2\nabla\eta\cdot\nabla h + \eta\Delta h \right), \tag{2.22}$$

in the sense that

$$\int_\Omega \nabla v \cdot \nabla\psi\, \phi_\kappa^2 dx + \lambda_\kappa \int_\Omega v\, \psi\, \phi_\kappa^2 dx = \int_\Omega \left(f_0 + \left(\Delta\eta + \kappa\frac{\eta}{d^2}\right) h + 2\nabla\eta\cdot\nabla h \right) \psi\, \phi_\kappa dx - \int_\Omega \nabla h \cdot \nabla(\eta\psi\phi_\kappa)\, dx \quad \forall \psi \in C_0^\infty(\Omega). \tag{2.23}$$

Let $\psi \in C_0^\infty(\Omega_\beta)$. By an argument similar to the one in the proof of Lemma 2.7 we have

$$\int_\Omega \psi^2 dx = \int_{\Omega_\beta} \psi^2 dx = \int_{\Omega_\beta} \mathrm{div}(d\nabla d)|\psi|^2 dx - \int_{\Omega_\beta} d\Delta d|\psi|^2 dx,$$

which implies

$$\int_{\Omega_\beta} \psi^2 dx \leq c'_{10} \int_{\Omega_\beta} d^2|\nabla\psi|^2 dx \leq c_{11} \int_{\Omega_\beta} d^{\alpha_+}|\nabla\psi|^2 dx. \tag{2.24}$$

Now

$$\left|\int_{\Omega_\beta} \left(\left(\Delta\eta + \kappa\frac{\eta}{d^2}\right) h + 2\nabla\eta\cdot\nabla h\right)\psi\,\phi_\kappa dx\right| \leq c_{12}\int_{\Omega_\beta}\psi^2 dx,$$

and

$$\left|\int_{\Omega_\beta} \nabla h\cdot\nabla\left(\eta\psi\,\phi_\kappa\right) dx\right| \leq c_{13}\left(\int_{\Omega_\beta}|\nabla h|^2 dx + \iint_{\Omega_\beta} d^{\alpha_+}|\nabla\psi|^2 dx + \iint_{\Omega_\beta}\psi^2 dx\right).$$

By (2.24) we can take $\psi \in H^1(\Omega, d^{\alpha_+}(x)dx)$ for test function. Thus we derive that there exists a weak solution $v \in H^1(\Omega, d^{\alpha_+}(x)dx)$ of (2.23).

To prove (2.21) we first obtain that if $\psi \in C_0^\infty(\Omega_\varepsilon)$

$$\int_\Omega \psi dx = -\int_{\Omega_\varepsilon} d\nabla d\cdot\nabla\psi dx - \int_{\Omega_\varepsilon} d\Delta d\psi dx.$$

Since

$$\begin{aligned}\left|\int_{\Omega} \left(\left(\Delta\eta + \kappa\frac{\eta}{d^2}\right) h + 2\nabla\eta\cdot\nabla h + \eta\Delta h\right)\psi\,\phi_\kappa dx\right| &\leq c_{14}\|h\|_{C^2(\overline{\Omega})}\int_\Omega|\psi|dx\\ &\leq \frac{1}{2}\int_{\Omega_\varepsilon} d^{\alpha_+}|\nabla\psi|^2 dx + c_{15}(\Omega,\kappa)\|h\|_{C^2(\overline{\Omega})},\end{aligned}$$

we use again (2.9) and Moser's iterative scheme as in Proposition 2.8, and we obtain

$$\|v\|_{L^\infty(\Omega)} \leq c_9\left(\|h\|_{C^2(\overline{\Omega})} + \|f_0\|_{L^q(\Omega,\phi_\kappa^2 dx)}\right),$$

where $c_9 = c_9(\Omega, q, \kappa) > 0$. From inequality it follows that v is Hölder continuous up to the boundary and the uniform convergence holds. □

Proposition 2.10. *Assume $\kappa = \frac{1}{4}$. If $f_0 \in L^2(\Omega)$ and $h \in H^1(\Omega)$, there exists a unique function u in $H^1_{loc}(\Omega)$ weak solution of*

$$\mathcal{L}_{\frac{1}{4}} u = f_0$$

verifying $d^{-\frac{1}{2}}(u - d^{\frac{1}{2}}|\log d|h) \in H^1(\Omega,\ d(x)dx)$. Furthermore, if $f_1 := \frac{f_0}{\phi_{\frac{1}{4}}} \in L^q(\Omega)$, $q > \frac{n+1}{2}$ and $h \in C^2(\overline{\Omega})$, then there exists $0 < \beta < 1$ such that

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega} \frac{u}{d^{\frac{1}{2}}|\log d|}(x) = h(y) \quad \forall y\in\partial\Omega,$$

uniformly with respect to y,

$$\left\|\frac{u}{\sqrt{d}\left|\log\frac{d}{D_0}\right|}\right\|_{L^\infty(\Omega)} \leq c_{16}\left(\|h\|_{C^2(\overline{\Omega})} + \|f_1\|_{L^q\left(\Omega,\phi_{\frac{1}{4}}^2 dx\right)}\right)$$

where $D_0 = 2\sup_{x\in\Omega} d(x)$. Finally there holds

$$\sup_{x,y\in\Omega,\ x\neq y}|x-y|^{-\beta}\left|\frac{u(x)}{\sqrt{d(x)}\left|\log\frac{d(x)}{D_0}\right|} - \frac{u(y)}{\sqrt{d(y)}\left|\log\frac{d(y)}{D_0}\right|}\right| < c_{17}\left(\|h\|_{C^2(\overline{\Omega})} + \|f_1\|_{L^q(\Omega,\phi_{\frac{1}{4}}^2 dx)}\right). \tag{2.25}$$

Proof. Using again Lemma 2.7, we know that $u - d^{\frac{1}{2}}|\log d|h \in W_0^{1,p}(\Omega)$, $\forall p < 2$. The proof is very similar to the proof of Proposition 2.9. The only differences are we impose $\eta = d^{\frac{1}{2}}|\log d|$ in Ω_β and we use the fact that $|\log d| \in L^p(\Omega)$, $\forall p \geq 1$. □

In the next result we prove that the boundary Harnack inequality holds, provided the vanishing property of a solution is understood in an appropriate way.

Proposition 2.11. *Let $\delta > 0$ be small enough, $\xi \in \partial\Omega$ and $u \in H^1_{loc}(B_\delta(\xi)\cap\Omega)\cap C(B_\delta(\xi)\cap\overline{\Omega})$ be a positive $\mathcal{L}_{\frac{1}{4}}$-harmonic function in $B_\delta(\xi)\cap\Omega$ vanishing on $\partial\Omega\cap B_\delta(\xi)$ in the sense that*

$$\lim_{\mathrm{dist}(x,K)\to 0}\frac{u(x)}{d^{\frac{1}{2}}(x)|\log d(x)|} = 0 \quad \forall K \subset \partial\Omega\cap B_\delta(\xi),\ K \text{ compact}. \tag{2.26}$$

Then there exists a constant $c_{18} = c_{18}(N,\Omega,\kappa) > 0$ such that

$$\frac{u(x)}{\phi_{\frac{1}{4}}(x)} \leq c_{18}\frac{u(y)}{\phi_{\frac{1}{4}}(y)} \quad \forall x,y \in \Omega\cap B_{\frac{\delta}{2}}(\xi).$$

Proof. We already know that $u \in C^2(\Omega)$. Let $\delta \leq \min(\beta_0, \frac{1}{2})$ such that $B_\delta(\xi)\cap\Omega \subset \Omega_\delta \subset \Omega_{\beta_0}$.

By [3, Lemma 2.8] there exists a positive supersolution $\zeta \in C^2(\Omega_\delta)$ of (1.3) in Ω_δ with the following behavior

$$\zeta(x) \approx d^{\frac{1}{2}}(x)\log\frac{1}{d(x)}\left(1 + c_{19}\left(\log\frac{1}{d(x)}\right)^{-\beta}\right),$$

for some $\beta \in (0,1)$ and $c_{19} = c_{19}(\Omega) > 0$. Set $v = \zeta^{-1}u$, then it satisfies

$$-\zeta^{-2}\mathrm{div}(\zeta^2\nabla v) \leq 0 \quad \text{in } B_\delta(\xi)\cap\Omega. \tag{2.27}$$

Let $\eta \in C_0^\infty(B_\delta(\xi))$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_{\frac{3\delta}{4}}(\xi)$. We set $v_s = \eta^2(v-s)_+$. Since by assumption v_s has compact support in $B_\delta(\xi)\cap\Omega$, we can use it as a test function in (2.27) and we get

$$\int_{B_\delta(\xi)\cap\Omega}\zeta^2\nabla v\cdot\nabla v_s dx = \int_{B_\delta(\xi)\cap\Omega}\zeta^2\nabla(v-s)_+\cdot\nabla v_s dx \leq 0, \tag{2.28}$$

which yields

$$\int_{B_\delta(\xi)\cap\Omega}|\nabla(v-s)_+|^2\zeta^2\eta^2 dx \leq 4\int_{B_\delta(\xi)\cap\Omega}|\nabla\eta|^2(v-s)_+^2\zeta^2 dx.$$

Letting $s \to 0$ we derive

$$\int_{B_\delta(\xi)\cap\Omega}|\nabla v|^2\zeta^2\eta^2 dx \leq 4\int_{B_\delta(\xi)\cap\Omega}|\nabla\eta|^2 v^2\zeta^2 dx.$$

Since

$$|\nabla(v-s)_+|^2\zeta^2\eta^2 \uparrow |\nabla v|^2\zeta^2\eta^2 \quad \text{as } s \to 0,$$

and convergence of $\nabla(v-s)_+$ to ∇v holds a.e. in Ω, it follows by the monotone convergence theorem

$$\lim_{s\to 0}\int_{B_\delta(\xi)\cap\Omega}|\nabla(v-(v-s)_+)|^2\zeta^2\eta^2 dx = 0. \tag{2.29}$$

Finally $\zeta v_s \to \eta^2\zeta v$ in $H^1(B_\delta(\xi)\cap\Omega)$, which yields in particular $\eta^2 u = \eta^2\zeta v \in H_0^1(B_\delta(\xi)\cap\Omega)$.

Step 2. By [3, Lemma 2.8] there exists a positive subsolution $h \in C^2(\Omega_\delta)$ of (1.3) in Ω_δ with the following behavior

$$h(x) \approx d^{\frac{1}{2}}(x)\log\frac{1}{d(x)}\left(1 - c_{20}\left(\log\frac{1}{d(x)}\right)^{-\beta}\right),$$

where $\beta \in (0,1)$ and $c_{20} = c_{20}(\Omega) > 0$. Set $w = h^{-1}u$ and $w_s = \eta^2(w-s)_+$. Then $w_s \to \eta^2 w$ in $H^1(B_\delta(\xi)\cap\Omega)$ by Step 1. Put $u_s = hw_s$, thus, for $0 < s, s'$, we have

$$\int_{B_\delta(\xi)\cap\Omega}|\nabla(u_s - u_{s'})|^2 dx - \frac{1}{4}\int_{B_\delta(\xi)}\frac{|u_s - u_{s'}|^2}{d^2(x)}dx$$

$$
\begin{aligned}
&= \int_{B_\delta(\xi)\cap\Omega} h^2|\nabla(w_s - w_{s'})|^2 dx + \int_{B_\delta(\xi)\cap\Omega} |\nabla h|^2 |w_s - w_{s'}|^2 dx \\
&\quad + \int_{B_\delta(\xi)\cap\Omega} h\nabla h\cdot\nabla(u_s - u_{s'})^2 dx - \frac{1}{4}\int_{B_\delta(\xi)\cap\Omega} \frac{h^2|w_s - w_{s'}|^2}{d^2(x)} dx \\
&\leq \int_{B_\delta(\xi)\cap\Omega} h^2|\nabla(w_s - w_{s'})|^2 dx,
\end{aligned} \tag{2.30}
$$

where, in the last inequality, we have performed by parts integration and then used the fact that h is a subsolution. Thus we have by (2.29) that

$$
\lim_{s,s'\to 0} \int_{B_\delta(\xi)} |\nabla(u_s - u_{s'})|^2 dx - \frac{1}{4}\int_{B_\delta(\xi)} \frac{|u_s - u_{s'}|^2}{d^2(x)} dx = 0. \tag{2.31}
$$

Step 3. Let $\mathbf{W}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the space of functions ϕ satisfying

$$
\|\phi\|_H^2 := \int_\Omega |\nabla\Phi|^2 dx - \frac{1}{4}\int_\Omega \frac{|\Phi|^2}{d^2(x)} dx < \infty.
$$

Thus $\eta^2 u \in \mathbf{W}(\Omega)$, which implies

$$
\frac{\eta u}{\phi_{\frac14}} \in H_0^1(\Omega,\ d(x)dx).
$$

Next we set $\tilde{v} = \phi_{\frac14}^{-1} u$; then $\tilde{v} \in H^1(B_{\frac{3\delta}{4}}(\xi), d(x)dx)$ and it satisfies

$$
-\phi_{\frac14}^{-2}\mathrm{div}\left(\phi_{\frac14}^2\nabla\tilde{v}\right) + \lambda_{\frac14}\tilde{v} = 0.
$$

Put $\tilde{v}^*(x,t) = e^{t\lambda_{\frac14}}\tilde{v}$, then $\tilde{v}^*$ satisfies

$$
\tilde{v}_t^* - \phi_{\frac14}^{-2}\mathrm{div}\left(\phi_{\frac14}^2\nabla\tilde{v}^*\right) = 0 \tag{2.32}
$$

in the weak sense of [14, Definition 2.9]. By [14, Theorem 1.5], $\tilde{v}^*$ satisfies a Harnack inequality up to the boundary of Ω in the sense that

$$
\text{ess sup}\left\{\tilde{v}^*(y,t) : (y,t) \in \mathcal{B}_{\frac{r}{2}}(\xi) \times \left[\frac{r^2}{4}, \frac{r^2}{2}\right]\right\} \leq C\,\text{ess inf}\left\{\tilde{v}^*(y,t) : (y,t) \in \mathcal{B}_{\frac{r}{2}}(\xi) \times \left[\frac{3r^2}{4}, r^2\right]\right\} \tag{2.33}
$$

where $\mathcal{B}_{\frac{r}{2}}(\xi)$ is a Lipschitz deformed Euclidean ball (see [14, p. 244] and Definition A.6). Since r is bounded and $\tilde{v}$ satisfies the same estimate up to a constant depending on Ω and finally there exists a constant $c_{18} = c_{18}(\Omega) > 0$ such that

$$
v(x) \leq c_{18} v(y) \quad \forall x, y \in B_{\frac{\delta}{2}}(\xi).
$$

The result follows. □

In the case $\kappa < \frac14$, the result holds with minor modifications.

Proposition 2.12. *Let $\delta > 0$ be small enough, $\xi \in \Omega,\ 0 < \kappa < \frac14$ and $u \in H_{loc}^1(B_\delta(\xi)\cap\Omega)\cap C(B_\delta(\xi)\cap\overline{\Omega})$ be a nonnegative $\mathcal{L}_\kappa$-harmonic in $B_\delta(\xi)$ vanishing on $\partial\Omega\cap B_\delta(\xi)$ in the sense that*

$$
\lim_{\mathrm{dist}(x,K)\to 0} \frac{u(x)}{(d(x))^{\frac{\alpha_-}{2}}} = 0 \quad \forall K \subset \partial\Omega\cap B_\delta(\xi),\ K\ \text{compact}. \tag{2.34}
$$

Then there exists $c_{21} = c_{21}(\Omega,\kappa) > 0$ such that

$$
\frac{u(x)}{\phi_\kappa(x)} \leq c_{21}\frac{u(y)}{\phi_\kappa(y)} \quad \forall x, y \in \overline{\Omega}\cap B_{\frac{\delta}{2}}(\xi).
$$

Proof. As in the previous proof we apply [3, Lemma 2.8], we consider a super-solution $\zeta \approx d^{\alpha_-}(1 + c_{19}d^\beta)$ and a sub-$h \approx d^{\alpha_-}(1 - c_{20}d^\beta)$ where $\beta \in (0, \sqrt{1-4\kappa})$. Thus

$$
\frac{\eta u}{\phi_\kappa} \in H_0^1(\Omega, d^{\alpha_+}(x)dx),
$$

where η is a cut-off function adapted to $B_r(\xi)$. The function $\tilde{v} = \phi_\kappa^{-1} u$ satisfies

$$-\phi_\kappa^{-2} \operatorname{div}(\phi_\kappa^2 \nabla \tilde{v}) + \lambda_\kappa \tilde{v} = 0,$$

and $\tilde{v} \in H_0^1(B_{\frac{3\delta}{4}}(\xi), d^{\alpha_+}(x)dx)$. Then the proof follows as in the previous proposition. □

Proposition 2.13. *Let $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ be a $\mathcal{L}_{\frac{1}{4}}$-subharmonic function such that*

$$\limsup_{d(x)\to 0} \frac{u(x)}{d^{\frac{1}{2}}(x)|\log d(x)|} \le 0.$$

Then $u \le 0$.

Proof. We set $v = \max(u, 0)$ and we proceed as in the Step 1 of the proof of Proposition 2.11 with $\eta = 1$. The result follows by letting $s \to 0$. □

Similarly we have

Proposition 2.14. *Let $u \in H^1_{loc}(\Omega) \cap C(\Omega)$ be a $\mathcal{L}_\kappa$-subharmonic function such that*

$$\limsup_{d(x)\to 0} \frac{u(x)}{(d(x))^{\frac{\alpha_-}{2}}} \le 0.$$

Then $u \le 0$.

The two next statements shows that comparison holds provided comparable boundary data are achieved in way which takes into account the specific form of the $\mathcal{L}_\kappa$-harmonic functions.

Proposition 2.15. *Assume $\kappa < \frac{1}{4}$ and $h_i \in H^1(\Omega)$ $(i = 1, 2)$. Let $u_i \in H^1_{loc}(\Omega)$ be two $\mathcal{L}_\kappa$-harmonic functions such that $d^{-\frac{\alpha_+}{2}}\left(u_i - d^{\frac{\alpha_-}{2}} h_i\right) \in H^1(\Omega,\ d^{\alpha_+}(x)dx)$. Then*
If $h_1 \le h_2$ a.e. in Ω, there holds

$$u_1(x) \le u_2(x) \quad \forall x \in \Omega.$$

If $h_1 - h_2 \in H^1_0(\Omega)$, there holds

$$u_1(x) = u_2(x) \quad \forall x \in \Omega.$$

Proof. Set $w = \phi_\kappa^{-1}(u_1 - u_2)$, then $w \in H^1(\Omega, \phi_\kappa^2 dx)$ and

$$-\operatorname{div}(\phi_\kappa^2 \nabla w) + \lambda_\kappa \phi_\kappa^2 w = 0.$$

Since $H^1(\Omega, \phi_\kappa^2 dx) = H^1_0(\Omega, \phi_\kappa^2 dx)$ by (2.5) we derive that w and w_+ belong to $H^1_0(\Omega, \phi_\kappa^2 dx)$ and, integrating by part, we derive $w_+ = 0$. The proof of the second statement is similar. □

In the same way we have in the case $\kappa = \frac{1}{4}$.

Proposition 2.16. *Assume $\kappa = \frac{1}{4}$. Let $h_i \in H^1(\Omega)$ $(i = 1, 2)$ and let $u_i \in H^1_{loc}(\Omega)$ be two $\mathcal{L}_{\frac{1}{4}}$-harmonic functions such that $d^{-\frac{1}{2}}(u_i - d^{\frac{1}{2}}|\log d|h_i) \in H^1(\Omega,\ d(x)dx)$.*

(i) *If $h_1 \le h_2$ a.e. in Ω, then*

$$u_1(x) \le u_2(x) \quad \forall x \in \Omega.$$

(ii) *If $h_1 - h_2 \in H^1_0(\Omega)$, then*

$$u_1(x) = u_2(x) \quad \forall x \in \Omega.$$

We end with existence and uniqueness results for solving the Dirichlet problem associated to $\mathcal{L}_\kappa$.

Proposition 2.17. *Assume $\kappa = \frac{1}{4}$. For any $h \in C(\partial\Omega)$ there exists a unique $\mathcal{L}_{\frac{1}{4}}$-harmonic function u belonging to $H^1_{loc}(\Omega)$ satisfying*

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega} \frac{u(x)}{d^{\frac{1}{2}}(x)|\log d(x)|} = h(y) \quad \text{uniformly for } y \in \partial\Omega.$$

Furthermore there exists a constant $c_{16} = c_{16}(\Omega) > 0 > 0$

$$\left\| \frac{u}{d^{\frac{1}{2}} \left|\log \frac{d}{D_0}\right|} \right\|_{L^\infty(\Omega)} \leq c_{24} \|h\|_{C(\partial\Omega)},$$

where $D_0 = 2\sup_{x\in\Omega} d(x)$.

Proof. Uniqueness is a consequence of Proposition 2.13. For existence let $m \in \mathbb{N}$ and h_n be smooth functions such that $h_m \to h$ in $L^\infty(\partial\Omega)$. Then we can find a function $H_m \in C^2(\overline{\Omega})$ with value h_m on $\partial\Omega$, and $\|H_m\|_{L^\infty(\Omega)} \leq \|h_m\|_{L^\infty(\partial\Omega)}$. By Proposition 2.10 there exists a unique weak solution u_m of $\mathcal{L}_{\frac{1}{4}} u = 0$ satisfying

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega} \frac{u_m}{d^{\frac{1}{2}}|\log d|}(x) = h_m(y) \quad \text{uniformly for } y \in \partial\Omega.$$

By Proposition 2.10 we have

$$\left\| \frac{u_m - u_n}{d^{\frac{1}{2}} \left|\log \frac{d}{D_0}\right|} \right\|_{L^\infty(\Omega)} \leq c_{16} \|h_m - h_n\|_{C(\partial\Omega)}.$$

Thus there exists u such that

$$\lim_{m\to\infty} \left\| \frac{u_m - u}{d^{\frac{1}{2}} \left|\log \frac{d}{D_0}\right|} \right\|_{L^\infty(\Omega)} = 0$$

and u is a solution of $L_{\frac{1}{4}} u = 0$.

Let $x \in \Omega$, with $d(x) < \frac{1}{2}$ and $y \in \partial\Omega$

$$\left|\frac{u}{d^{\frac{1}{2}}|\log d|}(x) - h(y)\right| \leq \left|\frac{u}{d^{\frac{1}{2}}|\log d|}(x) - \frac{u_m}{d^{\frac{1}{2}}|\log d|}(x)\right| + \left|\frac{u_m}{d^{\frac{1}{2}}|\log d|}(x) - h_m(y)\right| + |h(y) - h_m(y)|.$$

The result follows by letting successively $x \to y$ and $m \to \infty$. □

Similarly we have

Proposition 2.18. *Assume* $\kappa < \frac{1}{4}$. *Then for any* $h \in C(\partial\Omega)$ *there exists a unique* $\mathcal{L}_\kappa$*-harmonic function* $u \in H^1_{loc}(\Omega)$ *satisfying*

$$\lim_{x\in\Omega,\ x\to y\in\partial\Omega} \frac{u}{d^{\frac{\alpha_-}{2}}}(x) = h(y) \quad \text{uniformly for } y \in \partial\Omega.$$

Furthermore there exists a constant $c_9 = c_9(\Omega, \kappa) > 0$ *such that*

$$\left\| \frac{u}{d^{\alpha_-}} \right\|_{L^\infty(\Omega)} \leq c_9 \|h\|_{C(\partial\Omega)}.$$

A useful consequence of [3, Lemma 2.8] and Propositions 2.9 and 2.10 is the following local existence result.

Proposition 2.19. *There exists a positive* $\mathcal{L}_\kappa$*-harmonic function* $Z_\kappa \in C(\overline{\Omega_{\beta_0}}) \cap C^2(\Omega_{\beta_0})$ *satisfying*

$$\lim_{d(x)\to 0} \frac{Z_{\frac{1}{4}}(x)}{\sqrt{d(x)}|\ln d(x)|} = 0 \tag{2.35}$$

if $\kappa = \frac{1}{4}$, *and*

$$\lim_{d(x)\to 0} \frac{Z_\kappa(x)}{(d(x))^{\frac{\alpha_-}{2}}} = 0 \tag{2.36}$$

if $0 < \kappa < \frac{1}{4}$.

2.3. $\mathcal{L}_\kappa$-harmonic measure

Let $x_0 \in \Omega$, $h \in C(\partial\Omega)$ and denote $L_{\kappa,x}(h) := v_h(x_0)$ where v_h is the solution of the Dirichlet problem (see Propositions 2.17 and 2.18)

$$\begin{aligned} \mathcal{L}_\kappa v &= 0 \quad \text{in } \Omega \\ v &= h \quad \text{in } \partial\Omega, \end{aligned} \tag{2.37}$$

where v takes the boundary data in the sense of Propositions 2.17 and 2.18. By Propositions 2.14 and 2.13, the mapping $h \mapsto L_{\kappa,x_0}(h)$ is a linear positive functional on $C(\partial\Omega)$. Thus there exists a unique Borel measure on $\partial\Omega$, called $\mathcal{L}_\kappa$*-harmonic measure* in Ω, denoted by ω^{x_0}, such that

$$v_h(x_0) = \int_{\partial\Omega} h(y)d\omega^{x_0}(y).$$

Thanks to Harnack inequality the measures ω^x and ω^{x_0}, x_0, $x \in \Omega$ are mutually absolutely continuous. For every fixed x we denote the Radon–Nikodym derivative by

$$K_{\mathcal{L}_\kappa}(x,y) := \frac{dw^x}{dw^{x_0}}(y) \quad \text{for } \omega^{x_0}\text{-almost all } y \in \partial\Omega.$$

It is classical that the following formula is an equivalent definition of the $\mathcal{L}_\kappa$-harmonic measure: for any closed set $E \subset \partial\Omega$

$$\omega^{x_0}(E) = \inf\left\{\psi :\ \psi \in C_+(\Omega),\ \mathcal{L}_\kappa\text{-superharmonic in } \Omega \text{ s.t. } \liminf_{x\to E}\frac{\psi(x)}{W(x)} \geq 1\right\},$$

where

$$W(x) = \begin{cases} d^{\frac{\alpha_-}{2}}(x) & \text{if } \kappa < \dfrac{1}{4}, \\ d^{\frac{1}{2}}(x)|\log d(x)| & \text{if } \kappa = \dfrac{1}{4}. \end{cases}$$

The extension to open sets is standard. Let $\xi \in \partial\Omega$. We set $\Delta_r(\xi) = \partial\Omega \cap B_r(\xi)$ and $x_r = x_r(\xi) \in \Omega$, such that $d(x_r) = |x_r - \xi| = r$. Also $x_r(\xi) = \xi - r\mathbf{n}_\xi$ where $\mathbf{n}_\xi$ is the unit outward normal vector to $\partial\Omega$ at ξ. We recall that $\beta_0 = \beta_0(\Omega) > 0$ has been defined in Lemma 2.7.

Lemma 2.20. *There exists a constant $c_{25} > 0$ which depends only on Ω and κ such that if $0 < r \leq \beta_0$ and $\xi \in \partial\Omega$, there holds*

$$\frac{\omega^x(\Delta_r(\xi))}{W(x)} \geq c_{25} \quad \forall x \in \Omega \cap B_{\frac{r}{2}}(\xi). \tag{2.38}$$

Proof. Let $h \in C(\partial\Omega)$ be a function with compact support in $\Delta_r(\xi)$, $0 \leq h \leq 1$ and $h = 1$ on $\overline{\Delta_{\frac{3r}{4}}(\xi)}$. And let v_h, v_1 the corresponding $\mathcal{L}_\kappa$-harmonic functions with respective boundary data (in the sense of Propositions 2.17 and 2.18) h and 1. Then $v_1(x) \geq v_h(x) \geq 0$ and

$$\lim_{x\in\Omega,\ x\to x_0} \frac{v_1(x) - v_h(x)}{W(x)} = 0 \quad \forall x_0 \in \Omega \cap B_{\frac{3r}{4}}(\xi).$$

By Propositions 2.12 and 2.11, and $\phi_\kappa \approx d^{\frac{\alpha_+}{2}}$, there exists $c_{26} = c_{26}(\Omega,\kappa) > 0$ such that

$$\frac{v_1(x) - v_h(x)}{d^{\frac{\alpha_+}{2}}(x)} \leq c_{26}\frac{v_1(y) - v_h(y)}{d^{\frac{\alpha_+}{2}}(y)} \quad \forall x, y \in \Omega \cap \overline{B_{\frac{r}{2}}}(\xi).$$

We consider first the case $\kappa = \frac{1}{4}$. By Proposition 2.10, we have

$$0 \leq \frac{v_1(y) - v_h(y)}{d^{\frac{1}{2}}(y)} \leq \frac{v_1(y)}{d^{\frac{1}{2}}(y)} \leq c_{24}|\log d(y)|.$$

Thus, combining all above we have that

$$\frac{v_1(x)}{d^{\frac{1}{2}}(x)|\log d(x)|} - c_{27}\frac{|\log d(y)|}{|\log d(x)|} \leq \frac{v_h(x)}{d^{\frac{1}{2}}(x)|\log d(x)|}.$$

Now by Proposition 2.10, there exists $\varepsilon_0 > 0$ such that

$$\frac{v_1(x)}{d^{\frac{1}{2}}(x)|\log d(x)|} > \frac{1}{2} \quad \forall x \in \Omega_{\varepsilon_0}.$$

Thus if we choose y such that $d(y)=\frac{r}{4}$, there exists a constant $c_{27}=c_{27}(\Omega,\kappa)>0$ such that

$$c_{27}\frac{|\log d(y)|}{|\log d(x)|}=c_{27}\frac{\left|\log\frac{r}{4}\right|}{|\log d(x)|}\leq c_{27}\frac{\left|\log\frac{r}{4}\right|}{\left|\log\frac{r}{D_0}\right|}\leq\frac{1}{4}\quad\forall x\in\Omega_{\frac{r}{D_0}},$$

thus

$$\frac{v_h(x)}{d^{\frac{1}{2}}(x)|\log d(x)|}\geq\frac{1}{4}\quad\forall x\in\overline{B_{\frac{r}{2}}}(\xi)\cap\Omega_{\frac{r}{D_0}}.\tag{2.39}$$

In particular

$$\frac{v_h(x_{a^*r}(\xi))}{\sqrt{a^*r}|\log(a^*r)|}\geq\frac{1}{4},\tag{2.40}$$

where $a^*=(\max\{2,D_0\})^{-1}$. If $D_0\leq 2$ we obtain the claim. If $D_0>2$, set $k^*=\mathbb{E}[\frac{D_0}{2}]+1$ (we recall that $\mathbb{E}[x]$ denotes the largest integer less or equal to x). If $x\in\overline{B_{\frac{r}{2}}}(\xi)\cap\Omega'_{\frac{r}{D_0}}$ there exists a chain of at most $4k^*$ points $\{z_j\}_{j=0}^{j=j_0}$ such that $z_j\in\overline{B_{\frac{r}{2}}}(\xi)\cap\Omega$, $d(z_j)\geq a^*r$, $z_0=x_{a^*r}(\xi)$, $z_{j_0}=x$ and $|z_j-z_{j+1}|\leq\frac{a^*r}{4}$. By Harnack inequality (applied j_0-times)

$$v_h(x_{a^*r}(\xi))\leq c_{28}v_h(x).\tag{2.41}$$

Since

$$W(x_{a^*r}(\xi))\geq\left(a^*\right)^{\frac{1}{2}}W(x),$$

we obtain finally

$$\frac{1}{4}\leq\frac{\omega^{x_{a^*r}(\xi)}(\Delta_r(\xi))}{\sqrt{a^*r}|\log(a^*r)|}\leq c_{28}\left(\frac{1}{a^*}\right)^{\frac{1}{2}}\frac{\omega^x(\Delta_r(\xi))}{W(x)}\quad\forall x\in\Omega\cap B_{\frac{r}{2}}(\xi).\tag{2.42}$$

In the case $\kappa<\frac{1}{4}$, the proof is simpler since no log term appears and we omit it. □

The next result is a Carleson type estimate valid for positive $\mathcal{L}_\kappa$-harmonic functions.

Lemma 2.21. *There exists a constant c_{29} which depends on Ω and κ such that for any $\xi\in\partial\Omega$ and $0<r\leq s\leq\beta_0$,*

$$\frac{\omega^x(\Delta_r(\xi))}{W(x)}\leq c_{29}\frac{\omega^{x_s(\xi)}(\Delta_r(\xi))}{W(x_s(\xi))}\quad\forall x\in\Omega\setminus B_s(\xi).\tag{2.43}$$

Proof. Let $h\in C(\partial\Omega)$ with compact support in $\Delta_r(\xi)$ and $0\leq h\leq 1$. We denote by v_h, v_1, the solutions of (2.37) with boundary data h and 1 respectively. By Propositions 2.17 and 2.18 there exists a constant $c_{30}>0$ such that for $0<r<\beta_0$,

$$\frac{v_h}{W(x)}\leq\frac{\omega^x(\Delta_r(\xi))}{W(x)}\leq\frac{\omega^x(\partial\Omega)}{W(x)}\leq c_{30}\quad\forall x\in\Omega.\tag{2.44}$$

By Propositions 2.17 and 2.18, there holds

$$\lim_{d(x)\to 0}\frac{v_1(x)}{W(x)}=1,\tag{2.45}$$

thus we can replace W by v_1 in (2.43). Since $w_h=\frac{v_h(x)}{v_1(x)}$ is Hölder continuous in $\overline{\Omega}$ and satisfies

$$\begin{aligned}-\operatorname{div}(v_1^2\nabla w_h)&=0 &&\text{in }\Omega\setminus\overline{B_s}(\xi)\\ 0\leq w_h&\leq 1 &&\text{in }\Omega\setminus\overline{B_s}(\xi)\\ w_h&=0 &&\text{in }\partial\Omega\setminus\overline{B_s}(\xi),\end{aligned}\tag{2.46}$$

the maximum of w_h is achieved on $\Omega\cap\partial B_s(\xi)$, therefore it is sufficient to prove the Carleson estimate

$$w_h(x)\leq c_{29}w_h(x_s(\xi))\quad\forall x\in\Omega\cap\partial B_s(\xi).\tag{2.47}$$

If x such that $|x-\xi|=s$ is "far" from $\partial\Omega$, $w_h(x)$ is "controlled" by $w_h(x_s(\xi))$ thanks to Harnack inequality, while if it is close to $\partial\Omega$, $w_h(x)$ is "controlled" by the fact that it vanishes on $\partial\Omega\cap\partial B_s(\xi)$.

We also note that (2.38) can be written under the form

$$w_h(x)\geq c_{25}\quad\forall x\in\Omega\cap B_{\frac{r}{2}}(\xi).\tag{2.48}$$

Step 1: $r \leq s \leq 4r$. By Lemma 2.20, (2.44) and the above inequality we have that

$$w_h\left(x_{\frac{r}{2}}(\xi)\right) \geq \frac{c_{25}}{c_{30}} w_h(x) \quad \forall x \in \Omega.$$

Applying Harnack inequality to w_h in the balls $B_{\frac{(2+j)r}{4}}(x_{\frac{(2+j)r}{4}}(\xi))$ for $j = 0, \dots, j_0 \leq 14$, we obtain

$$w_h\left(x_{\frac{(2+j)r}{4}}(\xi)\right) \geq c_{31}^j w_h\left(x_{\frac{r}{2}}(\xi)\right) \quad \text{for } j = 1, \dots, j_0.$$

This implies

$$w_h(x_s(\xi)) \geq c_{32} w_h(x) \quad \forall x \in \Omega. \tag{2.49}$$

Step 2: $\beta_0 \geq s > 4r$. We apply Propositions 2.11, 2.12 to w_h in $B_{\frac{s}{2}}(\xi_1) \cap \Omega$ where $\xi_1 \in \partial\Omega$ is such that $|\xi - \xi_1| = s$ and we get

$$w_h(x) \leq c_{18} w_h\left(x_{\frac{s}{4}}(\xi_1)\right) \quad \forall x \in B_{\frac{s}{4}}(\xi_1) \cap \Omega. \tag{2.50}$$

Then we apply six times Harnack inequality to w_h between $x_{\frac{s}{4}}(\xi_1)$ and $x_s(\xi)$ and obtain

$$w_h\left(x_{\frac{s}{4}}(\xi_1)\right) \leq c_{33} w_h(x_s(\xi_1)). \tag{2.51}$$

Combining (2.50) and (2.51) we derive (2.47).

Step 3. For $\epsilon > 0$, set $z_h = w_h - c_{33} w_h(x_s(\xi)) - \epsilon$. Then z_h^+ has compact support in $\Omega \setminus B_s(\xi)$ and thus belongs to $H_0^1(\Omega \setminus B_s(\xi))$. Integration by parts in (2.46) leads to

$$\int_{\Omega \setminus B_s(\xi)} v_1^2 |\nabla z_h^+|^2 dx = 0. \tag{2.52}$$

Then $z_h^+ = 0$ by letting $\epsilon \to 0$. Combining with (2.49) and $h \uparrow \chi_{\Delta_r(\xi)}$ implies (2.43). $\square$

Theorem 2.22. *There exists a constant c_{34} which depends on Ω and κ such that, for any $0 < r \leq \beta_0$ and $\xi \in \partial\Omega$, there holds*

$$\frac{1}{c_{34}} r^{N-1-\frac{1}{2}} |\log r| G_{\mathcal{L}_{\frac{1}{4}}}(x_r(\xi), x) \leq \omega^x(\Delta_r(\xi)) \leq c_{34} r^{N-1-\frac{1}{2}} |\log r| G_{\mathcal{L}_{\frac{1}{4}}}(x_r(\xi), x) \quad \forall x \in \Omega \setminus B_{4r}(\xi). \tag{2.53}$$

Proof. Let $\eta \in C_0^\infty(B_{2r}(\xi))$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_r(\xi)$. We set

$$u = \eta(-\ln d)\sqrt{d} := \eta\psi,$$

(we assume that $4r < 1$), in order to have

$$\lim_{x \to x_0} \frac{u(x)}{\psi(x)} = \eta\lfloor_{\partial\Omega}(x_0) = \zeta(x_0) \quad \forall x_0 \in \partial\Omega,$$

uniformly with respect to x_0. Since

$$-\Delta\psi - \frac{1}{4}\frac{\psi}{d^2(x)} = \frac{2 + \ln d}{2\sqrt{d}} \Delta d = -(N-1)\frac{2 + \ln d}{2\sqrt{d}} K,$$

where K is the mean curvature of $\partial\Omega$. We have also

$$|\nabla\eta| \leq c_0 \chi_{\Omega \cap B_{2r}(\xi)} \frac{1}{r} \quad \text{and} \quad |\Delta\eta(x)| \leq c_0 \chi_{\Omega \cap B_{2r}(\xi)} \frac{1}{r^2} \leq c_0 \chi_{\Omega \cap B_{2r}(\xi)} \frac{1}{r} d^{-1}(x),$$

thus u satisfies

$$\begin{aligned} -\Delta u - \frac{1}{4}\frac{u}{d^2(x)} = -\psi\Delta\eta + \frac{2 + \ln d}{2\sqrt{d}} (2\nabla d \cdot \nabla\eta - (N-1)K\eta) &:= f \quad \text{in } \Omega \\ u &= \zeta \quad \text{on } \partial\Omega. \end{aligned}$$

Furthermore $|f| \leq \frac{c_{35}}{r}(-\frac{\ln d}{\sqrt{d}})\chi_{\Omega \cap B_{2r}(\xi)}$ since η vanishes outside $B_{2r}(\xi)$. We have by the representation formula [14]

$$0 = u(x) = \int_\Omega G_{\mathcal{L}_{\frac{1}{4}}}(x, y) f dy + \int_{\partial\Omega} h(y) d\omega^x(y) \quad \forall x \in \Omega \setminus B_{2r}(\xi). \tag{2.54}$$

By Proposition 2.1, we have that for any $x \in \Omega \setminus B_{4r}(\xi)$ and $y \in B_{2r}(\xi)$

$$G_{\mathcal{L}_{\frac{1}{4}}}(x,y) \leq c_{36} G_{\mathcal{L}_{\frac{1}{4}}}(x, x_r(\xi)),$$

thus

$$\begin{aligned}\omega^x(\Delta_r(\xi)) &\leq \int_{\Omega\cap B_{2r}(\xi)} G_{\mathcal{L}_{\frac{1}{4}}}(x,y)|f(y)|dy \\ &\leq \frac{c_{37}}{r} G_{\mathcal{L}_{\frac{1}{4}}}(x, x_r(\xi)) \int_{\Omega\cap B_{2r}(\xi)} \frac{|\ln d(y)|}{\sqrt{d(y)}} dy \\ &\leq c_{38} G_{\mathcal{L}_{\frac{1}{4}}}(x, x_r(\xi)) r^{N-1-\frac{1}{2}} |\ln r|,\end{aligned} \tag{2.55}$$

since

$$\int_{\Omega\cap B_{2r}(\xi)} \frac{|\ln d(y)|}{\sqrt{d(y)}} dy \leq c_{39} r^{N-1} \int_0^{2r} \frac{|\ln t| dt}{\sqrt{t}} \leq 2c_{39} r^{N-\frac{1}{2}} |\ln r|.$$

This implies the right-hand side part of (2.53). For the opposite inequality we observe that if $x \in \partial B_{4r}(\xi) \cap \Omega$, there holds by (2.38)

$$\begin{aligned} r^{N-1-\frac{1}{2}} |\log r| G_{\mathcal{L}_{\frac{1}{4}}}(x_r(\xi), x) &\leq c_{40} r^{N-1-\frac{1}{2}} |\log r| \min\left\{ \frac{1}{|x - x_r(\xi)|^{N-2}}, \frac{\sqrt{d(x)}\sqrt{d(x_r(\xi))}}{|x - x_r(\xi)|^{N-1}} \right\} \\ &\leq c_{41} \sqrt{d(x)} |\log r| \\ &\leq c_{42} W(x) \\ &\leq \frac{c_{42}}{c_{25}} \omega^{x_{\frac{r}{8}}(\xi)}(\Delta_r(\xi)). \end{aligned}$$

We end the proof by Harnack inequality between $\omega^{x_{\frac{r}{8}}(\xi)}(\Delta_r(\xi))$ and $\omega^{x_{4r}(\xi)}(\Delta_r(\xi))$ and by Harnack inequality between $\omega^x(\Delta_r(\xi))$ and $\omega^{x_{4r}(\xi)}(\Delta_r(\xi))$ on $\partial B_{4r}(\xi)$ and an argument like in the step 3 in Lemma 2.21. □

Replacing, in the last proof, the function $\psi = \sqrt{d}(-\ln d)$ by $\tilde{\psi} = d^{\frac{\alpha_-}{2}}$, we obtain similarly the following two-side estimate.

Theorem 2.23. *Assume $\kappa < \frac{1}{4}$. There exists a constant c_{42} which depends only on Ω and κ such that, for any $0 < r \leq \beta_0$ and $\xi \in \partial\Omega$, there holds*

$$\frac{1}{c_{42}} r^{N-2+\frac{\alpha_-}{2}} G_{\mathcal{L}_\kappa}(x_r(\xi), x) \leq \omega^x(\Delta_r(\xi)) \leq c_{42} r^{N-2+\frac{\alpha_-}{2}} G_{\mathcal{L}_\kappa}(x_r(\xi), x) \quad \forall x \in \Omega \setminus B_{4r}(\xi).$$

As a consequence of Theorems 2.22 and 2.23 and the Harnack inequality, the harmonic measure for $\mathcal{L}_\kappa$ possesses the doubling property.

Theorem 2.24. *Let $0 < \kappa \leq \frac{1}{4}$. There exists a constant c_{42} which depends only on Ω, κ such that for any $0 < r \leq \beta_0$, there holds*

$$\omega^x(\Delta_{2r}(\xi)) \leq c_{42} \omega^x(\Delta_r(\xi)) \quad \forall x \in \Omega \setminus B_{4r}(\xi).$$

The next result will be useful in the study of the Poisson kernel of $\mathcal{L}_\kappa$.

Lemma 2.25. *Let $0 < r \leq \beta_0$ and u be a positive $\mathcal{L}_\kappa$-harmonic function such that*

(i) $u \in C(\overline{\Omega \setminus B_r(\xi)})$,

(ii)

$$\lim_{x\to x_0} \frac{u(x)}{W(x)} = 0 \quad \forall x_0 \in \Omega \setminus \overline{B_r(\xi)},$$

uniformly with respect to x_0.

Then

$$c_{42}^{-1} \frac{u(x_r(\xi))}{W(x_r(\xi))} w^x(\Delta_r(\xi)) \leq u(x) \leq c_{42} \frac{u(x_r(\xi))}{W(x_r(\xi))} w^x(\Delta_r(\xi)) \quad \forall x \in \Omega \setminus \overline{B_{2r}(\xi)},$$

with c_{42} depends only on κ and Ω.

Proof. It follows from Propositions 2.11 and 2.12 that there exists $C > 0$ such that

$$\frac{1}{C}\frac{u(x_{2r}(\xi))}{w^{x_{2r}(\xi)}(\Delta_r(\xi))} \le \frac{u(x)}{w^x(\Delta_r(\xi))} \le C\frac{u(x_{2r}(\xi))}{w^{x_{2r}(\xi)}(\Delta_r(\xi))} \quad \forall x \in \Omega \cap \partial B_{2r}(\xi).$$

Applying Harnack inequality between $x_{2r}(\xi)$ and $x_r(\xi)$ we obtain

$$\frac{1}{C}\frac{u(x_r(\xi))}{w^{x_r(\xi)}(\Delta_r(\xi))} \le \frac{u(x)}{w^x(\Delta_r(\xi))} \le C\frac{u(x_r(\xi))}{w^{x_r(\xi)}(\Delta_r(\xi))} \quad \forall x \in \Omega \cap \partial B_{2r}(\xi).$$

Also by Harnack inequality we have that

$$w^{x_r(\xi)}(\Delta_r(\xi)) \ge Cw^{x_{\frac{r}{2}}(\xi)}(\Delta_r(\xi)) > C_0 W(x_r(\xi)),$$

where in the last inequality above we have used Lemma 2.20.

Combining all the above inequalities, we derive

$$C^{-1}\frac{u(x_r(\xi))}{W(x_r(\xi))}w^x(\Delta_r(\xi)) \le u(x) \le C\frac{u(x_r(\xi))}{W(x_r(\xi))}w^x(\Delta_r(\xi)) \quad \forall x \in \Omega \cap \partial B_{2r}(\xi).$$

The result follows by an argument similar to step 3 in Lemma 2.21. □

2.4. The Poisson kernel of $\mathcal{L}_\kappa$

In this section we establish some properties of the Poisson kernel associated to $\mathcal{L}_\kappa$.

Definition 2.26. Fix $\xi \in \partial\Omega$. A function K defined in Ω is called a kernel function at ξ with pole at $x_0 \in \Omega$ if

(i) $K(\cdot,\xi)$ is $\mathcal{L}_\kappa$-harmonic in Ω,
(ii) $K(\cdot,\xi) \in C(\overline{\Omega}\setminus\{\xi\})$ and for any $\eta \in \partial\Omega\setminus\{\xi\}$

$$\lim_{x\to\eta}\frac{K(x,\xi)}{W(x)} = 0,$$

(iii) $K(x,\xi) > 0$ for each $x \in \Omega$ and $K(x_0,\xi) = 1$.

Proposition 2.27. *There exists one and only one kernel function for $\mathcal{L}_\kappa$ at ξ with pole at x_0.*

Proof. The proof is similar as the one of [8, Th. 3.1] and we indicate it for the sake of completeness. Set

$$u_n(x) = \frac{w^x(\Delta_{2^{-n}}(\xi))}{w^{x_0}(\Delta_{2^{-n}}(\xi))}.$$

Since

$$u_n \ge 0,$$

$\mathcal{L}_\kappa u_n = 0$ in Ω and $u_n(x_0) = 1$ the sequence $\{u_n\}$ is locally bounded in Ω by Harnack inequality. Hence we can find a subsequence, again denoted by $\{u_n\}$, which converges to a function u, locally uniformly in Ω.

It is clear that $u \ge 0$ in Ω and $\mathcal{L}_\kappa u = 0$ in Ω. Since $u(x_0) = 1$, u is strictly positive in Ω. Now fix $P \in \partial\Omega$ and $P \ne \xi$. Let $n_0 \in \mathbb{N}$ be such that $P \in \Omega\setminus\overline{B_{2^{n+1}}(\xi)}$, $\forall n \ge n_0$. By Lemma 2.25 if we take n_0 sufficiently large, we have

$$u_n(x) \le c_{42}\frac{u_n(x_{2^{-n_0}}(\xi))}{W(x_{2^{-n_0}}(\xi))}w^x(\Delta_{2^{-n_0}}(\xi)) \quad \forall x \in \Omega\setminus\overline{B_{2^{-n_0+1}}(\xi)},$$

which implies

$$u(x) \le c_{42}\frac{u(x_{2^{-n_0}}(\xi))}{W(x_{2^{-n_0}}(\xi))}w^x(\Delta_{2^{-n_0}}(\xi)) \quad \forall x \in \Omega\setminus\overline{B_{2^{-n_0+1}}(\xi)},$$

and thus

$$\lim_{x\to P}\frac{u(x)}{W(x)} = 0.$$

We now turn to the question of uniqueness of the kernel function. Let us consider two arbitrary kernel functions f and g for $\mathcal{L}_\kappa$ in Ω at ξ. By Lemma 2.25 and the properties of $f,\ g$ there holds

$$\frac{1}{c_{42}^2}\frac{f(x_r(\xi))}{g(x_r(\xi))} \le \frac{f(x)}{g(x)} \le c_{42}^2\frac{f(x_r(\xi))}{g(x_r(\xi))} \quad \forall x \in \Omega\setminus\overline{B_{2r}(\xi)}.$$

In particular we can obtain if we take $x = x_0$

$$\frac{f(x_r(\xi))}{g(x_r(\xi))} \leq c_{42}^2,$$

and we obtain, using again Harnack,

$$\frac{f(x)}{g(x)} \leq c_{42}^3 := c \quad \forall x \in \Omega.$$

We derive that *for any two kernel functions* f and g for $\mathcal{L}_\kappa$ at ξ there holds

$$f(x) \leq cg(x) \leq c^2 f(x) \quad \forall x \in \Omega. \tag{2.56}$$

Obviously $c \geq 1$. If $c = 1$ the result is proved. If $c > 1$ then $f + A(f - g)$ is also a Kernel function for $\mathcal{L}_\kappa$ at ξ with $A = \frac{1}{c-1}$. Since (2.56) holds for any kernel functions,

$$g \leq c(f + A(f - g)),$$

and therefore

$$f + A(f - g) + A(f + A(f - g)),$$

is a kernel function at ξ. Proceeding in the above manner and by induction we conclude that for each positive integer k there exists nonnegative numbers $a_{1k}, \ldots, a_{kk}$ such that

$$f + \left(kA + \sum_{i=1}^{k} a_{ik}\right)(f - g)$$

is a kernel function at ξ. Hence

$$f + \left(kA + \sum_{i=1}^{k} a_{ik}\right)(f - g) \leq c^2 f.$$

This last inequality can hold for all k only if $f \equiv g$. □

We recall here that we denote by

$$K_{\mathcal{L}_\kappa}(x, \xi) := \frac{dw^x}{dw^{x_0}}(\xi) \quad \text{for } \omega^{x_0}\text{-almost all } \xi \in \partial\Omega,$$

the kernel function in Ω. Also in view of the proof of Proposition 2.27 and by uniqueness we can write

$$K_{\mathcal{L}_\kappa}(x, \xi) = \lim_{r \to 0} \frac{w^x(\Delta_r(\xi))}{w^{x_0}(\Delta_r(\xi))} \quad \text{for } \omega^{x_0}\text{-almost all } \xi \in \partial\Omega.$$

Proposition 2.28. *For any $x \in \Omega$, the function $\xi \mapsto K_{\mathcal{L}_\kappa}(x, \xi)$ is continuous on $\partial\Omega$.*

Proof. The proof is an adaptation of the one of [8, Corollary 3.2]. Suppose that $\xi_n \to \xi$ as $n \to \infty$. Then the sequence, $K(\cdot, \xi_n)$, of positive solutions of $\mathcal{L}_\kappa u = 0$ has a subsequence which converges locally uniformly in Ω to a function which must be a positive solution of $\mathcal{L}_\kappa u = 0$ in Ω. Outside any fixed neighborhood, B, of ξ, $\frac{K_{\mathcal{L}_\kappa}(x,\xi_n)}{W(x)}$ converges to zero uniformly in n as $x \to P \in \partial\Omega \setminus B$. Hence the limit function of the subsequence is the kernel function for $\mathcal{L}_\kappa$ at ξ. By uniqueness of the kernel function we conclude that the convergence

$$K_{\mathcal{L}_\kappa}(x, \xi_n) \to K_{\mathcal{L}_\kappa}(x, \xi)$$

holds for the entire sequence $\{\xi_n\}$. □

We can now identify the Martin boundary and topology with their classical analogues. We begin by recalling the definitions of the Martin boundary and related concepts. For $x, \; y \in \Omega$ we set

$$\mathcal{K}_\kappa(x, y) := \frac{G_{\mathcal{L}_\kappa}(x, y)}{G_{\mathcal{L}_\kappa}(x_0, y)}.$$

Consider the family of sequences $\{y_k\}_{k\geq 1}$ of points of Ω without cluster points in Ω for which $\mathcal{K}_\kappa(x, y_k)$ converges in Ω to a harmonic function, denoted $\mathcal{K}_\kappa(x, \{y_k\})$. Two such sequences y_k and y'_k are called equivalent if $\mathcal{K}_\kappa(x, \{y_k\}) = \mathcal{K}_\kappa(x, \{y'_k\})$ and each equivalence class is called an element of the Martin boundary Γ. If Y is such an equivalence class (i.e., $Y \in \Gamma$) then

$\mathcal{K}_\kappa(x,Y)$ will denote the corresponding harmonic limit function. Thus each $Y\in\Omega\cup\Gamma$ is associated with a unique function $\mathcal{K}_\kappa(x,Y)$. The Martin topology on $\Omega\cup\Gamma$ is given by the metric

$$\rho(Y,Y')=\int_A\frac{|\mathcal{K}_\kappa(x,Y)-\mathcal{K}_\kappa(x,Y')|}{1+|\mathcal{K}_\kappa(x,Y)-\mathcal{K}_\kappa(x,Y')|}dx\quad Y,Y'\in\Omega\cup\Gamma,$$

where A is a small enough neighborhood of x_0. $\mathcal{K}_\kappa(x,Y)$ is a ρ-*continuous* function of $Y\in\Omega\cup\Gamma$ for $xin\Omega$ fixed, $\Omega\cup\Gamma$ is compact and complete with respect to ρ, $\Omega\cup\Gamma$ is the ρ-closure of Ω and the ρ-topology is equivalent to the Euclidean topology in Ω. We have the following results.

Proposition 2.29. *There is a one-to-one correspondence between the Martin boundary of Ω and the Euclidean boundary $\partial\Omega$. If $Y\in\Gamma$ corresponds to $\xi\in\partial\Omega$ then $\mathcal{K}_\kappa(x,Y)=K_{\mathcal{L}_\kappa}(x,\xi)$. The Martin topology on $\Omega\cup\Gamma$ is equivalent to the Euclidean topology on $\Omega\cup\partial\Omega$.*

Proof. The proof is similar as the one of Theorem 4.2 in [19] and we recall it for the sake of completeness. By uniqueness of the kernel function we have that

$$\mathcal{K}_\kappa(x,\{y_k\})=K_{\mathcal{L}_\kappa}(x,\xi),$$

where $\{y_k\}$ is a sequence in Ω such that $y_k\to\xi\in\partial\Omega$. It follows that each point of Γ may be associated with a point of $\partial\Omega$. Lemma 2.25 clearly shows that $K_{\mathcal{L}_\kappa}(\cdot,\xi)\neq K_{\mathcal{L}_\kappa}(\cdot,\xi')$ if $\xi\neq\xi'$. Hence, the functions $\mathcal{K}_\kappa(x,y_k)$ cannot converge if the sequence y_k has more than one cluster point on $\partial\Omega$ and different points of $\partial\Omega$ must be associated with different points of Γ. This gives a one-to-one correspondence between $\partial\Omega$ and Γ with $\mathcal{K}_\kappa(x,Y)=K_{\mathcal{L}_\kappa}(x,\xi)$ when $Y\in\Gamma$ corresponds to $\xi\in\partial\Omega$. If $y_k\to\xi$ in the Euclidean topology then $\mathcal{K}_\kappa(x,Y_k)\to\mathcal{K}_\kappa(x,Y)$ and, therefore, $Y_k\to Y$ in the ρ-topology by Lebesgue's dominated convergence theorem. On the other hand suppose $Y_k\to Y$ in the ρ-topology. If ξ_k does not converge to ξ in the Euclidean topology there is a subsequence ξ_{k_j} such that $\xi_{k_j}\to\xi'\neq\xi$ in the Euclidean topology. Then $Y_{k_j}\to Y'$ and $Y_{k_j}\to Y$ in the ρ-*topology* with $Y\neq Y'$, which is impossible. Therefore, the Martin ρ-topology on $\Omega\cup\Gamma$ is equivalent to the Euclidean topology on $\Omega\cap\partial\Omega$. $\square$

By Propositions 2.29 and 2.1 we have the following result.

Theorem 2.30. *Assume $0<\kappa\leq\frac{1}{4}$. There exists a positive constant c_{43} such that*

$$\frac{1}{c_{43}}\frac{d^{\frac{\alpha_+}{2}}(y)}{|\xi-y|^{N+\alpha_+-2}}\leq K_{\mathcal{L}_\kappa}(y,\xi)\leq c_{43}\frac{d^{\frac{\alpha_+}{2}}(y)}{|\xi-y|^{N+\alpha_+-2}}.\tag{2.57}$$

Let us give a lemma that we will use to prove the representation formula.

Lemma 2.31. *Let $\xi\in\partial\Omega$, $r>0$ be small enough and u be a positive $\mathcal{L}_\kappa$-harmonic function in Ω. There exists a super $\mathcal{L}_\kappa$-harmonic function V such that*

$$V(x)=\begin{cases}v(x) & \text{in }\Omega\setminus B_r(\xi)\\ u(x) & \text{in }\Omega\cap\overline{B}_r(\xi),\end{cases}$$

where v satisfies

$$\begin{aligned}\mathcal{L}_\kappa v&=0 && \text{in }\Omega\setminus B_r(\xi)\\ \lim_{x\to y}v(x)&=u(y) && \forall y\in\partial B_r(\xi)\cap\Omega\\ \lim_{x\to y}\frac{v(x)}{W(x)}&=0 && \forall y\in\partial\Omega\setminus\overline{B}_r(\xi).\end{aligned}\tag{2.58}$$

Proof. The function u is C^2 in Ω since it is $\mathcal{L}_\kappa$-harmonic. Let $\xi_0\in B_r(\xi)\cap\Omega$, and r_0 be such that $B_{r_0}(\xi_0)\subset\Omega$. We consider the problem

$$\begin{aligned}&\mathcal{L}_\kappa w=0,\quad \text{in }\Omega\setminus\overline{B}_r(\xi)\\ &\lim_{x\to y}w(x)=\eta(y)w(y)\quad\forall y\in\partial B_r(\xi)\cap\Omega\\ &\lim_{x\to y}\frac{w(x)}{W(x)}=0,\quad\forall y\in\partial\Omega\setminus\overline{B}_r(\xi),\end{aligned}$$

where $\eta\in C_0^\infty(B_{\frac{r_0}{2}}(\xi_0))$, $0\leq\eta\leq1$. In view of the proof of Propositions 2.9 and 2.10 we can find a positive solution of the above problem w. Also we note here that $w\leq u$, and by Harnack inequalities 2.11 and 2.12, we have that for any $\zeta\in\partial\Omega$

$$\frac{w(x)}{\phi_\kappa(x)}\leq C(\kappa,N,\Omega)\frac{w(y)}{\phi_\kappa(y)}\quad\forall x,y\in B_\rho(\zeta),$$

where $\rho \leq \frac{1}{2}\mathrm{dist}(\zeta, \partial B_r(\xi))$. Thus we derive

$$\frac{w(x)}{\phi_\kappa(x)} \leq C(\kappa, N, \Omega)\frac{u(y)}{\phi_\kappa(y)} \quad \forall x, y \in B_\rho(\zeta).$$

The remaining of the proof is standard and we omit it. □

We consider a increasing sequence of bounded smooth domains $\{\Omega_n\}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\cup_n \Omega_n = \Omega$ and $\mathcal{H}^{N-1}(\Omega_n) \to \mathcal{H}^{N-1}(\Omega)$. Such a sequence is a *smooth exhaustion* of Ω. For each n, the operator $\mathcal{L}_\kappa^{\Omega_n}$ defined by

$$\mathcal{L}_\kappa^{\Omega_n} u = -\Delta u - \frac{\kappa}{d^2(x)}u \tag{2.59}$$

is uniformly elliptic and coercive in $H_0^1(\Omega_n)$ and its first eigenvalue $\lambda_\kappa^{\Omega_n}$ is larger than λ_κ. If $h \in C(\partial\Omega_n)$ the following problem

$$\begin{aligned} \mathcal{L}_\kappa^{\Omega_n} v &= 0 \qquad \text{in } \Omega_n \\ v &= h \qquad \text{on } \partial\Omega_n, \end{aligned} \tag{2.60}$$

admits a unique solution which allows to define the $\mathcal{L}_\kappa^{\Omega_n}$-harmonic measure on $\partial\Omega_n$ by

$$v(x_0) = \int_{\partial\Omega_n} h(y) d\omega_{\Omega_n}^{x_0}(y). \tag{2.61}$$

Thus the Poisson kernel of $\mathcal{L}_\kappa^{\Omega_n}$ is

$$K_{\mathcal{L}_\kappa^{\Omega_n}}(x, y) = \frac{d\omega_{\Omega_n}^{x}}{d\omega_{\Omega_n}^{x_0}}(y) \quad \forall y \in \partial\Omega_n. \tag{2.62}$$

Proposition 2.32. *Assume* $0 < \kappa \leq \frac{1}{4}$ *and let* $x_0 \in \Omega_1$. *Then for every* $Z \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z(x)W(x) d\omega_{\Omega_n}^{x_0}(x) = \int_{\partial\Omega} Z(x) d\omega^{x_0}(x). \tag{2.63}$$

Proof. We recall that $d \in C^2(\overline{\Omega}_\varepsilon)$ for any $0 < \varepsilon \leq \beta_0$ and let $n_0 \in \mathbb{N}$ be such that

$$\mathrm{dist}(\partial\Omega_n, \partial\Omega) < \frac{\beta_0}{2} \quad \forall n \geq n_0.$$

For $n \geq n_0$ let w_n be the solution of

$$\begin{aligned} \mathcal{L}_\kappa^{\Omega_n} w_n &= 0 \qquad \text{in } \Omega_n \\ w_n &= W \qquad \text{on } \partial\Omega_n. \end{aligned} \tag{2.64}$$

It is straightforward to see that the proof of Propositions 2.17 and 2.18 it is inferred that there exists a positive constant $c_{44} = c_{44}(\Omega, \kappa)$ such that

$$\|w_n\|_{L^\infty(\Omega_n)} \leq c_{44} \quad \forall n \geq n_0.$$

Furthermore

$$w_n(x_0) = \int_{\partial\Omega_n} W(x) d\omega_{\Omega_n}^{x_0}(x) < c_{45}. \tag{2.65}$$

We extend $\omega_{\Omega_n}^{x_0}$ as a Borel measure on $\overline{\Omega}$ by setting $\omega_{\Omega_n}^{x_0}(\overline{\Omega} \setminus \Omega_n) = 0$, and keep the notation $\omega_{\Omega_n}^{x_0}$ for the extension. Because of (2.65) the sequence $\{W\omega_{\Omega_n}^{x_0}\}$ is bounded in the space $\mathfrak{M}_b(\overline{\Omega})$ of bounded Borel measures in $\overline{\Omega}$. Thus there exists a subsequence still denoted by $\{W(x)\omega_{\Omega_n}^{x_0}\}$ which converges narrowly to some positive measure, say $\widetilde{\omega}$ which is clearly supported by $\partial\Omega$ and satisfies $\|\widetilde{\omega}\|_{\mathfrak{M}_b} \leq c_{45}$ as in (2.65). For every $Z \in C(\overline{\Omega})$ there holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Z(x)W d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} Z d\widetilde{\omega}.$$

Let $\zeta := Z\lfloor_{\partial\Omega}$ and

$$z(x) := \int_{\partial\Omega} K_{\mathcal{L}_\kappa}(x, y)\zeta(y) d\omega^{x_0}(y).$$

Then

$$\lim_{d(x)\to 0}\frac{z(x)}{W(x)}=\zeta \quad \text{and} \quad z(x_0)=\int_{\partial\Omega}\zeta d\omega^{x_0}.$$

By Propositions 2.17 and 2.18, $\frac{z}{W}\in C(\overline{\Omega})$. Since $\frac{z}{W}\lfloor_{\partial\Omega_n}$ converges uniformly to ζ as $n\to\infty$, there holds

$$z(x_0)=\int_{\partial\Omega_n} z\lfloor_{\partial\Omega_n} d\omega^{x_0}_{\Omega_n}=\int_{\partial\Omega_n} W\frac{z\lfloor_{\partial\Omega_n}}{W}d\omega^{x_0}_{\Omega_n}\to\int_{\partial\Omega}\zeta d\tilde{\omega} \quad \text{as } n\to\infty.$$

It follows

$$\int_{\partial\Omega}\zeta d\widetilde{\omega}=\int_{\partial\Omega}\zeta d\omega^{x_0} \quad \forall\zeta\in C(\partial\Omega).$$

Consequently $\widetilde{\omega}=d\omega^{x_0}$. Because the limit does not depend on the subsequence it follows that the whole sequence $W(x)d\omega^{x_0}_{\Omega_n}$ converges weakly to w. This implies (2.63). □

Theorem 2.33. *Let u be a positive $\mathcal{L}_\kappa$-harmonic in the domain Ω. Then $u\in L^1_{\phi_\kappa}(\Omega)$ and there exists a unique Radon measure μ on $\partial\Omega$ such that*

$$u(x)=\int_{\partial\Omega}K_{\mathcal{L}_\kappa}(x,\xi)d\mu(\xi).$$

Proof. The proof which is presented below follows the ideas of the one of [19, Th 4.3]. Let B be a relatively closed subset of Ω. We define

$$R^B_u(x):=\inf\{\psi(x):\ \psi \text{ is nonnegative superharmonic in } \Omega \text{ with } \psi\geq u \text{ on } B\}.$$

For a closed subset F of $\partial\Omega$, we define

$$\mu^x(F):=\inf\{R^{\Omega\cap\overline{G}}_u(x):\ F\subset G,\ G \text{ open in } \mathbb{R}^N\}.$$

The set function $F\mapsto\mu^x(F)$ defines a regular Borel measure on $\partial\Omega$ for each fixed $x\in\Omega$. Since $x\mapsto\mu^x(F)$ is a positive $\mathcal{L}_\kappa$-harmonic function in Ω the measures μ^x are absolutely continuous with respect to μ^{x_0} by Harnack's inequality. Hence,

$$\mu^x(F)=\int_F d\mu^x(F)(y)=\int_F\frac{d\mu^x(F)}{d\mu^{x_0}(F)}d\mu^{x_0}(y).$$

We assert that $\frac{d\mu^x(y)}{d\mu^{x_0}(y)}=K_{\mathcal{L}}(x,y)$ for μ^{x_0} a.e. $y\in\partial\Omega$. By Besicovitch's theorem,

$$\frac{d\mu^x(y)}{d\mu^{x_0}(y)}=\lim_{r\to 0}\frac{\mu^x(\Delta_r(y))}{\mu^{x_0}(\Delta_r(y))},$$

for μ^{x_0} a.e. $y\in\partial\Omega$.

By Lemma 2.31 and in view of the proof of Proposition 2.27 we have that $\frac{d\mu^x(F)}{d\mu^{x_0}(F)}$ is a kernel function, and by uniqueness of Kernel functions the proof of the assertion follows. Hence

$$\mu^x(A)=\int_A K_{\mathcal{L}}(x,y)d\mu^{x_0}(y),$$

for all Borel $A\subset\partial\Omega$ and in particular

$$u(x)=\mu^x(\partial\Omega)=\int_{\partial\Omega}K_{\mathcal{L}}(x,y)d\mu^{x_0}(y).$$

Suppose now

$$u(x)=\int_{\partial\Omega}K_{\mathcal{L}}(x,y)d\nu(y),$$

for a Borel measure ν on $\partial\Omega$. For a closed set $F\subset\partial\Omega$ we will show that $\nu(F)=\mu^{x_0}(F)$.

Choose a sequence of open set $\{G_k\}$ in $\mathbb{R}^N$ such that $\cap_{k=1}^\infty G_k=F$ and

$$\mu^{x_0}(F)=\lim_{k\to\infty}R^{\overline{G}_k}_u(x).$$

Since

$$R^B_u(x)\leq R^A_u(x) \quad \text{if } B\subset A,$$

we can choose G_k such that $\overline{G}_{k+1} \subset G_k$, $\forall k \geq 1$ and $\Omega \setminus G_k$ to be a C^2 domain for all $k \geq 1$. We consider a increasing sequence of bounded smooth domains $\{\Omega_k\}$ such that $\overline{\Omega_k} \subset \Omega_{k+1}, \cup\Omega_k = \Omega, \Omega_k \cap G_k = \emptyset$, $\mathcal{H}^{N-1}(\Omega_k) \to \mathcal{H}^{N-1}(\Omega)$ and

$$\mathcal{H}^{N-1}(\overline{\Omega}_k \cap \overline{G}_k) \to \mathcal{H}^{N-1}(F).$$

Let $w^{x_0}_{\Omega_k}(y)$ be the $\mathcal{L}_\kappa$-harmonic measure in $\partial\Omega_k$ (see (2.59)–(2.62)). Then

$$\begin{aligned} R^{\overline{G}_k}_u(x) &= \int_{\partial\Omega_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) \\ &= \int_{\partial\Omega_k\cap\partial G_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) + \int_{\partial\Omega_k\setminus\partial G_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) \\ &\geq \int_{\partial\Omega_k\cap\partial G_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y). \end{aligned}$$

Now, by Lemma 2.31

$$\begin{aligned} \int_{\partial\Omega_k\cap\partial G_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) &= \int_{\partial\Omega_k\cap\partial G_k} u(y)dw^{x_0}_{\Omega_k}(y) \\ &= \int_{\partial\Omega_k\cap\partial G_k}\int_{\partial\Omega} K_{\mathcal{L}}(y,\xi)d\nu(\xi)dw^{x_0}_{\Omega_k}(y) \\ &= \int_{\partial\Omega}\int_{\partial\Omega_k\cap\partial G_k} K_{\mathcal{L}}(y,\xi)dw^{x_0}_{\Omega_k}(y)d\nu(\xi) \\ &\geq \int_{F_n}\int_{\partial\Omega_k\cap\partial G_k} K_{\mathcal{L}}(y,\xi)dw^{x_0}_{\Omega_k}(y)d\nu(\xi), \end{aligned}$$

where $F_n \subset F$, $\cup F_n = F$ and $\operatorname{dist}(F_n, \partial\Omega\setminus F) > \frac{1}{n}$. If $\xi \in F_n$ we have

$$K(x_0,\xi) = \int_{\partial\Omega_k\cap\partial G_k} K_{\mathcal{L}}(y,\xi)dw^{x_0}_{\Omega_k}(y) + \int_{\partial\Omega_k\setminus G_k} K_{\mathcal{L}}(y,\xi)dw^{x_0}_{\Omega_k}(y).$$

But

$$K(y,\xi) \leq \frac{C}{n^{N+\alpha_+-2}} d^{\frac{\alpha_+}{2}}(y) \quad \forall y \in \partial\Omega_k \setminus G_k,$$

thus by Proposition 2.32 we have that

$$\lim_{k\to\infty}\int_{\partial\Omega_k\setminus G_k} K_{\mathcal{L}}(y,\xi)dw^{x_0}_{\Omega_k}(y) = 0.$$

Combining all the above inequality and using Lebesgue's dominated convergence theorem we obtain

$$\mu^{x_0}(F) = \lim_{k\to\infty} R^{\overline{G}_k}_u(x) \geq \int_{F_n}\int_{\partial\Omega_k\cap\partial G_k} K_{\mathcal{L}}(x_0,\xi)d\nu(\xi) = \nu(F_n),$$

which implies

$$\mu^{x_0}(F) \geq \nu(F).$$

For the opposite inequality, let $m \leq k-1$, $k \geq 2$ then

$$\begin{aligned} R^{\overline{G}_k}_u(x) &= \int_{\partial\Omega_k} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) \\ &= \int_{\partial\Omega_k\cap\partial G_m} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) + \int_{\partial\Omega_k\setminus\partial G_m} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y). \end{aligned}$$

In view of the proof of Lemma 2.31, we have that

$$R^{\overline{G}_k}_u(x) \leq C d^{\frac{\alpha_+}{2}}(x) \quad \forall x \in \Omega \setminus G_m.$$

Thus by Proposition 2.32 we have

$$\lim_{k\to\infty}\int_{\partial\Omega_k\setminus\partial G_m} R^{\overline{G}_k}_u(y)dw^{x_0}_{\Omega_k}(y) = 0,$$

and

$$\begin{aligned}\int_{\partial\Omega_k\cap\partial G_m} R_u^{\overline{G}_k}(y)dw_{\Omega_k}^{x_0}(y) &\leq \int_{\partial\Omega_k\cap\partial G_m} u(y)dw_{\Omega_k}^{x_0}(y)\\ &= \int_{\partial\Omega_k\cap\partial G_m}\int_{\partial\Omega} K_{\mathcal{L}}(y,\xi)d\nu(\xi)dw_{\Omega_k}^{x_0}(y)\\ &= \int_{\partial\Omega}\int_{\partial\Omega_k\cap\partial G_m} K_{\mathcal{L}}(y,\xi)dw_{\Omega_k}^{x_0}(y)d\nu(\xi).\end{aligned}$$

If $\xi\in\partial\Omega\setminus\overline{G}_m$ we have again by Proposition 2.32 that

$$\lim_{k\to\infty}\int_{\partial\Omega_k\cap\partial G_m} K_{\mathcal{L}}(y,\xi)dw_{\Omega_k}^{x_0}(y)=0.$$

If $\xi\in\partial\Omega\cap\overline{G}_m$, then

$$\int_{\partial\Omega_k\cap\partial G_m} K_{\mathcal{L}}(y,\xi)dw_{\Omega_k}^{x_0}(y)\leq K_{\mathcal{L}_\kappa}(x_0,\xi).$$

Combining all the above inequalities, we obtain

$$\mu^{x_0}(F)=\lim_{k\to\infty}R_u^{\overline{G}_k}(x)\leq\int_{\partial\Omega\cap\overline{G}_m} K_{\mathcal{L}}(x_0,\xi)d\nu(\xi)=\nu(\partial\Omega\cap\overline{G}_m),$$

which implies

$$\mu^{x_0}(F)\leq\nu(F),$$

and the proof of Theorem follows. □

Actually the measure μ is the boundary trace of u. This boundary trace can be achieved in a *dynamic way* as in [26, Section 2]. In the same way as the one they develop therein, we have

Proposition 2.34. *Let $x_0\in\Omega_1$ and $\mu\in\mathfrak{M}(\partial\Omega)$. Put*

$$v:=\int_{\partial\Omega} K_{\mathcal{L}_\kappa}(x,y)d\mu(y),$$

then for every $Z\in C(\overline{\Omega})$,

$$\lim_{n\to\infty}\int_{\partial\Omega_n} Z(x)v d\omega_{\Omega_n}^{x_0}=\int_{\partial\Omega} Z(x)d\mu. \tag{2.66}$$

Proof. The proof is same as the proof of Lemma 2.2 in [26] and we omit it. □

The next result is an analogous of the Green formula for positive $\mathcal{L}_\kappa$-harmonic functions.

Proposition 2.35. *Let v be a positive $\mathcal{L}_\kappa$-harmonic function in Ω with boundary trace μ. Let $Z\in C^2(\overline{\Omega})$ and $\tilde{G}\in C(\Omega)$ which coincides with $G_{\mathcal{L}_\kappa}(x_0,.)$ in Ω_δ for some $0<\delta<\beta_0$ and some $x_0\notin\overline{\Omega}_{\beta_0}$. Assume*

$$|\nabla\tilde{G}\cdot\nabla Z|\leq c'_{45}\phi_\kappa. \tag{2.67}$$

Then, if we set $\zeta=Z\tilde{G}$, there holds

$$\int_\Omega v\mathcal{L}_\kappa\zeta dx=\int_{\partial\Omega} Zd\mu. \tag{2.68}$$

Proof. Let $\{\Omega_j\}$ be a smooth exhaustion of Ω with Green kernel $G_{\mathcal{L}_\kappa}^{\Omega_j}$ and Poisson kernel $P_{\mathcal{L}_\kappa}^{\Omega_j}=-\partial_{\mathbf{n}}G_{\mathcal{L}_\kappa}^{\Omega_j}$. We assume that $j\geq j_0$ where $\overline{\Omega}'_\delta\subset\Omega_j$. Set $\zeta_j=Z\tilde{G}_j$, where the functions $\tilde{G}_j$ are C^∞ in Ω_j, coincide with $G_{\mathcal{L}_\kappa}^{\Omega_j}(x_0,.)$ in $\Omega_j\cap\overline{\Omega}_\delta$ and satisfy $\tilde{G}_j\to\tilde{G}$ in $C^2(\Omega)$-loc and such that $|\nabla\tilde{G}_j\cdot\nabla Z|\leq c'_{45}\phi_\kappa$.

$$\int_{\Omega_j} v\mathcal{L}_\kappa\zeta_j dx=-\int_{\partial\Omega_j} v\frac{\partial\zeta_j}{\partial\mathbf{n}}dS=-\int_{\partial\Omega_j} vZ\frac{\partial\tilde{G}_j}{\partial\mathbf{n}}dS=\int_{\partial\Omega_n} vZP_{\mathcal{L}_\kappa}^{\Omega_j}(x_0,.)dS=\int_{\partial\Omega_j} vZd\omega_{\Omega_j}^{x_0}.$$

By (2.66)

$$\int_{\partial\Omega_j} vZd\omega_{\Omega_j}^{x_0} \to \int_{\partial\Omega} Z(x)d\mu \quad \text{as } j\to\infty.$$

Next

$$\mathcal{L}_\kappa\zeta_j = Z\mathcal{L}_\kappa\tilde{G}_j + \tilde{G}_j\Delta Z + 2\nabla\tilde{G}_j\cdot\nabla Z.$$

Since $v\in L^1_{\phi_\kappa}(\Omega)$, the proof follows. □

Similarly we can prove

Proposition 2.36. *Let v be a positive $\mathcal{L}_\kappa$-harmonic function in Ω with boundary trace μ. Let $0\le Z\in C^2(\overline{\Omega})$ satisfy*

$$|\nabla\tilde{\phi}_\kappa\cdot\nabla Z|\le c'_{45}\phi_\kappa.$$

Then, if we set $\zeta = Z\phi_\kappa$, there holds

$$\int_\Omega v\mathcal{L}_\kappa\zeta\,dx \ge c_0\int_{\partial\Omega} Zd\mu,$$

where the constant $c_0>0$ depends on Ω, N and κ.

3. The nonlinear problem with measures data

3.1. The linear boundary value problem with L^1 data

In the sequel we denote by $\omega=\omega^{x_0}$ the $\mathcal{L}_\kappa$-harmonic measure in Ω, for some fixed $x_0\in\Omega$ and by $\mathfrak{M}_{\phi_\kappa}(\Omega)$ be the space of Radon measures ν in Ω such that $\phi_\kappa d|\nu|$ is a bounded measure. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of Radon measures on $\partial\Omega$ with respective norms $\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}$ and $\|\mu\|_{\mathfrak{M}(\partial\Omega)}$. Their respective positive cones are denoted by $\mathfrak{M}^+_{\phi_\kappa}(\Omega)$ and $\mathfrak{M}^+(\partial\Omega)$. By Fubini's theorem and (2.10), for any $\nu\in\mathfrak{M}_{\phi_\kappa}(\Omega)$ we can define

$$\mathbb{G}_{\mathcal{L}_\kappa}[\nu](x)=\int_\Omega G_{\mathcal{L}_\kappa}(x,y)d\nu(y),$$

and we have

$$\|\mathbb{G}_{\mathcal{L}_\kappa}[\nu]\|_{L^1_{\phi_\kappa}(\Omega)}\le c_{46}\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}. \tag{3.1}$$

If $\mu\in\mathfrak{M}(\partial\Omega)$, we set

$$\mathbb{K}_{\mathcal{L}_\kappa}[\mu](x)=\int_{\partial\Omega} K_{\mathcal{L}_\kappa}(x,y)d\mu(y),$$
$$\|\mathbb{K}_{\mathcal{L}_\kappa}[\mu]\|_{L^1_{\phi_\kappa}(\Omega)}\le c_{47}\|\mu\|_{\mathfrak{M}(\partial\Omega)}. \tag{3.2}$$

In the above inequalities c_{46} and c_{47} are positive constants depending on Ω and κ.

For $0<\kappa\le\frac14$, we define the space of test functions $\mathbf{X}(\Omega)$ by

$$\mathbf{X}(\Omega)=\left\{\eta\in H^1_{loc}(\Omega):\frac{\eta}{d^{\frac{\alpha_+}{2}}}\in H^1(\Omega,d^{\alpha_+}dx),\ (\phi_\kappa)^{-1}\mathcal{L}_\kappa\eta\in L^\infty(\Omega)\right\}. \tag{3.3}$$

The next statement follows immediately from Propositions 2.9 and 2.10.

Lemma 3.1. *Let $0<\kappa\le\frac14$. Let $m\in L^\infty(\Omega)$ and η_m be the solution of*

$$\begin{aligned}\mathcal{L}_\kappa\eta_m&=m\phi_\kappa &&\text{in }\Omega\\ \eta_m&=0&&\text{on }\partial\Omega,\end{aligned} \tag{3.4}$$

obtained by Propositions 2.9 and 2.10 with $f_0=m$ and $h=0$. Then η_m belongs to $\mathbf{X}(\Omega)$. Furthermore

$$-\frac{\|m_-\|_{L^\infty(\Omega)}}{\lambda_\kappa}\phi_\kappa\le-\eta_{m_-}\le\eta_m\le\eta_{m_+}\le\frac{\|m_+\|_{L^\infty(\Omega)}}{\lambda_\kappa}\phi_\kappa. \tag{3.5}$$

In the next proposition we give some key estimates satisfied by weak solutions of

$$\begin{aligned}\mathcal{L}_\kappa u&=f &&\text{in }\Omega\\ u&=h&&\text{on }\partial\Omega.\end{aligned} \tag{3.6}$$

Proposition 3.2. *For any* $(f, h) \in L^1_{\phi_\kappa}(\Omega) \times L^1(\partial\Omega, d\omega)$ *there exists a unique* $u := u_{f,h} \in L^1_{\phi_\kappa}(\Omega)$ *such that*

$$\int_\Omega u\mathcal{L}_\kappa \eta dx = \int_\Omega f\eta dx + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[h\omega]\mathcal{L}_\kappa \eta dx \quad \forall \eta \in \mathbf{X}(\Omega). \tag{3.7}$$

There holds

$$u = \mathbb{G}_{\mathcal{L}_\kappa}[f] + \mathbb{K}_{\mathcal{L}_\kappa}[h\omega], \tag{3.8}$$

and

$$\|u\|_{L^1_{\phi_\kappa}(\Omega)} \le c_{46}\|f\|_{L^1_{\phi_\kappa}(\Omega)} + c_{47}\|h\|_{L^1(\partial\Omega,d\omega)}. \tag{3.9}$$

Furthermore, for any $\eta \in \mathbf{X}(\Omega)$, $\eta \ge 0$, *we have*

$$\int_\Omega |u|\mathcal{L}_\kappa \eta dx \le \int_\Omega f\eta\, \mathrm{sgn(u)dx} + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[|\mathrm{h}|\omega]\mathcal{L}_\kappa \eta \mathrm{dx}, \tag{3.10}$$

and

$$\int_\Omega u_+\mathcal{L}_\kappa \eta dx \le \int_\Omega f\eta\, \mathrm{sgn}_+\mathrm{(u)dx} + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mathrm{h}_+\omega]\mathcal{L}_\kappa \eta \mathrm{dx}. \tag{3.11}$$

Proof. *Step* 1: *proof of estimate* (3.9). Assume u satisfies (3.7). If $\eta = \eta_{\mathrm{sgn(u)}}$, we have

$$\int_\Omega |u|\phi_\kappa dx = \int_\Omega u\mathcal{L}_\kappa \eta dx = \int_\Omega f\eta dx + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[h\omega]\mathrm{sgn(u)}\phi_\kappa \mathrm{dx}.$$

By (3.1), (3.2)

$$\int_\Omega f\eta dx \le \frac{1}{\lambda_\kappa}\int_\Omega |f|\phi_\kappa dx,$$
$$\int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[h\omega]\mathrm{sgn(u)}\phi_\kappa \mathrm{dx} \le \mathrm{c}_{47}\int_{\partial\Omega} |\mathrm{h}|\mathrm{d}\omega,$$

which implies (3.9) and uniqueness.

Step 2: *proof of existence*. If f and h are bounded, existence follows from Propositions 2.9 and 2.10. In the general case let $\{(f_n, h_n)\}$ be a sequence of bounded measurable functions in Ω and $\partial\Omega$ which converges to $\{(f, h)\}$ in $L^1_{\phi_\kappa}(\Omega) \times L^1(\partial\Omega, d\omega)$. Let $\{u_n\} = \{u_{f_n,h_n}\}$ be the sequence weak solutions of (3.6). By estimate (3.9) it is a Cauchy sequence in $L^1_{\phi_\kappa}(\Omega)$ which converges to u. Letting $n \to \infty$ in identity

$$\int_\Omega u_n\mathcal{L}_\kappa \eta dx = \int_\Omega f_n\eta dx + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[h_n\omega]\mathcal{L}_\kappa \eta dx, \tag{3.12}$$

where $\eta \in \mathbf{X}(\Omega)$ implies that $u = u_{f,h}$.

Step 3: *proof of estimates* (3.10), (3.11). We first assume that f is bounded and h is $C^2(\overline{\Omega})$. Set $\Omega_n = \Omega'_{\frac{1}{n}}$, let u_n be the unique solution of

$$\begin{aligned} \mathcal{L}_\kappa u_n &= f && \text{in } \Omega_n \\ v_n &= Wh && \text{on } \partial\Omega_n. \end{aligned} \tag{3.13}$$

Then u_n can be written in the form

$$u_n = \mathbb{G}^n_{\mathcal{L}_\kappa}[f](x) + w_n,$$

where w_n satisfies

$$\begin{aligned} \mathcal{L}_\kappa v &= 0 && \text{in } \Omega_n \\ v &= Wh && \text{on } \partial\Omega_n, \end{aligned} \tag{3.14}$$

and

$$\mathbb{G}^n_{\mathcal{L}_\kappa}[f](x) = \int_\Omega G^n_{\mathcal{L}_\kappa}(x, y)f(y)dy,$$

where $G^n_{\mathcal{L}_\kappa}$ denotes the Green Kernel of $\mathcal{L}_\kappa$ in Ω_n. Now note that $G^n_{\mathcal{L}_{\frac{1}{4}}}(x, y) \le G_{\mathcal{L}_{\frac{1}{4}}}(x, y) := G^\Omega_{\mathcal{L}_{\frac{1}{4}}}$, and for any $x, y \in \Omega,\ x \ne y$

$$G^n_{\mathcal{L}_{\frac{1}{4}}}(x, y) \uparrow G_{\mathcal{L}_{\frac{1}{4}}}(x, y). \tag{3.15}$$

Also, in view of the proof of Proposition 2.32, there exists $c_0 > 0$ which depends on $\Omega,\ N,\ \kappa,\ \|h\|_{C^2(\overline{\Omega})}$ such that

$$\sup_{x\in\Omega_n} |w_n| < c_0, \quad \forall n \in \mathbb{N},$$

and $w_n \to \mathbb{K}_{\mathcal{L}_\kappa}[h\omega]$. Thus by the properties of Green kernel that we described above, there exists a constant $c_{01}\ \Omega,\ N,\ \kappa,\ \|h\|_{C^2(\overline{\Omega})},\ \|f\|_{L^\infty(\Omega)}$, such that

$$\sup_{x\in\Omega_n} |u_n| < c_0, \quad \forall n \in \mathbb{N},$$

and

$$u_n \to u = \mathbb{G}_{\mathcal{L}_\kappa}[f] + \mathbb{K}_{\mathcal{L}_\kappa}[h\omega].$$

Let $\eta \in \mathbf{X}(\Omega)$ be nonnegative function and let η_n be the solution of the problem

$$\begin{aligned} \mathcal{L}_\kappa v &= \mathcal{L}_\kappa \eta \quad && \text{in } \Omega_n \\ v &= 0 && \text{on } \partial\Omega_n. \end{aligned}$$

Then there exists $c_0 = c_0(\|\Delta\eta\|_{L^\infty(\Omega)}, \kappa, N, \Omega)$ such that $|\eta_n| \le c_0\phi_\kappa$ and

$$\mathcal{L}_\kappa \eta_n \to L_{c_0}\eta, \quad \eta_n \to \eta.$$

Let z_n be the solution of

$$\begin{aligned} \mathcal{L}_\kappa v &= \operatorname{sgn}(\eta_n)\mathcal{L}_\kappa \eta \quad && \text{on } \partial\Omega_n \\ v &= 0 && \text{on } \partial\Omega_n. \end{aligned}$$

Then $z_n \ge \max(\eta_n, 0)$ since

$$\mathcal{L}_\kappa |\eta_n| \le \operatorname{sgn}(\eta_n)\mathcal{L}_\kappa \eta_n = \operatorname{sgn}(\eta_n)\mathcal{L}_\kappa \eta,$$

and $|z_n| \le c_0\phi_\kappa$,

$$\mathcal{L}_\kappa z_n \to L_{c_0}\eta, \quad z_n \to \eta.$$

Now note that $z_n \ge 0$ and $z_n \in C^1(\overline{\Omega}_n)$. Also, the following inequality holds (see e.g. [30]),

$$\begin{aligned} \int_\Omega |u_n| L_{c_0} z_n dx &\le \int_\Omega f z_n \operatorname{sgn}(u_n) - \int_{\partial\Omega} \frac{\partial z_n}{\partial \nu}|h| W dx \\ &= \int_\Omega f z_n \operatorname{sgn}(u_n) + \int_\Omega \widetilde{w}_n L_{c_0} z_n dx, \end{aligned} \tag{3.16}$$

where $\widetilde{w}_n$ is the solution of

$$\begin{aligned} \mathcal{L}_\kappa v &= 0 \quad && \text{in } \Omega_n \\ v &= W|h| && \text{on } \partial\Omega_n. \end{aligned} \tag{3.17}$$

In view of the proof of Proposition 2.32 there exists $c_{02} > 0$ which depends on $\Omega,\ N,\ \kappa,\ \|h\|_{C^2(\overline{\Omega})}$ such that

$$\sup_{x\in\Omega_n} |\widetilde{w}_n| < c_0, \quad \forall n \in \mathbb{N},$$

and $\widetilde{w}_n \to \mathbb{K}_{\mathcal{L}_\kappa}[|h|\omega]$ as $n \to \infty$. Thus combining all above and taking the limit in (3.16) we have the proof of (3.10) in the case that f is bounded and $h \in C^2(\overline{\Omega})$. We note here that for any $h \in C^2(\partial\Omega)$ there exists $H_m \in C^2(\overline{\Omega})$, such that $\|H_m\|_{C^2(\overline{\Omega})} \le c_{03}\|h\|_{L^\infty(\partial\Omega)}$, for some constant c_{03} which depends only on Ω, and $H_m \to h$ in $L^\infty(\partial\Omega)$. Thus it is not hard to prove that (2.32) is valid if f is bounded and $h \in C^2(\partial\Omega)$. In the general case we consider a sequence $(f_n, h_n) \subset L^\infty(\Omega) \times C^2(\partial\Omega)$ which converges to (f, h) in $L^1(\Omega) \times L^1(\partial\Omega, d\omega)$. Since u_{f_n,h_n} converges to $u_{f,h}$ in $L^1_{\phi_\kappa}(\Omega)$ we obtain (3.10) from the inequality verified by any $\eta \in \mathbf{X}(\Omega)$

$$\int_\Omega |u_{f_n,h_n}|\mathcal{L}_\kappa \eta dx \le \int_\Omega f_n \eta \operatorname{sgn}(\mathrm{u})\mathrm{dx} + \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[|\mathrm{h_n}|\omega]\mathcal{L}_\kappa \eta \mathrm{dx}. \quad \square$$

The proof of (3.11) is follows by adding (3.7) and (3.10).

3.2. General nonlinearities

Throughout this section Ω is a smooth bounded domain and κ a real number in the interval $(0, \frac{1}{4}]$. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a nondecreasing continuous function, vanishing at 0 for simplicity. The problem under consideration is the following

$$\begin{aligned} -\Delta u - \frac{\kappa}{d^2}u + g(u) &= \nu \quad \text{in } \Omega \\ u &= \mu \quad \text{in } \partial\Omega, \end{aligned} \tag{3.18}$$

where ν and μ are Radon measures respectively in Ω and $\partial\Omega$.

Definition. Let $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$ and $\mu \in \mathfrak{M}(\partial\Omega)$. We say that u is a solution of (3.18) if $u \in L^1_{\phi_\kappa}(\Omega)$, $g(u) \in L^1_{\phi_\kappa}(\Omega)$ and for any $\eta \in \mathbf{X}(\Omega)$ there holds

$$\int_\Omega (u\mathcal{L}_\kappa \eta + g(u)\eta)\,dx = \int_\Omega \left(\eta d\nu + \mathbb{K}_{\mathcal{L}_\kappa}[\mu]\mathcal{L}_\kappa \eta\right) dx. \tag{3.19}$$

Our main existence result for *subcritical nonlinearities* is the following.

Theorem 3.3. *Assume g satisfies*

$$\int_1^\infty (g(s) - g(-s))\, s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}\, ds < \infty. \tag{3.20}$$

Then for any $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \in \mathfrak{M}(\partial\Omega)$ problem (3.18) admits a unique solution $u = u_{\nu,\mu}$. Furthermore the mapping $(\nu, \mu) \mapsto u_{\nu,\mu}$ is increasing and stable in the sense that if $\{(\nu_n, \mu_n)\}$ converge to (ν, μ) in the weak sense of measures, $\{u_{\nu_n,\mu_n}\}$ converges to $u_{\nu,\mu}$ in $L^1_{\phi_\kappa}(\Omega)$.

The proof is based upon estimates of $\mathbb{M}_{\mathcal{L}_\kappa}$ and $\mathbb{K}_{\mathcal{L}_\kappa}$ into Marcinkiewicz spaces.

Lemma 3.4. *Let $\nu \in \mathfrak{M}^+_{\phi_\kappa}(\Omega)$, $\mu \in \mathfrak{M}^+(\partial\Omega)$ and for $s > 0$, $E_s(\nu) = \{x \in \Omega : \mathbb{G}_{\mathcal{L}_\kappa}[\nu](x) > s\}$ and $F_s(\mu) = \{x \in \Omega : \mathbb{K}_{\mathcal{L}_\kappa}[\mu](x) > s\}$. If we denote*

$$\mathcal{E}_s(\nu) = \int_{E_s(\nu)} \phi_\kappa dx \quad \textit{and} \quad \mathcal{F}_s(\mu) = \int_{F_s(\mu)} \phi_\kappa dx,$$

there holds

$$\mathcal{E}_s(\nu) + \mathcal{F}_s(\mu) \le c_{47} \left(\frac{\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + \|\mu\|_{\mathfrak{M}(\partial\Omega)}}{s} \right)^{\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}. \tag{3.21}$$

Proof. *Step* 1: *estimate of* $\mathcal{F}_s(\nu)$. By estimate (2.57), for any $\xi \in \partial\Omega$,

$$F_s(\delta_\xi) \subset \tilde{F}_s(\delta_\xi) := \left\{ x \in \Omega : \frac{d^{\frac{\alpha_+}{2}}(x)}{|x-\xi|^{N+\alpha_+-2}} \ge \frac{s}{c_{43}} \right\} \subset B_{\left(\frac{c_{43}}{s}\right)^\theta}(\xi),$$

with $\theta = \frac{1}{N-2+\frac{\alpha_+}{2}}$. From (2.2), (2.3)

$$\mathcal{F}_s(\delta_\xi) \le \int_{B_{\left(\frac{c_{43}}{s}\right)^\theta}(\xi)} \phi_\kappa dx \le c_{49} \int_{B_{\left(\frac{c_{43}}{s}\right)^\theta}(\xi)} |x-\xi|^{\frac{\alpha_+}{2}} dx = c_{50} s^{-\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}.$$

Therefore, for any $s_0 > 0$ and any Borel set $G \subset \Omega$

$$\begin{aligned} \int_G K_{\mathcal{L}_\kappa}(x,\xi)\phi_\kappa dx &\le s_0 \int_G \phi_\kappa dx + \int_{F_{s_0}(\delta_\xi)} K_{\mathcal{L}_\kappa}(x,\xi)\phi_\kappa dx \\ &\le s_0 \int_G \phi_\kappa dx - \int_{s_0}^\infty s d\mathcal{F}_s(\delta_\xi) \\ &\le s_0 \int_G \phi_\kappa dx + c_{50} \int_{s_0}^\infty s^{-\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} ds \\ &\le s_0 \int_G \phi_\kappa dx + c_{51} s_0^{-\frac{2}{N-2+\frac{\alpha_+}{2}}}. \end{aligned}$$

Next we choose s_0 so that the two terms in the right part of the last inequality are equal and we get

$$\int_G K_{\mathcal{L}_\kappa}(x,\xi)\phi_\kappa dx \leq c_{52}\left(\int_G \phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}}. \tag{3.22}$$

Henceforth, for any $\mu \in \mathfrak{M}(\partial\Omega)$, there holds by Fubini's theorem,

$$\int_G \mathbb{K}_{\mathcal{L}_\kappa}[|\mu|]\phi_\kappa dx = \int_\Omega\int_G K_{\mathcal{L}_\kappa}(x,\xi)\phi_\kappa(x)dxd|\mu|(\xi) \leq c_{52}\|\mu\|_{\mathfrak{M}(\partial\Omega)}\left(\int_G \phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}}. \tag{3.23}$$

If we take in particular $G = F_s(|\mu|)$, we derive

$$s\mathcal{F}_s(|\mu|) \leq c_{52}\|\mu\|_{\mathfrak{M}(\partial\Omega)}(\mathcal{F}_s(|\mu|))^{\frac{2}{N+\frac{\alpha_+}{2}}},$$

which yields to (3.21) with $\nu = 0$.

Step 2: *estimate of* $\mathcal{E}_s(\nu)$. By estimate (2.10), for any $y \in \Omega$,

$$E_s(\delta_y) \subset \tilde{E}_s(\delta_y) := \left\{x \in \Omega : \frac{d^{\frac{\alpha_+}{2}}(y)d^{\frac{\alpha_+}{2}}(x)}{|x-y|^{N+\alpha_+-2}} \geq \frac{s}{c_3}\right\} \bigcap \left\{x \in \Omega : \frac{1}{|x-y|^{N-2}} \geq \frac{s}{c_3}\right\}.$$

A simple geometric verification shows that there exists an open domain $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$ such that $y \in \mathcal{O}$, $\operatorname{dist}(y,\mathcal{O}^c) > \lambda_1 d(y)$, $\mathcal{O} \subset B_{\lambda_2 d(y)}(y)$ for some $0 < \lambda_1 < \lambda_2 < 1$ independent of y with the following properties

$$x \in \mathcal{O} \Longrightarrow \frac{d^{\frac{\alpha_+}{2}}(y)d^{\frac{\alpha_+}{2}}(x)}{|x-y|^{N+\alpha_+-2}} \geq \frac{1}{|x-y|^{N-2}}$$

$$x \in \mathcal{O}^c \Longrightarrow \frac{d^{\frac{\alpha_+}{2}}(y)d^{\frac{\alpha_+}{2}}(x)}{|x-y|^{N+\alpha_+-2}} \leq \frac{1}{|x-y|^{N-2}}.$$

Notice that if $\Omega = \mathbb{R}^N_+$ then $\mathcal{O} = B_{\frac{\sqrt{5}}{2}}(\tilde{y})$ where $d(\tilde{y}) = \frac{3}{2}d(y)$. Set

$$\tilde{E}^1_s(\delta_y) = \left\{x \in \Omega : \frac{1}{|x-y|^{N-2}} \geq \frac{s}{c_3}\right\} \cap \mathcal{O},$$

and

$$\tilde{E}^2_s(\delta_y) = \left\{x \in \Omega\setminus\mathcal{O} : \frac{d^{\frac{\alpha_+}{2}}(y)d^{\frac{\alpha_+}{2}}(x)}{|x-y|^{N+\alpha_+-2}} \geq \frac{s}{c_3}\right\}.$$

We can easily prove

$$\begin{aligned}\mathcal{E}_s(\delta_y) &= \int_{E_s(\delta_y)} \phi_\kappa dx \leq \int_{\tilde{E}_s(\delta_y)} \phi_\kappa dx \\ &\leq \int_{\tilde{E}^1_s(\delta_y)} \phi_\kappa dx + \int_{\tilde{E}^2_s(\delta_y)} \phi_\kappa dx \leq c_{53}s^{-\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}(d(y))^{\frac{\alpha_+\left(N+\frac{\alpha_+}{2}\right)}{2N-4+\alpha_+}}.\end{aligned}$$

As in step 1, for any Borel subset $\Theta \subset \Omega$, we write

$$\begin{aligned}\int_\Theta G_{\mathcal{L}_\kappa}(x,y)\phi_\kappa dx &\leq s_0\int_\Theta \phi_\kappa dx + \int_{E_{s_0}(\delta_y)} G_{\mathcal{L}_\kappa}(x,y)\phi_\kappa dx \\ &\leq s_0\int_\Theta \phi_\kappa dx - \int_{s_0}^\infty s d\mathcal{E}_s(\delta_y) \\ &\leq s_0\int_\Theta \phi_\kappa dx + c_{53}(d(y))^{\frac{\alpha_+\left(N+\frac{\alpha_+}{2}\right)}{2N-4+\alpha_+}}\int_{s_0}^\infty s^{-\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}}ds \\ &\leq s_0\int_\Theta \phi_\kappa dx + c_{54}(d(y))^{\frac{\alpha_+\left(N+\frac{\alpha_+}{2}\right)}{2N-4+\alpha_+}}s_0^{-\frac{2}{N-2+\frac{\alpha_+}{2}}}.\end{aligned}$$

Then

$$\int_{\Theta} G_{\mathcal{L}_\kappa}(x,y)\phi_\kappa dx \leq c_{55}(d(y))^{\frac{\alpha_+}{2}}\left(\int_G \phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}} \leq c_{56}\phi_\kappa(y)\left(\int_G \phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}}. \tag{3.24}$$

Thus, for any $\nu \in \mathfrak{M}_{\phi_\kappa}(\Omega)$, we have

$$\int_{\Theta} \mathbb{G}_{\mathcal{L}_\kappa}[|\nu|]\phi_\kappa dx = \int_{\Omega}\int_{\Theta} G_{\mathcal{L}_\kappa}(x,y)\phi_\kappa(x)dxd|\nu|(y) \leq c_{55}\|\nu\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}\left(\int_{\Theta}\phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}}. \tag{3.25}$$

Thus (3.21) holds. $\square$

Proof of Theorem 3.3. *Step* 1: *existence and uniqueness*. Let $\{(\nu_n,\mu_n)\} \subset C(\overline{\Omega}) \times C^1(\partial\Omega)$ which converges to (ν,μ) in the weak sense of measures in $\mathfrak{M}_{\phi_\kappa}(\Omega) \times \mathfrak{M}(\partial\Omega)$. Set $v_n = \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n\omega]$, then $v_n \in L^\infty(\Omega)$ and it is $\mathcal{L}_\kappa$-harmonic. Set $\widetilde{g}(t,x) = g(t+v_n(x)) - g(v_n(x))$ and $\widetilde{f}(x) = \nu_n(x) - g(v_n(x))$. Let J_κ be the functional defined in $L^2(\Omega)$ by the expression

$$\mathcal{J}_\kappa(w) = \frac{1}{2}\int_{\Omega}\left(|\nabla w|^2 - \frac{\kappa}{d^2}w^2 + 2J(w)\right)dx - \int_{\Omega}\widetilde{f}w\phi_\kappa dx, \tag{3.26}$$

where $J(w) = \int_0^w \widetilde{g}(t)dt$ with domain

$$D(\mathcal{J}_\kappa) = \{w \in \mathbf{H}_\kappa(\Omega) : J(w) \in L^1(\Omega)\},$$

(see definition in 2.1–5). By (2.8), $\mathcal{J}_\kappa$ is a convex lower semicontinuous and coercive functional over $L^2(\Omega)$. Let $w_n = w_{\nu_n,\mu_n}$ be its minimum, then $u_n = u_{\nu_n,\mu_n} = w_n + v_n$ is the solution of

$$\begin{aligned} \mathcal{L}_\kappa u_n + g(u_n) &= \nu_n && \text{in } \Omega \\ u_n &= \mu_n && \text{in } \partial\Omega, \end{aligned} \tag{3.27}$$

and for any $\eta \in \mathbf{X}(\Omega)$, there holds

$$\int_{\Omega}(u_n\mathcal{L}_\kappa\eta + g(u_n)\eta)\,dx = \int_{\Omega}\left(\nu_n\eta + \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n\omega]\mathcal{L}_\kappa\eta\right)dx. \tag{3.28}$$

By Proposition 3.2 (3.10), there holds, with $\eta = \phi_\kappa$,

$$\begin{aligned} \int_{\Omega}(\lambda_\kappa|u_n| + |g(u_n)|)\,\phi_\kappa dx &\leq \int_{\Omega}\left(|\nu_n| + \mathbb{K}_{\mathcal{L}_\kappa}[|\mu_n|\omega]\right)\phi_\kappa dx \\ &\leq c_{46}\|\nu_n\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + c_{47}\|\mu_n\|_{\mathfrak{M}(\partial\Omega)} \\ &\leq c_{57}. \end{aligned} \tag{3.29}$$

Moreover

$$-\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^-] - \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^-\omega] \leq u_n \leq \mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^+] + \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^+\omega]. \tag{3.30}$$

By using the local L^1 regularity theory for elliptic equations we obtain that the sequence $\{u_n\}$ is relatively compact in the L^1-local topology in Ω and that there exist a subsequence still denoted by $\{u_n\}$ and a function $u \in L^1_{\phi_\kappa}(\Omega)$ such that $u_n \to u$ a.e. in Ω. By (3.30)

$$|g(u_n)| \leq g\left(\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^+] + \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^+\omega]\right) - g\left(-\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^-] - \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^-\omega]\right). \tag{3.31}$$

We prove the convergence of $\{g(u_n)\}$ to $g(u)$ in $L^1_{\phi_\kappa}(\Omega)$ by the uniform integrability in the following way: let $G \subset \Omega$ be a Borel subset. Then for any $s_0 > 0$

$$\begin{aligned} \int_G |g(u_n)|\phi_\kappa dx &\leq \int_G\left(g\left(\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^+]\right) + g\left(\mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^+\omega]\right) - g\left(-\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^-]\right) - g\left(-\mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^-\omega]\right)\right)\phi_\kappa dx \\ &\leq s_0\int_G\phi_\kappa dx + \int_{E_s(\nu^+)} g\left(\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^+]\right)\phi_\kappa dx + \int_{F_s(\mu^+)} g\left(\mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^+]\right)\phi_\kappa dx \\ &\quad - \int_{E_s(\nu^-)} g\left(-\mathbb{G}_{\mathcal{L}_\kappa}[\nu_n^-]\right)\phi_\kappa dx - \int_{F_s(\mu^-)} g\left(-\mathbb{K}_{\mathcal{L}_\kappa}[\mu_n^-]\right)\phi_\kappa dx \\ &\leq s_0\int_G\phi_\kappa dx - \int_{s_0}^\infty g(s)(d\mathcal{E}_s(\nu_n^+) + d\mathcal{F}_s(\mu_n^+)) + \int_{s_0}^\infty g(-s)(d\mathcal{E}_s(\nu_n^-) + d\mathcal{F}_s(\mu_n^-)). \end{aligned}$$

But

$$
\begin{aligned}
-\int_{s_0}^{\infty} g(s)d\mathcal{E}_s(\nu_n^+) &= g(s_0)\mathcal{E}_{s_0}(\nu_n^+) + \int_{s_0}^{\infty} \mathcal{E}_s(\nu_n^+)dg(s) \\
&\leq g(s_0)\mathcal{E}_{s_0}(\nu_n^+) + c_{47}\left(\|\nu_n^+\|_{\mathfrak{M}_{\phi_\kappa}}\right)^{\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} \int_{s_0}^{\infty} s^{-\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} dg(s) \\
&\leq \frac{2N+\alpha_+}{2N-4+\alpha_+} c_{47}\left(\|\nu_n^+\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}\right)^{\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} \int_{s_0}^{\infty} s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} g(s)ds.
\end{aligned}
$$

All the other terms yields similar estimates which finally yields to

$$
\begin{aligned}
&\int_G |g(u_n)|\phi_\kappa dx \leq s_0 \int_G \phi_\kappa dx \\
&\quad + c_{58}\left(\|\nu_n\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)}\right)^{\frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} \int_{s_0}^{\infty} s^{-2\frac{N-1+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}} (g(s)-g(-s))ds.
\end{aligned} \tag{3.32}
$$

Since $\|\nu_n\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)} + \|\mu_n\|_{\mathfrak{M}(\partial\Omega)}$ is bounded independently of n, we obtain easily, using (3.20) and fixing s_0 first, that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$
\int_G \phi_\kappa dx \leq \delta \Longrightarrow \int_G |g(u_n)|\phi_\kappa dx \leq \epsilon. \tag{3.33}
$$

Since

$$
|u_n| \leq \mathbb{G}_{\mathcal{L}_\kappa}[|\nu_n|] + \mathbb{K}_{\mathcal{L}_\kappa}[|\mu_n|\omega],
$$

we have by (3.23), (3.25)

$$
\int_G |u_n|\phi_\kappa dx \leq \left(c_{52}\|\mu_n\|_{\mathfrak{M}(\partial\Omega)} + c_{55}\|\nu_n\|_{\mathfrak{M}_{\phi_\kappa}(\Omega)}\right)\left(\int_G \phi_\kappa dx\right)^{\frac{2}{N+\frac{\alpha_+}{2}}}. \tag{3.34}
$$

This implies the uniform integrability of the sequence $\{u_n\}$. Letting $n \to \infty$ in identity (3.28), we conclude that (3.19) holds. Uniqueness, as well as the monotonicity of the mapping $(\nu, \mu) \mapsto u_{\nu,\mu}$, is an immediate consequence of (3.10), (3.11) and the monotonicity of g.

Step 2: *stability*. The stability is a direct consequence of inequalities (3.32) and (3.34) which show the uniform integrability of the sequence $(u_n, g(u_n))$ in $L^1_{\phi_\kappa}(\Omega) \times L^1_{\phi_\kappa}(\Omega)$. □

Because of the uniqueness of the solution $u_{\mu,\nu}$ of problem (3.18) and the fact that $g(u_{\mu,\nu}) \in L^1_{\phi_\kappa}(\Omega)$ the following representation statement is valid, and its proof is obtained by approximation of the measures as is [28, Lemma 3.2, Definition 3.3].

Proposition 3.5. *Let $(\nu, \mu) \in \mathfrak{M}_{\phi_\kappa}(\Omega) \times \in \mathfrak{M}(\partial\Omega)$ such that problem (3.18) admits a solution $u_{\mu,\nu}$. Then*

$$
u_{\mu,\nu} = -\mathbb{G}_{\mathcal{L}_\kappa}[g(u_{\mu,\nu})] + \mathbb{K}_{\mathcal{L}_\kappa}[\mu]. \tag{3.35}
$$

Conversely, if $u \in L^1_{\phi_\kappa}(\Omega)$ such that $g(u) \in L^1_{\phi_\kappa}(\Omega)$ satisfies (3.35), it coincides with the solution $u_{\mu,\nu}$ of problem (3.18).

3.3. The power case

In this section we study in particular the following boundary value problem with $\mu \in \mathfrak{M}(\partial\Omega)$

$$
\begin{aligned}
\mathcal{L}_\kappa u + |u|^{q-1}u &= 0 \quad \text{in } \Omega \\
u &= \mu \quad \text{in } \partial\Omega.
\end{aligned} \tag{3.36}
$$

A Radon measure for which this problem has a solution (always unique) is called a *good measure*. The solution, whenever it exists, is unique and denoted by u_μ. For such a nonlinearity, the condition (3.20) is fulfilled if and only if

$$
0 < q < q_c := \frac{N+\frac{\alpha_+}{2}}{N-2+\frac{\alpha_+}{2}}. \tag{3.37}
$$

On the contrary, in the *supercritical case* i.e. if $q \geq q_c$, a continuity condition with respect to some Besov capacity is needed in order a measure be good. We recall some notations concerning Besov space. For $\sigma > 0$, $1 \leq p < \infty$, we denote by $W^{\sigma,p}(\mathbb{R}^d)$

the Sobolev space over $\mathbb{R}^d$. If σ is not an integer the Besov space $B^{\sigma,p}(\mathbb{R}^d)$ coincides with $W^{\sigma,p}(\mathbb{R}^d)$. When σ is an integer we denote $\Delta_{x,y}f = f(x+y) + f(x-y) - 2f(x)$ and

$$B^{1,p}(\mathbb{R}^d) = \left\{ f \in L^p(\mathbb{R}^d) : \frac{\Delta_{x,y}f}{|y|^{1+\frac{d}{p}}} \in L^p(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

with norm

$$\|f\|_{B^{1,p}} = \left(\|f\|_{L^p}^p + \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\Delta_{x,y}f|^p}{|y|^{p+d}} dxdy \right)^{\frac{1}{p}}.$$

Then

$$B^{m,p}(\mathbb{R}^d) = \left\{ f \in W^{m-1,p}(\mathbb{R}^d) : D_x^\alpha f \in B^{1,p}(\mathbb{R}^d)\ \forall \alpha \in \mathbb{N}^d\ |\alpha| = m-1 \right\},$$

with norm

$$\|f\|_{B^{m,p}} = \left(\|f\|_{W^{m-1,p}}^p + \sum_{|\alpha|=m-1} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|D_x^\alpha \Delta_{x,y}f|^p}{|y|^{p+d}} dxdy \right)^{\frac{1}{p}}.$$

These spaces are fundamental because they are stable under the real interpolation method developed by Lions and Petree. For $\alpha \in \mathbb{R}$ we defined the Bessel kernel of order α by $G_\alpha(\xi) = \mathcal{F}^{-1}(1+|\cdot|^2)^{-\frac{\alpha}{2}}\mathcal{F}(\xi)$, where $\mathcal{F}$ is the Fourier transform of moderate distributions in $\mathbb{R}^d$. The Bessel space $L_{\alpha,p}(\mathbb{R}^d)$ is defined by

$$L_{\alpha,p}(\mathbb{R}^d) = \{f = G_\alpha * g : g \in L^p(\mathbb{R}^d)\},$$

with norm

$$\|f\|_{L_{\alpha,p}} = \|g\|_{L^p} = \|G_{-\alpha} * f\|_{L^p}.$$

It is known that if $1 < p < \infty$ and $\alpha > 0$, $L_{\alpha,p}(\mathbb{R}^d) = W^{\alpha,p}(\mathbb{R}^d)$ if $\alpha \in \mathbb{N}$ and $L_{\alpha,p}(\mathbb{R}^d) = B^{\alpha,p}(\mathbb{R}^d)$ if $\alpha \notin \mathbb{N}$, always with equivalent norms. The Bessel capacity is defined for compact subset $K \subset \mathbb{R}^d$ by

$$C_{\alpha,p}^{\mathbb{R}^d} = \inf\{\|f\|_{L_{\alpha,p}}^p, f \in \mathcal{S}'(\mathbb{R}^d), f \geq \chi_K\}.$$

It is extended to open set and then any set by the fact that it is an outer measure. Our main result is the following.

Theorem 3.6. *Assume* $0 < \kappa \leq \frac{1}{4}$. *Then* $\mu \in \mathfrak{M}^+(\partial\Omega)$ *is a good measure if and only if it is absolutely continuous with respect to the Bessel capacity* $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$ *where* $q' = \frac{q}{q-1}$, *that is*

$$\forall E \subset \partial\Omega, \quad E \text{ Borel},\ C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(E) = 0 \Longrightarrow \mu(E) = 0. \tag{3.38}$$

The striking aspect of the proof is that it is based upon potential estimates which have been developed by Marcus and Véron in the study of the supercritical boundary trace problem in polyhedral domains [28]. Before proving this result we need a key potential estimate.

Theorem 3.7. *Assume* $0 < \kappa \leq \frac{1}{4}$ *and* $q \geq q_c$. *There exists a constant* $c_{59} > 1$ *depending on* Ω, q, *and* κ *such that for any* $\mu \in \mathfrak{M}^+(\partial\Omega)$ *there holds*

$$\frac{1}{c_{59}} \|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq \int_\Omega \left(\mathbb{K}_{\mathcal{L}_\kappa}[\mu]\right)^q \phi_\kappa dx \leq c_{59} \|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}}. \tag{3.39}$$

Proof. *Step* 1: *local estimates*. Denote by $\xi = (\xi_1, \xi')$ the coordinates in $\mathbb{R}^N_+$, $\xi_1 > 0$, $\xi' \in \mathbb{R}^{N-1}$ The ball of radius $R > 0$ and center a in $\mathbb{R}^{N-1}$ is denoted by $B'_R(a)$ (by B'_R if $a = 0$). Let $R > 0$, $\nu \in \mathfrak{M}^+(\mathbb{R}^{N-1}_+)$ with support in $B'_{\frac{R}{2}}$ and

$$\mathbf{K}[\nu](\xi) = \int_{B'_{\frac{R}{2}}} \frac{d\nu(\zeta')}{(\xi_1^2 + |\xi' - \zeta'|^2)^{\frac{N-2+\alpha_+}{2}}}. \tag{3.40}$$

Then, by [28, Th 3.1],

$$\frac{1}{c_{60}}\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq \int_0^R\int_{B'_R}\xi_1^{(q+1)\frac{\alpha_+}{2}}\left(\int_{B'_{\frac{R}{2}}}\frac{d\nu(\zeta')}{(\xi_1^2+|\xi'-\zeta'|^2)^{\frac{N-2+\alpha_+}{2}}}\right)^q d\xi' d\xi_1$$
$$\leq c_{60}\left(1+R^{(q+1)\frac{\alpha_+}{2}}\right)\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}}. \tag{3.41}$$

There exists $R>0$ such that for any $y_0\in\partial\Omega$, there exists a C^2 diffeomorphism $\Theta := \Theta_{y_0}$ from $B_R(y_0)$ into $\mathbb{R}^N$ such that $\Theta(y_0)=0$, $\Theta_{y_0}(B_R(y_0))=B_R$ and

$$\Theta(\Omega\cap B_R(y_0)) = B_R^+ := B_R\cap\mathbb{R}^N_+, \qquad \Theta(\partial\Omega\cap B_{\frac{R}{2}}(y_0)) = B'_{\frac{R}{2}}, \qquad \Theta(\partial\Omega\cap B_R(y_0)) = B'_R.$$

Moreover, Θ has bounded distortion, in the sense that since

$$\phi_\kappa(x)\int_{\partial\Omega\cap B_R(y_0)}\frac{d\mu(z)}{|x-z|^{N-2+\alpha_+}} = \phi_\kappa\circ\Theta^{-1}(\xi)\int_{B'_R}\frac{d(\mu\circ\Theta^{-1})(\zeta)}{|\Theta^{-1}(\xi)-\Theta^{-1}(\zeta)|^{N-2+\alpha_+}},$$

there holds

$$\frac{\xi_1^{\frac{\alpha_+}{2}}}{c_{61}}\int_{B'_{\frac{R}{2}}}\frac{d(\mu\circ\Theta^{-1})(\zeta)}{(\xi_1^2+|\xi'-\zeta'|^2)^{\frac{N-2+\alpha_+}{2}}} \leq \phi_\kappa\circ\Theta^{-1}(\xi)\int_{B'_{\frac{R}{2}}}\frac{d(\mu\circ\Theta^{-1})(\zeta)}{|\Theta^{-1}(\xi)-\Theta^{-1}(\zeta)|^{N-2+\alpha_+}}$$
$$\leq c_{61}\xi_1^{\frac{\alpha_+}{2}}\int_{B'_{\frac{R}{2}}}\frac{d(\mu\circ\Theta^{-1})(\zeta)}{(\xi_1^2+|\xi'-\zeta'|^2)^{\frac{N-2+\alpha_+}{2}}}.$$

Since $\mu\mapsto\mu\circ\Theta^{-1}$ is a C^2 diffeomorphism between $\mathfrak{M}^+(\partial\Omega\cap B_{\frac{R}{2}}(y_0))\cap B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega\cap B_{\frac{R}{2}}(y_0))$ and $\mathfrak{M}^+(B'_{\frac{R}{2}})\cap B^{-2+\frac{2+\alpha_+}{2q'},q}(B'_{\frac{R}{2}})$, we derive, using (2.57) and (3.41),

$$\frac{1}{c_{62}}\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq \int_{\Omega\cap B_R(y_0)}(\mathbb{K}_{\mathcal{L}_\kappa}[\mu])^q\phi_\kappa dx \leq c_{62}\|\mu\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}}. \tag{3.42}$$

Clearly the left-hand side inequality (3.39) follows. Combining Harnack inequality and boundary Harnack inequality we obtain

$$\int_\Omega(\mathbb{K}_{\mathcal{L}_\kappa}[\mu])^q\phi_\kappa dx \leq c_{63}\int_{\Omega\cap B_R(y_0)}(\mathbb{K}_{\mathcal{L}_\kappa}[\mu])^q\phi_\kappa dx, \tag{3.43}$$

which implies the left-hand side inequality (3.39) when μ has it support in a ball $B_{\frac{R}{2}}(y_0)\cap\partial\Omega$.

Step 2: global estimates. We write $\mu=\sum_{j=1}^{j_0}\mu_j$ where the μ_j are positive measures on $\partial\Omega$ with support in some ball $B_{\frac{R}{2}}(y_j)$ with $y_j\in\partial\Omega$ and such that

$$\frac{1}{c_{64}}\|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq \|\mu_j\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq c_{64}\|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}}.$$

Then

$$\|\mathbb{K}_{\mathcal{L}_\kappa}[\mu]\|_{L^q_{\phi_\kappa}} \leq \sum_{j=1}^{j_0}\|\mathbb{K}_{\mathcal{L}_\kappa}[\mu_j]\|_{L^q_{\phi_\kappa}} \leq c_{59}^{\frac{1}{q}}\sum_{j=1}^{j_0}\|\mu_j\|^q_{B^{-2+\frac{2+\alpha_+}{2q'},q}} \leq j_0c_{64}c_{59}^{\frac{1}{q}}\|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}}.$$

On the opposite side

$$\|\mathbb{K}_{\mathcal{L}_\kappa}[\mu]\|_{L^q_{\phi_\kappa}} \geq \max_{1\leq j\leq j_0}\|\mathbb{K}_{\mathcal{L}_\kappa}[\mu_j]\|_{L^q_{\phi_\kappa}}$$
$$\geq \frac{1}{c_{59}^{\frac{1}{q}}}\max_{1\leq j\leq j_0}\|\mu_j\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}}$$

$$\geq \frac{1}{j_0 c_{59}^{\frac{1}{q}}} \sum_{j=1}^{j_0} \|\mu_j\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}}$$
$$\geq \frac{1}{c_{64} c_{59}^{\frac{1}{q}}} \|\mu\|_{B^{-2+\frac{2+\alpha_+}{2q'},q}},$$

which ends the proof. □

Proof of Theorem 3.6. *The condition is sufficient.* Let μ be a boundary measure such that $|\mathbb{K}_{\mathcal{L}_\kappa}[\mu]|^q \in L^1_{\phi_\kappa}(\Omega)$. For $k > 0$ set $g_k(u) = \operatorname{sgn}(u)\min\{|u|^q, k^q\}$ and let u_k be the solution of

$$\begin{aligned} \mathcal{L}_\kappa u_k + g_k(u_k) &= 0 \quad \text{in } \Omega \\ u_k &= \mu \quad \text{in } \partial\Omega, \end{aligned} \tag{3.44}$$

which exists a is unique by Theorem 3.3. Furthermore $k \mapsto u_k$ is decreasing,

$$0 \leq u_k \leq \mathbb{K}_{\mathcal{L}_\kappa}[\mu],$$

and

$$0 \leq g_k(u_k) \leq g_k(\mathbb{K}_{\mathcal{L}_\kappa}[\mu]) \leq (\mathbb{K}_{\mathcal{L}_\kappa}[\mu])^q,$$

and the first terms on the right of the two previous inequalities are integrable for the measure $\phi_\kappa dx$ by Theorem 3.7. Finally for any $\eta \in \mathbf{X}_\kappa(\Omega)$, there holds

$$\int_\Omega (u_k \mathcal{L}_\kappa \eta + g_k(u_k)\eta)\, dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu]\mathcal{L}_\kappa \eta dx.$$

Since u_k and $g_k(u_k)$ converge respectively to u and $g(u)$ a.e. and in $L^1_{\phi_\kappa}(\Omega)$; we conclude that

$$\int_\Omega \left(u\mathcal{L}_\kappa \eta + u^q \eta\right) dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu]\mathcal{L}_\kappa \eta dx.$$

If μ is a positive measure which vanishes on Borel sets $E \subset \partial\Omega$ with $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$-capacity zero, there exists an increasing sequence of positive measures in $B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)$ $\{\mu_n\}$ which converges to μ (see [9,13]). Let u_{μ_n} be the solution of (3.36) with boundary data μ_n. The sequence $\{u_{\mu_n}\}$ is increasing with limit u. Since, by taking ϕ_κ as test function, we obtain

$$\int_\Omega \left(\lambda_\kappa u_{\mu_n} + g(u_{\mu_n})\right)\phi_\kappa dx = \lambda_\kappa \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu_n]\phi_\kappa dx,$$

it follows that $u, g(u) \in L^1_{\phi_\kappa}(\Omega)$. Thus

$$\int_\Omega (u\mathcal{L}_\kappa \eta + g(u)\eta)\, dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu]\mathcal{L}_\kappa \eta dx \quad \forall \eta \in \mathbf{X}_\kappa(\Omega),$$

and therefore $u = u_\mu$.

Definition. A smooth lifting is a continuous linear operator $R[.]$ from $C^2_0(\partial\Omega)$ to $C^2_0(\overline{\Omega})$ satisfying

$$\begin{aligned} &\text{(i) } 0 \leq \eta \leq 1 \Longrightarrow 0 \leq R[\eta] \leq 1, \quad R[\eta]\lfloor_{\partial\Omega} = \eta, \\ &\text{(ii) } |\nabla\phi_\kappa \cdot \nabla R[\eta]| \leq c_{65}\phi_\kappa, \end{aligned} \tag{3.45}$$

where c_{65} depends on the C^1-norm of η.

Our proof are based upon a modification of an argument developed by Marcus and Véron in [23].

Lemma 3.8. *Assume there exists a solution u_μ of (3.36) with $\mu \geq 0$. For $\eta \in C^2(\Omega)$, $0 \leq \eta \leq 1$ set $\zeta = \phi_\kappa (R[\eta])^{q'}$ where R is a smooth lifting. Then*

$$\left(\int_{\partial\Omega} \eta d\mu\right)^{q'} \leq c_{67}\int_\Omega u^q \zeta dx + c_{67}\left(\int_\Omega u^q\zeta dx\right)^{\frac{1}{q}} \left(\left(\int_\Omega \phi_\kappa dx\right)^{\frac{1}{q'}} + q'\left(\int_\Omega (L[\eta])^{q'} dx\right)^{\frac{1}{q'}}\right), \tag{3.46}$$

where

$$L[\eta] = (R[\eta])^{q'-1}\left(2\phi_\kappa^{-\frac{1}{q}}|\nabla\phi_\kappa \cdot \nabla R[\eta]| + \phi_\kappa^{\frac{1}{q'}}|\Delta R[\eta]|\right), \tag{3.47}$$

and c_{67} depends on Ω, λ_κ, q, κ, N.

Proof. There holds

$$\mathcal{L}_\kappa\zeta = \lambda_\kappa(R[\eta])^{q'}\phi_\kappa - 2q'(R[\eta])^{q'-1}\nabla\phi_\kappa\cdot\nabla R[\eta] - q'(R[\eta])^{q'-2}\phi_\kappa\left(R[\eta]\Delta R[\eta] - (q'-1)|\nabla R[\eta]|^2\right).$$

Then $\zeta \in \mathbf{X}_\kappa(\Omega)$ because of (3.45)-(ii) and by Proposition 2.36

$$c_{66}\int_{\partial\Omega}\eta^{q'}d\mu \le \int_\Omega \left(u\mathcal{L}_\kappa\zeta + u^q\zeta\right)dx.$$

Since

$$u\mathcal{L}_\kappa\zeta \le u\left(\lambda_\kappa(R[\eta])^{q'}\phi_\kappa + 2q'(R[\eta])^{q'-1}|\nabla\phi_\kappa\cdot\nabla R[\eta]| + q'(R[\eta])^{q'-1}\phi_\kappa|\Delta R[\eta]|\right),$$

we obtain

$$\int_\Omega u\mathcal{L}_\kappa\zeta\,dx \le \left(\int_\Omega u^q\zeta\,dx\right)^{\frac{1}{q}}\left(\left(\int_\Omega \phi_\kappa\,dx\right)^{\frac{1}{q'}} + q'\left(\int_\Omega (L[\eta])^{q'}dx\right)^{\frac{1}{q}}\right),$$

where $L[\eta]$ is defined by (3.47). $\square$

Lemma 3.9. *There exist a smooth lifting R such that $\eta\mapsto L[\eta]$ is continuous from $B^{2-\frac{2+\alpha_+}{2q'},q'}(\partial\Omega)$ into $L^{q'}(\Omega)$. Furthermore,*

$$\|L[\eta]\|_{L^{q'}(\Omega)} \le c'_{66}\|\eta\|^{q'-1}_{L^\infty(\partial\Omega)}\|\eta\|_{B^{2-\frac{2+\alpha_+}{2q'},q'}(\partial\Omega)}. \tag{3.48}$$

Proof. The construction of the lifting is originated into [26, Sec 1]. For $0 < \delta \le \beta_0$, we set $\Sigma_\delta = \{x \in \Omega : d(x) = \delta\}$ and we identify $\partial\Omega$ with $\Sigma := \Sigma_0$. The set $\{\Sigma_\delta\}_{0<\delta\le\beta_0}$ is a smooth foliation of $\partial\Omega$. For each $\delta \in (0,\beta_0]$ there exists a unique $\sigma(x) \in \Sigma_\delta$ such that $d(x) = \delta$ and $|x-\sigma(x)| = \delta$. The set of couples (δ,σ) defines a system of coordinates in Ω_{β_0} called the flow coordinates. The Laplacian obtain the following expression in this system

$$\Delta = \frac{\partial^2}{\partial\delta^2} + b_0\frac{\partial}{\partial\delta} + \Lambda_\Sigma, \tag{3.49}$$

where Λ_Σ is a linear second-order elliptic operator on Σ with C^1 coefficients. Furthermore $b_0 \to K$ and $\Lambda_\Sigma \to \Delta_\Sigma$, where K is the mean curvature of Σ and Δ_Σ the Laplace–Beltrami operator on Σ. If $\eta \in B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)$, we denote by $H := H[\eta]$ the solution of

$$\begin{aligned}\frac{\partial H}{\partial s} + \Delta_\Sigma H = 0 &\quad \text{in } (0,\infty)\times\Sigma\\ H(0,.) = \eta &\quad \text{in } \Sigma.\end{aligned} \tag{3.50}$$

Let $h \in C^\infty(\mathbb{R}_+)$ such that $0 \le h \le 1$, $h' \le 0$, $h \equiv 1$ on $[0,\frac{\beta_0}{2}]$, $h \equiv 0$ on $[\beta_0,\infty]$. The lifting we consider is expressed by

$$R[\eta](x) = \begin{cases} H[\eta](\delta^2,\sigma(x))h(\delta) & \text{if } x \in \overline{\Omega}_{\beta_0}\\ 0 & \text{if } x \in \Omega'_{\beta_0},\end{cases} \tag{3.51}$$

with $x \approx (\delta,\sigma) := (d(x),\sigma(x))$. Mutatis mutandis, we perform the same computation as the one in [23, Lemma 1.2], using local coordinates $\{\sigma_j\}$ on Σ and obtain

$$\nabla R[\eta] = 2\delta h(\delta)\frac{\partial H}{\partial\delta}(\delta^2,\sigma)\nabla\delta + \sum_{j=1}^{N-1}h(\delta)\frac{\partial H}{\partial\sigma_j}(\delta^2,\sigma)\nabla\sigma_j + h'(\delta)H(\delta^2,\sigma)\nabla\delta.$$

Then there holds in $\Omega_{\frac{\beta_0}{2}}$,

$$\nabla R[\eta]\cdot\nabla\phi_\kappa = 2\delta h(\delta)\frac{\partial H}{\partial\delta}(\delta^2,\sigma)\nabla\phi_\kappa\cdot\nabla\delta + \sum_{j=1}^{N-1}h(\delta)\frac{\partial H}{\partial\sigma_j}(\delta^2,\sigma)\nabla\sigma_j\cdot\nabla\phi_\kappa + h'(\delta)H(\delta^2,\sigma)\nabla\delta\cdot\nabla\phi_\kappa. \tag{3.52}$$

Moreover $\phi_\kappa(x) \le c_2(d(x))^{\frac{\alpha_+}{2}} = c_2\delta^{\frac{\alpha_+}{2}}$ and $|\nabla\phi_\kappa(x)| \le c'_2(d(x))^{\frac{\alpha_+}{2}-1} = c'_2\delta^{\frac{\alpha_+}{2}-1}$. Similarly as in [23, (1.13)]

$$\nabla\phi_\kappa = \frac{\partial\phi_\kappa}{\partial\delta}\nabla d + \sum_{j=1}^{N-1}\frac{\partial\phi_\kappa}{\partial\sigma_j}(\delta^2,\sigma)\nabla\sigma_j,$$

thus

$$|\nabla\phi_\kappa\cdot\nabla\sigma_j|\leq c_{68}\delta^{\frac{\alpha_+}{2}},$$

$$\phi_\kappa^{-\frac{1}{q}}|\nabla R[\eta]\cdot\nabla\phi_\kappa|\leq c_{69}\delta^{\frac{\alpha_+}{2q'}}\left(\left|\frac{\partial H}{\partial\delta}(\delta^2,\sigma)\right|+\sum_{j=1}^{N-1}\left|\frac{\partial H}{\partial\sigma_j}(\delta^2,\sigma)\right|-\frac{h'(\delta)}{\delta}H(\delta^2,\sigma)\right).$$

Thus

$$\int_\Omega\phi_\kappa^{-\frac{q'}{q}}|\nabla R[\eta]\cdot\nabla\phi_\kappa|^{q'}dx\leq c_{70}\int_{\Omega_{\beta_0}}\delta^{\frac{\alpha_+}{2}}\left|\frac{\partial H}{\partial\delta}(\delta^2,\sigma)\right|^{q'}dx+c_{70}\sum_{j=1}^{N-1}\int_{\Omega_{\beta_0}}\delta^{\frac{\alpha_+}{2}}\left|\frac{\partial H}{\partial\sigma_j}(\delta^2,\sigma)\right|^{q'}dx$$
$$+c_{70}\int_{\Omega_{\beta_0}\setminus\Omega_{\frac{\beta_0}{2}}}\delta^{\frac{\alpha_+}{2}}H^{q'}(\delta^2,\sigma)dx.$$

Then

$$\begin{aligned}\int_\Omega\phi_\kappa^{-\frac{q'}{q}}|\nabla R[\eta]\cdot\nabla\phi_\kappa|^{q'}dx&\leq c_{71}\int_0^{\beta_0}\delta^{\frac{\alpha_+}{2}}\int_\Sigma\left|\frac{\partial H}{\partial\delta}(\delta^2,\sigma)\right|^{q'}dSd\delta\\&\leq c_{71}\int_0^{\beta_0^2}\int_\Sigma\left(t^{\frac{2+\alpha_+}{4q'}}\left\|\frac{\partial H}{\partial t}(t,.)\right\|_{L^{q'}(\Sigma)}\right)^{q'}\frac{dt}{t}\\&\leq c_{72}\|\eta\|^{q'}_{B^{2-\frac{2+\alpha_+}{2q'},q'}(\Sigma)},\end{aligned}\tag{3.53}$$

by using the classical real interpolation identity

$$\left[W^{2,q'}(\Sigma),L^{q'}(\Sigma)\right]_{1-\frac{2+\alpha_+}{4q'},q'}=B^{2-\frac{2+\alpha_+}{2q'},q'}(\Sigma).\tag{3.54}$$

Similarly (see [23, (1.17), (1.19)])

$$\sum_{j=1}^{N-1}\int_{\Omega_{\beta_0}}\delta^{\frac{\alpha_+}{2}}\left|\frac{\partial H}{\partial\sigma_j}(\delta^2,\sigma)\right|^{q'}dx+\int_{\Omega_{\beta_0}\setminus\Omega_{\frac{\beta_0}{2}}}\delta^{\frac{\alpha_+}{2}}H^{q'}(\delta^2,\sigma)dx\leq c_{72}\|\eta\|^{q'}_{W^{2-\frac{2+\alpha_+}{2q'},q'}(\Sigma)}.\tag{3.55}$$

Next we consider the second term. Adapting in a straightforward manner the computation in [23, p. 886–887] we obtain the following instead of [23, (1.21)]

$$\begin{aligned}\int_\Omega\phi_\kappa|\Delta R[\eta]|^{q'}dx&\leq c_{72}\int_0^{\beta_0}\int_\Sigma\left|\delta^{2+\frac{\alpha_+}{2q'}}\frac{\partial^2H[\eta]}{\partial\delta^2}\right|^{q'}(\delta^2,\sigma)d\sigma d\delta\\&\quad+c_{72}\int_0^{\beta_0}\int_\Sigma\delta^{\frac{\alpha_+}{2}}\left(\left|\frac{\partial H[\eta]}{\partial\delta}\right|^{q'}+|H|^{q'}+|\Lambda_\Delta-\Lambda_\Sigma|^{q'}\right)(\delta^2,\sigma)dx.\end{aligned}\tag{3.56}$$

Furthermore

$$\begin{aligned}\int_0^{\beta_0}\int_\Sigma\left|\delta^{2+\frac{\alpha_+}{2q'}}\frac{\partial^2H[\eta]}{\partial\delta^2}\right|^{q'}(\delta^2,\sigma)d\sigma d\delta&=\int_0^{\beta_0^2}\int_\Sigma\left|t^{2\left(1-\frac{4q'-\alpha_+-2}{8q'}\right)}\frac{\partial^2H[\eta]}{\partial t^2}\right|^{q'}d\sigma\frac{dt}{t}\\&\leq c_{73}\|\eta\|^{q'}_{B^{2-\frac{2+\alpha_+}{2q'},q'}(\Sigma)},\end{aligned}\tag{3.57}$$

as a consequence of the real interpolation identity

$$\left[W^{4,q'}(\Sigma),L^{q'}(\Sigma)\right]_{\frac{4q'-\alpha_+-2}{8q'},q'}=B^{2-\frac{2+\alpha_+}{2q'},q'}(\Sigma).\tag{3.58}$$

The other term in the right-hand side of (3.56) yields to the same inequality as in (3.55). □

Proof of Theorem 3.6. *The condition is necessary*. Let $K\subset\partial\Omega$ be a compact set and $\eta\in C_0^2(\partial\Omega)$ such that $0\leq\eta\leq 1$ and $\eta=1$ on K. Then, by (3.46)

$$\begin{aligned}(\mu(K))^{q'}&\leq c_{67}\int_\Omega u^q(R[\eta])^{q'}\phi_\kappa dx\\&+c_{67}\left(\int_\Omega u^q(R[\eta])^{q'}\phi_\kappa dx\right)^{\frac{1}{q}}\left(\left(\int_\Omega\phi_\kappa dx\right)^{\frac{1}{q'}}+c'_{66}q'\|\eta\|_{B^{2-\frac{2+\alpha_+}{2q'},q'}(\partial\Omega)}\right).\end{aligned}\tag{3.59}$$

From this inequality, we obtain classically the result since if $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(K)=0$ there exists a sequence $\{\eta_n\}$ in $C_0^2(\partial\Omega)$ with the following properties:

$$0\leq\eta_n\leq 1,\quad \eta_n=1\text{ in a neighborhood of }K\text{ and }\eta_n\to 0\text{ in }B^{2-\frac{2+\alpha_+}{2q'},q'}(\partial\Omega)\text{ as }n\to\infty.\tag{3.60}$$

This implies that $u^q(R[\eta_n])^{q'}\to 0$ in $L^1_{\phi_\kappa}(\Omega)$. Therefore the right-hand side of (3.59) tends to 0 if we substitute η_n to η and thus $\mu(K)=0$ for any K compact with zero capacity and this relation holds for any Borel subset. □

Definition. We say that a compact set $K\subset\partial\Omega$ is *removable* if any positive solution $u\in C(\overline{\Omega}\setminus K)$ of

$$\mathcal{L}_\kappa u+|u|^{q-1}u=0\quad\text{in }\Omega,\tag{3.61}$$

such that

$$\int_\Omega(u\mathcal{L}_\kappa\eta+|u|^{q-1}u\eta)dx=0\quad\forall\eta\in\mathbf{X}^K_\kappa(\Omega),\tag{3.62}$$

where $\mathbf{X}^K_\kappa(\Omega)=\{\eta\in\mathbf{X}_\kappa(\Omega):\ \text{s.t. }\eta=0\text{ in a neighborhood of }K\}$, is identically zero.

Theorem 3.10. *Assume $0<\kappa\leq\frac{1}{4}$ and $q\geq 1$. A compact set $K\subset\partial\Omega$ is removable if and only if $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q}(K)=0$.*

Proof. The condition is clearly necessary since, if a compact boundary set K has positive capacity, there exists a capacitary measure $\mu_k\in\mathfrak{M}_+(\partial\Omega)\cap B^{-2+\frac{2+\alpha_+}{2q'},q}(\partial\Omega)$ with support in K (see e.g. [1]). For such a measure there exists a solution u_{μ_K} of (3.36) with $\mu=\mu_K$ by Theorem 3.6. Next we assume that $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q}(K)=0$. Then there exists a sequence $\{\eta_n\}$ in $C_0^2(\partial\Omega)$ satisfying (3.60). In particular, there exists a decreasing sequence $\{\mathcal{O}_n\}$ of relatively open subsets of $\partial\Omega$, containing K such that $\eta_n=1$ on $\mathcal{O}_n$ and thus $\eta_n=1$ on $K_n:=\overline{\mathcal{O}}_n$. We set $\tilde\eta_n=1-\eta_n$ and $\tilde\zeta_n=\phi_\kappa(R[\tilde\eta_n])^{2q'}$ where R is defined by (3.51). Then $0\leq\tilde\eta_n\leq 1$ and $\tilde\eta_n=0$ on K_n. Therefore

$$\tilde\zeta_n(x)\leq\phi_\kappa\min\left\{1,c_{74}(d(x))^{1-N}e^{-(4d(x))^{-2}(\mathrm{dist}(x,K_n^c))^2}\right\}.\tag{3.63}$$

Furthermore

$$\begin{aligned}&\text{(i) }|\nabla R[\tilde\eta_n]|\leq c_{75}\min\left\{1,(d(x))^{-2-N}e^{-(4d(x))^{-2}(\mathrm{dist}(x,K_n^c))^2}\right\},\\&\text{(ii) }|\Delta R[\tilde\eta_n]|\leq c_{75}\min\left\{1,(d(x))^{-4-N}e^{-(4d(x))^{-2}(\mathrm{dist}(x,K_n^c))^2}\right\}.\end{aligned}\tag{3.64}$$

Step 1. We claim that

$$\int_\Omega\left(u\mathcal{L}_\kappa\tilde\zeta_n+u^q\tilde\zeta_n\right)dx=0.\tag{3.65}$$

By Proposition A.3 there exists $c_{74}>0$ such that

$$\begin{aligned}&\text{(i) }u(x)\leq c_{76}(d(x))^{\frac{\alpha_+}{2}}(\mathrm{dist}(x,K))^{-\frac{2}{q-1}-\frac{\alpha_+}{2}},\\&\text{(ii) }|\nabla u(x)|\leq c_{76}(d(x))^{\frac{\alpha_+}{2}-1}(\mathrm{dist}(x,K))^{-\frac{2}{q-1}-\frac{\alpha_+}{2}},\end{aligned}\tag{3.66}$$

for all $x\in\Omega$. As in the proof of Lemma 3.8,

$$|u\mathcal{L}_\kappa\tilde\zeta_n|\leq c_{77}(R[\tilde\eta_n])^{2q'-2}u\left(\phi_\kappa R^2[\tilde\eta_n]+R[\tilde\eta_n]|\nabla\phi_\kappa\cdot\nabla R[\tilde\eta_n]|+\phi_\kappa(R[\tilde\eta_n]|\Delta R[\tilde\eta_n]|+|\nabla R[\tilde\eta_n]|^2)\right).\tag{3.67}$$

Let $\mathcal{O}$ be a relatively open neighborhood of K such that $\overline{\mathcal{O}} \subset \mathcal{O}_n$. We set $G_{\mathcal{O},\beta_0} = \{x \in \Omega_{\beta_0} : \sigma(x) \in \mathcal{O}\}$ and $G_{\mathcal{O}^c,\beta_0} = \Omega_{\beta_0} \setminus G_{\mathcal{O}}$. If $x \in G_{\mathcal{O}}$, $\mathrm{dist}(x, K_n^c) \geq \tau > 0$. Then, by (3.66)-(i) and (3.63), $u^q\tilde{\zeta}_n \in L^q(G_{\mathcal{O}})$. Since $u(x) = \circ(W(x))$ in $G_{\mathcal{O}^c}$ it follows that $u^q\tilde{\zeta}_n \in L^1(\Omega_{\beta_0})$ and thus $u^q\tilde{\zeta}_n$ is integrable in Ω. Similarly, using (N22-1)-(i) and (ii), $u\mathcal{L}_\kappa\tilde{\zeta}_n \in L^1(\Omega)$. Since $\tilde{\zeta}_n$ does not vanish in a neighborhood of K, we introduce a cut-off function $\theta_\epsilon \in C^2(\overline{\Omega})$ for $0 < \epsilon \leq \frac{\beta_0}{2}$, with the following properties,

$$0 \leq \theta_\epsilon \leq 1,\ \theta_\epsilon(x) = 0\ \forall x \in G_{\mathcal{O},\epsilon}, \quad \theta_\epsilon(x) = 1\ \forall x \in \overline{\Omega} \text{ s.t. } \mathrm{dist}(x, G_{\mathcal{O},\epsilon}) \geq \epsilon$$
$$|\nabla\theta_\epsilon| \leq c_{78}\epsilon^{-1}\chi_{G_{\mathcal{O}_\epsilon,\epsilon}\setminus G_{\mathcal{O},\epsilon}} \quad \text{and} \quad |D^2\theta_\epsilon| \leq c_{78}\epsilon^{-2}\chi_{G_{\mathcal{O}_\epsilon,\epsilon}\setminus G_{\mathcal{O},\epsilon}},$$

where we have taken ϵ small enough so that

$$G_{\mathcal{O}_\epsilon,\epsilon} := \{x \in \Omega : \mathrm{dist}(x, G_{\mathcal{O},\epsilon}) \leq \epsilon\} \subset G_{K_n,2\epsilon} = \{x \in \Omega_{2\epsilon} : \sigma(x) \in K_n\}.$$

Clearly $\theta_\epsilon\tilde{\zeta}_n \in \mathbf{X}_\kappa^K(\Omega)$, thus

$$\int_\Omega \left(u\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n) + u^q\theta_\epsilon\tilde{\zeta}_n\right)dx = 0.$$

Next

$$\int_\Omega \left(u\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n) + u^q\theta_\epsilon\tilde{\zeta}_n\right)dx = \int_{\Omega\setminus G_{\mathcal{O}_\epsilon,\epsilon}} \left(u\mathcal{L}_\kappa(\zeta_n) + u^q\tilde{\zeta}_n\right)dx + \int_{G_{\mathcal{O}_\epsilon,\epsilon}} \left(u\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n) + u^q\theta_\epsilon\tilde{\zeta}_n\right)dx$$
$$= I_\epsilon + II_\epsilon.$$

Clearly

$$\lim_{\epsilon\to 0} I_\epsilon = \int_\Omega \left(u\mathcal{L}_\kappa\tilde{\zeta}_n + u^q\tilde{\zeta}_n\right)dx,$$

and

$$\lim_{\epsilon\to 0}\int_{G_{\mathcal{O}_\epsilon,\epsilon}} u^q\theta_\epsilon\tilde{\zeta}_n dx = 0.$$

Finally, since $\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n) = \theta_\epsilon\mathcal{L}_\kappa\tilde{\zeta}_n + \tilde{\zeta}_n\Delta\theta_\epsilon + 2\nabla\theta_\epsilon\cdot\nabla\tilde{\zeta}_n$, θ_ϵ is constant outside $G_{\mathcal{O}_\epsilon,\epsilon}\setminus G_{\mathcal{O},\epsilon}$ and $\mathrm{dist}(G_{\mathcal{O}_\epsilon,\epsilon}\setminus G_{\mathcal{O},\epsilon}, F_n^c) \geq \tau > 0$, independent of ϵ there holds, by (3.63)

$$|\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n)| \leq c_{79}\epsilon^{-N+4}e^{-\frac{\tau}{\epsilon^2}}.$$

Using (3.66)-(i) we derive

$$\lim_{\epsilon\to 0}\int_{G_{\mathcal{O}_\epsilon,\epsilon}} u\mathcal{L}_\kappa(\theta_\epsilon\tilde{\zeta}_n)dx = 0,$$

which yields to (3.65).

Step 2. We claim that

$$\int_\Omega u^q\phi_\kappa dx < \infty. \tag{3.68}$$

Using the expression of $\mathcal{L}_\kappa\zeta_n$ in (3.65) where replace η_n by $\tilde{\eta}_n$, we derive

$$\begin{aligned}\int_\Omega u^q\tilde{\zeta}_n dx &= \int_\Omega \Big(-\lambda_\kappa(R[\tilde{\eta}_n])^{2q'}\phi_\kappa + 4q'(R[\tilde{\eta}_n])^{2q'-1}\nabla\phi_\kappa\cdot\nabla R[\tilde{\eta}_n] \\ &\quad + 2q'(R[\tilde{\eta}_n])^{2q'-2}\phi_\kappa\left(R[\tilde{\eta}_n]\Delta R[\tilde{\eta}_n] + (2q'-1)|\nabla R[\tilde{\eta}_n]|^2\right)\Big)udx \\ &\leq c_{79}\left(\int_\Omega u^q\tilde{\zeta}_n dx\right)^{\frac{1}{q}}\left(\int_\Omega(\tilde{L}[\eta_n])^{q'}dx\right)^{\frac{1}{q'}},\end{aligned} \tag{3.69}$$

where we have set

$$\tilde{L}[\eta] = (\phi_\kappa)^{-\frac{1}{q}}\nabla\phi_\kappa\cdot\nabla R[\eta_n] + (\phi_\kappa)^{\frac{1}{q'}}|\Delta R[\tilde{\eta}_n]| + (\phi_\kappa)^{\frac{1}{q'}}|\nabla R[\tilde{\eta}_n]|^2. \tag{3.70}$$

By Lemma 3.9 we know that

$$\int_\Omega(\phi_\kappa)^{-\frac{q'}{q}}|\nabla\phi_\kappa\cdot\nabla R[\eta_n]|^{q'} + \phi_\kappa|\Delta R[\tilde{\eta}_n]|^{q'}dx \leq (c_{72}+c_{73})\|\eta_n\|^{q'}_{B^{2-\frac{2+\alpha_+}{2q'},2}(\partial\Omega)}. \tag{3.71}$$

The last term is estimated in the following way

$$\int_\Omega \phi_\kappa |\nabla R[\tilde{\eta}_n]|^{2q'} dx \le c_{80} \int_0^{\beta_0^2} \int_\Sigma s^{q'+\frac{\alpha_++2}{4}} \left|\frac{\partial H[\eta_n]}{\partial s}\right|^{2q'} dS \frac{ds}{s} + c_{80} \int_0^{\beta_0^2} \int_\Sigma s^{\frac{\alpha_++2}{4}} \left(|\nabla_\Sigma H[\eta_n]|^{2q'} + (H[\eta_n])^{2q'}\right) dS \frac{ds}{s}, \tag{3.72}$$

where ∇_Σ denotes the covariant gradient on Σ. Since the following interpolation identity holds

$$\left[W^{2,2q'}(\Sigma), L^{2q'}(\Sigma)\right]_{1-\frac{\alpha_++2}{8q'},2q'} = B^{1-\frac{\alpha_++2}{4q'},2q'}(\Sigma),$$

we obtain

$$\int_0^{\beta_0^2} \int_\Sigma s^{q'+\frac{\alpha_++2}{4}} \left|\frac{\partial H[\eta_n]}{\partial s}\right|^{2q'} \frac{ds}{s} \le c_{81} \|\eta_n\|^{2q'}_{B^{1-\frac{\alpha_++2}{4q'},2q'}(\Sigma)}.$$

By the Gagliardo–Nirenberg inequality

$$\|\eta_n\|^{2q'}_{B^{1-\frac{\alpha_++2}{4q'},2q'}(\Sigma)} \le c_{82} \|\eta_n\|^{q'}_{B^{2-\frac{\alpha_++2}{2q'},q'}(\Sigma)} \|\eta\|^{q'}_{L^\infty(\Sigma)} = c_{82} \|\eta_n\|^{q'}_{B^{2-\frac{\alpha_++2}{2q'},q'}(\Sigma)}. \tag{3.73}$$

By the same inequality

$$\int_\Sigma \left(|\nabla_\Sigma H[\eta_n]|^{2q'} + (H[\eta_n])^{2q'}\right) dS \le c_{82} \|H[\eta_n]\|^{q'}_{L^\infty(\Sigma)} \int_\Sigma \left(|\Delta_\Sigma H[\eta_n]|^{q'} + (H[\eta_n])^{q'}\right) dS. \tag{3.74}$$

Using the estimates on $L[\eta]$ in Lemma 3.9 and the fact that $0 \le H[\eta_n] \le 1$, we conclude that

$$\int_0^{\beta_0^2} \int_\Sigma s^{\frac{\alpha_++2}{4}} \left(|\nabla_\Sigma H[\eta_n]|^{2q'} + (H[\eta_n])^{2q'}\right) dS \frac{ds}{s} \le c_{83} \|\eta_n\|^{q'}_{B^{2-\frac{\alpha_++2}{2q'},q'}.(\Sigma)}.$$

It follows from (3.69)

$$\int_{\Omega_{\frac{\beta_0}{2}}} u^q (R[\tilde{\eta}_n])^{2q'} \phi_\kappa dx \le c_{84} \int_{\Omega_{\beta_0}} (\tilde{L}\eta_n)^{q'} dx \le c_{85} \|\eta_n\|^{q'}_{B^{2-\frac{\alpha_++2}{2q'},q'}.(\Sigma)}. \tag{3.75}$$

Letting $n \to \infty$ and using the fact that $\eta_n \to 0$, we obtain by Fatou's lemma that

$$\int_{\Omega_{\frac{\beta_0}{2}}} u^q \phi_\kappa dx = 0.$$

Combining this with the fact that u is bounded in $\Omega'_{\frac{\beta_0}{2}}$ we obtain (3.68). Notice that $\|u\|_{L^q_{\phi_\kappa}}(\Omega)$ is bounded independently of u.

Step 3. End of the proof. Since $u^q \in L^1_{\phi_\kappa}(\Omega)$, by Proposition 3.2 there exists a unique weak solution $v \in L^1_{\phi_\kappa}(\Omega)$ of

$$\begin{aligned} \mathcal{L}_\kappa v &= u^q \quad \text{in } \Omega \\ v &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{3.76}$$

and $v \ge 0$. Then $w = u + v$ is $\mathcal{L}_\kappa$-harmonic in Ω, and by Theorem 2.33 there exists a unique positive Radon measure τ on $\partial\Omega$ such that $w = \mathbb{K}_{\mathcal{L}_\kappa}[\tau]$. Since v and u vanish respectively on $\partial\Omega$ and $\partial\Omega \setminus K$, it follows from Propositions 2.34 and 2.35 that the support of τ is included in K. By Theorem 3.6, τ vanishes on Borel subsets with zero $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$-capacity. Since $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}(K) = 0$, $\tau = 0$. This implies that u is a weak solution of

$$\begin{aligned} \mathcal{L}_\kappa u + u^q &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{3.77}$$

and therefore $u = 0$. □

Remark. Using the fact that u^+ and u_- are subsolutions of (3.61), it is easy to check that Theorem 3.10 remains valid for any signed solution of (3.61).

Remark. If $1 < q < q_c$ (see (3.37)) it follows from Sobolev embedding theorem that only the empty set has zero $C^{\mathbb{R}^{N-1}}_{2-\frac{2+\alpha_+}{2q'},q'}$-capacity. As a consequence of the previous result, if $q \geq q_c$ any isolated boundary singularity of a solution of (3.61) is removable.

4. Isolated boundary singularities

We denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ the canonical basis in $\mathbb{R}^N = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}\}$ and by (r, σ) the spherical coordinates therein. Then $\mathbb{R}^N_+ = \{= (x', x_N) :, x' \in \mathbb{R}^{N-1}, x_N > 0\}$. We although denote by S^{N-1} and S^{N-1}_+ the unit sphere and the upper hemisphere of $\mathbb{R}^N_+$, i.e. $S^{N-1} : \cap \mathbb{R}^N_+$. In this section we study the behavior near 0 of solutions of

$$-\Delta u - \frac{\kappa}{d^2}u + |u|^{q-1}u = 0 \tag{4.1}$$

in a bounded convex domain Ω of $\mathbb{R}^N$ with a smooth boundary containing 0 where d is the distance function to the boundary, κ a constant in $(0, \frac{1}{4}]$ and $q > 1$. Although it is not bounded, the model case is $\Omega = \mathbb{R}^N_+ = \{= (x', x_N) :, x' \in \mathbb{R}^{N-1}, x_N > 0\}$ which is represented by $(r, \sigma), r > 0, \sigma \in S^{N-1}_+$ in spherical coordinates. Then

$$\mathcal{L}_\kappa u = -u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta_{S^{N-1}}u - \frac{\kappa}{r^2(\mathbf{e}_N \cdot \sigma)^2}u + |u|^{q-1}u. \tag{4.2}$$

We also denote by ∇' the covariant gradient on S^{N-1} in the metric of S^{N-1} obtained by the embedding into $\mathbb{R}^N$.

4.1. The spherical $\mathcal{L}_\kappa$-harmonic problem

It is straightforward to check that the Poisson kernel $K_{\mathcal{L}_\kappa}$ of $\mathcal{L}_\kappa$ in $\mathbb{R}^N_+$ has the following expression

$$K_{\mathcal{L}_\kappa}(x, \xi) = c_{N,\kappa}\frac{x_N^{\frac{\alpha_+}{2}}}{|x-\xi|^{N+\alpha_+-2}}. \tag{4.3}$$

In spherical coordinates

$$K_{\mathcal{L}_\kappa}(x, 0) = c_{N,\kappa}r^{2-N-\frac{\alpha_+}{2}}\psi(\sigma) \quad r > 0, \ \sigma \in S^{N-1}_+$$

where $\psi_\kappa(\sigma) = \frac{x_N}{|x|}\lfloor^{\frac{\alpha_+}{2}}_{S^{N-1}_+} = (\mathbf{e}_N \cdot \sigma)^{\frac{\alpha_+}{2}}$ solves

$$\begin{aligned} -\Delta_{S^{N-1}}\psi_\kappa - \mu_\kappa\psi_\kappa - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2}\psi_\kappa &= 0 \quad \text{in } S^{N-1}_+ \\ \psi_\kappa &= 0 \quad \text{in } \partial S^{N-1}_+, \end{aligned} \tag{4.4}$$

and

$$\mu_\kappa = \frac{\alpha_+}{2}\left(N + \frac{\alpha_+}{2} - 2\right). \tag{4.5}$$

Notice that Eq. (4.4) admits a unique positive solution with supremum 1. We could have defined the first eigenvalue μ_κ of the operator

$$\phi \mapsto \mathcal{L}'_\kappa w := -\Delta_{S^{N-1}}w - \frac{\kappa}{(\mathbf{e}_N \cdot \sigma)^2}w,$$

by

$$\mu_\kappa = \inf\left\{\frac{\int_{S^{N-1}_+}\left(|\nabla w|^2 - \kappa(\mathbf{e}_N \cdot \sigma)^{-2}w^2\right)dS}{\int_{S^{N-1}_+}w^2 dS} : w \in H^1_0(S^{N-1}_+), w \neq 0\right\}. \tag{4.6}$$

By [10, Th 6.1] the infimum exists since $\rho(\sigma) = x_N\lfloor_{S^{N-1}_+} = \mathbf{e}_N \cdot \sigma$ is the first eigenfunction of $-\Delta_{S^{N-1}}$ in $H^1_0(S^{N-1}_+)$. The minimizer ψ_κ belongs to $H^1_0(S^{N-1}_+)$ only if $1 < \kappa < \frac{1}{4}$. Furthermore

$$\psi_\kappa \in \mathbf{Y}(S^{N-1}_+) := \{\phi \in H^1_{loc}(S^{N-1}) : \rho^{-\frac{\alpha_+}{2}}\phi \in H^1(S^{N-1}_+, \rho^{\alpha_+})\}. \tag{4.7}$$

We can also define μ_k by

$$\mu_k = \inf\left\{\int_{S_+^{N-1}} |\nabla'(\rho^{-\frac{\alpha_+}{2}}\omega)|^2\rho^{\alpha_+}dS : \omega \in \mathbf{Y}(S_+^{N-1}),\ \int_{S_+^{N-1}}\omega^2 dS = 1\right\}. \tag{4.8}$$

We can use the symmetry of the operator to obtain the second eigenvalue and eigenfunction of $\mathcal{L}'_\kappa$ on S_+^{N-1}. We first notice that for $j = 1, \dots, N-1$, the function

$$x \mapsto \frac{x_N^{\frac{\alpha_+}{2}} x_j}{|x|^{N+\alpha_+}}, \tag{4.9}$$

is $\mathcal{L}_\kappa$-harmonic in $\mathbb{R}_+^{N-1}$, positive (resp. negative) on $\{x = (x_1, \dots, x_N) : x_j > 0, x_N > 0\}$ (resp. $\{x = (x_1, \dots, x_N) : x_j < 0, x_N > 0\}$) and vanishes on $\{x = (x_1, \dots, x_N) : x_j = 0, x_N = 0\}$.

Proposition 4.1. *For any $j = 1, .., N-1$ the function*

$$\sigma \mapsto \psi_{\kappa,j}(\sigma) = (\mathbf{e}_N\cdot\sigma)^{\frac{\alpha_+}{2}}\mathbf{e}_j\cdot\sigma,$$

satisfies

$$\mathcal{L}'_\kappa\psi_{\kappa,j} = (\mu_\kappa + N - 1 + \alpha_+)\rho_{\kappa,j} \tag{4.10}$$

in S_+^{N-1}. It is positive (resp. negative) on $S_+^{N-1}\cap\{x = (x_1, \dots, x_N) = x_j > 0\}$ (resp. $S_+^{N-1}\cap\{x = (x_1, \dots, x_N) = x_j < 0\}$) and it vanishes on $\partial S_+^{N-1}\cap\{x = (x_1, \dots, x_N) = x_j = 0\}$. The real number

$$\mu_{\kappa,2} = \mu_\kappa + N - 1 + \alpha_+ = \left(\frac{\alpha_+}{2}+1\right)\left(N + \frac{\alpha_+}{2} - 1\right)$$

is the second eigenvalue of $\mathcal{L}'_\kappa$ in $\mathbf{Y}(S_+^{N-1})$.

Proof. There holds

$$\begin{aligned}\mathcal{L}'_\kappa\psi_{\kappa,j} &= \mathbf{e}_j\cdot\sigma\mathcal{L}_\kappa\psi_\kappa + \psi_\kappa\Delta_{S^{N-1}}\mathbf{e}_j\cdot\sigma + 2\nabla'\psi_\kappa\cdot\nabla'\mathbf{e}_j\cdot\sigma\\ &= (\mu_\kappa + N - 1)\psi_{\kappa,j} - \alpha_+(\mathbf{e}_N\cdot\sigma)^{\frac{\alpha_+}{2}-1}\nabla'(\mathbf{e}_j\cdot\sigma)\cdot\nabla'(\mathbf{e}_N\cdot\sigma).\end{aligned}$$

Now

$$\nabla\left(\frac{x_j}{r}\right) = \left(\frac{x_j}{r}\right)_r\frac{x}{r} + \frac{1}{r}\nabla'\left(\frac{x_j}{r}\right) = \frac{1}{r}\nabla'\left(\frac{x_j}{r}\right) = \frac{1}{r}\mathbf{e}_j - \frac{x_j}{r^3}x,$$

thus

$$\nabla\left(\frac{x_j}{r}\right)\cdot\nabla\left(\frac{x_N}{r}\right) = -\frac{x_jx_N}{r^4} = \frac{1}{r^2}\nabla'\left(\frac{x_j}{r}\right)\cdot\nabla'\left(\frac{x_N}{r}\right) = \frac{1}{r^2}\nabla'(\mathbf{e}_j\cdot\sigma)\cdot\nabla'(\mathbf{e}_N\cdot\sigma),$$

which implies

$$\nabla'(\mathbf{e}_j\cdot\sigma)\cdot\nabla'(\mathbf{e}_N\cdot\sigma) = -\frac{x_jx_N}{r^2} = -(\mathbf{e}_j\cdot\sigma)(\mathbf{e}_N\cdot\sigma),$$

and finally

$$\mathcal{L}_\kappa\psi_{\kappa,j} = (\mu_\kappa + N - 1 + \alpha_+)\psi_{\kappa,j}. \tag{4.11}$$

Since $S_+^{N-1} = \{(\sigma'\sin\theta, \cos\theta) : \sigma' \in S^{N-2}, \theta \in [0, \frac{\pi}{2}]\}$, $\mathbf{e}_N\cdot\sigma = \cos\theta$, $\mathbf{e}_j\cdot\sigma = \mathbf{e}_j\cdot\sigma'\sin\theta$ and $dS = (\sin\theta)^{N-2}dS'd\theta$ where dS and dS' are the volume elements of S^{N-1} and S^{N-2} respectively, we derive from the fact that $\sigma' \mapsto \mathbf{e}_j\cdot\sigma'$ is an odd function on S^{N-2},

$$\begin{aligned}\int_{S_+^{N-1}}\psi_{\kappa,j}\psi_\kappa dS &= \int_{S_+^{N-1}}(\mathbf{e}_N\cdot\sigma)^{\alpha_+}\mathbf{e}_j\cdot\sigma dS\\ &= \int_0^{\frac{\pi}{2}}\left(\int_{S^{N-2}}\mathbf{e}_j\cdot\sigma' dS'\right)(\cos\theta)^{\alpha_+}(\sin\theta)^{N-1}d\theta\\ &= 0.\end{aligned}$$

Hence $\psi_{\kappa,j}$ is an eigenvalue of $\mathcal{L}'_\kappa$ in $\mathbf{Y}(S_+^{N-1})$ with two nodal domains and the space the $\psi_{\kappa,j}$ span is $(N-1)$-dimensional and any linear combination of the $\psi_{\kappa,j}$ has exactly two nodal domains since

$$\sum_{j=1}^{N-1} a_j\psi_{\kappa,j} = (\mathbf{e}_N\cdot\sigma)^{\frac{\alpha_+}{2}}\left(\sum_{j=1}^{N-1} a_j\mathbf{e}_j\right)\cdot\sigma.$$

This implies that $\mu_{\kappa,2}$ is the second eigenvalue. □

4.2. The nonlinear eigenvalue problem

If we look for separable solutions under the form

$$u(x) = u(r,\sigma) = r^\alpha\omega(\sigma),$$

then necessarily $\alpha = -\frac{2}{q-1}$ and ω is a solution of

$$\begin{aligned} -\Delta_{S^{N-1}}\omega - \ell_{q,N}\omega - \frac{\kappa}{(\mathbf{e}_N\cdot\sigma)^2}\omega + |\omega|^{q-1}\omega &= 0 \quad \text{in } S_+^{N-1} \\ \omega &= 0 \quad \text{in } \partial S_+^{N-1}, \end{aligned} \tag{4.12}$$

$$\ell_{q,N} = \frac{2}{q-1}\left(\frac{2}{q-1}+2-N\right), \tag{4.13}$$

and (4.6) is transformed accordingly. We denote by

$$\mathcal{E}_\kappa = \left\{\omega\in\mathbf{Y}(S_+^{N-1})\cap L^{q+1}(S_+^{N-1}) \text{ s.t. (4.12) holds}\right\} \tag{4.14}$$

and by $\mathcal{E}_\kappa^+$ the set of the nonnegative ones. We also recall that $q_c := \frac{2N+\alpha_+}{2N-4+\alpha_+}$ and we define a second critical value $q_e := \frac{2N+2+\alpha_+}{2N-2+\alpha_+}$.

The following result holds.

Theorem 4.2. *Assume* $0<\kappa\le\frac{1}{4}$ *and* $q>1$, *then*

(i) *If* $q\ge q_c$, $\mathcal{E}_\kappa=\{0\}$.
(ii) *If* $1<q<q_c$, $\mathcal{E}_\kappa^+$ *is contains exactly two elements:* 0 *and* ω_κ. *Furthermore* ω_κ *depends only on the azimuthal angle* θ.
(iii) *If* $q_e\le q<q_c$, $\mathcal{E}_\kappa$ *contains three elements:* 0, ω_κ *and* $-\omega_\kappa$.

Proof. We recall that $q\ge q_c \iff \ell_{q,N}\le\mu_\kappa$. Then non-existence follows by multiplying by ω and integrating on S_+^{N-1}. For existence, we consider the functional

$$J_\kappa(w) = \int_{S_+^{N-1}}\left(|\nabla'(w)|^2 + (\mu_\kappa-\ell_{q,N})w^2 + \frac{2}{q+1}\psi_\kappa^{q-1}|w|^{q+1}\right)\psi_\kappa^2 dS, \tag{4.15}$$

defined in $H^1(S_+^{N-1},\psi_\kappa^2dS)\cap L^{q+1}(S_+^{N-1},\psi_\kappa^{q+1}dS)$. Since $\mu_\kappa-\ell_{q,N}<0$, there exists a nontrivial minimum $\omega_\kappa>0$, which satisfies

$$-\mathrm{div}(\psi_\kappa^2\nabla' w_\kappa) + (\mu_\kappa-\ell_{q,N})\psi_\kappa^2 w_\kappa + \psi_\kappa^{q+1}w_\kappa^q = 0. \tag{4.16}$$

If we set $\omega_\kappa = \psi_\kappa w_\kappa$, then ω_κ satisfies

$$\mathcal{L}'_\kappa\omega_\kappa - \ell_{q,N}\omega_\kappa + \omega_\kappa^q = 0 \quad \text{in } S_+^{N-1}. \tag{4.17}$$

By monotonicity we derive that $\omega_\kappa\in L^p(S_+^{N-1})$ for any $1<p<\infty$ and finally, that ω_κ satisfies the regularity estimates of Propositions 2.9 and 2.10. Moreover $\omega_\kappa>0$ by the maximum principle.

In the case $q\ge q_c$ or equivalently $\mu_\kappa-\ell_{q,N}\ge 0$, the nonexistence of nontrivial solution is clear from (4.16).

Uniqueness. By Proposition 2.8 $\omega_\kappa(x)\le c_{86}(\rho(x))^{\frac{\alpha_+}{2}}$ and by standard scaling techniques $|\nabla\omega_\kappa(x)|\le c_{87}(\rho(x))^{\frac{\alpha_+}{2}-1}$. Assume now that two different positive solutions of (4.12) ω_κ and ω'_κ exist. Since $\max\{\omega_\kappa,\omega'_\kappa\}$ and $\omega_\kappa+\omega'_\kappa$ are respectively a subsolution and a supersolution and they are ordered, we can assume that $\omega'_\kappa<\omega_\kappa<c\omega'_\kappa$ for some $c>1$. Let $\epsilon>0$ and $\epsilon'=c^{-1}\epsilon$, then $\epsilon\omega'_\kappa\ge\epsilon'\omega_\kappa$. Set

$$\vartheta_\epsilon = \frac{((\omega'_\kappa+\epsilon')^2-(\omega_\kappa+\epsilon)^2)_+}{\omega_\kappa+\epsilon}, \qquad \vartheta_{\epsilon'} = \frac{((\omega'_\kappa+\epsilon')^2-(\omega_\kappa+\epsilon)^2)_+}{\omega'_\kappa+\epsilon'},$$

and $S_{\epsilon,\epsilon'}=\{\sigma\in S_+^{N-1}:\omega'_\kappa+\epsilon'>\omega_\kappa+\epsilon\}$. The assume that $S_{\epsilon,\epsilon'}\neq\emptyset$ for any $\epsilon>0$. Then

$$\int_{S_{\epsilon,\epsilon'}}\left(\nabla\omega'_\kappa\cdot\nabla\vartheta_{\epsilon'}-\nabla\omega_\kappa\cdot\nabla\vartheta_\epsilon-\left(\ell_{q,N}+\frac{\kappa}{\rho^2}\right)(\omega'_\kappa\cdot\vartheta_{\epsilon'}-\omega_\kappa\cdot\vartheta_\epsilon)+\omega'^q_\kappa\vartheta_{\epsilon'}-\omega^q_\kappa\vartheta_\epsilon\right)dS=0.$$

The first integrand on the l.h. side is equal to

$$\int_{S_{\epsilon,\epsilon'}}\left(\left|\nabla\omega'_\kappa-\frac{\omega'_\kappa+\epsilon'}{\omega_\kappa+\epsilon}\nabla\omega_\kappa\right|^2+\left|\nabla\omega_\kappa-\frac{\omega_\kappa+\epsilon}{\omega'_\kappa+\epsilon'}\nabla\omega'_\kappa\right|^2\right)dS\geq 0.$$

Since $\epsilon\omega'_\kappa<\epsilon'\omega_\kappa$ and $(\omega'_\kappa+\epsilon')^2>(\omega_\kappa+\epsilon)^2$, the second integrand on the l.h. side is equal to

$$-\int_{S_{\epsilon,\epsilon'}}\left(\ell_{q,N}+\frac{\kappa}{\rho^2}\right)\left(\frac{\omega'_\kappa}{\omega'_\kappa+\epsilon'}-\frac{\omega_\kappa}{\omega_\kappa+\epsilon}\right)((\omega'_\kappa+\epsilon')^2-(\omega_\kappa+\epsilon)^2)dS\geq 0.$$

At end, the last integrand is

$$\int_{S_{\epsilon,\epsilon'}}\left(\frac{\omega'^q_\kappa}{\omega'_\kappa+\epsilon'}-\frac{\omega^q_\kappa}{\omega_\kappa+\epsilon}\right)((\omega'_\kappa+\epsilon')^2-(\omega_\kappa+\epsilon)^2)dS.$$

If we let $\epsilon\to 0$, we derive

$$\int_{S_+^{N-1}}\left(\omega'^{q-1}_\kappa-\omega^{q-1}_\kappa\right)(\omega'^2_\kappa-\omega^2_\kappa)_+dS\leq 0.$$

This yields a contradiction. Therefore uniqueness holds.

Case $q_e\leq q<q_c$. Assume ω_κ is a solution. Using the representation of S_+^{N-1} already introduced in the proof of Proposition 4.1, with $\sigma=(\sigma',\theta)$ and

$$\Delta_{S^{N-1}}\omega_\kappa=\frac{1}{(\sin\theta)^{N-2}}\frac{\partial}{\partial\theta}\left((\sin\theta)^{N-2}\frac{\partial\omega_\kappa}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\Delta_{S^{N-2}}\omega_\kappa,$$

where $\Delta_{S^{N-2}}$ is the Laplace–Beltrami operator on S^{N-2}, we set

$$\bar\omega_\kappa(\theta)=\frac{1}{|S^{N-2}|}\int_{S^{N-2}}\omega_\kappa(\sigma',\theta)dS'(\sigma').$$

Then $\bar\omega_\kappa$ is independent of $\sigma'\in S^{N-2}$ and furthermore

$$\int_{S_+^{N-1}}(\omega_\kappa-\bar\omega_\kappa)\psi_\kappa dS=\int_0^{\frac{\pi}{2}}\left(\int_{S^{N-2}}(\omega_\kappa-\bar\omega_\kappa)dS'\right)(\sin\theta)^{N-2}(\cos\theta)^{\frac{\alpha_+}{2}}d\theta=0,$$

thus $\bar\omega_\kappa$ is the projection of ω_κ onto the first eigenspace of $\mathcal{L}_\kappa$ and

$$\int_{S_+^{N-1}}(\omega_\kappa-\bar\omega_\kappa)\mathcal{L}_\kappa(\omega_\kappa-\bar\omega_\kappa)dS\geq\mu_{\kappa,2}\int_{S_+^{N-1}}(\omega_\kappa-\bar\omega_\kappa)^2dS.$$

At end, noting that

$$\int_{S_+^{N-2}}(\overline{g_q\circ\omega_\kappa}-g_q\circ\bar\omega_\kappa)(\omega_\kappa-\bar\omega_\kappa)dS'=0,$$

where we have set $g_q\circ u=|u|^{q-1}u$ for brevity, and thus

$$\begin{aligned}\int_{S_+^{N-1}}(g_q\circ\omega_\kappa-\overline{g_q\circ\omega_\kappa})(\omega_\kappa-\bar\omega_\kappa)dS&=\int_0^{\frac{\pi}{2}}\int_{S_+^{N-2}}(g_q\circ\omega_\kappa-\overline{g_q\circ\omega_\kappa})(\omega_\kappa-\bar\omega_\kappa)dS'(\sin\theta)^{N-2}d\theta\\&=\int_0^{\frac{\pi}{2}}\int_{S_+^{N-2}}(g_q\circ\omega_\kappa)-(g_q\circ\bar\omega_\kappa)(\omega_\kappa-\bar\omega_\kappa)dS'(\sin\theta)^{N-2}d\theta\\&\geq 2^{1-q}\int_{S_+^{N-1}}|\omega_\kappa-\bar\omega_\kappa|^{q+1}dS,\end{aligned}$$

we derive that $w=\omega_\kappa-\bar\omega_\kappa$, satisfies

$$\int_{S_+^{N-1}}\left((\mu_{\kappa,2}-\ell_{N,q})(\omega_\kappa-\bar\omega_\kappa)^2+2^{1-q}|\omega_\kappa-\bar\omega_\kappa|^{q+1}\right)dS\leq 0,$$

which implies $\omega_\kappa = \bar{\omega}_\kappa$ and it satisfies

$$\frac{1}{(\sin\theta)^{N-2}}\frac{d}{d\theta}\left((\sin\theta)^{N-2}\frac{d\omega_\kappa}{d\theta}\right)+\left(\ell_{q,N}+\frac{\kappa}{\cos^2\theta}\right)\omega_\kappa - g_q\circ\omega_\kappa = 0. \tag{4.18}$$

Because $\mu_{\kappa,1} < \ell_{q,N} \leq \mu_{\kappa,2}$, by [5, Th. 4, Corol. 1], this equation admits the three solutions, ω_κ, $-\omega_\kappa$ and 0. □

Remark. For $\epsilon > 0$ small enough the function $\epsilon\psi_\kappa$ is a subsolution for problem (4.12). This implies

$$\omega_\kappa(\sigma) \geq \epsilon\psi_\kappa(\sigma) \quad \forall\sigma \in S_+^{N-1}. \tag{4.19}$$

4.3. Isolated boundary singularities

Throughout this section we assume that $\Omega \subset \mathbb{R}_+^N$, $0 \in \partial\Omega$ the tangent plane to $\partial\Omega$ at 0 is $\partial\mathbb{R}_+^N$ and that $1 < q < q_c$.

Lemma 4.3. *There holds*

$$\lim_{|x|\to 0}\frac{\mathbb{G}_{\mathcal{L}_\kappa}[(K_{\mathcal{L}_\kappa}(.,0))^q](x)}{K_{\mathcal{L}_\kappa}(x,0)} = 0. \tag{4.20}$$

Proof. We recall the following estimates (1.9), (2.57)

$$\text{(i) } G_{\mathcal{L}_\kappa}(x,y) \leq c_3 \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{(d(x))^{\frac{\alpha_+}{2}}(d(y))^{\frac{\alpha_+}{2}}}{|x-y|^{N+\alpha_+-2}}\right\},$$

$$\text{(ii) } c_3^{-1}\frac{(d(x))^{\frac{\alpha_+}{2}}}{|x|^{N+\alpha_+-2}} \leq K_{\mathcal{L}_\kappa}(x,0) \leq c_3\frac{(d(x))^{\frac{\alpha_+}{2}}}{|x|^{N+\alpha_+-2}}.$$

Then

$$\begin{aligned}\frac{\mathbb{G}_{\mathcal{L}_\kappa}[K^q_{\mathcal{L}_\kappa}(.,0)](x)}{K_{\mathcal{L}_\kappa}(x,0)} &\leq c_3^{q+2}|x|^{N+\alpha_+-2}\int_\Omega \frac{(d(y))^{\frac{(q+1)\alpha_+}{2}}dy}{|x-y|^{N+\alpha_+-2}|y|^{q(N+\alpha_+-2)}}\\ &\leq c_3^{q+2}|x|^{N+\frac{\alpha_+}{2}-q\left(N+\frac{\alpha_+}{2}-2\right)}\int_{\mathbb{R}^N}\frac{d\eta}{|e_x-\eta|^{N+\alpha_+-2}|\eta|^{q(N+\alpha_+-2)}},\end{aligned}$$

where $e_x = |x|^{-1}x$. This last integral is finite and independent of x. Since $q < q_c$, (4.20) follows. □

Corollary 4.4. *Let $u_{k\delta_0}$ be the unique solution of*

$$\begin{aligned}\mathcal{L}_\kappa u + |u|^{q-1}u &= 0 && \text{in } \Omega\\ u &= k\delta_0 && \text{in } \partial\Omega.\end{aligned} \tag{4.21}$$

Then

$$\lim_{x\to 0}\frac{u_{k\delta_0}}{K_{\mathcal{L}_\kappa}(x)} = k. \tag{4.22}$$

Proof. This is a consequence of (4.20) and the inequality

$$k\mathbb{K}_{\mathcal{L}_\kappa}[\delta_0](x) - k^q\mathbb{G}[(\mathbb{K}_{\mathcal{L}_\kappa}[\delta_0])^q](x) \leq u_{k\delta_0}(x) \leq k\mathbb{K}_{\mathcal{L}_\kappa}[\delta_0](x). \quad □ \tag{4.23}$$

Proposition 4.5. *There exists $u_{\infty,0} = \lim_{k\to\infty} u_{k\delta_0}$ and there holds*

$$\lim_{\substack{x\to 0, x\in\Omega\\ x|x|^{-1}\to\sigma}} |x|^{\frac{2}{q-1}}u_{\infty,0}(x) = \omega_\kappa(\sigma), \tag{4.24}$$

uniformly on compact subsets of S_+^{N-1}.

Proof. The correspondence $k \mapsto u_{k\delta_0}$ is increasing and, by the Keller–Osserman estimate, it converges, when $k \to \infty$ to some smooth function $u_{\infty,0}$ defined in Ω where it satisfies (1.1). By Proposition A.1, for any $R \in (0, R_0)$, the function $u_{k\delta_0}$, and also $u_{\infty,0}$, vanishes on any compact subset of $\partial\Omega \setminus \{0\}$. Furthermore

$$u_{\infty,0}(x) \leq \begin{cases} c_{K,\gamma,\kappa}(\mathrm{dist}(x,K))^{\gamma} \quad \forall \gamma \in \left(\frac{\alpha_-}{2}, \frac{\alpha_+}{2}\right) & \text{if } 0 < \kappa < \frac{1}{4}, \\ c_K\sqrt{\mathrm{dist}(x,K)}\sqrt{\ln\left(\frac{\mathrm{diam}(\Omega)}{\mathrm{dist(x,K)}}\right)} & \text{if } \kappa = \frac{1}{4}, \end{cases}$$

for all compact set $K \subset \partial\Omega \setminus \{0\}$. Combining this estimate with Proposition A.3 we obtain

$$u_{\infty,0}(x) \leq c_{90}(d(x))^{\frac{\alpha_+}{2}}|x|^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall x \in \Omega, \tag{4.25}$$

and

$$|\nabla u_{\infty,0}(x)| \leq c_{90}(d(x))^{\frac{\alpha_+}{2}-1}|x|^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall x \in \Omega. \tag{4.26}$$

Let $\ell_0 > 0$ be small enough such that $\ell\mathbf{e} \in \Omega$ for any $0 < \ell < \ell_0$, where $\mathbf{e} = (0, \ldots, 0, 1)$. Then by (1.9), (2.57) and (4.23) we can easily prove that there exist positive constants c_{01} and c_{02} such that

$$\ell^{\frac{2}{q-1}}u_{\infty,0}(\ell\mathbf{e}) \geq c_{01}k\ell^{\frac{2}{q-1}-N-\frac{\alpha_+}{2}+2} - c_{02}k^q\ell^{2-q\left(N+\frac{\alpha_+}{2}-2\right)+\frac{2}{q-1}} \quad \forall k > 0.$$

Now we set $k = \frac{1}{M\ell^{\frac{2}{q-1}-N-\frac{\alpha_+}{2}+2}}$, then there holds

$$\ell^{\frac{2}{q-1}}u_{\infty,0}(\ell\mathbf{e}) \geq \frac{c_{01}}{M} - \frac{c_{02}}{M^q}.$$

Thus if we choose M big enough, we can easily show that there exists $c_{03} > 0$ which depends on κ, Ω, q, N such that

$$\ell^{\frac{2}{q-1}}u_{\infty,0}(\ell\mathbf{e}) \geq c_{03} > 0 \quad \forall 0 < \ell < \ell_0. \tag{4.27}$$

For $\ell > 0$, we put $T_\ell[v](x) = \ell^{\frac{2}{q-1}}v(\ell x)$, $\Omega_\ell = \ell^{-1}\Omega$, $d_\ell(y) = \mathrm{dist}(y, \partial\Omega_\ell)$. If v satisfies (4.1) in Ω and vanishes on $\partial\Omega \setminus \{0\}$, $T_\ell[v]$ vanishes on $\partial\Omega_\ell \setminus \{0\}$ and satisfies

$$-\Delta T_\ell[v] - \frac{\kappa}{d_\ell^2}T_\ell[v] + |T_\ell[v]|^{q-1}T_\ell[v] = 0 \quad \text{in } \Omega_\ell. \tag{4.28}$$

In order to avoid ambiguity, we set $u_{k\delta_0} = u^{\Omega}_{k\delta_0}$, $v_{k\delta_0} = v^{\Omega}_{k\delta_0}$, $u_{\infty,0} = u^{\Omega}_{\infty,0}$ and $v_{\infty,0} = v^{\Omega}_{\infty,0}$. Since inequalities (4.25) and (4.26) are invariant under the scaling transformation T_ℓ, the standard elliptic equations regularity theory yields the following estimates

$$u^{\Omega_\ell}_{\infty,0}(y) \leq c_{92}(d_\ell(y))^{\frac{\alpha_+}{2}}|y|^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall y \in \Omega_\ell, \tag{4.29}$$

and

$$|\nabla u^{\Omega_\ell}_{\infty,0}(y)| \leq c_{92}(d_\ell(y))^{\frac{\alpha_+}{2}-1}|y|^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall y \in \Omega_\ell, \tag{4.30}$$

valid for any $0 < \ell \leq 1$. If we let $k \to \infty$, we obtain $T_\ell[u^{\Omega}_{\infty,0}] = u^{\Omega_\ell}_{\infty,0}$ and because of the group property of the transformations $\{T_\ell\}_{\ell>0}$, there holds $T_{\ell'}[u^{\Omega_\ell}_{\infty,0}] = u^{\Omega_{\ell'\ell}}_{\infty,0}$ for any $\ell, \ell' > 0$. Estimates (4.29) and (4.30) imply that $\{u^{\Omega_\ell}_{\infty,0}\}$ is relatively compact for the topology of convergence on compact subsets of $\mathbb{R}^N_+$. Therefore there exist a sequence $\{\ell_n\}$ tending to 0 and a function U such that $\{u^{\Omega_{\ell_n}}_{\infty,0}\}$ converges to U uniformly on any compact subset of $\mathbb{R}^N_+$. By (4.27) this function is identically equal to zero. Therefore U is a weak solution of

$$-\Delta U - \frac{\kappa}{y_N^2}U + U^q = 0 \quad \text{in } \mathbb{R}^N_+. \tag{4.31}$$

Furthermore

$$u^{\mathbb{R}^N_+}_{\infty,0}(y) \leq c_{92}y_N^{\frac{\alpha_+}{2}}|y|^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall y \in \mathbb{R}^N_+. \tag{4.32}$$

Since $T_{\ell'}[u^{\Omega_{\ell_n}}_{\infty,0}] = u^{\Omega_{\ell'\ell_n}}_{\infty,0}$, we derive $T_{\ell'}[U] = U$ for any $\ell' > 0$, thus U is self similar. Set $\omega(\frac{y}{|y|}) = U(\frac{y}{|y|})$. If we set $\sigma = \frac{y}{|y|}$, there holds

$$\omega(\sigma) \leq c_{92}\psi_\kappa(\sigma) \quad \forall \sigma \in S^{N-1}_+. \tag{4.33}$$

Therefore ω satisfies (4.12) and it coincides with the unique positive element ω_κ of $\mathcal{E}_\kappa$, since by (4.27) $U(\mathbf{e}) \geq c_{03} > 0$. Thus $u^{\Omega_\ell}_{\infty,0}$ converges to U on compact subsets of $\mathbb{R}^N_+$. In particular (4.24) holds on compact subsets of S^{N-1}_+. □

5. The boundary trace of positive solutions

As before we assume that $0 < \kappa \leq \frac{1}{4}$, $q > 1$ and Ω is a bounded smooth domain, convex if $\kappa = \frac{1}{4}$. Although the construction of the boundary trace can be made in a more general framework, we restrict ourselves to the class $\mathcal{U}_+(\Omega)$ of positive smooth functions u satisfying

$$\mathcal{L}_\kappa u + |u|^{q-1}u = 0 \tag{5.1}$$

in Ω.

Lemma 5.1. *Let $f \in L^1_{\phi_\kappa}(\Omega)$. If u is a nonnegative solution of*

$$\mathcal{L}_\kappa u = f \quad \text{in } \Omega \tag{5.2}$$

there exists $\mu \in \mathfrak{M}_+(\partial\Omega)$ such that u admits μ for boundary trace and

$$u = \mathbb{G}_{\mathcal{L}_\kappa}[f] + \mathbb{K}_{\mathcal{L}_\kappa}[\mu]. \tag{5.3}$$

Proof. Let $v = \mathbb{G}_{\mathcal{L}_\kappa}[f]$, then $u - v$ is $\mathcal{L}_\kappa$-harmonic and positive thus the result follows. □

Definition. Let $G \subset \Omega$ be a domain. A function $u \in L^q_{loc}(G)$ is a supersolution (resp. subsolution) of (5.1) if

$$\mathcal{L}_\kappa u + |u|^{q-1}u \geq 0 \quad (\text{resp. } \mathcal{L}_\kappa u + |u|^{q-1}u \leq 0) \tag{5.4}$$

in the sense of distributions in G.

The following comparison principle holds [3, Lemma 3.2].

Proposition 5.2. *Let $G \subset \Omega$ be a smooth domain and $\bar{u}$, $\underline{u}$ a pair of nonnegative supersolution and subsolution respectively in G.*

(i) *If there holds*

$$\limsup_{\text{dist}(x,\partial G)\to 0} (\bar{u}(x) - \underline{u}(x)) < 0, \tag{5.5}$$

then $\underline{u} < \bar{u}$ in G.

(ii) *Assume $\overline{G} \subset \Omega$ and $\bar{u}$ and $\underline{u}$ belong to $H^1(G) \cap C(\overline{G})$. If $\underline{u} \leq \bar{u}$ in ∂G, then $\underline{u} \leq \bar{u}$ in G.*

5.1. Construction of the boundary trace

We use the notations of [25].

Proposition 5.3. *Let v be a non-negative function in $C(\Omega)$.*

(i) *If v is a subsolution of (5.1), there exists a minimal solution u_* dominating v, i.e. $v \leq u_* \leq U$ for any solution $U \geq v$.*
(ii) *If v is a supersolution of (5.1), there exists a maximal solution u^* dominated by v, i.e. $U \leq u^* \leq v$ for any solution $U \leq v$.*

Proof. (i) Let $\{\Omega_n\}$ be a smooth exhaustion Ω and for each $n \in \mathbb{N}$, u_n the positive solution of

$$\begin{aligned} \mathcal{L}_\kappa u + |u|^{q-1}u &= 0 \quad \text{in } \Omega_n \\ u &= v \quad \text{in } \partial\Omega_n. \end{aligned} \tag{5.6}$$

By the comparison principle $u_n \geq v$, which implies $u_{n+1}(x) \geq u_n(x)$ $\forall x \in \Omega_n$. Since $\{u_n\}$ is uniformly bounded on compact subsets of Ω and thus in C^2 by standard regularity arguments that $u_n \uparrow u_*$ which is a positive solution of (5.1). Furthermore, if U is any solution of (5.1) dominating v, it dominates u_n in Ω_n and thus $u_* \leq U$.

The proof of (ii) is similar: we construct a decreasing sequence $\{u'_n\}$ of nonnegative solutions of (5.1) in Ω_n coinciding with v on $\partial\Omega_n$ and dominated by v. It converges to some u^* which satisfies $U \leq u^* \leq v$ for any solution U dominated by v. □

Proposition 5.4. *Let $0 \leq u, v \in C(\Omega)$.*

(i) *If u and v are subsolutions (resp. supersolutions) then $\max(u, v)$ is a subsolution (resp. $\min(u, v)$ is a supersolution).*
(ii) *If u and v are supersolutions then $u + v$ is a supersolution.*
(iii) *If u is a subsolution and v is a supersolution then $(u - v)_+$ is a subsolution.*

Proof. The first two statements follow Kato's inequality. The last statement is verified using that

$$-\Delta(u-v)_+ \le \mathrm{sign}_+(u-v)(-\Delta(u-v)) \le -\mathrm{sign}_+(u-v)(u^q-v^q)+\kappa\frac{(u-v)_+}{d^2(x)}$$
$$\le -(u-v)_+^q+\kappa\frac{(u-v)_+}{d^2(x)}. \quad \square$$

Notation 5.5. *Let u, v be nonnegative continuous functions in Ω.*

(a) *If u is a subsolution, $[u]_\dagger$ denotes the smallest solution dominating u.*
(b) *If u is a supersolution, $[u]^\dagger$ denotes the largest solution dominated by u.*
(c) *If u, v are subsolutions then $u \vee v := [\max(u,v)]_\dagger$.*
(d) *If u, v are supersolutions then $u \wedge v := [\inf(u,v)]^\dagger$ and $u \oplus v = [u+v]^\dagger$.*
(e) *If u is a subsolution and v is a supersolution then $u \ominus v := [(u-v)_+]_\dagger$.*

The next result based upon local uniform estimates is due to Dynkin [12].

Proposition 5.6. (i) *Let $\{u_k\} \subset C(\Omega)$ be a sequence of positive subsolutions (resp. supersolutions) of (5.1). Then $U := \sup u_k$ (resp. $U := \inf u_k$) is a subsolution (resp. supersolution).*
(ii) *Let $\mathcal{T} \subset C(\Omega)$ be a family of positive solutions of (5.1). Suppose that, for every pair $u_1, u_2 \in \mathcal{T}$ there exists $v \in \mathcal{T}$ such that*

$$\max(u_1,u_2) \le v \quad (\text{resp. } \min(u_1,u_2) \ge v).$$

Then there exists a monotone sequence $\{u_n\} \subset \mathcal{T}$ such that

$$u_n \uparrow \sup \mathcal{T} \qquad (\text{resp. } u_n \downarrow \inf T).$$

Furthermore $\sup \mathcal{T}$ (resp. $\inf \mathcal{T}$) is a solution.

Definition 5.7. Let $F \subset \partial\Omega$ be a closed set. We set

$$U_F := \sup\left\{u \in \mathcal{U}_+(\Omega) : \lim_{x\to\xi}\frac{u(x)}{W(x)} = 0,\ \forall\xi \in \partial\Omega\setminus F\right\}, \tag{5.7}$$

and

$$[u]_F = \sup\left\{v \in \mathcal{U}_+(\Omega) : v \le u,\ \lim_{x\to\xi}\frac{v(x)}{W(x)} = 0,\ \forall\xi \in \partial\Omega\setminus F\right\}. \tag{5.8}$$

Notice that $F \mapsto U_F$ and $F \mapsto [u]_F$ are increasing with respect to the inclusion order relation in $\partial\Omega$, $[u]_F = u \wedge U_F$. As a consequence of Proposition A.3, U_F satisfies

$$\lim_{x\to\xi}\frac{U_F(x)}{W(x)} = 0, \quad \forall\xi \in \partial\Omega\setminus K. \tag{5.9}$$

Proposition 5.8. *Let $E, F \subset \partial\Omega$ be closed sets. Then*

(i) *$U_E \wedge U_F = U_{E\cap F}$.*
(ii) *If $F_n \subset \partial\Omega$ is a decreasing sequence of closed sets there holds*

$$\lim_{n\to\infty} U_{F_n} = U_F \quad \text{where } F = \cap F_n.$$

Proof. (i) $U_E \wedge U_F$ is the largest solution dominated by $\inf(U_E, U_F)$ and therefore, by definition, it is the largest solution which vanishes outside $E \cap F$.

(ii) If $V := \lim U_{F_n}$ then $U_F \le V$. But supp $(V) \subset F_n$ for each $n \in \mathbb{N}$ and consequently $V \le U_F$. $\square$

For $\beta > 0$, we recall that Ω_β, Σ_β and the mapping $x \mapsto (d(x), \sigma(x))$ have been defined in the proof of Lemma 3.9. We also set $\Omega'_\beta = \Omega \setminus \overline{\Omega_\beta}$ and, if $Q \subset \partial\Omega$, $\Sigma_\beta(Q) = \{x \in \Omega_\beta : \sigma(x) \in Q\}$.

Proposition 5.9. *Let $u \in \mathcal{U}(\Omega)$.*

(i) *If $A, B \subset \partial\Omega$ are closed sets. Then*

$$[[u]_A]_B = [[u]_B]_A = [u]_{A\cap B}. \tag{5.10}$$

(ii) *If* $\{F_n\}$ *is a decreasing sequence of closed subsets of* $\partial\Omega$ *and* $F=\cap F_n$, *then*

$$[u]_{F_n}\downarrow [u]_F.$$

(iii) *If* $A,B\subset\partial\Omega$ *are closed sets. Then*

$$[u]_A\leq [u]_{A\cap B}+[u]_{\overline{A\setminus B}}. \tag{5.11}$$

Proof. (i) It follows directly from definition that,

$$[[u]_A]_B\leq \inf(u,U_A,U_B).$$

The largest solution dominated by u and vanishing on $A^c\cup B^c$ is $[u]_{A\cap B}$. Thus

$$[[u]_A]_B\leq [u]_{A\cap B}.$$

On the other hand

$$[u]_{A\cap B}=[[u]_{A\cap B}]_B\leq [[u]_A]_B,$$

this proves (5.10).

(ii) If $F_n\downarrow F$, it follows by Proposition 5.8-(ii) that $U_{F_n}\to U_F$, thus

$$[u]_F\leq \lim_{n\to\infty}[u]_{F_n}=\lim_{n\to\infty}u\wedge U_{F_n}\leq \lim_{n\to\infty}\inf(u,U_{F_n})\leq \inf(u,U_F).$$

Since $[u]_F$ is the largest solution dominated by $\inf(u,U_F)$, $[u]_{F_n}$ is the largest solution dominated by $\inf(u,U_{F_n})$ and $U_{F_n}\downarrow U_F$ by Proposition 5.8, the function $v=\lim_{n\to\infty}[u]_{F_n}$ is a solution of (5.1) dominated by $\inf(u,U_F)$, thus $v\leq [u]_F$ and the proof of (ii) is complete.

(iii) Without loss of generality we assume that $A\cap B\neq\emptyset$. Let $O,O'\subset\partial\Omega$ be a relatively open set such that $A\cap B\subset O$ and $\overline{A\cap B^c}\subset O'$ Set $v=[u]_A$ and let v^1_β be the solution of

$$\begin{aligned}\mathcal{L}_\kappa w+|w|^{q-1}w&=0 &&\text{in } \Omega'_\beta\\ w&=\chi_{\Sigma_\beta(\overline{O})}v &&\text{on } \Sigma_\beta.\end{aligned}$$

Also we denote by v^2_β and v^3_β the solutions of the above problem with respective boundary data $\chi_{\Sigma(\overline{O'})}v$ and $\chi_{\Sigma(O^c\cap O'^c)}v$. Then $v^i_\beta\leq v\lfloor_{\Omega'_\beta}\leq v^1_\beta+v^2_\beta+v^3_\beta$, $i=1,2,3$. Let now $\{\beta_j\}$ be a decreasing sequence converging to 0 and such that

$$v^i_{\beta_j}\to v^i\leq v\leq v^1+v^2+v^3,\quad i=1,2,3 \text{ locally uniformly in } \Omega.$$

By definition of v^i and Proposition A.1, we have that $v^1\leq [v]_{\overline{O}}$, $v^2\leq [v]_{\overline{O'}}$ and $v^3\leq [v]_{O^c\cap O'^c}$. But by (i) we have

$$[v]_{O^c\cap O'^c}=[[u]_A]_{O^c\cap O'^c}=[u]_{A\cap O^c\cap O'^c}=0.$$

Thus

$$v\leq [v]_{\overline{O}}+[v]_{\overline{O'}}.$$

We can consider decreasing sequences $\{O_n\}$ and $\{O'_n\}$ such that $\cap\overline{O_n}=A\cap B$ and $\cap\overline{O'_n}=\overline{A\cap B^c}$. By (ii) we obtain

$$v\leq [[u]_A]_{A\cap B}+[[u]_A]_{\overline{A\cap B^c}}\leq [u]_{A\cap B}+[u]_{\overline{A\cap B^c}}$$

which is (iii). □

Remark. Since any $u\in\mathcal{U}_+(\Omega)$ is dominated by $u_{\partial\Omega}$, it follows from (iii) that for any set $A\subset\partial\Omega$, there holds

$$u=[u]_{\partial\Omega}\leq [u]_{\overline{A}}+[u]_{\overline{\partial\Omega\setminus\overline{A}}}\leq [u]_{\overline{A}}+[u]_{\overline{\partial\Omega\setminus A}}. \tag{5.12}$$

Proposition 5.10. *Let u be a positive solution of* (5.1). *If* $u\in L^q_{\phi_\kappa}(\Omega)$ *it possesses a boundary trace* $\mu\in\mathfrak{M}(\partial\Omega)$, *i.e., u is the solution of the boundary value problem* (3.36) *with this measure* μ.

Proof. If $v:=\mathbb{G}_{\mathcal{L}_\kappa}[u^q]$ then $v\in L^1_{\phi_\kappa}(\Omega)$ and $u+v$ is a positive $\mathcal{L}_\kappa$-harmonic function. Hence $u+v\in L^1_{\phi_\kappa}(\Omega)$ and there exists a non-negative measure $\mu\in\mathfrak{M}(\partial\Omega)$ such that $u+v=\mathbb{K}_{\mathcal{L}_\kappa}[\mu]$. By Proposition 3.5 this implies the result. □

Proposition 5.11. *Let u be a positive solution of* (5.1) *and* $\mu\in\mathfrak{M}(\partial\Omega)$. *If for an exhaustion* $\{\Omega_n\}$ *of* Ω, *we have*

$$\lim_{n\to\infty}\int_{\partial\Omega_n}Z(x)u\,d\omega^{x_0}_{\Omega_n}=\int_{\partial\Omega}Z(x)d\mu\quad\forall Z\in C(\overline{\Omega}),$$

where $\omega^{x_0}_{\Omega_n}$ *is the* $\mathcal{L}_\kappa$-*harmonic measure of* Ω_n *relative to a point* $x_0\in\Omega_1$, *then u and* $|u|^p$ *belong to* $L^1_{\phi_\kappa}(\Omega)$. *Furthermore u possesses the boundary trace* $\mu\in\mathfrak{M}(\partial\Omega)$, *i.e. u is the solution of the boundary value problem* (3.36) *with this measure* μ.

Proof. Let $\mathbb{G}^n_{\mathcal{L}_\kappa}$ be the green function of $\mathcal{L}_\kappa$ in Ω_n, then

$$\mathbb{G}^n_{\mathcal{L}_\kappa}(x,y) \leq \mathbb{G}^{n+1}_{\mathcal{L}_\kappa}(x,y) \quad \forall x,y \in \Omega_n$$

and

$$\mathbb{G}^n_{\mathcal{L}_\kappa} \uparrow \mathbb{G}_{\mathcal{L}_\kappa}.$$

Since

$$\int_{\partial\Omega_n} u d\omega^{x_0}_{\Omega_n} = u(x_0) + \int_{\Omega_n} \mathbb{G}^n_{\mathcal{L}_\kappa}(x,x_0)|u(x)|^q dx,$$

we derive, as $n \to \infty$,

$$\mu(\partial\Omega) = u(x_0) + \int_{\Omega_n} \mathbb{G}_{\mathcal{L}_\kappa}(x,x_0)|u(x)|^q dx.$$

By Proposition 2.1 this implies $|u|^q \in L^1_{\phi_\kappa}(\Omega)$, and the result follows by Proposition 5.10. □

Proposition 5.12. *If $F \subset \partial\Omega$ is a closed set and u a positive solution of (5.1) with boundary trace $\mu \in \mathfrak{M}(\partial\Omega)$, then $[u]_F$ has boundary trace $\mu\chi_F$.*

Proof. The function $[u]_F$ belongs to $\mathcal{U}_+(\Omega)$ and is dominated by u which satisfies (5.1), thus $[u]_F \in L^q_{\Phi_\kappa}(\Omega)$ and $[u]_F$ admits a boundary trace $\mu_F \leq \mu$ by Proposition 5.10. Let v be the solution of (3.36) with boundary data $\mu\chi_F$. Let $O \subset \partial\Omega$ relatively open such that $F \subset O$. By (5.12) we have

$$v \leq [v]_{\overline{O}} + [v]_{\overline{O^c}}.$$

Let A be an open set such that $F \subset A \subset \overline{A} \subset O$, and for exhaustion we take $\Omega_n = \Omega'_{\frac{1}{n}}$ which is smooth for n large enough, and $\partial\Omega_n = \Sigma_{\frac{1}{n}}$. Then

$$\int_{\partial\Omega_n} [v]_{\overline{O^c}} d\omega^{x_0}_{\Omega_n} = \int_{\Sigma_{\frac{1}{n}}(A)} [v]_{\overline{O^c}} d\omega^{x_0}_{\Omega_n} + \int_{\partial\Omega_n\setminus\Sigma_{\frac{1}{n}}(A)} [v]_{\overline{O^c}} d\omega^{x_0}_{\Omega_n}.$$

But

$$\int_{\Sigma_{\frac{1}{n}}(A)} [v]_{\overline{O^c}} d\omega^{x_0}_{\Omega_n} \leq \int_{\Sigma_{\frac{1}{n}}(A)} v d\omega^{x_0}_{\Omega_n} \to 0$$

and

$$\int_{\partial\Omega_n\setminus\Sigma_{\frac{1}{n}}(A)} [v]_{\overline{O^c}} d\omega^{x_0}_{\Omega_n} \leq \int_{\partial\Omega_n\setminus\Sigma_{\frac{1}{n}}(A)} U_{\overline{O^c}} d\omega^{x_0}_{\Omega_n} \to 0,$$

as $n \to \infty$, thus $[v]_{\overline{O^c}} = 0$ by Proposition 5.11 and therefore $v \leq [v]_{\overline{O}} \leq [u]_{\overline{O}}$. Since O be an arbitrary open set, take a sequence of open set $\{O_n\}$ such that $F \subset O_n \subset \overline{O}_n \subset O_{n-1}$ and $\cap O_n = F$. Using Proposition 5.9 we derive

$$v \leq [u]_F,$$

and thus $\mu\chi_F \leq \mu_F$. Conversely, let $Z \in C(\overline{\Omega})$, $Z \geq 0$,

$$\begin{aligned}\int_{\partial\Omega_n} Z[u]_F d\omega^{x_0}_{\Omega_n} &= \int_{\partial\Omega_n\cap\Sigma_{\frac{1}{n}}(A)} Z[u]_F d\omega^{x_0}_{\Omega_n} + \int_{\partial\Omega_n\setminus\Sigma_{\frac{1}{n}}(A)} Z[u]_F d\omega^{x_0}_{\Omega_n} \\ &\leq nt_{\partial\Omega_n\cap\Sigma_{\frac{1}{n}}(A)} Zud\omega^{x_0}_{\Omega_n} + \int_{\partial\Omega_n\setminus\Sigma_{\frac{1}{n}}(A)} ZU_F d\omega^{x_0}_{\Omega_n} \\ &\leq I_n + II_n.\end{aligned}$$

Because of (5.9), $II_n \to 0$ as $n \to \infty$, thus

$$\int_{\partial\Omega} Z d\mu_F \leq \int_{\partial\Omega} Z\chi_F d\mu \Longrightarrow \mu_F \leq \mu\chi_O,$$

and the result follow by regularity since O is arbitrary. □

The next result shows that the boundary trace has a local character.

Proposition 5.13. *Let $u \in \mathcal{U}_+(\Omega)$ and $\xi \in \partial\Omega$. We assume that there exists $\rho > 0$ such that*

$$\int_{B_\rho(\xi)\cap\Omega} u^q(x)\phi_\kappa(x)dx < \infty. \tag{5.13}$$

(i) *Then*

$$[u]_F^q \in L^1_{\phi_\kappa}(\Omega) \quad \forall F \subset \partial\Omega \cap B_\rho(\xi),\ F \text{ closed}.$$

Thus $[u]_F$ possesses a boundary trace $\mu_F \in \mathfrak{M}(\partial\Omega)$, and $\operatorname{supp}(\mu_F) \subset F$.

(ii) *There exists a nonnegative Radon measure μ_ρ on $B_\rho(\xi)$ such that for any closed set $F \subset B_\rho(\xi) \cap \partial\Omega$*

$$\mu_F = \mu_\rho \chi_F,$$

and for any exhaustion $\{\Omega_n\}$ of Ω and any $Z \in C(\overline{\Omega})$ such that $\operatorname{supp}(Z) \cap \partial\Omega \subset \partial\Omega \cap B_\rho(\xi)$

$$\lim_{n\to\infty} \int_{\partial\Omega_n} u(x)Z(x)d\omega^{x_0}_{\Omega_n} = \int_{\partial\Omega} u(x)Z(x)d\mu_\rho. \tag{5.14}$$

Proof. (i) Let F be a closed set and $0 < \rho' < \rho$ be such that

$$F \subset \partial\Omega \cap B_{\rho'}(\xi).$$

Since $[u]_F \le \inf(u, U_F)$ and $U_F \in C(\overline{\Omega} \setminus F)$, we have

$$\int_\Omega [u]_F^q \phi_\kappa(x)dx \le \int_{B_\rho(\xi)\cap\Omega} |u|^p\phi_\kappa(x)dx + \int_{\Omega\setminus B_\rho(\xi)} |U_F|^p\phi_\kappa(x)dx < \infty.$$

(ii) Let $0 < \rho_1 < \rho_2 < \rho$, then

$$[u]_{\overline{B}_{\rho_2}(\xi)\cap\partial\Omega} \le u \le [u]_{\overline{B}_{\rho_2}(\xi)\cap\partial\Omega} + U_{\overline{\partial\Omega\setminus\overline{B}_{\rho_2}(\xi)}}.$$

The function $[u]_{\overline{B}_{\rho_2}(\xi)\cap\partial\Omega}$ which belongs $L^q_{\phi_\kappa}(\Omega)$ admits a boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$ and

$$\lim_{n\to\infty} \int_{\partial\Omega_n} U_{\overline{\partial\Omega\setminus\overline{B}_{\rho_2}(\xi)}} Z(x)d\omega^{x_0}_{\Omega_n} = 0,$$

for any $Z \in C(\overline{\Omega})$ such that $\operatorname{supp}(Z) \cap \partial\Omega \subset \partial\Omega \cap B_{\rho_1}(\xi)$. Combined with Proposition 5.12 it follows identity (5.14) and finally statement (ii). □

Using a partition of unity it is easy to prove the following extension of the previous result.

Proposition 5.14. *The set $\mathcal{R}_u$ of points ξ such that there exists $r > 0$ such that (5.14) holds is relatively open. For any compact set $F \subset \mathcal{R}_u$ and any open set $G \subset \mathbb{R}^N$ such that $F \subset G \cap \partial\Omega \subset \overline{G \cap \partial\Omega} \subset \mathcal{R}_u$, there holds*

$$\int_{G\cap\Omega} u^q(x)\phi_\kappa(x)dx < \infty. \tag{5.15}$$

Then $[u]_F \in L^1_{\phi_\kappa}(\Omega)$, $[u]_F$ possesses a boundary trace $\mu_F \in \mathfrak{M}(\partial\Omega)$ with support in F. There exists a unique positive Radon measure μ_u on $\mathcal{R}_u$ such that

$$\mu_F = \mu_u \chi_F, \tag{5.16}$$

and for any $Z \in C(\overline{\Omega})$ such that supp $(Z) \cap \partial\Omega \subset \mathcal{R}_u$, there holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} u(x)Z(x)d\omega^{x_0}_{\Omega_n} = \int_{\partial\Omega} u(x)Z(x)d\mu_u. \tag{5.17}$$

Definition 5.15. The set $\mathcal{S}_u := \partial\Omega \setminus \mathcal{R}_u$ is closed. The couple $(\mathcal{S}_u, \mu_u)$ is the boundary trace of u, denoted by $\mathrm{Tr}_{\partial\Omega}(u)$. The measure μ_u is the regular part of $\mathrm{Tr}_{\partial\Omega}(u)$, the set $(\mathcal{S}_u)$ is its singular part.

Proposition 5.16. *Let u be a positive solution in Ω and let $\{\Omega_n\}$ be an exhaustion of Ω. If $y \in \mathcal{S}_u$ then for every nonnegative $Z \in C(\overline{\Omega})$ such that $Z(y) > 0$ we have*

$$\lim_{n\to\infty} \int_{\partial\Omega_n} Zud\omega^{x_0}_{\Omega_n} = \infty.$$

Proof. Let $Z \in C(\overline{\Omega})$, $Z \geq 0$, such that $Z(y) \neq 0$ and

$$\liminf_{n\to\infty} \int_{\partial\Omega_n} Zud\omega_{\Omega_n}^{x_0} < \infty.$$

There exists a subsequence n_j such that

$$\lim_{j\to\infty} \int_{\partial\Omega_{n_j}} Zud\omega_{\Omega_{n_j}}^{x_0} = M < \infty.$$

Let r be such that $Z(x) > \frac{Z(y)}{2}$, $\forall x \in B_r(y) \cap \overline{\Omega}$, then for any $r' < r$ we have that

$$\limsup_{j\to\infty} \int_{\partial\Omega_{n_j}} [u]_{\overline{B_{r'}(y)\cap\partial\Omega}} d\omega_{\Omega_n}^{x_0} < \infty.$$

In view of the proposition of 5.11 the last fact implies that $[u]^q_{\overline{B_{r'}(y)}} \in L_{\phi_\kappa}(\Omega)$, which implies that $u \in L^q_{\phi_\kappa}(B_{r''}(y))$ for all $r'' < r'$, which is clearly a contradiction, by Proposition 5.13. □

Proposition 5.17. *Let u be a positive solution of (5.1) in Ω with boundary trace $(\mathcal{S}_u, \mu_u)$. If F is a closed subset of $\mathcal{R}_u$, then*

$$\int_\Omega (u\mathcal{L}_\kappa \zeta + u^q\zeta)dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu_u\chi_F]\mathcal{L}_\kappa\zeta\, dx,$$

for any $\zeta \in \mathbf{X}(\Omega)$ such that supp $(\zeta) \cap \partial\Omega \subset F$.

Proof. The proof is an adaptation to our situation of [26, Th 4.6]. Consider the function $\zeta \in \mathbf{X}(\Omega)$ such that $\mathrm{supp}(\zeta) \cap \partial\Omega \subset F$. For $\epsilon > 0$, set

$$O_\varepsilon = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, F) < \varepsilon\},$$

and let $\varepsilon_0 > 0$ be small enough such that

$$\overline{O_\varepsilon} \cap \partial\Omega \subset \mathcal{R}_u, \quad \forall\, 0 < \varepsilon \leq \varepsilon_0.$$

Let $\varepsilon < \frac{\varepsilon_0}{4}$ and η be a cut off function such that $\eta \in C_0^\infty(O_\varepsilon)$, $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\overline{O}_{\frac{\varepsilon}{2}}$. For $0 < \beta \leq \beta_0$, let v_β be the solution of

$$\begin{aligned} \mathcal{L}_\kappa w + |w|^{q-1}w &= 0 \quad \text{in } \Omega'_\beta \\ w &= \eta u \quad \text{on } \Sigma_\beta. \end{aligned}$$

Since v_β remains eventually locally uniformly bounded in Ω, there exists a sequence $\{\beta_j\}$ decreasing to 0 such that $v_{\beta_j} \to v$ locally uniformly, and

$$v \leq [u]_{\partial\Omega\cap\overline{O_\varepsilon}}.$$

Thus v has boundary trace μ_0 such that

$$\mu_0 \leq \mu_u\chi_{\partial\Omega\cap\overline{O_\varepsilon}}.$$

Let v_β^1 and v_β^2 be the solutions of

$$\begin{aligned} \mathcal{L}_\kappa w + |w|^{q-1}w &= 0 \quad \text{in } \Omega'_\beta \\ w &= \eta[u]_{\partial\Omega\cap\overline{O}_{2\varepsilon}} \quad \text{on } \Sigma_\beta \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_\kappa w + |w|^{q-1}w &= 0 \quad \text{in } \Omega'_\beta \\ w &= \eta U_{\partial\Omega\setminus O_{2\varepsilon}} \quad \text{on } \Sigma_\beta, \end{aligned}$$

respectively. Since $u \leq [u]_{\partial\Omega\cap\overline{O}_{2\varepsilon}} + U_{\partial\Omega\setminus O_{2\varepsilon}}$ we have that

$$v_\beta \leq v_\beta^1 + v_\beta^2 \leq [u]_{\partial\Omega\cap\overline{O}_{2\varepsilon}} + v_\beta^2.$$

Notice that $[u]^q_{\partial\Omega\cap\overline{O}_{2\varepsilon}} \in L^1_{\phi_\kappa}(\Omega)$. From estimate (A.20) we derive

$$\eta(x)U_{\partial\Omega\setminus O_{2\varepsilon}}(x) \leq c_{90} d^{\frac{\alpha_+}{2}}(x) \quad \forall x \in \Omega,$$

where $c_{90} > 0$ depends on N, q, κ and dist(supp(η), $\partial\Omega \setminus O_\epsilon$). Thus $v_\beta^2(x) \leq c_{90} d^{\frac{\alpha_+}{2}}(x)$ and

$$v_\beta \leq [u]_{\partial\Omega\cap\bar{O}_{2\varepsilon}} + c_{90} d^{\frac{\alpha_+}{2}}(x), \quad \forall x \in \Omega'_\beta. \tag{5.18}$$

Let w_β be the solution of

$$\begin{aligned} \mathcal{L}_\kappa w + |w|^{q-1} w = 0 \qquad & \text{in } \Omega'_\beta \\ w = \chi_{\Sigma_\beta\left(\overline{\partial\Omega\setminus O_{\frac{\varepsilon}{2}}}\right)}[u]_F \qquad & \text{on } \Sigma_\beta. \end{aligned}$$

Then

$$[u]_F \leq v_\beta + w_\beta \quad \text{in } \Omega'_\beta.$$

We have that $w_{\beta_j} \to 0$ locally uniformly in Ω, which implies that

$$[u]_F \leq v.$$

Thus we have

$$\mu_u \chi_F \leq \mu_0 \leq \mu_u \chi_{\partial\Omega\cap\overline{O_\varepsilon}}. \tag{5.19}$$

Let ζ_β be the solution of

$$\begin{aligned} \mathcal{L}_\kappa w = \mathcal{L}_\kappa \zeta \qquad & \text{in } \Omega'_\beta \\ w = 0 \qquad & \text{on } \Sigma_\beta. \end{aligned}$$

Since $\zeta \in \mathbf{X}(\Omega)$, there exists a constant c_{91} such that $\zeta_\beta \leq c_{91}\phi_\kappa$ in Ω'_β. Thus there exists a decreasing sequence $\{\beta_j\}$ converging to 0 such that $\zeta_{\beta_j} \to \zeta$ locally uniformly. For simplicity we will denote it by $\{\beta\}$. Now,

$$\begin{aligned} \int_{\Omega'_\beta} (u\mathcal{L}_\kappa \zeta_\beta + u^q \zeta_\beta) dx &= -\int_{\partial\Omega'_\beta} \frac{\partial \zeta_\beta}{\partial \mathbf{n}} \eta u dS \\ &= \int_{\Omega'_\beta} (v_\beta \mathcal{L}_\kappa \zeta_\beta + v_\beta^q \zeta_\beta) dx \end{aligned} \tag{5.20}$$

which yields, by the definition of ζ_β and v_β,

$$\int_{\Omega'_\beta} (u\mathcal{L}_\kappa \zeta + u^q \zeta_\beta) dx = \int_{\Omega'_\beta} (v_\beta \mathcal{L}_\kappa \zeta + v_\beta^q \zeta_\beta) dx. \tag{5.21}$$

Since supp $(\zeta) \cap \partial\Omega \subset F$, then for β small enough $u \in L^q_{\phi_\kappa}(\Omega \cap O_\epsilon)$. Furthermore $v_\beta \leq u_{\lfloor \Omega'_\beta}$, therefore, it follows the following convergence relations by the dominated convergence theorem, (5.17) and (3.5):

$$\lim_{\beta\to 0} \int_{\Omega'_\beta} u^q \zeta_\beta dx = \int_\Omega u^q \zeta dx \quad \text{and} \quad \lim_{\beta\to 0} \int_{\Omega'_\beta} v_\beta^q \zeta_\beta dx = \int_\Omega v^q \zeta dx,$$

and

$$\lim_{\beta\to 0} \int_{\Omega'_\beta} u\mathcal{L}_\kappa \zeta dx = \int_\Omega u\mathcal{L}_\kappa \zeta dx \quad \text{and} \quad \lim_{\beta\to 0} \int_{\Omega'_\beta} v_\beta \mathcal{L}_\kappa \zeta dx = \int_{\Omega'_\beta} v\mathcal{L}_\kappa \zeta.$$

This implies

$$\int_\Omega (u\mathcal{L}_\kappa \zeta + u^q \zeta) dx = \int_\Omega (v\mathcal{L}_\kappa \zeta + v^q \zeta) dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu_0] \mathcal{L}_\kappa \zeta dx$$

by (3.19). Letting $\varepsilon \to 0$ we have the desired result from (5.19). □

5.2. Subcritical case

We recall that

$$q_c = \frac{N + \frac{\alpha_+}{2}}{N + \frac{\alpha_+}{2} - 2}$$

is the critical exponent for the equation. If $1 < q < q_c$, we have seen in Section 4 that for any $a \in \partial\Omega$ and $k \geq 0$ there exists $u_{k\delta_a}$ and $\lim_{k\to\infty} u_{k\delta_a} = u_{\infty,a}$. Furthermore, by Proposition 5.16, $Tr_{\partial\Omega}(u_{\infty,a}) = (\{a\}, 0)$.

Theorem 5.18. *Assume $1 < q < q_c$ and $a \in \mathcal{S}_u$. Then*

$$u(x) \geq u_{\infty,a}(x) \quad \forall x \in \Omega. \tag{5.22}$$

For proof of the above inequality uses some ideas of the proof of Theorem 7.1 in [24] and needs several intermediate lemmas.

Lemma 5.19. *Assume $1 < q < q_c$. Let $\{\xi^n\}$ be a sequence of points in Ω converging to $a \in \partial\Omega$ and let $l \in (0, 1)$. We define the sets*

$$\Omega_n := \Omega'_{d(\xi^n)} = \{x \in \Omega : d(x) > d(\xi^n)\} \quad \text{and} \quad \Sigma_n := \partial\Omega_n. \tag{5.23}$$

Let $x_0 \in \Omega'_1$ and denote by $\omega_n := \omega^{x_0}_{\Omega_n}$ the $\mathcal{L}_\kappa$-harmonic measure in Ω_n relative to x_0. Put

$$V_n = B_{r_n}(\xi^n) \cap \partial\Omega_n \quad \text{with } r_n = d(\xi_n).$$

Let $h_n \in L^\infty(\Sigma_n)$, $n = 1, 2, \ldots$, and suppose that there exist numbers c and k such that

$$supp(h_n) \subset V_n \quad \text{and} \quad 0 \leq h_n \leq c r_n^{-N-\frac{\alpha_+}{2}+2}, \tag{5.24}$$

and

$$\lim_{n\to\infty} \int_{\Sigma_n} h_n \phi d\omega^{x_0}_{\Omega_n} = k\phi(a) \quad \forall \phi \in C(\overline{\Omega}).$$

Let w_n be the solution of the problem

$$\begin{aligned} \mathcal{L}_\kappa w_n + |w_n|^{q-1} w_n &= 0 \qquad \text{in } \Omega_n \\ w_n &= h_n \qquad \text{on } \partial\Sigma_n. \end{aligned}$$

Then

$$w_n \to u_{k,a} \quad \text{locally uniformly in } \Omega.$$

Proof. Let $\eta^n \in \partial\Omega$ be such that $d(\xi^n) = |\xi^n - \eta^n|$. By Theorem 2.30 we have

$$\mathbb{K}_{\mathcal{L}_\kappa}(x, \eta^n) \geq \frac{1}{c_{43}} r_n^{-N-\frac{\alpha_+}{2}+2} \geq \frac{1}{c_{43}} h_n(x), \quad \forall x \in \Sigma_n, \tag{5.25}$$

by the maximum principle,

$$\mathbb{K}_{\mathcal{L}_\kappa}(x, \eta^n) \geq \frac{1}{c_{43}} w_n(x) \quad \forall x \in \Omega_n. \tag{5.26}$$

Moreover

$$\int_\Omega \mathbb{K}^q_{\mathcal{L}_\kappa}(x, y) d^{\frac{\alpha_+}{2}}(x) dx \leq c(q, \Omega) \quad \forall 1 < q < q_c,$$

where $c(q, \Omega)$ is a constant independent of y. Since q is subcritical, it follows that the sequences $\{\mathbb{K}^q_{\mathcal{L}_\kappa}(\cdot, \eta^n)\}$ and $\{\mathbb{K}_{\mathcal{L}_\kappa}(\cdot, \eta^n)\}$ are uniformly integrable in $L^1_{\phi_\kappa}(\Omega)$. Let $\overline{w}_n$ denotes the extension of w_n to Ω defined by $\overline{w}_n = 0$ in $\Omega \setminus \Omega_n$. In view of (5.25) we conclude that the sequences $\{\overline{w}^q_n\}$ and $\{\overline{w}_n\}$ are uniformly integrable in $L^1_{\phi_\kappa}(\Omega)$, and locally uniformly bounded in Ω By regularity results for elliptic equations there exists a subsequence of $\{\overline{w}_n\}$, say again $\{\overline{w}_n\}$ that converges locally uniformly in Ω to a solution w of (5.1). This fact and the uniform integrability mentioned above imply that

$$w_n \to w \quad \text{in } L^q_{\phi_\kappa}(\Omega) \cap L^1_{\phi_\kappa}(\Omega).$$

Since $w \in L^q_{\phi_\kappa}(\Omega)$ by Proposition 5.10 there exists $\mu \in \mathfrak{M}(\Omega)$ such that

$$\int_\Omega w \mathcal{L}_\kappa \eta dx + \int_\Omega |w|^{q-1} w \eta dx = \int_\Omega \mathbb{K}_{\mathcal{L}_\kappa}[\mu] \mathcal{L}_\kappa \eta dx \quad \forall \eta \in \mathbf{X}(\Omega).$$

Furthermore, using (5.25) we prove below that measure μ is concentrated at a. Let $\phi_{\kappa,n}$ be the first eigenfunction of $\mathcal{L}_\kappa$ in Ω_n normalized by $\phi_{\kappa,n}(x_0) = 1$ for some $x_0 \in \Omega_1$. Let $\eta \in \mathbf{X}(\Omega)$ be nonnegative function and let η_n be the solution of the problem

$$\begin{aligned} \mathcal{L}_\kappa \eta_n &= \frac{\phi_{\kappa,n}}{\phi_\kappa} \mathcal{L}_\kappa \eta \qquad \text{in } \Omega_n \\ \eta_n &= 0 \qquad \text{in } \partial\Omega_n. \end{aligned}$$

Then $\eta_n \in C^2(\overline{\Omega}_n)$ and since $\phi_{\kappa,n} \to \phi_\kappa$,

$$\mathcal{L}_\kappa \eta_n \to \mathcal{L}_\kappa \eta \quad \text{and} \quad \eta_n \to \eta \quad \text{as } n \to \infty.$$

Then we have

$$\int_{\Omega_n} w_n \mathcal{L}_\kappa \eta_n dx + \int_\Omega |w_n|^{q-1} w \eta dx = \int_\Omega v_n \mathcal{L}_\kappa \eta_n dx, \tag{5.27}$$

where v_n solves

$$\begin{aligned} &\mathcal{L}_\kappa v_n = 0 && \text{in } \Omega_n \\ &v_n = h_n && \text{on } \partial\Sigma_n. \end{aligned}$$

By the same arguments as above there exists a subsequence of $\{v_n \chi_{\Omega_n}\}$, that we still denote by $\{v_n \chi_{\Omega_n}\}$, converging to a nonnegative $\mathcal{L}_\kappa$-harmonic function v in $L^1_{\phi_\kappa}(\Omega)$. By (5.25) we have

$$cc_{43} K_{\mathcal{L}_\kappa}(x, a) \geq v(x) \quad \forall x \in \Omega. \tag{5.28}$$

Thus there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$, concentrated at a such that v solves

$$\begin{aligned} &\mathcal{L}_\kappa v = 0 && \text{in } \Omega \\ &v = \nu && \text{on } \partial\Omega. \end{aligned}$$

But

$$k = \lim_{n\to\infty} \int_{\Sigma_n} h_n d\omega^{x_0}_{\Omega_n} = \lim_{n\to\infty} v_n(x_0) = v(x_0) = \int_{\partial\Omega} d\nu,$$

the results follows if we let n tend to ∞ in (5.27). □

Lemma 5.20. *For every $l \in (0, 1)$ there exists a constant $c_l = c(N, \kappa, q, l)$ such that, for every positive solution u of (5.1) in Ω and every $x_0 \in \Omega$,*

$$u(x) \leq c_l u(y) \quad \forall x, y \in B_{lr_0}(x_0)\ r_0 = d(x_0). \tag{5.29}$$

Proof. Put $r_1 = \frac{1+l}{2} r_0$. Then u satisfies

$$\mathcal{L}_\kappa u + u^q = 0 \quad \text{in } B_{r_1}(x_0).$$

Denote by Ω_{r_0} the domain

$$\Omega_{r_0} = \{y \in \mathbb{R}^n : r_0 y \in \Omega\}.$$

Set $v(y) = u(r_0 y)$, and $y_0 = \frac{x_0}{r}$, then $v(y)$ satisfies

$$-\Delta v - \kappa \frac{v}{\operatorname{dist}^2(y, \partial\Omega_{y_0})} + r_0^2 |v|^{q-1} v = 0 \quad \text{in } B_{\frac{1+l}{2}}(y_0).$$

Now note that

$$\frac{1}{\operatorname{dist}^2(y, \partial\Omega_{y_0})} \leq \frac{4}{(1-l)^2} \quad \forall y \in B_{\frac{1+l}{2}}(y_0),$$

and by Keller–Osserman condition

$$r_0^2 |v(y)|^{q-1} = r_0^2 |u(r_0 y)|^{q-1} \leq C(\Omega, \kappa, N) r_0^2 \frac{1}{d^2(r_0 y)} \leq C(\Omega, \kappa, N) B_{\frac{1+l}{2}}(y_0).$$

Thus, by Harnack inequality, there exists a constant $c_l > 0$ such that

$$v(z) \leq c_l v(y) \quad \forall z, y \in B_l(y_0),$$

and the results follows. □

For the proof of the next lemma we need some notations. Let $\beta > 0$ and $\xi \in \Sigma_\beta := \partial\Omega'_\beta$. We set $\Delta^\beta_r(\xi) = \Sigma_\beta \cap B_r(\xi)$ and, for $0 < r < \beta < 2r$, $x^\beta_r = x^\beta_r(\xi) \in \overline{\Omega}_\beta$, such that $d(x^\beta_r) = |x^\beta_r - \xi| = r$. Also we denote by $\omega^x_{\Omega'_\beta}$ the $\mathcal{L}_\kappa$-harmonic measure in $\Omega'_\beta := \Omega \setminus \overline{\Omega}_\beta$ relative to x.

Lemma 5.21. *Let $r_0 = r_0(\Omega) > 0$ be small enough and $0 < r \le \frac{r_0}{4}$. Then there exists a constant c_{95} which depends only on Ω, N such that*

$$\omega^x_{\Omega'_\beta}(\Delta_r(\xi)) > c_{95} \quad \forall x \in \Omega \cap B_{\frac{r}{2}}(\xi). \tag{5.30}$$

Proof. Since $x \mapsto \omega^x_{\Omega'_\beta}$ is positive and $\mathcal{L}_\kappa$-harmonic in Ω'_β, it is a positive superharmonic function (relative to the Laplacian) in Ω'_β. Thus

$$\omega^x_{\Omega'_\beta} \ge \upsilon^x_{\Omega'_\beta} \quad \forall x \in \Omega'_\beta.$$

The result follows by [8, Lemma 2.1]. □

Lemma 5.22. *Let $\kappa = \frac{1}{4}$, $\varepsilon \in (0, 1)$ and $x_0 \in \Omega_1$. Let $\{\xi^n\}$ be a sequence of points in Ω converging to $a \in \partial\Omega$. Then there exist $n_0 = n_0(\varepsilon, \Omega) \in \mathbb{N}$ and $c_{96} = c_{96}(\Omega, N, \varepsilon)$ such that*

$$\omega^{x_0}_{\Omega_n}(B_{d(\xi^n)}(\xi^n) \cap \partial\Omega_n) \ge c_{96} d(\xi^n)^{N+\frac{1}{2}-2}(-\log d(\xi^n))^{1-\varepsilon} \quad \forall n \ge n_0. \tag{5.31}$$

Proof. We recall that for any $n \in \mathbb{N}$ Ω_n is defined by (5.23), $G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}} \le G_{\mathcal{L}_{\frac{1}{4}}} := G^{\Omega}_{\mathcal{L}_{\frac{1}{4}}}$, and for a fixed point $y_0 \in \Omega_1$

$$G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}} \chi_{\Omega_n}(x) \uparrow G_{\mathcal{L}_{\frac{1}{4}}}(x, y_0) \quad \text{locally uniformly in } \Omega \setminus y_0. \tag{5.32}$$

Set $x(\xi^n) = x^{2r_n}_{\frac{r_n}{2}}(\xi^n)$, with $r_n = \frac{d(\xi^n)}{2}$. By (2.10) we have

$$r_n^{N-2} G^n_{\mathcal{L}_{\frac{1}{4}}}(x, x(\xi^n)) < c_{97} \quad \forall x \in \Omega_n \cap \partial B_{r_n}(\xi^n),$$

and by Lemma 5.21 there exists $r_0 = r_0(\Omega) > 0$ such that for any $r_n \le \frac{r_0}{4}$

$$r_n^{N-2} G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}}(x, x(\xi^n)) \le c_{98} \omega^x_{\Omega_n}(\partial\Omega_n \cap B_{r_n}(\xi^n)) \quad \forall x \in \Omega_n \cap \partial B_{r_n}(\xi^n).$$

Since if $|x - y| > \varepsilon > 0$ there holds

$$G^{\Omega_n}_{L_{\frac{1}{4}}}(x, y) \approx c_{99}(\varepsilon, \Omega_n) \mathrm{dist}(x, \partial\Omega_n) \mathrm{dist}(y, \partial\Omega_n),$$

thus we have by the maximum principle and properties of the Green function

$$r_n^{N-2} G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}}(x, x(\xi^n)) \le c_{100} \omega^x_{\Omega_n}(\partial\Omega_n \cap B_{r_n}(\xi^n)) \quad \forall x \in \Omega_n \setminus B_{r_n}(\xi^n). \tag{5.33}$$

By [3, Lemma 2.8] there exists $\beta_0 = \beta_0(\Omega, \varepsilon) > 0$ such that the function

$$h_1(x) = d^{\frac{1}{2}}(x)(-\log d(x)) \left(1 + (-\log d(x))^{-\varepsilon}\right),$$

is a supersolution in Ω_{β_0} and the function

$$h_2(x) = d^{\frac{1}{2}}(x)(-\log d(x)) \left(1 - (-\log d(x))^{-\varepsilon}\right),$$

is a subsolution in Ω_{β_0}. Set

$$c_{101} = \frac{1 - (-\log d(\xi_n))^{-\varepsilon}}{1 + (-\log d(\xi_n))^{-\varepsilon}}$$

and

$$H(x) = h_2(x) - c_{101} h_1(x).$$

Let $n_0 \in \mathbb{N}$ such that $r_n \le \frac{\beta_0}{4}$, $\forall n \ge n_0$. Then the function $H(x)$ is a nonnegative subsolution in $\Omega_n \setminus \Omega'_{\beta_0}$, and $H(x) = 0$, $\forall x \in \partial\Omega_n$. By (5.32) we can choose $n_1 \in \mathbb{N}$ such that

$$G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}}(x_0, x) \ge c(\Omega, N, \kappa)\beta_0^{\frac{1}{2}} \quad \forall x \in \partial\Omega'_{\beta_0}.$$

Thus we can find a constant $c_{102} = c_{102}(\beta_0) > 0$ such that

$$c_{102} H(x) \le G^{\Omega_n}_{\mathcal{L}_{\frac{1}{4}}}(x_0, x) \quad \forall x \in \partial\Omega'_{\beta_0}.$$

Since H vanishes on $\partial\Omega_n$ it follows by the maximum principle that

$$c_{102}H(x) \leq G_{\mathcal{L}_{\frac{1}{4}}}(x_0, x) \quad \forall x \in \overline{\Omega}_n \setminus \Omega'_{\beta_0}. \tag{5.34}$$

But

$$H(x(\xi^n)) \geq c_{103}(\beta_0) \geq c_{104}(\Omega, N) r_n^{\frac{1}{2}} (-\log r_n)^{1-\varepsilon},$$

thus the result follows by the above inequality combined with inequalities (5.34) and (5.33). □

Lemma 5.23. *Let $\kappa < \frac{1}{4}$, $\varepsilon \in \left(0, \sqrt{1-4\kappa}\right)$ and $x_0 \in \Omega_1$. Let $\{\xi^n\}$ be a sequence of points in Ω converging to $a \in \partial\Omega$. Then there exists $n_0 = n_0(\varepsilon, \Omega) \in \mathbb{N}$ such that*

$$\omega_{\Omega_n}^{x_0}\left(B_{d(\xi^n)}(\xi^n) \cap \partial\Omega'_n\right) \geq c_{105}(\Omega, N, \kappa, \varepsilon) d(\xi^n)^{N+\frac{\alpha_-}{2}+\varepsilon-2} \quad \forall n \geq n_0,$$

where Ω_n is defined by (5.23).

Proof. The proof is similar as the one of Lemma 5.22. The only difference is that we use $d^{\alpha_-}(1-d^{\varepsilon})$ and the supersolution $d^{\alpha_-}(1+d^{\varepsilon})$ as a subsolution. □

Proof of Theorem 5.18. *Step* 1: if

$$\limsup_{x\in\Omega, x\to a} (d(x))^{N+\frac{\alpha_+}{2}-2} u(x) < \infty, \tag{5.35}$$

then $a \in \mathcal{R}_u$. Thus we have to prove that there exists $r_0 > 0$ such that $u \in L^q_{\phi_\kappa}(\Omega \cap B_{r_0}(a))$. By (5.35) there exists $r_1 > 0$ such that

$$\sup_{x\in\Omega\cap B_{r_1}(a)} d^{N+\frac{\alpha_+}{2}-2}(x) u(x) = M < \infty.$$

Let U be a smooth open domain such that

$$\Omega \cap B_{\frac{r_1}{2}}(a) \subset U \subset \Omega \cap B_{r_1}(a),$$

and

$$\overline{U} \cap \partial\Omega \subset \partial\Omega \cap B_{r_1}(a).$$

For $\beta > 0$, set

$$d_U(x) = \mathrm{dist}(x, \partial U) \quad \forall x \in U,\ U_\beta = \{x \in U : d_U(x) > \beta\},\ V_\beta = U \setminus U_\beta.$$

Let $\beta_0 > 0$ be small enough such that $d_U \in C^2(\overline{U}_{\beta_0})$. Let $0 < \beta < \beta_0$ and $\zeta(x) = d_U(x) - \beta$. Then u satisfies

$$\int_{\partial V_\beta} u dS = \int_{V_\beta \setminus V_{\beta_0}} (u\mathcal{L}_\kappa \zeta + u^q \zeta) dx - \int_{\partial V_{\beta_0}} \frac{\partial u}{\partial \mathbf{n}} \zeta dS.$$

Now

$$\left| \int_{\partial V_{\beta_0}} \frac{\partial u}{\partial \mathbf{n}} \zeta dS \right| \leq c_{106}(\beta_0 - \beta),$$

where c_{106} depends on $q, \kappa, \Omega, \beta_0$,

$$\int_{V_\beta \setminus V_{\beta_0}} u\mathcal{L}_\kappa \zeta dx \leq -\int_{V_\beta \setminus V_{\beta_0}} u\Delta\zeta dx \leq c_{107} \int_{V_\beta \setminus V_{\beta_0}} u dx,$$

and by (5.35)

$$u^{q-1}(x) \leq c_{108}(d(x))^{-(q-1)(N+\frac{\alpha_+}{2}-2)} \leq c_{108}(d_U(x))^{-(q-1)(N+\frac{\alpha_+}{2}-2)} \quad \forall x \in U.$$

Combining the above inequalities, we derive

$$\int_{\partial V_\beta} u dS \leq c_{109} \left(\int_\beta^{\beta_0} \left(\sigma^{1-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)} + 1 \right) \int_{\partial V_\sigma} u(x) dS d\sigma + 1 \right).$$

Multiplying the above inequality by $\beta^{\frac{\alpha_+}{2}}$ we get

$$\int_{\partial V_\beta} u d_U^{\frac{\alpha_+}{2}} dS \leq c_{109}\left(\int_\beta^{\beta_0}\left(\sigma^{1-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)}+1\right)\int_{\partial V_\sigma} d_U^{\frac{\alpha_+}{2}}(x)u(x)dSd\sigma+1\right).$$

Set

$$U(\sigma)=\int_{\partial V_\sigma} d_U^{\frac{\alpha_+}{2}}(x)u(x)dS,$$

Then we have

$$U(\beta)\leq c_{110}\left(\int_\beta^{\beta_0}\left(\sigma^{1-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)}+1\right)U(\sigma)d\sigma+1\right). \tag{5.36}$$

Set

$$W(\beta)=\int_\beta^{\beta_0}\left(\sigma^{1-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)}+1\right)U(\sigma)d\sigma+1,$$

then

$$W'(\beta)=-\left(\beta^{1-(p-1)\left(N+\frac{\alpha_+}{2}-2\right)}+1\right)U(\beta)=-h(\beta)U(\beta).$$

Thus inequality (5.36) becomes

$$-W'(\beta)\leq c_{110}h(\beta)W(\beta)\Longleftrightarrow (H(\beta)W(\beta))'\geq 0,$$

where

$$H(\beta)=e^{-c_{110}\int_\beta^{\beta_0}h(s)ds}.$$

Thus we have

$$W(\beta)\leq \frac{1}{H(\beta)}W(\beta_0)\quad \forall 0<\beta<\beta_0.$$

But

$$\frac{1}{H(\beta)}=e^{c_{110}\int_\beta^{\beta_0}h(s)ds}=e^{c_{110}\int_\beta^{\beta_0}\sigma^{1-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)}+1ds}<\infty$$

if and only if

$$2-(q-1)\left(N+\frac{\alpha_+}{2}-2\right)>0\Longleftrightarrow q<q_c.$$

Thus we have proved that

$$\int_U u^q(d_U(x))^{\frac{\alpha_+}{2}}dx<\infty,$$

which implies the existence of a $r_2>0$ such that

$$\int_{\Omega\cap B_{r_2}(a)} u^q(d(x))^{\frac{\alpha_+}{2}}dx<\infty,$$

i.e. $a\in\mathcal{R}_u$, which is the claim.

Step 2. Since $a\in\mathcal{S}_u$ the previous statement implies that there exists a sequence $\{\xi^n\}\subset\Omega$ such that

$$\xi^n\to a\quad\text{and}\quad \limsup_{n\to\infty}(d(\xi^n))^{N+\frac{\alpha_+}{2}-2}u(\xi^n)=\infty. \tag{5.37}$$

By Lemma 5.20, there exists a constant c_l such that

$$u(x)\leq c_l u(y)\quad \forall x,y\in B_{\frac{r_n}{2}}(\xi^n),\ r_n=d(\xi^n). \tag{5.38}$$

Put $V_n:=B_{\frac{r_n}{2}}(\xi^n)\cap\partial\Omega'_{r_n}$, and, for $k>0$, $h_{n,k}:=\frac{k}{b_n}u\chi_{V_n}$.

Case 1: $\kappa = \frac{1}{4}$. By (5.38) and Lemma 5.22 there exists a constant $c_{111} > 0$ such that

$$b_n := \int_{V_n} udS \geq c_{111} A_n r_n^{N+\frac{1}{2}-2} (-\log r_n)^{1-\varepsilon}, \qquad A_n := \sup_{x \in B_{\frac{r_n}{2}}(\xi^n)} u(x).$$

Then

$$\int_{\partial\Omega'_n} h_{n,k} dS = k, \quad h_{n,k} \leq \frac{k}{c_2} r_n^{2-\frac{\alpha_+}{2}-N} \chi_{V_n} \; \forall n \geq n_0. \tag{5.39}$$

By (5.37),

$$b_n \to \infty, \quad r_n \to 0. \tag{5.40}$$

Hence, for every $k > 0$ there exists n_k such that

$$u \geq h_{n,k} \quad \text{on } \partial\Omega'_n \; \forall n \geq n_k. \tag{5.41}$$

Let $w_{n,k}$ be defined as in Lemma 5.19 with h_n replaced by $h_{n,k}$. By (5.39) and (5.40), the sequence $\{h_{n,k}\}_{n=1}^{\infty}$ satisfies (5.24) for every fixed $k > 0$. Therefore by Lemma 5.19

$$\lim_{n \to \infty} w_{n,k} = u_{k\delta_a} \quad \text{locally uniformly in } \Omega.$$

By (5.41), $u \geq w_{n,k}$ in $x \in \Omega : d(x) > r_n$. Hence $u \geq u_{k\delta_a}$ for every $k > 0$. The proof in the case $0 < \kappa < \frac{1}{4}$ is similar. □

As a consequence we provide a full classification of positive solution of (4.1) with a boundary isolated singularity.

Theorem 5.24. *Assume $1 < q < q_c$ and $u \in C(\overline{\Omega} \setminus \{0\})$ is a positive solution of (4.1) which satisfies*

$$\lim_{x \in \Omega,\, x \to \xi} \frac{u(x)}{W(x)} = 0 \quad \forall \xi \in \partial\Omega \setminus \{0\}.$$

Then the following alternative holds:

(i) *either there exists $k \geq 0$ such that*

$$\lim_{\substack{x \to 0, x \in \Omega \\ x|x|^{-1} \to \sigma}} |x|^{N+\frac{\alpha_+}{2}-2} u(x) = k\psi_1(\sigma) \tag{5.42}$$

and u solves

$$\begin{aligned} -\Delta u - \frac{\kappa}{d^2} u + u^q = 0 \qquad & \text{in } \Omega \\ u = k\delta_0 \qquad & \text{in } \partial\Omega, \end{aligned} \tag{5.43}$$

(ii) *or*

$$\lim_{\substack{x \to 0, x \in \Omega \\ x|x|^{-1} \to \sigma}} |x|^{\frac{2}{q-1}} u(x) = \omega_\kappa(\sigma) \tag{5.44}$$

locally uniformly on S_+^{N-1}.

The result is a consequence of the following result.

Lemma 5.25. *Assume $1 < q < q_c$, $a \in \partial\Omega$ and $F_\epsilon(a) = \partial\Omega \cap \overline{B_\epsilon(a)}$. Then*

$$\lim_{\epsilon \to 0} U_{F_\epsilon(a)} = u_{\infty,a}. \tag{5.45}$$

Proof. Without loss of generality, we can assume $a = 0$. Clearly, $U_{\{0\}} := \lim_{\epsilon \to 0} U_{F_\epsilon(0)}$ is a solution of (5.1) which satisfies

$$\lim_{x \to \xi} \frac{U_{\{0\}}}{W(x)} = 0 \quad \forall \xi \in \partial\Omega \setminus \{0\}$$

locally uniformly on $\partial\Omega \setminus \{0\}$. By (A.20) it verifies

$$U_{\{0\}}(x) \leq c|x|^{-\frac{2}{q-1}} \left(\frac{d(x)}{|x|} \right)^{\frac{\alpha_+}{2}}. \tag{5.46}$$

By Proposition 4.5 and (A.24), we can follow the same argument like in the proof of Theorem 3.4.6-(ii) in [27] to prove that: there exists $c_0 = c_{112}(N, \kappa, q) > 1$ such that

$$\frac{1}{c_0}|x|^{-\frac{2}{q-1}}\left(\frac{d(x)}{|x|}\right)^{\frac{\alpha_+}{2}} \leq u_{\infty,0}(x) \leq U_{\{0\}}(x) \leq c_0|x|^{-\frac{2}{q-1}}\left(\frac{d(x)}{|x|}\right)^{\frac{\alpha_+}{2}}$$

which implies

$$U_{\{0\}}(x) \leq cu_{\infty,0}(x) \quad \forall x \in \Omega, \tag{5.47}$$

where $c = c_{122}(N, \kappa, q) > 1$.

Assume $U_{\{0\}} \neq u_{\infty,0}$, thus $U_{\{0\}}(x) > u_{\infty,0}(x)$ for all $x \in \Omega$ and put $\tilde{u} = u_{\infty,0} - \frac{1}{2c}(U_{\{0\}} - u_{\infty,0})$. By convexity $\tilde{u}$ is a supersolution of (5.1) which is smaller than $u_{\infty,0}$. Now $\frac{c+1}{2c}u_{\infty,0}$ is a subsolution, thus there exists a solution $\underline{u}$ of (5.1) in Ω which satisfies

$$\frac{c+1}{2c}u_{\infty,0}(x) \leq \underline{u}(x) \leq \tilde{u}(x) < u_{\infty,0}(x) \quad \forall x \in \Omega. \tag{5.48}$$

This implies that $Tr_{\partial\Omega}(\underline{u}) = (\{0\}, 0)$, and by Theorem 5.18, $\underline{u} \geq u_{\infty,0}$, which is a contradiction.

Proof of Theorem 5.24. Assume $a = 0$ without loss of generality. If $a \in \mathcal{S}_u$, then for any $\epsilon > 0$, $u \leq U_{F_\epsilon(0)}$ which is a maximal solution which vanishes on $\partial\Omega \setminus F_\epsilon(0)$. Thus, using (5.45)

$$u \leq \lim_{\epsilon\to 0} U_{F_\epsilon(0)} = U_{\{0\}} = u_{\infty,0}.$$

If $0 \in \mathcal{R}_u$, this implies that $Tr_{\partial\Omega}(u) = (\emptyset, k\delta_0)$ for some $k \geq 0$ and we conclude with Corollary 4.4. □

The next result can be proven by using the same approximation methods as in [24, Th 9.6].

Theorem 5.26. *Assume $\mathcal{S} \subset \partial\Omega$ is closed and ν is a positive Radon measure on $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$. Then there exists a positive solution of* (4.1) *in Ω with boundary trace $(\mathcal{S}, \mu)$.*

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Appendix. Barriers and a priori estimates

A.1. Barriers

Following a localization principle introduced in [24] we the following lemma is at the core of the a priori estimates construction

Proposition A.1. *Let $\Omega \subset \mathbb{R}^N$ be a C^2 domain $0 < \kappa \leq \frac{1}{4}$ and $p > 1$. Then there exists $R_0 > 0$ such that for any $z \in \partial\Omega$ and $0 < R \leq R_0$, there exists a super solution $f := f_{R,z}$ of* (4.1) *in $\Omega \cap B_R(z)$ such that $f \in C(\overline{\Omega} \cap B_R(z))$, $f(x) \to \infty$ when $\operatorname{dist}(x, K) \to 0$, for any compact subset $K \subset \Omega \cap \partial B_R(z)$ and which vanishes on $\partial\Omega \cap B_R(z)$, and more precisely*

$$f(x) = \begin{cases} c_{\beta,\gamma,\kappa,q}(R^2 - |x-z|^2)^{-\beta} d^\gamma(x) \quad \forall \gamma \in \left(\frac{\alpha_-}{2}, \frac{\alpha_+}{2}\right) & \text{if } 0 < \kappa < \dfrac{1}{4} \\ c_{\beta,\gamma,q}(R^2 - |x-z|^2)^{-\beta}\sqrt{d(x)}\sqrt{\ln\left(\dfrac{\operatorname{diam}(\Omega)}{d(x)}\right)} & \text{if } \kappa = \dfrac{1}{4} \end{cases} \tag{A.1}$$

for $\beta \geq \max\{\frac{2}{q-1} + \gamma, \frac{N-2}{2}, 1\}$.

Proof. We assume $z = 0$

Step 1: $\kappa < \frac{1}{4}$. Set $f(x) = \Lambda(R^2 - |x|^2)^{-\beta}(d(x))^\gamma$ where $\beta, \gamma > 0$ to be chosen later on. Then, with $r = |x|$,

$$\Lambda^{-1}\mathcal{L}_\kappa f = -(R^2 - r^2)^{-\beta}\left(\Delta d^\gamma + \kappa d^{\gamma-2}\right) - d^\gamma \Delta(R^2 - r^2)^{-\beta} - 2\nabla(R^2 - r^2)^{-\beta} \cdot \nabla d^\gamma.$$

Since $\Delta d(x) = (N-1)H_d$ where H_d is the mean curvature of the foliated set $\Sigma_d := \{x \in \Omega : d(x) = d\}$ and $|\nabla d|^2 = 1$,

$$\begin{aligned}
&\Delta d^\gamma = (N-1)\gamma H_d d^{\gamma-1} + \gamma(\gamma-1)d^{\gamma-2}\\
&\Delta d^\gamma + \kappa d^{\gamma-2} = (N-1)\gamma H_d d^{\gamma-1} + (\gamma(\gamma-1)+\kappa)\, d^{\gamma-2}\\
&\nabla d^\gamma = \gamma d^{\gamma-1}\nabla d,\\
&\nabla(R^2-r^2)^{-\beta} = 2\beta(R^2-r^2)^{-\beta-1}x,
\end{aligned}$$

thus

$$\begin{aligned}
\nabla(R^2-r^2)^{-\beta}\cdot\nabla d^\gamma &= 2\beta\gamma d^{\gamma-1}(R^2-r^2)^{-\beta-1}x\nabla d\\
\Delta(R^2-r^2)^{-\beta} &= 2N\beta(R^2-r^2)^{-\beta-1} + 4\beta(\beta+1)(R^2-r^2)^{-\beta-2}r^2\\
&= 2\beta(R^2-r^2)^{-\beta-2}\left(NR^2 + (2\beta+2-N)r^2\right).
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda^{-1}\mathcal{L}_\kappa f = -(R^2-r^2)^{-\beta-2}d^{\gamma-2}\big[&(R^2-r^2)^2\left((N-1)\gamma H_d d + \gamma(\gamma-1)+\kappa\right)\\
&+2\beta d^2\left(NR^2+(2\beta+2-N)r^2\right) + 4\beta\gamma d(R^2-r^2)x\cdot\nabla d\big].
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{L}_\kappa f + f^q = \Lambda(R^2-r^2)^{-\beta-2}d^{\gamma-2}\big[&\Lambda^{q-1}(R^2-r^2)^{-(q-1)\beta+2}d^{(q-1)\gamma+2}\\
&-(R^2-r^2)^2\left((N-1)\gamma H_d d+\gamma(\gamma-1)+\kappa\right)\\
&-2\beta d^2\left(NR^2+(2\beta+2-N)r^2\right)+4\beta\gamma d(R^2-r^2)x\cdot\nabla d\big].
\end{aligned}\tag{A.2}$$

If we fix $\beta \geq \max\{\frac{2}{q-1}+\gamma, \frac{N-2}{2}, 1\}$, there holds

$$2\beta d^2\left(NR^2+(2\beta+2-N)r^2\right)+4\beta\gamma d(R^2-r^2)x\cdot\nabla d \leq 4d^2\beta(\beta+1)NR^2 + 4\beta\gamma dR(R^2-r^2).$$

We choose $\frac{\alpha_-}{2} < \gamma < \frac{\alpha_+}{2}$ so that $\gamma(\gamma-1)+\kappa < 0$. There exist $\delta_0, \epsilon_0 > 0$ such that

$$(N-1)\gamma H_d d + \gamma(\gamma-1)+\kappa < -\epsilon_0 < -1$$

provided $d(x) \leq \delta_0$. We set

$$A = \left\{x \in \Omega\cap B_R : d(x) \leq \frac{\epsilon_0(R^2-r^2)}{16\beta R}\right\} \quad \text{and} \quad B := A\cap\{x\in\Omega\cap B_R : d(x)\leq\delta_0\}.$$

Then, if $x \in B$, there holds

$$\begin{aligned}
&-(R^2-r^2)^2\left((N-1)\gamma H_d d+\gamma(\gamma-1)+\kappa\right) - 2\beta d^2\left(NR^2+(2\beta+2-N)r^2\right)\\
&\quad+4\beta\gamma d(R^2-r^2)x\cdot\nabla d \geq \frac{(R^2-r^2)^2\epsilon_0}{2}.
\end{aligned}$$

Finally, assume $x \in A^c \cap \{x\in\Omega\cap B_R : d(x)\leq\delta_0\}$ and thus

$$d \geq c_1\frac{R^2-r^2}{R}.$$

In order to have

$$\begin{aligned}
&\text{(i) } \Lambda^{q-1}(R^2-r^2)^{2-(q-1)\beta}d^{(q-1)\gamma+2} \geq d^2R^2\\
&\text{(ii) } \Lambda^{q-1}(R^2-r^2)^{2-(q-1)\beta}d^{(q-1)\gamma+2} \geq dR(R^2-r^2)
\end{aligned}\tag{A.3}$$

or equivalently

$$\begin{aligned}
&\text{(i) } \Longleftrightarrow \Lambda^{\frac{1}{\gamma}}d \geq (R^2-r^2)^{\frac{\beta}{\gamma}}\\
&\text{(ii) } \Longleftrightarrow \Lambda^{\frac{q-1}{(q-1)\gamma+1}}d \geq R^{\frac{1}{(q-1)\gamma+1}}(R^2-r^2)^{\frac{(q-1)\beta-1}{(q-1)\gamma+1}}
\end{aligned}\tag{A.4}$$

it is sufficient to have, for (i)

$$c_1\Lambda^{\frac{1}{\gamma}}\frac{R^2-r^2}{R} \geq (R^2-r^2)^{\frac{\beta}{\gamma}} \quad \forall r\in(0,R) \Longleftrightarrow \Lambda \geq c_2R^{2\beta-\gamma}\tag{A.5}$$

and for (ii)

$$c_1 \Lambda^{\frac{q-1}{(q-1)\gamma+1}} \frac{R^2-r^2}{R} \geq R^{\frac{1}{(q-1)\gamma+1}} (R^2-r^2)^{\frac{(q-1)\beta-1}{(q-1)\gamma+1}} \quad \forall r \in (0,R)$$
$$\Longleftrightarrow \Lambda \geq c_2 R^{2\beta-\gamma-\frac{2}{q-1}} \tag{A.6}$$

where $c_2 = c_2(N,\gamma,\beta) > 0$ since $\beta > \gamma + \frac{2}{q-1}$.
At end, in the set $C := \{x \in \Omega : d(x) \geq \delta_0\}$, it suffices that

$$\Lambda \geq c_3 \max\left\{R^{2\beta}, R^{2\beta-\frac{1}{q-1}}\right\} \tag{A.7}$$

for some $c_3 = c_3(N,\gamma,\beta,\max|H_d|,\delta_0) > 0$ in order to insure

$$\begin{aligned}
&\text{(i) } \Lambda^{q-1}(R^2-r^2)^{-(q-1)\beta+2} d^{(q-1)\gamma+2} \geq (R^2-r^2)^2(N-1)\gamma|H_d|d \\
&\text{(ii) } \Lambda^{q-1}(R^2-r^2)^{-(q-1)\beta+2} d^{(q-1)\gamma+2} \geq 4d^2\beta(\beta+1)NR^2 \\
&\text{(iii) } \Lambda^{q-1}(R^2-r^2)^{-(q-1)\beta+2} d^{(q-1)\gamma+2} \geq 4\beta dR(R^2-r^2).
\end{aligned} \tag{A.8}$$

Noticing that $2\beta > 2\beta - \frac{1}{q-1}$, $2\beta - \gamma > 2\beta - \gamma - \frac{1}{q-1}$, we conclude that there exists a constant $c_4 = c_4(N,\gamma,\beta,\max|H_d|,\delta_0) > 0$ such that if

$$\Lambda \geq c_4 \max\left\{R^{2\beta}, R^{2\beta-\gamma-\frac{1}{q-1}}\right\} \tag{A.9}$$

there holds

$$\mathcal{L}_\kappa(f) + f^q \geq 0 \quad \text{in } \Omega. \tag{A.10}$$

Step 2: $\kappa = \frac{1}{4}$. Set $f(x) = \Lambda(R^2-r^2)^{-\beta}\sqrt{d}(\ln\frac{eR}{d})^{\frac{1}{2}}$ for some Λ, β to be fixed. Then

$$\begin{aligned}
\Delta\sqrt{d}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} &= \frac{1}{\sqrt{d}}\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\ln\frac{eR}{d}\right)^{-\frac{1}{2}}\right)\Delta d + \frac{1}{d^{\frac{3}{2}}}\left(-\frac{1}{4}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} - \frac{1}{4}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}}\right) \\
&= \frac{N-1}{\sqrt{d}}\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\ln\frac{eR}{d}\right)^{-\frac{1}{2}}\right)H_d + \frac{1}{d^{\frac{3}{2}}}\left(-\frac{1}{4}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} - \frac{1}{4}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta\sqrt{d}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} + \frac{\kappa}{d^2}\sqrt{d}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} &= \frac{N-1}{\sqrt{d}}\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\ln\frac{eR}{d}\right)^{-\frac{1}{2}}\right)H_d - \frac{1}{4d^{\frac{3}{2}}}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}} \\
&= \frac{1}{d^{\frac{3}{2}}}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}}\left[(N-1)dH_d\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^2 - \frac{1}{2}\left(\ln\frac{eR}{d}\right)\right) - \frac{1}{4}\right].
\end{aligned}$$

Further

$$\nabla(R^2-r^2)^{-\beta}\cdot\nabla\sqrt{d}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}} = \frac{\beta(R^2-r^2)^{-\beta-1}\left(\ln\frac{eR}{d}\right)^{-\frac{1}{2}}}{\sqrt{d}}\left(\left(\ln\frac{eR}{d}\right) - 1\right)x\cdot\nabla d.$$

Therefore

$$\begin{aligned}
\Lambda^{-1}\mathcal{L}_\kappa f &= -(R^2-r^2)^{-\beta-2}d^{-\frac{3}{2}}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}}\left[(R^2-r^2)^2\left[(N-1)dH_d\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^2 - \frac{1}{2}\left(\ln\frac{eR}{d}\right)\right) - \frac{1}{4}\right]\right. \\
&\quad \left. + 2\beta(R^2-r^2)d\left[\left(\ln\frac{eR}{d}\right)^2 - \left(\ln\frac{eR}{d}\right)\right]x\cdot\nabla d + 2\beta d^2\left(\ln\frac{eR}{d}\right)^2\left[NR^2 + (2\beta+2-N)r^2\right]\right].
\end{aligned}$$

Finally

$$\mathcal{L}_\kappa f + f^q = \Lambda(R^2-r^2)^{-\beta-2}d^{-\frac{3}{2}}\left(\ln\frac{eR}{d}\right)^{-\frac{3}{2}}\left[\Lambda^{q-1}(R^2-r^2)^{(1-q)\beta+2}d^{\frac{q+3}{2}}\left(\ln\frac{eR}{d}\right)^{\frac{1}{2}(q-1)+2}\right.$$

$$- (R^2-r^2)^2\left[(N-1)dH_d\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^2-\frac{1}{2}\left(\ln\frac{eR}{d}\right)\right)-\frac{1}{4}\right]$$
$$-2\beta(R^2-r^2)d\left[\left(\ln\frac{eR}{d}\right)^2-\left(\ln\frac{eR}{d}\right)\right]x\,.\,\nabla d-2\beta d^2\left(\ln\frac{eR}{d}\right)^2\left[NR^2+(2\beta+2-N)r^2\right]\Bigg]. \tag{A.11}$$

Notice that $\frac{eR}{d}\geq e$ thus $-\frac{1}{2}\leq(\ln\frac{eR}{d})^2-(\ln\frac{eR}{d})\leq(\ln\frac{eR}{d})^2$ If β is large enough, as in Step 1, there holds

$$\left|2\beta(R^2-r^2)d\left[\left(\ln\frac{eR}{d}\right)^2-\left(\ln\frac{eR}{d}\right)\right]x\,.\,\nabla d+2\beta d^2\left(\ln\frac{eR}{d}\right)^2\left[NR^2+(2\beta+2-N)r^2\right]\right|$$
$$\leq 4N\beta(\beta+1)\left(\ln\frac{R}{d}\right)^2\left((R^2-r^2)dR+d^2R^2\right).$$

There exists $\delta_0>0$ such that

$$(N-1)dH_d\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^2-\frac{1}{2}\left(\ln\frac{eR}{d}\right)\right)-\frac{1}{4}\leq-\frac{1}{8}<-1$$

if $d(x)\leq\delta_0$. If we define A, B by

$$A=\left\{x\in\Omega\cap B_R:d(x)\leq\frac{\epsilon_0(R^2-r^2)}{16\beta R\left(\ln\frac{eR}{d}\right)^2}\right\}\quad\text{and}\quad B:=A\cap\{x\in\Omega\cap B_R:d(x)\leq\delta_0\}$$

there holds if $x\in B$

$$-2\beta(R^2-r^2)d\left[\left(\ln\frac{eR}{d}\right)^2-\left(\ln\frac{eR}{d}\right)\right]x\cdot\nabla d-2\beta d^2\left(\ln\frac{eR}{d}\right)^2\left[NR^2+(2\beta+2-N)r^2\right]$$
$$-(R^2-r^2)^2\left[(N-1)dH_d\left(\frac{1}{2}\left(\ln\frac{eR}{d}\right)^2-\frac{1}{2}\left(\ln\frac{eR}{d}\right)\right)-\frac{1}{4}\right]\geq\frac{(R^2-r^2)^2}{16}.$$

If $x\in A^c\cap\{x\in\Omega\cap\Omega:d(x)\leq\delta_0\}$, then

$$d(x)\geq c_1\frac{R^2-r^2}{R\left(\ln\frac{eR}{d}\right)^2}. \tag{A.12}$$

In order to have

$$\begin{aligned}&\text{(i) }\Lambda^{q-1}(R^2-r^2)^{(1-q)\beta+2}d^{\frac{q+3}{2}}\left(\ln\frac{eR}{d}\right)^{\frac{q+3}{2}2}\geq\left(\ln\frac{eR}{d}\right)^2(R^2-r^2)dR\\&\text{(ii) }\Lambda^{q-1}(R^2-r^2)^{(1-q)\beta+2}d^{\frac{q+3}{2}}\left(\ln\frac{eR}{d}\right)^{\frac{q+3}{2}}\geq\left(\ln\frac{eR}{d}\right)^2d^2R^2\end{aligned} \tag{A.13}$$

or equivalently

$$\begin{aligned}&\text{(i) }\Lambda^{\frac{2q-2}{q+1}}d\left(\ln\frac{eR}{d}\right)^{\frac{q-1}{q+1}}\geq(R^2-r^2)^{\frac{2(q-1)\beta-2}{q+1}}R^{\frac{2}{q+1}}\\&\text{(ii) }\Lambda^2d\ln\frac{eR}{d}\geq R^{\frac{4}{q-1}}(R^2-r^2)^{2\beta-\frac{4}{q-1}}.\end{aligned} \tag{A.14}$$

Up to taking c_1 small enough, (A.12) is fulfilled if

$$\frac{eR}{d}\leq\frac{R^2}{R^2-r^2}\left(\ln\left(\frac{R^2}{R^2-r^2}\right)\right)^2\Longleftrightarrow d\geq\frac{e(R^2-r^2)}{R}\left(\ln\left(\frac{R^2}{R^2-r^2}\right)\right)^{-2}. \tag{A.15}$$

Inequality (A.13) (i) will be insured if

$$\Lambda^{\frac{2q-2}{q+1}}\geq\frac{1}{e}(R^2-r^2)^{2\frac{(q-1)\beta-1}{q+1}-1}R^{\frac{2}{q+1}+1}\left(\ln\left(\frac{R^2}{R^2-r^2}\right)\right)^{\frac{2}{q+1}}$$

which holds if, for any $\epsilon > 0$, we have for any $r \in (0, R)$

$$\Lambda^{\frac{2q-2}{q+1}} \geq C_\epsilon (R^2 - r^2)^{2\frac{(q-1)\beta-1}{q+1}-1} R^{\frac{2}{q+1}+1} \left(\frac{R^2}{R^2-r^2}\right)^{\epsilon}.$$

A sufficient condition for such a task is, with the help of (A.15),

$$\Lambda \geq c_3 R^{3\beta - \frac{2}{q-1}}. \tag{A.16}$$

As for (A.13) (ii), it will be insured if

$$\Lambda \geq c_4 R^{2\beta - \frac{2}{q-1} - \frac{1}{2}}. \tag{A.17}$$

Thus, if

$$\Lambda \geq c_5 \max\{R^{2\beta - \frac{2}{q-1} - \frac{1}{2}}, R^{3\beta - \frac{2}{q-1}}\} \tag{A.18}$$

for some $c_5 > 0 = c_5(N, \gamma, \beta, \delta_0, |H_d|)$, the function f satisfies (A.10). □

A.2. A priori estimates

By the Keller–Osserman estimate, it is clear that any solution u of (4.1) in Ω satisfies

$$u(x) \leq C(q, \Omega, N) d^{-\frac{2}{q-1}}(x) \quad \forall x \in \Omega. \tag{A.19}$$

This estimate is also a consequence of the following result [3, Prop 3.4].

Proposition A.2. *Let ϕ_* be the first positive eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. For $q > 1$, there exists $\gamma > 0$ and $\epsilon_0 > 0$ such that for any $0 \leq \epsilon \leq \epsilon_0$ the function $h_{+\epsilon} = \gamma(\phi_* - \epsilon)^{-\frac{2}{q-1}}$ is a supersolution of (4.1) in $\Omega_{\epsilon,\phi_*} := \{x \in \Omega : \phi_*(x) > \epsilon\}$.*

We recall here that

$$W(x) = \begin{cases} d^{\frac{\alpha_-}{2}}(x) & \text{if } \kappa < \frac{1}{4} \\ d^{\frac{1}{2}}(x)|\log d(x)| & \text{if } \kappa = \frac{1}{4}. \end{cases}$$

Proposition A.3. *Let Ω be a bounded open domain uniformly of class C^2 and let F be a compact subset of the boundary. Let u be a nonnegative solution of (5.1) in Ω such that*

$$\lim_{x\in\Omega,\, x\to\xi} \frac{u(x)}{W(x)} = 0 \quad \forall \xi \in \partial\Omega \setminus F,$$

locally uniformly in $\partial\Omega \setminus F$. Then there exists a constant C depending only on q, κ and Ω such that,

$$|u(x)| \leq C d^{\frac{\alpha_+}{2}}(x)\,(\mathrm{dist}(x, F))^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall x \in \Omega, \tag{A.20}$$

$$\left| \frac{u(x)}{d^{\frac{\alpha_+}{2}}(x)} - \frac{u(y)}{d^{\frac{\alpha_+}{2}}(y)} \right| \leq C|x-y|^{\beta}\,(\mathrm{dist}(x, F))^{-\frac{2}{q-1}-\beta-\frac{\alpha_+}{2}} \quad \forall (x, y) \in \Omega \times \Omega \tag{A.21}$$

such that $\mathrm{dist}(x, F) \leq \mathrm{dist}(y, F)$,

$$|\nabla u(x)| \leq C d^{\frac{\alpha_+}{2}-1}(x)\,(\mathrm{dist}(x, F))^{-\frac{2}{q-1}-\frac{\alpha_+}{2}} \quad \forall x \in \Omega. \tag{A.22}$$

Proof. The proof is based on the proof of Proposition 3.4.3 in [27]. Let $\xi \in \partial\Omega \setminus F$ and put $d_F(\xi) = \frac{1}{2}\mathrm{dist}(\xi, F)$. Denote by Ω^ξ the domain

$$\Omega^\xi = \{y \in \mathbb{R}^n : d_F(\xi)y \in \Omega\}.$$

If u is a positive solution of (5.1) in Ω, denote by u^ξ the function

$$u^\xi(y) = |d_F(\xi)|^{\frac{2}{q-1}} u(d_F(\xi)y), \quad \forall y \in \Omega^\xi.$$

Then,

$$-\Delta u^{\xi}-\kappa\frac{u}{|\mathrm{dist}(y,\partial\Omega^{\xi})|^{2}}+\left|u^{\xi}\right|^{q}=0 \quad \text{in } \Omega^{\xi}.$$

Let R_0 be the constant in Proposition A.1. First, we assume that

$$\mathrm{dist}(\xi,F)\leq\frac{1}{1+R_0}.$$

Set $r_0=\frac{3R_0}{4}$, then the solution $W_{r_0,\xi}$ mentioned in Proposition A.1 satisfies

$$u^{\xi}(y)\leq W_{r_0,\xi}(y)\quad \forall y\in B_{\frac{3R_0}{4}}(\xi)\cap\Omega^{\xi}.$$

Thus u^{ξ} is bounded in $B_{\frac{3R_0}{5}}(\xi)\cap\Omega^{\xi}$ by a constant $C>0$ depending only on n,q,κ and the C^2 characteristic of Ω^{ξ}. As $d_F(\xi)\leq 1$ a C^2 characteristic of Ω is also a C^2 characteristic of Ω^{ξ} therefore the constant C can be taken to be independent of ξ. We note here that the constant $0<R_0<1$ depends on C^2 characteristic of Ω.

Now we note that

$$\lim_{y\in\Omega^{\xi},\,y\to P}\frac{u^{\xi}(y)}{W(x)}=0\quad \forall P\in\partial\Omega^{\xi}\cap B_{\frac{3R_0}{5}}(\xi).$$

Thus in view of the proof of Propositions 2.11 and 2.12, by the above inequality and in view of the proof of Theorem 2.12 in [14], we have that there exists $C>0$ depending only on n,p,κ such that

$$u^{\xi}(y)\leq\left|\mathrm{dist}(y,\partial\Omega^{\xi})\right|^{\frac{\alpha_+}{2}}\quad \forall y\in B_{\frac{R_0}{2}}(\xi)\cap\Omega^{\xi}. \tag{A.23}$$

$$\frac{u^{\xi}(y)}{\left|\mathrm{dist}(y,\partial\Omega^{\xi})\right|^{\frac{\alpha_+}{2}}}\leq C\frac{u^{\xi}(x)}{\left|\mathrm{dist}(x,\partial\Omega^{\xi})\right|^{\frac{\alpha_+}{2}}}\quad \forall x,y\in B_{\frac{R_0}{2}}(\xi)\cap\Omega^{\xi}.$$

Hence

$$u^{\xi}(x)\leq d^{\frac{\alpha_+}{2}}(x)d_F(\xi)^{-\frac{2}{q-1}-\frac{\alpha_+}{2}}\quad \forall x\in B_{d_F(\xi)\frac{R_0}{2}}(\xi)\cap\Omega.$$

$$\frac{u(y)}{d^{\frac{\alpha_+}{2}}}(y)\leq C\frac{u^{\xi}(x)}{d^{\frac{\alpha_+}{2}}(x)}\quad \forall x,y\in B_{d_F(\xi)\frac{R_0}{2}}(\xi)\cap\Omega. \tag{A.24}$$

Let $x\in\Omega_{\frac{R_0}{2}}$ and assume that

$$d(x)\leq\frac{R_0}{2}d_F(x).$$

Let ξ be the unique point in $\partial\Omega\setminus F$ such that $|x-\xi|=d(x)$. Then we have

$$d_F(\xi)\leq d(x)+d_F(x)\leq(1+R_0)d_F(x)<1$$

and

$$|u(x)|\leq Cd^{\frac{\alpha_+}{2}}(x)\left((1+R_0)\mathrm{dist}(x,F)\right)^{-\frac{2}{q-1}-\frac{\alpha_+}{2}}.$$

If $d(x)>\frac{R_0}{4}d_F(x)$, then by (A.19) we have that

$$|u(x)|\leq Cd^{-\frac{2}{q-1}}(x)\leq Cd^{\frac{\alpha_+}{2}}(x)\left(\frac{R_0}{2}\mathrm{dist}(x,F)\right)^{-\frac{2}{q-1}-\frac{\alpha_+}{2}}.$$

Thus (A.20) holds for every $x\in\Omega_{\frac{R_0}{2}}$ such that $\mathrm{dist}(x,F)<\frac{1}{1+R_0}$.

Now we assume that $x\in\Omega_{\frac{R_0}{2}}$ and

$$\mathrm{dist}(x,F)\geq\frac{1}{1+R_0}.$$

Let ξ be the unique point in $\partial\Omega\setminus F$ such that $|x-\xi|=d(x)$. Similarly with the proof of (A.23) we can prove that

$$u(x)\leq Cd^{\frac{\alpha_+}{2}}(x)\leq d^{\frac{\alpha_+}{2}}(x)C\left((1+R_0)\mathrm{dist}(x,F)\right)^{-\frac{2}{q-1}-\frac{\alpha_+}{2}}\quad \forall x\in B_{\frac{R_0}{2}}(\xi)\cap\Omega.$$

Now if $x\in\Omega\setminus\Omega_{\frac{R_0}{2}}$, the proof of (A.20) follows by (A.19).

(ii) Let $x_0\in\Omega$. Set

$$\Omega^{x_0}=\{y\in\mathbb{R}^n: d(x_0)y\in\Omega\},$$

and $d_{x_0}(y)=\mathrm{dist}(y,\partial\Omega^{x_0})$. If $x\in B_{\frac{d(x_0)}{2}}(x_0)$ then $y=\frac{x}{d(x_0)}$ belongs to $B_{\frac12}(y_0)$, where $y_0=\frac{x_0}{d(x_0)}$. Also we have that $\frac12\leq d_{x_0}(y)\leq\frac32$ for each $y\in B_{\frac12}(y_0)$. Set now $v(y)=u(d(x_0)y),\ \forall y\in B_{\frac12}(y_0)$. Then v satisfies

$$-\Delta v-\kappa\frac{u}{|d_{x_0}(y)|^2}+d^2(x_0)\,|v|^q=0\quad\text{in } B_{\frac12}(y_0).$$

By standard elliptic estimate we have

$$\sup_{y\in B_{\frac14}(y_0)}|\nabla v|\leq C\left(\sup_{y\in B_{\frac13}(y_0)}|v|+\sup_{y\in B_{\frac13}(y_0)}d^2(x_0)|v|^q\right).$$

Now since $\nabla v(y)=d(x_0)\nabla u(d(x_0)y)$, by above inequality and (A.20) we have that

$$|\nabla u(x_0)|\leq C\left(d^{\frac{\alpha_+}{2}-1}(x_0)\,(\mathrm{dist}(x_0,F))^{-\frac{2}{q-1}-\frac{\alpha_+}{2}}+d^{\frac{q\alpha_+}{2}+1}(x_0)\,(\mathrm{dist}(x_0,F))^{-q\left(\frac{2}{q-1}-\frac{\alpha_+}{2}\right)}\right).$$

Using $\frac{2q}{q-1}=\frac{2}{q-1}+2$ and the fact that x_0 is arbitrary the result follows. □

Proposition A.4. *Let $O\subset\partial\Omega$ be a relatively open subset and $F=\overline{O}$. Let U_F be defined by (5.7) be the maximal solution of (5.1) which vanishes on $\partial\Omega\setminus F$. Then for any compact set $K\subset O$, there holds*

$$\lim_{\xi\to x}(d(\xi))^{\frac{2}{q-1}}U_F(\xi)=\ell_\kappa=\left(\frac{2(q+1)}{(q-1)^2}+\kappa\right)^{\frac{1}{q-1}}\quad\text{uniformly with respect to } x\in K.\tag{A.25}$$

Proof. *Step* 1. We claim that for any $\epsilon>0$ there exists $C_\epsilon,\tau_\epsilon>0$ such that for any $z\in O$ such that $\overline{B}_{2\tau_\epsilon}(z)\subset O$, there holds

$$u(x)\leq(\epsilon+\ell_\kappa^{q-1})^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}}+C_\epsilon\quad\forall\tau\in(0,\tau_\epsilon],\ \forall x\in\Sigma_\tau(\overline{B_{\tau_\epsilon}}(z)).\tag{A.26}$$

We recall that $\Sigma_\tau(\overline{B_{\tau_\epsilon}}(z))=\left\{x\in\Omega,\ x\approx(d(x),\sigma(x)),d(x)=\tau,\sigma(x)\in\overline{B}_{\tau_\epsilon}(z)\right\}$. Set $g(x)=\ell d^{-\frac{2}{q-1}}(x)$, then

$$\mathcal{L}_\kappa g+g^q=\frac{2(N-1)}{q-1}H_d d^{-\frac{q+1}{q-1}}+\left(\ell^{q-1}-\ell_\kappa^{q-1}\right)d^{-\frac{2q}{q-1}},\tag{A.27}$$

where H_d is the mean curvature of Σ_d. If Ω is convex we take $\ell=\ell_\kappa$ and g is a supersolution for $d(x)\leq R_0$ for some R_0. In the general case, we take $\ell=\ell(\epsilon)=(\epsilon+\ell_\kappa^{q-1})^{\frac{1}{q-1}}$, and $g=g_\epsilon=\ell(\epsilon)d^{-\frac{2}{q-1}}$ is a supersolution in the set Ω_{τ_ϵ} where

$$\tau_\epsilon=\max\left\{\tau:0<\tau\leq\frac{R_0}{2},\ \frac{2(N-1)}{q-1}\|H_\tau\|_{L^\infty(\Sigma_\tau)}+\epsilon>0\right\}.$$

Then $f_{2\tau_\epsilon,z}+g_\epsilon$ is a supersolution of (5.1) in $B_{2\tau_\epsilon}(z)\cap\Omega$ which tends to infinity on $\partial(B_{2\tau_\epsilon}(z)\cap\Omega)=\partial\Omega\cap B_{2\tau_\epsilon}(z)\cup\Omega\cap\partial B_{2\tau_\epsilon}(z)$. Since we can replace $g_\epsilon(x)$ by $g_{\epsilon,\tau}(x)=\ell(d(x)-\tau)^{-\frac{2}{q-1}}$ for $\tau\in(0,\rho_\epsilon)$, any positive solution u of (5.1) in Ω is bounded from above by $f_{2\tau_\epsilon,z}+g_{\epsilon,\tau}$ and therefore by $f_{2\tau_\epsilon,z}+g_\epsilon$. This implies (A.26) with $C_\epsilon=\max\{f_{2\tau_\epsilon,z}(y):|y-z|\leq\tau_\epsilon\}$, and it can be made explicit thanks to (A.1).

Step 2. With the same constants as in step 1, we claim that

$$U_F(x)\geq(\ell_\kappa^{q-1}-\epsilon)^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}}-C_\epsilon\quad\forall\tau\in(0,\tau_\epsilon],\ \forall x\in\Sigma_\tau(\overline{B_{\tau_\epsilon}}(z)).\tag{A.28}$$

If in the definition of the function g, we take $\ell=\ell(\epsilon)=(\ell_\kappa^{q-1}-\epsilon)^{\frac{1}{q-1}}$, then g is a subsolution in the same set Ω_{τ_ϵ}. Since $U_F+f_{2\tau_\epsilon,z}$ is a supersolution of (5.1) in $B_{2\tau_\epsilon}(z)\cap\Omega$ which tends to infinity on the boundary, it dominates the subsolution $g_{\epsilon,-\tau}=\ell(d(.)+\tau)^{-\frac{2}{q-1}}$ for $\tau\in(0,\rho_\epsilon)$ and thus, as $\tau\to0$, $g_\epsilon(x)\leq U_F(x)+f_{2\tau_\epsilon,z}(x)$. This implies (A.28) with the same constant C_ϵ.

Step 3. End of the proof. Since $K\subset O$ is precompact, for any $\epsilon>0$, there exists a finite number of points $z_j, j=1,\dots,k$ such that $K\subset\cup_{j=1}^k\overline{B_{\tau_\epsilon}}(z_j)$ with $\overline{B_{2\tau_\epsilon}}(z_j)\subset O$. Therefore

$$(\ell_\kappa^{q-1}-\epsilon)^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}}-C_\epsilon\leq U_F(x)\leq(\epsilon+\ell_\kappa^{q-1})^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}}+C_\epsilon\quad\forall\tau\in(0,\tau_\epsilon],\ \forall x\in\Sigma_\tau(K).\tag{A.29}$$

Since ϵ is arbitrary, it yields to

$$\lim_{\tau\to 0}\|\tau^{\frac{2}{q-1}}U_F-\ell_\kappa\|_{L^\infty(\Sigma_\tau(K))}=0 \tag{A.30}$$

which is (A.25). □

Corollary A.5. *Let $U_{\partial\Omega}$ be the maximal solution of (5.1) in Ω, then*

$$\lim_{d(x)\to 0}(d(x))^{\frac{2}{q-1}}U_{\partial\Omega}(x)=\ell_\kappa. \tag{A.31}$$

A.3. Moser iteration

In this subsection we always assume that Ω is a bounded smooth convex domain, $D=2\sup_{x,y\in\Omega}|x-y|$ and $f_0\in L^q(\Omega)$, $q>\frac{N+\alpha}{2}$. The main goal of this subsection is to prove Boundary Harnack inequality for positive solutions of the problem

$$-L_{\phi_\kappa}v:=-\frac{\operatorname{div}(\phi_\kappa^2\nabla v)}{\phi_\kappa^2}=\frac{f}{\phi_\kappa}\quad\text{in }\Omega, \tag{A.32}$$

where

$$|f(x)|\leq c_f\frac{\left|\log\frac{d(x)}{D}\right|}{\phi_\kappa}+f_0(x)\phi_\kappa\quad\forall x\in\Omega, \tag{A.33}$$

for some positive constant $c_f>0$.

In the sequel we will use the following local representation of the boundary of Ω. There exists a finite number m of coordinate systems $(y_i',y_n)\in\partial\Omega$, $y_i'=(y_{i1},\dots,y_{in-1})$ and the same number m of functions $a_i(y_i')$ defined on the closure cubs, $\Delta_i:=\{x\in\mathbb{R}^n:|y_{ij}-x_i|\leq b\}$, for $j=1,\dots,n$, and $i\in\{1,..,m\}$ so that for each point $x\in\partial\Omega$ there is at least i such that $x=(x_i',a_i(x_i'))$. The function a_i satisfies the Lipschitz condition on $\overline{\Delta}_i$ with constant $A>0$, that is

$$|a_i(y_i')-a_i(z_i')|\leq A|y_i'-z_i'|,$$

for $y_i',z_i'\in\overline{\Delta}_i$. Moreover there exists a positive constant $b<1$ such that the set B_i is defined for any $i\in\{1,..,m\}$ by the relation $B_i=\{(y_i',y_{in}):y_i'\in\Delta_i,\ a_i(y_i')\leq y_{in}\leq a_i(y_i')+b\}$ and $\Gamma_i=B_i\cap\partial\Omega=\{(y_i',y_{in}):y_i'\in\Delta_i,y_{in}=a_i(y_i')\}$. Furthermore, let us observe for any $y\in B_i$ where someone can make the following inequality on the distance function

$$(1+A)^{-1}(y_{in}-a_i(y_i'))\leq d(y)\leq y_{in}-a_i(y_i').$$

Finally let $x\in\partial B_i$ and $v\in C_0^1(\Omega)$. Set $x_i=y_i$ for $i=1,\dots,n-1$ and $x_n=y_n+a_i(y')$ then $\nabla_{y'}v=\nabla_{x'}v+v_{x_n}\nabla_{x'}a_i(x')$ and $v_{y_n}=v_{x_n}$, thus

$$C(A)|\nabla_xv|\leq|\nabla_yv|\leq c(A)|\nabla_xv|. \tag{A.34}$$

Let us now define the "balls" which we will use to prove some Poincaré, weighted Poincaré and Moser inequalities. More precisely we have the following definition.

Definition A.6. Let $\gamma\in(1,2)$. For any $x\in\Omega$ and for any $0<r<\frac{\min\{C_0,b\}}{2\gamma}$, we define the ball centered at x and having radius r as follows.

(i) If $d(x)\leq\gamma r$ then

$$\mathfrak{B}(x,r)=\{(y_i',y_{in}):|y_i'-x_i'|\leq r,d(x)-r\leq y_{in}-a_i(y_i')\leq r+d(x)\},$$

where $i\in\{1,\dots,m\}$ is uniquely defined by the point $\bar{x}\in\partial\Omega$ such that $|x-\bar{x}|=d(x)$, that is by the projection of the center x onto $\partial\Omega$.

(ii) If $d(x)\geq\gamma r$ then $\mathfrak{B}(x,r)=B(x,r)$ the Euclidean ball centered at x.

We also define by

$$V(x,r)=\int_{\mathfrak{B}(x,r)\cap\Omega}\phi_\kappa^2(y)dy,$$

the volume of the "ball" centered at x and having radius r.

We first recall some known results the proofs of which are in [14]. The first one [14, Lemma 2.2] is a two-sided estimates of $V(x,r)$.

Proposition A.7. *There exist positive constants d_1 and d_2 such that for any $x \in \Omega$ and $0 < r < \frac{\min\{C_0, b\}}{2\gamma}$, we have*

$$d_1 \max\{d^\alpha(x), r^\alpha\} r^N \leq V(x, r) \leq d_2 \max\{d^\alpha(x), r^\alpha\} r^N. \tag{A.35}$$

From the previous lemma it follows the *Doubling property* satisfied by $V(x, .)$.

Corollary A.8. *Let $N \geq 2$, $\alpha > 0$ and Ω be a smooth bounded domain. Then there exist positive constants $C(N, \gamma, \Omega, \alpha)$ and $\beta(\Omega, \gamma)$ such that for any $x \in \Omega$ and $0 < r < \beta$ we have*

$$V(x, 2r) \leq CV(x, r).$$

The *Local Poincaré inequality* is proved in [14, Theorem 2.5].

Proposition A.9. *There exist positive constants $C(N, \gamma, \Omega, \alpha_+)$ and $\beta(\Omega, \gamma)$ such that for any $x_0 \in \Omega$ and $r < \beta$ we have*

$$\inf_{\xi \in \mathbb{R}} \int_{\mathfrak{B}(x_0, r) \cap \Omega} |\widetilde{f}(y) - \xi|^2 \phi_\kappa^2 dy \leq Cr^2 \int_{\mathfrak{B}(x_0, r) \cap \Omega} |\nabla \widetilde{f}(y)|^2 \phi_\kappa^2 dy \quad \forall \widetilde{f} \in C^\infty(\mathfrak{B}(x_0, r) \cap \Omega).$$

As a consequence there holds a *local weighted Moser inequality* which is proved in [14, Th 2.6].

Proposition A.10. *There exist positive constants $C_M(N, \Omega, \alpha_+)$ and $\beta(\Omega)$ such that for any $\nu \geq N + \alpha$, $x_0 \in \Omega$, $r < \beta$ and $f \in C_0^\infty(\mathfrak{B}(x_0, r) \cap \Omega)$ we have*

$$\int_{\mathfrak{B}(x_0, r) \cap \Omega} |f(y)|^{2\left(1 + \frac{2}{\nu}\right)} \phi_\kappa^2(y) dy \leq C_M r^2 V(x, r)^{-\frac{2}{\nu}} \int_{\mathfrak{B}(x_0, r) \cap \Omega} |\nabla f(y)|^2 \phi_\kappa^2(y) dy \left(\int_{\mathfrak{B}(x_0, r) \cap \Omega} |f(y)|^2 \phi_\kappa^2(y) dy \right)^{\frac{2}{\nu}}.$$

Let us now make precise the notion of a weak solution.

Definition A.11. We will say that $v \in H_\phi^1(\mathfrak{B}(x, r) \cap \Omega)$ is a weak solution of $L_{\phi_\kappa} v = f$ in $\mathfrak{B}(x, r) \cap \Omega$, if for each $\Phi \in C_0^\infty(\mathfrak{B}(x, r) \cap \Omega)$, we have

$$\int_{\mathfrak{B}(x, r) \cap \Omega} \nabla v \cdot \nabla \Phi dm = \int_{\mathfrak{B}(x, r) \cap \Omega} f \Phi dm,$$

where $dm = \phi_\kappa^2 dx$ and $\sigma > 0$.

We denote here by $H_\phi^1(\mathfrak{B}(x, r) \cap \Omega)$ the space of all functions $u \in L^2_{\phi_\kappa}(\mathfrak{B}(x, r) \cap \Omega)$ such that $\nabla u \in L^2_{\phi_\kappa}(\mathfrak{B}(x, r) \cap \Omega)$, endowed with the norm

$$\|u\|_{H^1_{\phi_\kappa}(\mathfrak{B}(x, r) \cap \Omega)} = \left(\int_{\mathfrak{B}(x, r) \cap \Omega} |\nabla u|^2 \phi_\kappa^2 dx + \int_{\mathfrak{B}(x, r) \cap \Omega} u^2 \phi_\kappa^2 dx \right)^{\frac{1}{2}}.$$

Then we have the following Harnack inequality up to the boundary.

Theorem A.12. *Let v be a non-negative solution of $L_{\phi_\kappa} v = f$ in Ω where f satisfies (A.33). Then there exists a constant $A > 0$ such that the following estimate holds,*

$$v(y) \leq Av(x) \quad \forall x, y \in \Omega.$$

In order to prove Theorem A.12 we use the Moser iteration technique as it is adapted to degenerate elliptic operators in [17,18,29]. In this approach one inserts in the weak form of the equation $L_{\phi_\kappa} v = f$ suitable test functions Φ. One of the key ideas is to use test functions Φ of the form $\eta^2 v^q$, where v is the weak solution of the equation, η is a cut off function and $q \in \mathbb{R}$. To this end one has to check that $\eta^2 v^q$ is in the right space of test function. In this direction the following density theorem is crucial, the proof of which is [14, Th 2.11].

Theorem A.13. *Let $N \geq 2$, $\alpha \geq 1$ and $U \subset \mathbb{R}^n$ be a smooth bounded domain. Then we have*

$$H_0^1(U, d^\alpha(y) dy) = H^1(U, d^\alpha(y) dy)$$

where we have set

$$H_0^1(U, d^\alpha(y) dy) = \left\{ v = v(y) : \|v\|^2_{H_1^1} = \int_U d^\alpha (|\nabla v|^2 + v^2) dy < \infty \right\}.$$

We note here the above theorem allows us to take the cut of function $\eta \in C_0^\infty(\mathfrak{B}(x,r))$ instead of it as a usual taking in $\eta \in C_0^\infty(\mathfrak{B}(x,r)\cap\Omega)$. Clearly the two function spaces differ only if the "ball" intersects the boundary of Ω.

To explain what are the appropriate modifications of the standard iteration argument by Moser, we now present in detail the first step, which is the L^p; $p\geq 2$ mean value inequality for any positive local subsolution of $L_{\phi_\kappa}v\leq f$. Similarly with Definition A.11, we call a function $v\in H^1_\phi(\mathfrak{B}(x,r)\cap\Omega)$ subsolution of $L_{\phi_\kappa}v\leq f$ in $\mathfrak{B}(x,r)\cap\Omega$, if for each $0\leq\Phi\in C_0^\infty(\mathfrak{B}(x,r)\cap\Omega)$ we have

$$\int_{\mathfrak{B}(x,r)\cap\Omega}\nabla v\cdot\nabla\Phi\,\phi_\kappa^2dx\leq\int_{\mathfrak{B}(x,r)\cap\Omega}f\,\phi_\kappa^2dx. \tag{A.36}$$

Theorem A.14. *Let $\gamma\in(1,2)$ and $p\geq 2$. Then there exist positive constants $c_0(\Omega)$ and $C(\Omega,p,\kappa,c_0)$ such that for any $x\in\Omega$, $R<c_0$ and for any positive subsolution of $L_{\phi_\kappa}v\leq f$ in $\mathfrak{B}(x,r)\cap\Omega$, we have the estimate*

$$\sup_{\mathfrak{B}(x,\sigma R)\cap\Omega}|v|^p\leq\frac{C}{(1-\sigma)^\nu V(x,R)}\int_{\mathfrak{B}(x,R)\cap\Omega}|v|^p\phi_\kappa^2dx$$
$$+C\left(R^{2-\alpha_+}(\log R)c_f+R^{2-\frac{N+\alpha_+}{q}}\left(\int_{\mathfrak{B}(x,R)\cap\Omega}|f_0|^q\phi_\kappa^2dx\right)^{\frac{1}{q}}\right)$$

for each $0<\sigma<1$.

Proof. Let $\gamma\in(1,2)$ and $x_0\in\Omega$. First we assume that $d(x_0)<\gamma R$, in other case the proof is standard and we omit it. Let $R<\min(c_0,1)$ we denote by Ω^R the domain

$$\Omega^R=\{\xi\in\mathbb{R}^n:\ R\xi\in\Omega\}.$$

Set $x_0=Ry_0$, $\widetilde{\phi}_\kappa(y)=\phi_\kappa(Ry)$

$$\widetilde{V}(y,r)=\int_{\mathfrak{B}(y,r)\cap\Omega^R}\widetilde{\phi}_\kappa^2(x)dx,$$
$$\widetilde{d}(y)=\operatorname{dist}(y,\Omega^R)=\frac{d(Ry)}{R}.$$

As $R\leq 1$ a C^2 characteristic of Ω is also a C^2 characteristic of Ω^R therefore the constant C can be taken to be independent of y. We note here that the constant $0<c_0<1$ depends on C^2 characteristic of Ω.

Set $\widetilde{v}(y)=v(Ry)$, $c_{\widetilde{f}}=2R^{2-\alpha_+}(\log R)c_f$, $\widetilde{f}(y)=R^2f(Ry)$, $\widetilde{f}_0(y)=R^2f_0(Ry)$ $u=\widetilde{v}+k$, where $k=c_{\widetilde{f}}+\|\widetilde{f}_0\|_{L^q(\Omega^R,\widetilde{\phi}_\kappa^2dx)}$. Then u is bounded away from zero. Thus by (A.36) we have for any $\Phi\in C_0^\infty(\mathfrak{B}(y,1)\cap\Omega^R)$

$$\int_{\mathfrak{B}(y_0,1)\cap\Omega}\nabla u\cdot\nabla\Phi\phi_\kappa^2dx\leq\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\Phi\widetilde{f}_0\widetilde{\phi}_\kappa^{\,2}dx+c_{\widetilde{f}}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{R\widetilde{d}(x)}{D}\right|\Phi dx.$$

Let $\beta>0$, we set

$$u_m=\begin{cases}u & u\leq k+m\\ k+m & u>k+m\end{cases}$$

and $\Phi=\psi^2u_m^\beta u$. Due to Theorem A.13 there exists a sequence of functions Φ_k in $C^\infty(\overline{\mathfrak{B}(y_0,1)\cap\Omega^R})$ having compact support in Ω such that $\Phi_k\to\Phi$ in $H^1(\mathfrak{B}(y_0,1)\cap\Omega^R,d^{\alpha_+}dy)$. Since $\phi\sim d^{\frac{\alpha_+}{2}}$, we have that $\Phi_k\to\Phi$ in $H^1_{\phi_\kappa}(\mathfrak{B}(y_0,1)\cap\Omega^R)$. Hence for any $\forall\ \psi\in C_0^\infty(\mathfrak{B}(y_0,1))$ and $m\geq 1$ the function $\Phi=\psi^2u_m^\beta u$ is an admissible test function, that is, the following holds true:

$$\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\nabla u\cdot\nabla(\psi^2u_m^\beta u)\widetilde{\phi}_\kappa^2dx\leq\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2u_m^\beta u\widetilde{f}_0\widetilde{\phi}_\kappa^2dx+c_0\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{d(x)}{D}\right|\psi^2u_m^\beta udx$$
$$\leq\frac{1}{k}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2u_m^\beta u^2\widetilde{f}_0\widetilde{\phi}_\kappa^2dx+\frac{c_{\widetilde{f}}}{k}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{d(x)}{D}\right|\psi^2u_m^\beta u^2dx.$$

Thus by straightforward calculations and Hölder inequality we have

$$\frac{1}{2}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2dx+\beta\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u_m|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2dx$$
$$\leq c\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla\psi|^2u_m^\beta u^2\widetilde{\phi}_\kappa^2dx+\frac{1}{k}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2u_m^\beta u^2\widetilde{f}_0\widetilde{\phi}_\kappa^2dx+\frac{c_{\widetilde{f}}}{k}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{d(x)}{D}\right|\psi^2u_m^\beta u^2dx.$$

Now we have by Hölder inequality

$$\frac{1}{k}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R} u^2\psi^2u^\beta\widetilde{f}_0\widetilde{\phi}_\kappa dx \leq \frac{1}{k}\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\widetilde{f}_0|^q\widetilde{\phi}_\kappa^2 dx\right)^{\frac{1}{q}}\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|u_m^\beta u^2\psi^2|^{\frac{q}{q-1}}\widetilde{\phi}_\kappa^2 dx\right)^{\frac{q-1}{q}}.$$

Since $\frac{2(N+\alpha_+)}{N+\alpha_+-2} > \frac{2q}{q-1} > 2$ if $q > \frac{N+\alpha}{2}$, we have by interpolation inequality and (2.9)

$$\begin{aligned}\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|u_m^\beta u^2\psi^2|^{\frac{q}{q-1}}\widetilde{\phi}_\kappa^2 dx\right)^{\frac{q-1}{q}} &\leq \varepsilon\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|u_m^\beta u^2\psi^2|^{\frac{N+\alpha_+}{N+\alpha_+-2}}\widetilde{\phi}_\kappa^2 dx\right)^{\frac{N+\alpha_+-2}{N+\alpha_+}}\\ &\quad + C(N,\alpha_+,q)\varepsilon^{-\frac{N+\alpha_+}{2q-N+\alpha_+}}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|u_m^\beta u^2\psi^2|\widetilde{\phi}_\kappa^2 dx\\ &\leq \varepsilon\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\nabla\left(u_m^{\frac{\beta}{2}}u\psi\right)\right|^2\widetilde{\phi}_\kappa^2 dx\\ &\quad + C(N,\alpha_+,q)\varepsilon^{-\frac{N+\alpha_+}{2q-N+\alpha_+}}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|u_m^\beta u^2\psi^2|\widetilde{\phi}_\kappa^2 dx.\end{aligned}$$

Also

$$\begin{aligned}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\psi^2u_m^\beta u^2 dx &= -\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\widetilde{d}\nabla\widetilde{d}\cdot\nabla(\psi^2u_m^\beta u^2)dx\\ &\quad -\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\widetilde{d}\Delta\widetilde{d}\psi^2G(u_k)u dx\\ &\quad +\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2u_m^\beta u^2 dx.\end{aligned}$$

Let $0<\sigma<\sigma'<1$, we choose a function $\psi=\xi(|y_0'-x'|)\xi(|x_n-a(x')-\widetilde{d}(y_0)|)$, where $\xi\in C^\infty(\mathbb{R})$ and satisfies $0\leq\xi\leq 1$, $\xi(s)=1$ if $s\leq\frac{\sigma}{2}$ and $\xi(s)=0$ if $s>\sigma'$. Then clearly we have $|\nabla\psi|\leq\frac{C}{\sigma'-\sigma}$.

$$\begin{aligned}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right| d|\nabla\psi|u_m^\beta u^2 dx &\leq \frac{C}{\sigma'-\sigma}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\widetilde{d}\psi u_m^\beta u^2 dx\\ &= -\frac{C}{\sigma'-\sigma}\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\widetilde{d}^2\nabla\widetilde{d}\cdot\nabla\left(\left|\log\frac{\widetilde{d}(x)}{D}\right|\psi u_m^\beta u^2\right)dx\right)\\ &\quad -\frac{C}{\sigma'-\sigma}\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\widetilde{d}^2\Delta d\left(\left|\log\frac{\widetilde{d}(x)}{D}\right|\psi u_m^\beta u^2\right)dx\right).\end{aligned}$$

$$\begin{aligned}\beta\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\widetilde{d}\psi^2|\nabla u_m|u_m^\beta u dx &\leq \frac{\beta}{4}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2|\nabla u_m|^2u_m^\beta\widetilde{d}^{\alpha_+}dx\\ &\quad + C\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|^2\widetilde{d}^{2-\alpha_+}\psi^2u_m^\beta u^2 dx.\end{aligned}$$

Working as the last two inequalities and using the fact that $\widetilde{\phi}_\kappa\sim\widetilde{d}^{\frac{\alpha_+}{2}}$, we can prove that there exists $\varepsilon\in(0,2-\alpha_+)$, such that

$$\begin{aligned}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\left|\log\frac{\widetilde{d}(x)}{D}\right|\psi^2u_m^\beta u^2 dx &\leq \frac{\beta}{4}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2|\nabla u_m|^2u_m^\beta\widetilde{\phi}_\kappa^2 dx+\frac{1}{4}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}\psi^2|\nabla u|^2u_m^\beta\widetilde{\phi}_\kappa^2 dx\\ &\quad +\frac{C(\beta+1)^2}{(\sigma'-\sigma)^2}\int_{\mathfrak{B}(y,\sigma')\cap\Omega^R}\psi^2u_m^\beta u^2\widetilde{\phi}_\kappa^2 dx.\end{aligned}$$

Let $\beta\geq 2$, combining all above there exist $\delta=\delta(N,\alpha_+,q)>0$ and $C=C(N,\alpha_+,q)>0$ such that

$$\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2 dx+\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u_m|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2 dx\leq\frac{C\beta^\delta}{(R-r)^2}\int_{\mathfrak{B}(y,\sigma')\cap\Omega^R}u_m^\beta u^2\widetilde{\phi}_\kappa^2 dx.$$

Set now $w=u_m^{\frac{\beta}{2}}$, then

$$\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla(\psi w)|^2dx\leq C(\beta+1)\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2dx+\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla u_m|^2u_m^\beta\psi^2\widetilde{\phi}_\kappa^2dx\right).$$

Thus we get

$$\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla(\psi w)|^2dx\leq C\frac{\beta^{\delta+1}}{(R-r)^2}\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}w^2\widetilde{\phi}_\kappa^2dx. \tag{A.37}$$

Using the above inequality Proposition A.10 we obtain

$$\begin{aligned}\int_{\mathfrak{B}(y_0,\sigma)\cap\Omega^R}|w|^{2+\frac{4}{\nu}}\widetilde{\phi}_\kappa^2dx&\leq\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\psi w|^{2+\frac{4}{\nu}}\widetilde{\phi}_\kappa^2dx\\&\leq E\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\nabla(w\psi)|^2\widetilde{\phi}_\kappa^2dx\right)\left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R}|\psi w|^2\widetilde{\phi}_\kappa^2dx\right)^{\frac{2}{\nu}}\\&\leq EC\beta^{\delta+1}\left(\frac{1}{(\sigma'-\sigma)^2}\int_{\mathfrak{B}(x,\sigma')\cap\Omega}|w|^2\widetilde{\phi}_\kappa^2dx\right)^{1+\frac{2}{\nu}}\end{aligned} \tag{A.38}$$

where $E=C_M\widetilde{V}^{-\frac{2}{\nu}}(y_0,1)$ is the constant in Proposition A.10.

Set $\beta=p$ and let $m\to\infty$, then we have by (A.38) and the definition of w,

$$\int_{\mathfrak{B}(y_0,\sigma)\cap\Omega}|u|^{p\left(1+\frac{2}{\nu}\right)}\widetilde{\phi}_\kappa^2dx\leq A\left(\frac{p^{\delta+1}}{(\sigma'-\sigma)^2}\int_{B(x,\sigma')}|\psi u|^p\widetilde{\phi}_\kappa^2dx\right)^{1+\frac{2}{\nu}},$$

where $A=EC$ the constant in (A.38).

We note that by iteration with $p_0=p,\ p_1=p(1+\frac{1}{\nu}),\dots,p_i=p\left(1+\frac{1}{\nu}\right)^i$,

$$\int_{\mathfrak{B}(y_0,\sigma'')\cap\Omega}u^{p_i}\widetilde{\phi}_\kappa^2dxdt<\infty\quad\forall\, i\geq0\text{ and }\sigma''<r'.$$

Thus by the same argument as before we have

$$\int_{\mathfrak{B}(x,\sigma)\cap\Omega}u^{p_{i+1}}\widetilde{\phi}_\kappa^2dx\leq A\left(\frac{p_i^{\delta+1}}{(\sigma'-\sigma)^2}\int_{\mathfrak{B}(x,\sigma')\cap\Omega}u^{p_i}\widetilde{\phi}_\kappa^2dx\right)^{1+\frac{2}{\nu}}. \tag{A.39}$$

Now set $r_0=\sigma'$ and $r_i=\sigma'-(\sigma'-\sigma)\sum_{j=1}^i2^{-j}$. Then $r_i-r_{i+1}=(\delta'-\delta)2^{-i-1}$ and $p_{i+1}=p_i(1+\frac{2}{\nu})$, thus inequality (A.39) becomes

$$\begin{aligned}\int_{\mathfrak{B}(y_0,r_{i+1})\cap\Omega^R}u^{p_{i+1}}\widetilde{\phi}_\kappa^2dx\ &\leq\ A\frac{2^{2(i+1)}}{(\sigma'-\sigma)^2}\left(p_i^{\delta+1}\int_{\mathfrak{B}(y_0,r_i)\cap\Omega^R}u^{p_i}\widetilde{\phi}_\kappa^2dx\right)^{1+\frac{2}{\nu}}\\\Longleftrightarrow&\left(\int_{\mathfrak{B}(y_0,r_{i+1})\cap\Omega^R}u^{p_{i+1}}\widetilde{\phi}_\kappa^2dx\right)^{\frac{1}{p_{i+1}}}\\&\leq\ A^{\frac{1}{p_{i+1}}}\left(\frac{2^{2(i+1)}}{(\sigma'-\sigma)^2}\right)^{\frac{1}{p_{i+1}}}\left(p_i^{\delta+1}\int_{\mathfrak{B}(y_0,r_i)\cap\Omega^R}u^{p_i}\widetilde{\phi}_\kappa^2dx\right)^{\frac{1}{p_i}}\\&\leq\ \left(\frac{A}{(\sigma'-\sigma)^2}\right)^{\frac{1}{p_{i+1}}+\frac{1}{p_i}}2^{\frac{2(i+1)}{p_{i+1}}+\frac{2i}{p_i}}p_i^{\frac{\delta+1}{p_i}}p_{i-1}^{\frac{\delta+1}{p_{i-1}}}\left(\int_{\mathfrak{B}(y_0,r_{i-1})\cap\Omega^R}u^{p_{i-1}}\widetilde{\phi}_\kappa^2dx\right)^{\frac{1}{p_{i-1}}}\\&\leq\ \left(\frac{A}{(\sigma'-\sigma)^2}\right)^{\frac{1}{p}\sum_{j=1}^\infty\Theta^{-j}}4^{\frac{1}{p}\sum_{j=0}^\infty\frac{j+1}{\Theta^j}}e^{\frac{\delta+1}{2}\sum_{j=0}^\infty\Theta^{-j}\log(p_0\Theta^j)}\left(\int_{\mathfrak{B}(y_0,r_0)\cap\Omega^R}u^{p_0}\widetilde{\phi}_\kappa^2dx\right)^{\frac{1}{p_0}},\end{aligned}$$

where $\Theta=1+\frac{2}{\nu}$. Observe now that $r_i\to\delta$ as $i\to\infty$, all sum above are finite and $\sum_{j=0}^\infty\Theta^{-j}=\frac{\nu}{2}+1$.

Hence we have,

$$\sup_{\mathfrak{B}(y_0,\sigma)\cap\Omega^R}|u|^p\leq A^{\frac{\nu}{2}}\frac{1}{(\sigma'-\sigma)^\nu}\int_{\mathfrak{B}(y_0,\sigma')\cap\Omega^R}|u|^p\widetilde{\phi}_\kappa^2dx\quad\forall\, p\geq2$$

where $A=C_M\widetilde{V}^{-\frac{2}{\nu}}(x,1)$.

Thus we have

$$\sup_{\mathfrak{B}(y_0,\frac{1}{2})\cap\Omega^R} |\widetilde{v}|^p \leq A^{\frac{\nu}{2}} 2^{\nu} \left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R} |\widetilde{v}|^p \widetilde{\phi}_\kappa^2 dx + k \right) \quad \forall\, p \geq 2,$$

which implies

$$\sup_{\mathfrak{B}(y_0,\frac{R}{2})\cap\Omega^R} |v|^p \leq \frac{1}{V(x,R)} \left(\int_{\mathfrak{B}(y_0,1)\cap\Omega^R} |v|^p \phi_\kappa^2 dx + k \right) \quad \forall\, p \geq 2.$$

The estimate in $\mathfrak{B}(y_0, \sigma R) \cap \Omega$ can be obtained by applying the above result to $\mathfrak{B}(y_0, (1-\sigma)R) \cap \Omega$ for any $y \in \mathfrak{B}(y, \sigma R) \cap \Omega$. $\square$

Using Moser's iterative scheme we are now in a situation to prove.

Proposition A.15. *Let u be a weak solution of* (A.32). *Then there exist two constants* $C > 0$ *and* $\alpha \in (0, 1]$, *depending on* Ω, N *and* κ *such that*

$$\sup_{x,y\in\Omega\ x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C \left(c_f + \left(\int_{\mathfrak{B}(x,R)\cap\Omega} |f_0|^q \phi_\kappa^2 dx \right)^{\frac{1}{q}} \right).$$

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