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Geodesic stability for memoryless binary long-lived consensus ☆

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ABSTRACT

The determination of the (in-)stability of the long-lived consensus problem is a fundamental open problem in distributed systems. We concentrate on the memoryless binary case with geodesic paths. For this case, we offer a conjecture on the instability, measured by the parameter *inst*, exhibit two classes of colourings which attain the conjectured bound, and improve the known lower bounds for all colourings. We also introduce a related parameter, *winst*, which measures the stability only for certain geodesics, and for which we also prove lower bounds.

1. Introduction

The *consensus problem* in distributed systems consists of the following: given a set of values, each coming from a processor or sensor, decide on a representative value, meaning the consensus of the given values. The *long-lived* consensus problem consists of repeatedly solving related instances of the consensus problem. Dolev and Rajsbaum [5] introduced the concept of *stability* of long-lived consensus, where one wishes the representative values, produced by an algorithm for a sequence of input instances, to change as few times as possible (there might be some cost associated with a change). So the question is how to choose the outputs in a way that they are stable in time. In the case with memory, the algorithm may use the value produced for the previous instances in the sequence to decide on the value of the current instance. This is not allowed in the so-called *memoryless* case. See also [1].

We will consider binary-valued consensus, with the input sequences being a geodesic path. The case with memory is completely solved in [5] and also, for the memoryless case, some bounds for the minimum number of changes are shown. Related work includes multi-valued consensus [3], binary and ternary consensus with random walks instead of geodesic paths [2,8], and multi-valued consensus with oblivious paths (in which at most a certain number of components change) [4].

We need a few definitions in order to properly state the problem. The *n-hypercube* is $\mathcal{H}_n := \{0,1\}^n$. Write 0^n for $(0,0,\dots,0)$, and similar. The *ball* $B_t(0^n)$ *of radius* t around 0^n consists of all elements of $\mathcal{H}_n$ with at most t entries identical to 1. In the same way, we define $B_t(1^n)$.

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A *colouring* of $\mathcal{H}_n$ is a function $f : \mathcal{H}_n \to \{0, 1\}$. We say that a colouring f *respects* $B_t(0^n)$ and $B_t(1^n)$ if $f(x) = 0$ for each x in $B_t(0^n)$ and $f(x) = 1$ for each x in $B_t(1^n)$. Observe that if $n < 2t + 1$, the two balls $B_t(0^n)$ and $B_t(1^n)$ intersect, and no colouring can respect $B_t(0^n)$ and $B_t(1^n)$. As we are not interested in this case, we say t is *valid* (for n) if $n \geq 2t + 1$.

A *geodesic* P (in $\mathcal{H}_n$) is a sequence $(x_0, x_1, \ldots, x_n)$ with $x_i \in \mathcal{H}_n$ for $i = 0, 1, \ldots, n$, so that there is a permutation $(p_1, \ldots, p_n)$ of $(1, \ldots, n)$ such that the ℓth entry of x_j differs from the ℓth entry of x_{j-1} if and only if $j = p_\ell$. We then say that P *fixed* the ℓth entry at time j.

We denote by *inst*(f, P), for *instability*, the number of colour-jumps of P in the colouring f, that is, the number of indices i where $f(x_i) \neq f(x_{i-1})$. Any such index i shall be called a *jump* of P (in f). Let *inst*(f) be the maximum value of *inst*(f, P) over all geodesics P.

The connection of these concepts and the memoryless consensus problem in distributed systems is as follows. Each point of $\mathcal{H}_n$ represents a set of n input values (one from each sensor). A colouring of $\mathcal{H}_n$ corresponds to an assignment of a representative value for each possible set of input values. We prefer colourings that respect the balls of a certain radius as the output value should in some way be representative. A geodesic stands for a slowly changing system of inputs (one sensor at a time), and its instability is the number of changes of the representative value. We remark that, if one considers arbitrary paths instead of geodesics, there is no bound on the instability as the path might go back and forth between two points with a different output value (see [5]).

Now, a colouring that respects $B_t(0^n)$ and $B_t(1^n)$ and has low instability is a good candidate for a consensus algorithm. One is therefore interested in the lowest possible instability.

Problem 1.1. (See Dolev & Rajsbaum [5].) Given $n \in \mathbb{N}$, and t valid for n, find the minimum value *inst*(n, t) for *inst*(f) over all colourings f of $\mathcal{H}_n$ that respect $B_t(0^n)$ and $B_t(1^n)$.

Dolev and Rajsbaum [5] proved the following special cases: *inst*$(n, t) \geq 1$ for $n > 4t$, *inst*$(n, 0) = 1$, *inst*$(n, 1) = 3$, and *inst*$(2t + 1, t) = 2t + 1$. We establish a lower bound of $\lfloor \frac{t-1}{n-2t} \rfloor + \lceil \frac{t-1}{n-2t} \rceil + 3$ on *inst*(n, t) that holds for all values of n and $t \geq 1$ (cf. Theorem 4.1(b)). The idea of the proof is using a geodesic that visits the two balls alternately.

A similar lower bound holds for the related parameter *winst*(n, t), which measures the maximum instability of a colouring considering only a special class of geodesics, namely those that start or end next to one of the balls, but in the opposite colour.

We consider the special case of $n = 2t + 2$, which is simpler, as every point outside the balls is neighbouring simultaneously the two balls, and so has neighbours in both colours. For $t \geq 2$, we improve our bounds to *inst*$(2t + 2, t) \geq$ *winst*$(2t + 2, t) \geq t + 3$ (Theorem 4.2). The basic tool for this result is Lemma 4.3, which serves for extending lower bounds for *winst* for smaller values of t to larger values of t. This tool is generalised for arbitrary values of n in Proposition 4.5. We apply Proposition 4.5 to the case $n = 2t + 3$ to obtain a lower bound for *winst* and *inst* which again improves the ones given by Theorem 4.1.

As for upper bounds for *inst*, in [5] an example is given which shows that *inst*$(n, t) \leq 2t + 1$, and here, we provide more such examples (see below). We conjecture that the bound $2t + 1$ is indeed the correct value.

Conjecture 1.2 *(Main conjecture). Let $n \in \mathbb{N}$, and t be valid for n. Then inst$(n, t) = 2t + 1$.*

If one can solve Problem 1.1, it would be interesting to find all *optimal* colourings, i.e., all colourings for which the bound *inst*(n, t) is attained. We exhibit two new classes, $maj_t(k)$ and b_t^k, of colourings that have instability exactly $2t + 1$. The only earlier example ($maj_t(2t + 1)$ in our language) is the one from [5].

The paper is organised as follows. In Section 2, we exhibit the two new classes of colourings with instability $2t + 1$. The parameter *winst* is introduced and motivated in Section 3. In Section 4, we establish the new lower bound on *inst*(n, t), and in Sections 4.2 and 4.3 we present better lower bounds for the cases $n = 2t + 2$ and $n = 2t + 3$ respectively. Section 5 contains some final remarks.

2. Candidates for optimal colourings

We present two classes of colourings that respect the balls $B_t(0^n)$ and $B_t(1^n)$ and have instability $2t + 1$.

2.1. The majority colourings

For a positive odd value k, define $maj_t(k)$ to be the colouring that assigns to each point $x \in \mathcal{H}_n$ not in $B_t(0^n) \cup B_t(1^n)$ the colour that appears on the majority of the first k entries of x. The balls $B_t(0^n)$ and $B_t(1^n)$ are coloured canonically with 0 and 1, respectively.

For a positive even value k, define the auxiliary class $maj'_t(k)$ as the class of colourings f that assign to each point x outside $B_t(0^n)$ and $B_t(1^n)$ (which are coloured canonically) the colour that appears on the majority of the first k entries of x, if there is such a (strict) majority, and an arbitrary colour if both colours appear equally often in the first k entries of x. We say a colouring f in $maj'_t(k)$ is *symmetric* if, for every x and y outside $B_t(0^n) \cup B_t(1^n)$, $f(x) \neq f(y)$ whenever x and y, restricted to their first k entries, are the complement of each other. Let $maj_t(k)$ be the class of all symmetric colourings in $maj'_t(k)$.

Note that, for even k, the class $maj_t(k)$ is non-empty. Indeed, we can obtain a symmetric colouring f in $maj'_t(k)$ in the following way. For all points $x \in \mathcal{H}_k$ that have equally many 0's and 1's, and moreover start with a 0, we assign any colour c_x to all points outside $B_t(0^n) \cup B_t(1^n)$ that start with x. Then, we assign the complement colour $1 - c_x$ to all points outside $B_t(0^n) \cup B_t(1^n)$ that start with the complement of x. The total number of points of $\mathcal{H}_n$ coloured 0 in f equals the total number of points of $\mathcal{H}_n$ coloured 1 (which is also true for $maj_t(k)$ when k is odd).

In what follows, we often abuse notation and, for a positive even value k, write $maj_t(k)$ for an arbitrary element of $maj_t(k)$.

Proposition 2.1. *Let $k, t, n \in \mathbb{N}$, with $0 < k \leq 2t+1 \leq n$. Then $inst(maj_t(k)) = 2t+1$.*

The proof of Proposition 2.1 splits into two parts: in Lemma 2.2 we show that no geodesic jumps more than $2t+1$ in any $maj_t(k)$, and in Lemma 2.3 we present a geodesic that jumps at least $2t+1$ in any $maj_t(k)$.

Before we turn to these lemmas, let us remark that, for $k > 2t+1$ and odd, it is easy to find a geodesic that jumps k times in $maj_t(k)$. Indeed, we may start at the point $(01)^{\lfloor n/2 \rfloor}0$ and then at each step switch an entry, from the first to the last. Each of the k first steps is a jump. This shows that $maj_t(k)$, for k large and odd, has instability larger than $2t+1$.

The remainder of this section is devoted to the proof of Proposition 2.1, i.e., to Lemma 2.2 and Lemma 2.3. For a geodesic $P = (x_0, x_1, \dots, x_{n-1}, x_n)$, the path $Q = (x_n, x_{n-1}, \dots, x_1, x_0)$ is also a geodesic, and is called the *reverse* of P. Clearly, $inst(f, P) = inst(f, Q)$ for any colouring f.

Lemma 2.2. *If $0 < k \leq 2t+1 \leq n$, then $inst(maj_t(k)) \leq 2t+1$.*

Proof. Suppose otherwise. Then there is a geodesic $P = (x_0, x_1, x_2, \dots, x_n)$ in $\mathcal{H}^n$ with $inst(maj_t(k), P) \geq 2t+2$. Let m be so that the mth jump of P is the first jump that fixes one of the last $n-k$ entries (as $k < 2t+2$, there is such an m, $1 \leq m \leq n$). Suppose that, among all geodesics as above, P is chosen such that $m = m(P)$ is as large as possible. Our plan is to modify P to a geodesic P' with at least $2t+2$ jumps and $m(P') > m(P)$, thus obtaining a contradiction.

Let $i+1$ be the first jump of P, and let ℓ be the $(2t+2)$nd jump of P. We assume that $maj_t(k)(x_\ell) = 1$, and thus $maj_t(k)(x_i) = 1$. The other case is analogous.

As ℓ is the $(2t+2)$nd jump of P, there are $t+1$ jumps j with $j < \ell$ and $maj_t(k)(x_j) = 0$. Hence, P fixed $(t+1)$ 0's before time ℓ, and therefore $x_\ell \notin B_t(1^n)$. Thus, since $maj_t(k)(x_\ell) = 1$, the majority of the first k entries of x_ℓ is not 0: it is 1 or k is even and x_ℓ has as many 0's as 1's in its first k entries. We can use the same argument on the reverse of P to obtain that $x_i \notin B_t(1^n)$, and thus the majority of the first k entries of x_i is 1 or k is even and x_i has as many 0's as 1's in its first k entries. Thus we showed that

$$\text{the first } k \text{ entries of } x_i \text{ contain at least as many 1's as 0's, and the same holds for } x_\ell. \tag{1}$$

Because $maj_t(k)(x_i) = maj_t(k)(x_\ell) = 1$, the first k entries of x_i and of x_ℓ are not the complement of each other. So, by (1), at least one entry within the k first, say the first entry, is 1 in both x_i and x_ℓ. This implies that all x_j with $i \leq j \leq \ell$ start with a 1.

Let S be the set of those of the first k entries of x_i that do not change in P between x_i and x_ℓ. We have just seen that $s := |S| \geq 1$. Let z_1 be obtained from x_i by changing the first entry to 0, and for $1 < j \leq s$ let z_j be obtained from z_{j-1} by changing another of the entries in S. Then

$$\text{the first } k \text{ entries of } z_s \text{ are the complement of the first } k \text{ entries of } x_\ell. \tag{2}$$

Let h be the $(2t+1)$st jump of P. Then $maj_t(k)(x_{h-1}) = 1$ and $maj_t(k)(x_h) = 0$. There are t jumps $j \leq h-1$ with $maj_t(k)(x_j) = 1$, each fixing a 1 distinct from the first entry. Thus in total x_h and $x_{\ell-1}$ have at least $(t+1)$ 1's, and cannot be in $B_t(0^n)$. In the same way, we see that $x_{i+1} \notin B_t(0^n)$.

Consider $P' = (z_s, z_{s-1}, \dots, z_1, x_i, x_{i+1}, \dots, x_\ell, y_0, y_1, \dots, y_{n-s-\ell+i-1})$, where the y_j's are arbitrarily chosen to complete P' to a geodesic. Note that P' jumps at least $2t+2$ times, as it has the same jumps as P between x_i and x_ℓ. We claim that

$$P' \text{ has a jump in its first } s+1 \text{ steps.} \tag{3}$$

Then we are done because the first $m+1$ jumps of P' fix one of the first k entries, contradicting our choice of P.

It remains to prove (3). As $x_{i+1} \notin B_t(0^n)$ and $maj_t(k)(x_{i+1}) = 0$, there are at least as many 0's as 1's among the first k entries of x_{i+1}. So, since the first entry of x_i is 1, but the first entry of z_1 is 0, there are also at least as many 0's as 1's among the first k entries of z_1.

Now, as $x_i \notin B_t(1^n)$, also $z_1 \notin B_t(1^n)$. Hence, if the first k entries of z_1 contain more 0's than 1's, it follows that $maj_t(k)(z_1) = 0$. As $maj_t(k)(x_i)$=1, the geodesic P' has the jump x_i, which is as desired for (3). So we may assume that the first k entries of z_1 contain exactly as many 0's as 1's. (It is because of this possibility that we add not only z_1 but also $z_s, \dots, z_2$ to P'.) By (1) and (2), and by the definition of z_s, it follows that z_s has at least as many 0's as 1's in its first k entries, and so at least as many 0's as z_1 has. So, $z_1 \notin B_t(1^n)$ implies that $z_s \notin B_t(1^n)$ and hence, $maj_t(k)(z_s) = 0$. This finishes the proof of (3), and thus the proof of the lemma. □

For the proof of the second lemma, and later on, the following definition will turn out to be useful. We call a geodesic in $\mathcal{H}_n$ an *m-geodesic* if it starts in a point of $\mathcal{H}_n$ which has exactly m entries equal to 1. (It then ends in a point which has exactly m entries that equal 0.)

For even $k \le 2t+1$, the statement of the second lemma is slightly stronger than we first claimed. Indeed, we present a geodesic that jumps $2t+1$ times for a subclass of $maj'_t(k)$ larger than $maj_t(k)$. We say a colouring f in $maj'_t(k)$ is *k-defined* if $f(x) = f(y)$ whenever x and y coincide in the first k entries, for every x and y outside $B_t(0^n) \cup B_t(1^n)$. Let $\overline{maj}_t(k)$ denote the set of all k-defined colourings in $maj'_t(k)$ for which there is a point coloured 1 which has exactly $t+1$ entries equal to 1. Let us argue that $maj_t(k) \subseteq \overline{maj}_t(k)$.

First, observe that if f is symmetric then f is k-defined. Indeed, consider a point x in $\mathcal{H}_k$ with the same number of 1's and 0's, and its complement $\bar{x}$. As f is symmetric, $f(xy) \neq f(\bar{x}z)$ for every y and z in $\mathcal{H}_{n-k}$ such that xy and $\bar{x}z$ are not in $B_t(0^n) \cup B_t(1^n)$. Further, note that for every xy outside $B_t(0^n) \cup B_t(1^n)$ there is a point $\bar{x}z$ outside $B_t(0^n) \cup B_t(1^n)$, namely $\bar{x}\bar{y}$. Thus $f(xy) = f(xy')$ for every y and y' in $\mathcal{H}_{n-k}$ such that xy and xy' are not in $B_t(0^n) \cup B_t(1^n)$.

Second, for $k \le 2t+1 \le n$, let us show that, for any $maj_t(k)$, there is a point coloured 1 with exactly $t+1$ entries equal to 1. Consider the point

$$x = 1^{\lceil k/2 \rceil} 0^{\lfloor k/2 \rfloor} 1^{t+1-\lceil k/2 \rceil} 0^{n-t-1-\lfloor k/2 \rfloor}.$$

Note that this point is well-defined, as $\lceil k/2 \rceil \le t+1$ and $n \ge 2t+1 \ge t+1+\lfloor k/2 \rfloor$. Further, x has exactly $t+1$ entries equal to 1. If k is odd, then the majority of the first k entries is 1, and so x has colour 1. If k is even, then there is a tie on the first k entries. Consider the point

$$y = 0^{k/2} 1^{k/2} 1^{t+1-k/2} 0^{n-t-1-k/2}.$$

Note that also y has exactly $t+1$ entries equal to 1. By symmetry, one of x, y has colour 1, and is thus the point we were looking for.

Lemma 2.3. *If $0 < k \le 2t+1 \le n$, then $inst(maj_t(k)) \ge 2t+1$ and, if k is even, $inst(\overline{maj}_t(k)) \ge 2t+1$.*

Proof. Let $f = maj_t(k)$ if k is odd, and let f be an arbitrary colouring in $\overline{maj}_t(k)$ if k is even. We will prove the following stronger assertion.

For $0 < k \le 2t+1 \le n$, there exists a $(t+1)$-geodesic P such that

$$inst(f, P) \ge 2t+1 \tag{4}$$

and the first point of P is coloured 1.

We shall prove (4) by using induction on t, keeping k fixed, but letting n vary. More precisely, fix $k > 0$, then, at each step t, the assertion is shown to hold for t if $k \le 2t+1$, and for all choices of n which satisfy the inequality $2t+1 \le n$.

We start the induction with $t = \lfloor k/2 \rfloor$, that is, $k = 2t$ or $k = 2t+1$. Let x be a point coloured 1 with exactly $t+1$ entries equal to 1, and exactly $\lfloor k/2 \rfloor$ 0's within the first k entries. Say $x = 1^t 0^t 1 0^{n-2t-1}$. Consider the $(t+1)$-geodesic

$$\begin{array}{lll}
P = (& 1^t 0^t 1 0^{n-2t-1}, & [1] \\
& 1^t 0^t 0^{n-2t}, & [0] \\
& 1^{t+1} 0^{t-1} 0^{n-2t}, & [1] \\
& 0 1^t 0^{t-1} 0^{n-2t}, & [0] \\
& 0 1^{t+1} 0^{t-2} 0^{n-2t}, & [1] \\
& 0^2 1^t 0^{t-2} 0^{n-2t}, & [0] \\
& 0^2 1^{t+1} 0^{t-3} 0^{n-2t}, & [1] \\
& 0^3 1^t 0^{t-3} 0^{n-2t}, & [0] \\
& \dots & \\
& 0^t 1^t 0^{n-2t}, & [0] \\
& 0^t 1^t 0^{n-2t-1} 1, & [0] \\
& 0^t 1^t 0^{n-2t-2} 1^2, & [0] \\
& \dots & \\
& 0^t 1^t 0 1^{n-2t-1}), & [0].
\end{array}$$

Note that, within the first $2t+2$ points in P, every second point lies in $B_t(0^n)$ and thus has colour 0. The first point is x, and thus has colour 1. The remaining of the first $2t+2$ points are not in $B_t(0^n)$ and have a majority of 1's on their first k entries, so they are coloured 1. Hence P jumps $2t+1$ times within its first $2t+2$ points, and is a geodesic as desired.

So, for the induction step, suppose that $k \le 2t-1$. Then, $n \ge 2t+1 \ge k+2$. Consider f on

$$\tilde{\mathcal{H}}^n := \{x \in \mathcal{H}^n : x(n-1) = 0 \text{ and } x(n) = 1\},$$

and observe that this is equivalent to considering $maj_{t-1}(k-2)$ on $\mathcal{H}^{n-2}$ for odd k, or $\overline{maj}_{t-1}(k-2)$ on $\mathcal{H}^{n-2}$ for even k. Indeed, outside $B_t(0^n) \cup B_t(1^n)$, the majority on the first k entries rules and, for even k, if there is a tie on the first k entries, this still determines exactly one colour, and there is a point coloured 1 with exactly $t+1$ entries equal to 1, and exactly $k/2$ 0's within the first k entries.

Hence, by induction, we know that there exists a t-geodesic $\tilde{P}$ in $\mathcal{H}^{n-2}$ that is as in (4) for $k-2$ and $t-1$. In particular, $\tilde{P}$ jumps at least $2(t-1)+1 = 2t-1$ times. Abusing notation slightly, we shall consider $\tilde{P}$ as a path in $\tilde{\mathcal{H}}^n$.

Now we extend $\tilde{P}$ to a geodesic in $\mathcal{H}^n$ adding two more jumps. By (4), we know that $\tilde{P}$ starts at a point $y = (y(1), y(2), \ldots, y(n-2), 0, 1)$ with $f(y) = 1$, and with exactly $t+1$ entries equal to 1 (among these the last entry). We add the points $y'' := (y(1), y(2), \ldots, y(n-2), 1, 0)$ and $y' := (y(1), y(2), \ldots, y(n-2), 0, 0)$ to the beginning of $\tilde{P}$ and obtain a geodesic P as desired. Indeed, $y' \in B_t(0^n)$ as y' has exactly t entries equal to 1, hence $f(y') = 0$, and so we have our first extra jump. Note that y'' has exactly as many 1's as y (in particular, $y'' \notin B_t(0^n)$) and, moreover, y'' has the same first k entries as y. Thus, $f(y'') = 1$, and we have the second extra jump, implying that P is as desired for (4). $\square$

2.2. The partition colourings

We present a second class of colourings, the colourings b_t^k, which respect the balls $B_t(0^n)$ and $B_t(1^n)$ and have instability $2t+1$. Before that, we define the auxiliary colouring $a_j^{\mathcal{Q}}$ that will be used in the definition of b_t^k.

Let m, s, and t be such that $m \ge (s+1)(t+1)$. Let $\mathcal{Q}$ be a partition of $[m]$ into $s+1$ sets of size at least $t+1$ each. For $j = 0, 1$, let $a_j^{\mathcal{Q}}$ be the following colouring of $\mathcal{H}_n$.

We define the colouring $a_0^{\mathcal{Q}}$ by letting $a_0^{\mathcal{Q}}(x) = 0$ if and only if, in at least one of the sets in $\mathcal{Q}$, all entries are 0. As the sets in $\mathcal{Q}$ have size at least $t+1$, it is not difficult to see that $a_0^{\mathcal{Q}}$ respects $B_t(1^m)$ (because, for $a_0^{\mathcal{Q}}(x) = 0$, at least $t+1$ entries of x must be 0). Also, as $\mathcal{Q}$ has $s+1$ sets, $a_0^{\mathcal{Q}}$ respects $B_s(0^m)$ (because, for $a_0^{\mathcal{Q}}(x) = 1$, point x must have at least one entry 1 for each of the $s+1$ sets).

The second colouring, $a_1^{\mathcal{Q}}$, is defined by setting $a_1^{\mathcal{Q}}(x) = 1$ if and only if, in at least one of the sets in $\mathcal{Q}$, all entries are 1. Similarly as for $a_0^{\mathcal{Q}}$, we see that $a_1^{\mathcal{Q}}$ respects both $B_s(1^m)$ and $B_t(0^m)$.

Consider a geodesic $P = (x_0, x_1, \ldots, x_m)$ in $\mathcal{H}_m$. Note that, if i is a jump of P in $a_j^{\mathcal{Q}}$, then for some set Q in $\mathcal{Q}$ we have that $x_\ell(q) = j$ for all $q \in Q$ either for $\ell = j-1$ or for $\ell = j$, but not for both. We say that the jump i is associated with this set Q. Thus there are at most two jumps in P associated with the same set Q in $\mathcal{Q}$. This implies that $a_j^{\mathcal{Q}}$ jumps at most $2|\mathcal{Q}| = 2(s+1)$ times.

Now, let k, s, t, and n be such that k is odd, $s \ge -1$, $t = s + (k+1)/2$, and $n \ge (s+1)(t+1)+k$. Note that $k \le 2t+1$ because $s \ge -1$. Let $\mathcal{Q}$ be a partition of $[n-k]$ into $s+1$ sets of size at least $t+1$ each. (If $s = -1$, then $n = k$ and $\mathcal{Q} = \emptyset$.) We shall define the colouring $b_t^k = b_t^k(\mathcal{Q})$ using $a_0^{\mathcal{Q}}$ and $a_1^{\mathcal{Q}}$ in $\mathcal{H}_{n-k}$. We abuse notation and assume that $a_j^{\mathcal{Q}}(y) = 1 - j$ if $\mathcal{Q}$ or y is empty.

For each point x, if the majority of the first k entries of x is 1, then let $b_t^k(x) = a_0^{\mathcal{Q}}(x')$, where x' is x without the first k entries. If the majority of the first k entries of x is 0, then let $b_t^k(x) = a_1^{\mathcal{Q}}(x')$. In both cases, we sometimes abuse notation and write that $b_t^k = a_j^{\mathcal{Q}}$ in x.

It is not difficult to see that b_t^k respects the balls $B_t(0^n)$ and $B_t(1^n)$. Indeed, let us suppose the majority of the first k entries of some point x is 1, and hence $b_t^k = a_0^{\mathcal{Q}}$ (the other case is symmetric). If x has at most t entries equal to 0, clearly no set in $\mathcal{Q}$ can only consist of 0's, and so $b_t^k(x) = 1$. On the other hand, if x has at most t 1's, then x' has at most $t - (k+1)/2 = s$ 1's and therefore, as $|\mathcal{Q}| = s+1$, there is a set in $\mathcal{Q}$ that only consists of 0's. Thus $b_t^k(x) = 0$ in this case. Hence, in either case, $b_t^k(x)$ is as desired.

Observe that, for $t = 0$ and $k = 1$, we have $s = -1$, and hence $n = 1$. In this case, $b_0^1 = maj_0(1)$.

Proposition 2.4. *Let $k, t, n \in \mathbb{N}$ be such that k is odd, $k \le 2t+1$ and $n \ge (t+1-\frac{k+1}{2})(t+1)+k = \frac{(t+1)(2t+1)-k(t-1)}{2}$. Then $inst(b_t^k) = 2t+1$.*

Proof. Let P be a geodesic in $\mathcal{H}_n$. To prove that P jumps at most $2t+1$ times in b_k^t, first note that at most k jumps of P are associated with its first k entries. Second, note that P has at most two jumps associated with each set Q in $\mathcal{Q}$. Indeed, if P has one jump associated with Q while $b_t^k = a_j^{\mathcal{Q}}$, then P has at most one more jump associated with Q while $b_t^k = a_{1-j}^{\mathcal{Q}}$. Similarly, if P has two jumps associated with Q while $b_t^k = a_j^{\mathcal{Q}}$, then P has no jumps associated with Q while $b_t^k = a_{1-j}^{\mathcal{Q}}$.

Also, it is not hard to find a geodesic in $\mathcal{H}_n$ that jumps $2t+1$ times in b_t^k. Consider a point x_0 with $(k+1)/2$ 1's in the first k entries, and exactly one 1 in each of the sets in $\mathcal{Q}$. Then x_0 has exactly $t+1$ entries equal to 1. Take a geodesic that starts in x_0, and jumps k times by changing alternatively 1's to 0's and 0's to 1's within the first k entries. After that, we have that $b_t^k = a_1^{\mathcal{Q}}$. So we can jump twice per set Q in $\mathcal{Q}$ by changing all entries in Q to 1 first, and then changing the unique entry in Q that started with a 1 to a 0. $\square$

3. Well-ending geodesics and *k*-defined colourings

If we try to determine $inst(f)$ for some given colouring f, in general it is not necessary to calculate $inst(f, P)$ for *all* geodesics P. A geodesic P that starts deep inside $B_t(0^n)$, for instance, will jump at most as much as any geodesic P' we obtain from P by cutting off the first few steps, and prolonging it arbitrarily at the end. So we may restrict our attention to geodesics that start outside the ball $B_t(0^n)$, or on its border.

In this section, we will introduce an even more restricted class of geodesics, that are easier to handle, and might be useful for attacking Conjecture 1.2. We need some notation for this. Let f be some colouring of $\mathcal{H}_n$. We say f is a $(t+1)$-*colouring* of $\mathcal{H}_n$ if f respects $B_t(0^n)$ and $B_t(1^n)$ but f does not respect $B_{t+1}(0^n)$ and $B_{t+1}(1^n)$. In particular, if $f(0^n)=1$ or $f(1^n)=0$, then f is a 0-colouring of $\mathcal{H}_n$. Note that every colouring of $\mathcal{H}_n$ is a $(t+1)$-colouring for a unique t. Now, let f be a $(t+1)$-colouring. By our definition of an m-geodesic above (before Lemma 2.3), a $(t+1)$-geodesic starts in a point with exactly $t+1$ entries equal to 1, that is, right next to the ball $B_t(0^n)$.

If P is a geodesic whose first point is coloured 1 in f, or whose last point is coloured 0 in f, we say P *ends well* (in f). Let $winst(f)$ denote the maximum value of $inst(f, P)$, taken over all well-ending $(t+1)$-geodesics P, where t is such that f is a $(t+1)$-colouring of $\mathcal{H}_n$. In analogy to Problem 1.1, we ask the following.

Problem 3.1. Given t valid for n, which is the smallest value $winst(n,t)$ such that $winst(n,t) = winst(f)$ for some $(t+1)$-colouring f?

Observe that $winst(f) \le inst(f)$ for every colouring f. Moreover, $\max_{t\le s\le (n-1)/2}\{winst(n,s)\} \le inst(n,t)$ for all t valid for n.

Here we extend the definition of k-defined given in Subsection 2.1. Call a colouring f *k-defined*[3] if there are k indices such that $f(x) = f(y)$ for any two points $x, y \in \mathcal{H}_n \setminus (B_t(0^n) \cup B_t(1^n))$ that coincide in all entries given by these k indices. A k-defined colouring that is not $(k-1)$-defined is called *strictly k-defined*. For instance, $maj_t(k)$ is strictly k-defined and $a_0^{\mathcal{Q}}$ is strictly n-defined.

Let t be valid for n. For the next lemma, let $F^n(t)$ denote the set of all strictly n-defined $(t+1)$-colourings of $\mathcal{H}_n$, and let $F^{<n-2t}(t)$ denote the set of all strictly k-defined $(t+1)$-colourings of $\mathcal{H}_n$ with $0 \le k < n-2t$.

Lemma 3.2. *If $winst(f') \ge 2t'+1$ for all t' valid for n and all $f' \in F^n(t')$, then $winst(f) \ge 2t+1$ for all t valid for n and all $f \in F^{<n-2t}(t)$.*

This lemma might be used as a step towards a solution of Problem 1.1. Indeed, if we could prove that $winst(f) \ge 2t+1$ for every $(t+1)$-colouring f that is strictly k-defined with $k \ge n-2t$, then Lemma 3.2 would assure this bound holds for all colourings of $\mathcal{H}_n$, and thus imply Conjecture 1.2. The proof of a slightly more general version of Lemma 3.2 can be found in [7].

4. Lower bounds on *inst*(*n*, *t*) and *winst*(*n*, *t*)

4.1. The zig-zag bound

In this section, we prove lower bounds for $inst(n,t)$ and $winst(n,t)$. Recall that any lower bound on $winst(f)$ also serves as a lower bound for $inst(f)$. We start with a bound for all values of n and valid t, which we obtain from a zig-zag argument.

In Theorem 4.2 we will improve the bounds from Theorem 4.1 for the special case $n = 2t+2$ and, in Corollaries 4.7 and 4.9, Theorem 4.1 will be improved for $n = 2t+3$.

Theorem 4.1 *(The zig-zag bound). Let $n \in \mathbb{N}$ and let $t \ge 0$ be valid for n. Then*

(a) $winst(n,t) \ge \lfloor \frac{t}{n-2t} \rfloor + \lceil \frac{t}{n-2t} \rceil + 1$,
(b) $inst(n,t) \ge \lfloor \frac{t-1}{n-2t} \rfloor + \lceil \frac{t-1}{n-2t} \rceil + 3$, *if* $t \ge 1$.

[3] We remark that in [5, p. 39], *one-bit defined* colourings are introduced. This definition differs from ours (for $k=1$) as we canonically colour the balls $B_t(0^n)$ and $B_t(1^n)$. For instance, $maj_t(1)$ is 1-defined, but not one-bit defined.

We remark that Theorem 4.1(a) proves Conjecture 1.2 for $t = 0$ and Theorem 4.1(b) proves Conjecture 1.2 for $t = 1$. This has been shown earlier in [5].

We dedicate the rest of this subsection to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let f be a $(t+1)$-colouring with $t \geq 0$. For (a), our aim is to find a well-ending $(t+1)$-geodesic P that jumps at least $\lfloor \frac{t}{n-2t} \rfloor + \lceil \frac{t}{n-2t} \rceil + 1$ times in f.

As f is a $(t+1)$-colouring, there is a point $x \in \mathcal{H}^n$ that has exactly $(t+1)$ 1's or $(t+1)$ 0's, and that is coloured 1 or 0, respectively. Say the former holds for x (the other case is symmetric).

We let P start in x, then enter $B_t(0^n)$, then go to $B_t(1^n)$, come back to $B_t(0^n)$, go to $B_t(1^n)$ again, etc., until P has used up all of its entries. For example, if $x = 1^{t+1}0^{n-t-1}$, we let P pass next through $1^t 0^{n-t}$ and then through $1^t 0^t 1^{n-2t}$, through $1^{t-(n-2t)} 0^{n-t} 1^{n-2t}$, through $1^{t-(n-2t)} 0^t 1^{2n-4t}$, and so on.

We can do this until one of the following two things happens. Firstly, coming from $B_t(0^n)$, we might end in the complement of x with $(t+1)$ 0's (just before reaching $B_t(1^n)$). This will happen exactly when $n = \ell(n-2t)$ for some odd value ℓ, which is the case if and only if $n-2t$ divides t. Then we will have jumped at least ℓ times and

$$\ell = \frac{n}{n-2t} = \frac{2t}{n-2t} + 1 = \left\lfloor \frac{t}{n-2t} \right\rfloor + \left\lceil \frac{t}{n-2t} \right\rceil + 1.$$

Secondly, on our way from $B_t(1^n)$ to $B_t(0^n)$, we might reach a point of $\mathcal{H}_n \setminus (B_t(0^n) \cup B_t(1^n))$ which has no more unused 1's. This happens if and only if $n-2t$ does not divide t. Then we have to return in the direction of $B_t(1^n)$ to end in the complement of x (if we are not already there). In this case, we have jumped at least

$$1 + 2 \cdot \left\lfloor \frac{n-(n-2t)}{2(n-2t)} \right\rfloor + 1 = 2 \cdot \left\lfloor \frac{t}{n-2t} \right\rfloor + 2 = \left\lfloor \frac{t}{n-2t} \right\rfloor + \left\lceil \frac{t}{n-2t} \right\rceil + 1$$

times, because at least one jump is achieved during the $n-2t$ steps (the first step is a jump), then we get at least two jumps for every $2(n-2t)$ steps, and finally we jump at least once more in the last part of P when entering $B_t(1^n)$. Note that, by the construction of P, we have to end up in one of the two situations just described. This completes the proof of (a).

For (b), the proof is similar, the difference being that we let P start inside $B_t(0^n)$, have x as its second point, then re-enter $B_t(0^n)$, and then go on in a zig-zag fashion as before. We will obtain two jumps in the first two steps of P, at least one jump during the next $n-2t$ steps, and then two jumps every $2(n-2t)$ steps. Finally, we might ensure another jump depending on whether $n-2 = \ell(n-2t)$ for some odd value ℓ or not. More precisely, if $n-2 = \ell(n-2t)$ for some odd value ℓ, that is, if $n-2t$ divides $t-1$, then we get

$$\ell + 2 = \frac{n-2}{n-2t} + 2 = \left\lfloor \frac{t-1}{n-2t} \right\rfloor + \left\lceil \frac{t-1}{n-2t} \right\rceil + 3$$

jumps, and otherwise we also get

$$2 + 1 + 2 \cdot \lfloor \frac{n-2-(n-2t)}{2(n-2t)} \rfloor + 1 = \left\lfloor \frac{t-1}{n-2t} \right\rfloor + \left\lceil \frac{t-1}{n-2t} \right\rceil + 3$$

jumps, which is as desired. Clearly, we need here that $t \geq 1$, because otherwise we could not enter $B_t(0^n)$ twice in the beginning. $\square$

4.2. Better bounds for one strip

In this subsection we will concentrate on the case in which $\mathcal{H}_n$ contains, besides the balls, only one 'strip' of points which all have the same number of entries equal to 0 and equal to 1. That is, we treat the case $n = 2t + 2$.

From Theorem 4.1, we have that $inst(2t+2, t) \geq t+2$ for $t \geq 1$, and $winst(2t+2, t) \geq t+1$. The following result improves these bounds.

Theorem 4.2. *$inst(2t+2, t) \geq winst(2t+2, t) \geq t+3$ for all $t \geq 2$.*

We will prove Theorem 4.2 by combining the next two lemmas. The first of these is a tool for extending bounds for small values of t to larger values of t.

Lemma 4.3. *Let y_0, t_0 and $t \in \mathbb{N}$ with $t \geq t_0$. If $winst(2t_0+2, t_0) \geq y_0$ for some $t_0 \geq 0$ then $winst(2t+2, t) \geq y_0 + t - t_0$.*

Proof. We proceed by induction on t. The base, for $t = t_0$, follows directly from the hypothesis of the lemma. For $t > t_0$, consider a $(t+1)$-colouring f of the hypercube $\mathcal{H}_n$ of dimension $n = 2t+2$.

Define a colouring g of the hypercube $\mathcal{H}_{n-2}$ by assigning to each x' in $\mathcal{H}_{n-2}$ the value $g(x') = f(01x')$. Then g is a t-colouring of $\mathcal{H}_{n-2}$. Indeed, any point of $\mathcal{H}_{n-2} \setminus (B_{t-1}(0^{n-2}) \cup B_{t-1}(1^{n-2}))$ is a witness to this. We may thus apply the induction hypothesis to obtain a well-ending t-geodesic $\tilde{P}$ in $\mathcal{H}_{n-2}$ that jumps at least $y_0 + t - 1 - t_0$ times in g. Extending each point $\tilde{x}$ of $\tilde{P}$ to the point $01\tilde{x}$ of $\mathcal{H}_n$, we obtain a path P' in $\mathcal{H}_n$ that jumps at least $y_0 + t - 1 - t_0$ times in f.

Let $01a$ and $01z$ be the first and last point of P' respectively. We extend P' to P by adding to its beginning the points $00a$ and $10a$ if $g(01a) = 1$, and the points $11a$ and $10a$ otherwise. As we thus pass once more through either $B_t(0^n)$ or $B_t(1^n)$, our extension P of P' jumps at least once more than P', that is, $y_0 + t - t_0$ times in total. Clearly, P is a $(t+1)$-geodesic, and so is its reverse, because $n = 2t + 2$. Now at least one of the two, P or its reverse, has to be well-ending, which completes the proof of the lemma. □

The next lemma takes care of the base case $t = t_0$ for Lemma 4.3. Together with a previous upper bound of Dolev and Rajsbaum [5], it implies that $inst(2t + 2, t) = 2t + 1$ for $t = 0, 1, 2$. So it confirms Conjecture 1.2 for $n = 2t + 2$ and small values of t.

Lemma 4.4. *$winst(2t + 2, t) \geq 2t + 1$ for $t = 0, 1, 2$.*

Proof. The case $t = 0$ is trivial. For $t = 1$, let f be a 2-colouring of $\mathcal{H}_4$. Note that there are two points x and y in $\mathcal{H}_4$ with exactly $t + 1 = 2$ entries equal to 1, differing in exactly two entries (that is, such that $||x - y||^2 = 2$), and such that $f(x) = f(y)$. For example, two of the three points 1100, 1010, 1001 must have the same colour in f. Now it is easy to construct a well-ending 2-geodesic that starts in x, goes through $B_1((1 - f(x))^4)$, through y, and again through $B_1((1 - f(x))^4)$ and which jumps at least three times.

For $t = 2$, let f be a 3-colouring of $\mathcal{H}_6$. Observe that we only need to find three points x, y, z, all with exactly $t + 1 = 3$ entries equal to 1, such that $||x - y||^2 = ||y - z||^2 = 2$, $||x - z||^2 = 4$, and $f(x) = f(y) = f(z)$. Indeed, if we have such points, it is easy to construct a well-ending 3-geodesic that starts in x and jumps at least five times.

The proof of the existence of x, y and z is a case analysis. By rearranging the order of the entries, we may assume the points $x = 111000$ and $y = 110100$ have the same colour j in f. If one among $x' = 100110$, $y' = 100101$, and $z' = 010101$ has colour j, then we may take it as our third point z. If not, then x', y', and z' all have colour $1 - j$ and form a triple of points as desired. □

Proof of Theorem 4.2. The statement is an immediate consequence of Lemma 4.3 and Lemma 4.4 for $t = 2$. □

4.3. The extension method for more strips

We now extend the results from the previous subsection to the general case, when we have more 'strips'. The main result of this subsection, Proposition 4.5, is an extension of Lemma 4.3 for this case. We also include a version of the result for the parameter *inst* (Proposition 4.5(b)).

Proposition 4.5. *Let n, $y_0, t_0 \in \mathbb{N}$ and let $t \geq t_0$ be valid for n. Suppose $n - 2t$ divides $t - t_0$.*

(a) *If $winst(n, t_0) \geq y_0$, then $winst(n, t) \geq y_0 + 2\frac{t-t_0}{n-2t}$.*
(b) *If $inst(n, t_0) \geq y_0$, then $inst(n, t) \geq y_0 + 2\frac{t-t_0}{n-2t}$.*

Clearly, Proposition 4.5 can be used in the same way as Lemma 4.3 to improve Theorem 4.1. We will do so for part (a) of Theorem 4.1, which deals with the parameter *winst*. The next lemma takes care of the base case $t = t_0$ for Proposition 4.5(a), for the case $n = 2t + 3$. It also confirms Conjecture 1.2 for $n = 5$ and $t = 1$. The proof of Proposition 4.5 will be presented at the end of this subsection.

Lemma 4.6. *$winst(5, 1) \geq 3$, $winst(7, 2) \geq 4$, and $winst(9, 3) \geq 4$.*

Proof. We start by proving that $winst(5, 1) \geq 3$. Let f be a 2-colouring of $\mathcal{H}_5$. We say a point x in $\mathcal{H}_5$ is *good* (in f) if there is a $j = j(x) \in \{0, 1\}$ so that x has exactly two entries equal to j and $f(x) = j$. Also, we say that two points x and y in $\mathcal{H}_5$ are *neighbours* if $||x - y||^2 = 2$ and they have the same number of entries equal to 1.

First of all, we observe that, if there are two good points x and y that are neighbours, then it is easy to construct a well-ending 2-geodesic that jumps the required number of times (in the same way as in the proof of Lemma 4.4). So we may assume that

$$\text{if } x \text{ and } y \text{ are good in } f\text{, then they are not neighbours.} \tag{5}$$

Second, we may assume that

$$\text{if } x \text{ is good in } f \text{ then its complement is not good in } f. \tag{6}$$

Indeed, if a point x and its complement are good in f, then we may obtain a well-ending 2-geodesic as desired by starting out at x, going to $B_1(j(x)^5)$, then going to $B_1((1-j(x))^5)$, and then ending at the complement of x.

As f is a 2-colouring, there is a point w that is good in f. By symmetry, we can assume that $w = 00011$. Now, because of (5), at most one of the points 11000, 10100, and 01100 is good. So, one of them (in fact, two of them), say 11000, has colour 0 in f. Consider the 2-geodesic

$$(00011[1], 00001[0], 01001[0], 11001[?], 11000[0], 11100[1]).$$

Its third point has colour 0 because of (5), and its last point has colour 1 because of (6). So this well-ending 2-geodesic only jumps less than three times if $f(11001) = 0$.

But in this case, we use (5) to see that $f(11010) = 1$ and $f(10010) = 0$, and consider the well-ending 2-geodesic

$$(00011[1], 00010[0], 10010[0], 11010[1], 11000[0], 11100[1]),$$

that jumps $4 > 3$ times. This concludes the proof that $winst(5, 1) \geq 3$.

The idea for the other cases is similar to the one used in the proof of Lemma 4.3. We reduce the problem to 5 entries, obtaining as above a 'partial' geodesic that jumps at least three times, and extend it so that it jumps at least four times, as needed.

For $winst(7, 2)$, let f be a 3-colouring of $\mathcal{H}_7$. Let w be a point in $\mathcal{H}_7$ with exactly 3 entries equal to j and such that $f(w) = j$. By symmetry, we may assume that the first two entries of w are 01. Define a colouring g of the hypercube $\mathcal{H}_5$ by assigning to each x' in $\mathcal{H}_5$ the value $g(x') = f(01x')$. Then g is a 2-colouring of $\mathcal{H}_5$. Indeed, the point w' in $\mathcal{H}_5$ such that $w = 01w'$ serves as a witness to this.

As $winst(5, 1) \geq 3$, there is a well-ending 2-geodesic $\tilde{P}$ in $\mathcal{H}_5$ that jumps at least three times in g. Extending each point x' of $\tilde{P}$ to the point $01x'$ of $\mathcal{H}_7$, we obtain a path P' in $\mathcal{H}_7$ that jumps at least three times in f. If $\tilde{P}$ jumps exactly three times, then it ends well in both of its ends. Thus we can extend P' in one of its ends, passing by the neighbouring ball, so that it jumps once more, and the result will be a well-ending 3-geodesic as desired. If, on the other hand, $\tilde{P}$ jumps at least four times, then we just extend it in any way so that the resulting 3-geodesic is still well-ending. This completes the proof that $winst(7, 2) \geq 4$. The proof that $winst(9, 3) \geq 4$ is analogous, so we omit it. □

Corollary 4.7. *Let $t \geq 1$. Then $winst(2t+3, t) \geq \frac{2t+(t \bmod 3)}{3} + 2$.*

Proof. We obtain the bound by applying Proposition 4.5 to $n = 2t + 3$ and the base cases obtained from Lemma 4.6: $t_0 = 1$ with $y_0 = 3$, $t_0 = 2$ with $y_0 = 4$, and $t_0 = 3$ with $y_0 = 4$. □

This bound improves by one the bound from Theorem 4.1(a) for $n = 2t + 3$ and $t \bmod 3 = 0$ or 1, and by two for $t \bmod 3 = 2$.

Lemma 4.8. *$inst(7, 2) \geq 5$ and $inst(9, 3) \geq 5$.*

Proof. We first show that $inst(9, 3) \geq 5$. Let f be a 4-colouring of $\mathcal{H}_9$. Let w be a point in $\mathcal{H}_9$ with exactly 4 entries equal to j and such that $f(w) = j$. By symmetry, we may assume that the first two entries of w are 01. Define a colouring g of the hypercube $\mathcal{H}_7$ by assigning to each x' in $\mathcal{H}_7$ the value $g(x') = f(01x')$. Then g is a 3-colouring of $\mathcal{H}_7$. Indeed, the point w' in $\mathcal{H}_7$ with $w = 01w'$ serves as a witness to this.

As $winst(7, 2) \geq 4$ by Lemma 4.6, there is a well-ending 3-geodesic $\tilde{P}$ in $\mathcal{H}_7$ that jumps at least four times in g. Extending each point x' of $\tilde{P}$ to the point $01x'$ of $\mathcal{H}_9$, we obtain a path P' in $\mathcal{H}_9$ that jumps at least four times in f. As $\tilde{P}$ ends well in one of its ends, we can extend $\tilde{P}$ there, entering and exiting the neighbouring ball, so that it jumps at least once more, and the result is a geodesic. This completes the proof that $inst(9, 3) \geq 5$.

To prove that $inst(7, 2) \geq 5$, we use a similar argument and the fact that $winst(5, 1) \geq 3$ by Lemma 4.6. Now, we distinguish three cases, namely whether $\tilde{P}$ jumps exactly 3 times, exactly 4 times, or at least 5 times. In the latter case, we only need to extend $\tilde{P}$ to a geodesic in $\mathcal{H}_7$, and in the case that $\tilde{P}$ jumps exactly 4 times, we extend it as above at its well-ending end. In the first case, when $\tilde{P}$ jumps exactly three times, it has to end well in both its endpoints. Thus we may extend $\tilde{P}$ at both ends to obtain a geodesic in $\mathcal{H}_7$ that jumps five times. □

Corollary 4.9. *Let $t \geq 2$ be such that $t \bmod 3 \neq 1$. Then*

$$inst(2t+3, t) \geq \frac{2t + (t \bmod 3)}{3} + 3.$$

Proof. We obtain the bound by applying Proposition 4.5 to $n = 2t + 3$ and the base cases obtained from Lemma 4.8: $t_0 = 2$ and 3 with $y_0 = 5$. □

This bound improves by one the bound from Theorem 4.1(b) for $n = 2t + 3$ and $t \bmod 3 \neq 1$.

The rest of this section is dedicated to the proof of Proposition 4.5.

Proof of Proposition 4.5. We first show (a). We proceed by induction on $i = i(n,t) := \frac{t-t_0}{n-2t}$. The base, for $i = 0$ (i.e., $t = t_0$), follows directly from the hypothesis of the lemma. For $i > 0$, consider a $(t+1)$-colouring f of the hypercube $\mathcal{H}_n$.

As f is a $(t+1)$-colouring, there is an x in $\mathcal{H}_n$ with exactly $t+1$ entries equal to $f(x)$. As t is valid for n, we know that x has at least t entries equal to $1-f(x)$. So, as $n-2t \le t-t_0 \le t$, we may assume that $x = 0^{n-2t}1^{n-2t}x'$, where $x' \in \mathcal{H}_{n'}$ for $n' := n-2(n-2t)$.

Define a colouring g of the hypercube $\mathcal{H}_{n'}$ by assigning to each x'' in $\mathcal{H}_{n'}$ the value $g(x'') = f(0^{n-2t}1^{n-2t}x'')$. Then g is a $(t'+1)$-colouring of $\mathcal{H}_{n'}$, where $t' := t-(n-2t)$. Indeed, g respects the balls $B_{t'}(0^{n'})$ and $B_{t'}(1^{n'})$ because f respects the balls $B_t(0^n)$ and $B_t(1^n)$, and the point x' has exactly $t-(n-2t)+1 = t'+1$ entries equal to $g(x') = f(x)$.

Note that t' is valid for n' and that $n'-2t' = n-2t$ divides $t'-t_0$. Moreover,

$$i(n',t') = \frac{t-t_0-(n-2t)}{n-2t} = i(n,t)-1.$$

So, we may apply the induction hypothesis to $\mathcal{H}_{n'}$ and g to obtain a well-ending $(t'+1)$-geodesic $\tilde{P}$ in $\mathcal{H}_{n'}$ that jumps at least $y_0 + 2\frac{t-t_0}{n-2t} - 2$ times in g. We suppose that the first point $\tilde{a}$ of $\tilde{P}$ is such that $g(\tilde{a}) = 1$. In other words, we suppose that $\tilde{P}$ ends well in its first point. The other case is analogous.

Extending each point x'' of $\tilde{P}$ to the point $0^{n-2t}1^{n-2t}x''$ of $\mathcal{H}_n$, we obtain a path P' in $\mathcal{H}_n$ that jumps at least $y_0 + 2\frac{t-t_0}{n-2t} - 2$ times in f. Let $z = 0^{n-2t}1^{n-2t}z'$ be the last point of P'. If $f(z) = 0$, then we extend P' to P by adding to its end the points

$$\begin{aligned}
&0^{n-2t-1}1^{n-2t+1}z',\\
&0^{n-2t-1}101^{n-2t-1}z',\\
&0^{n-2t-1}10^21^{n-2t-2}z',\\
&\dots\\
&0^{n-2t-1}10^{n-2t}z',\\
&10^{n-2t-2}10^{n-2t}z',\\
&1^20^{n-2t-3}10^{n-2t}z',\\
&\dots\\
&1^{n-2t}0^{n-2t}z'.
\end{aligned}$$

As we thus pass once through $B_t(1^n)$, and then through $B_t(0^n)$, our geodesic P jumps at least two times more than P'.

On the other hand, if $f(z) = 1$, then we extend P' to P by adding to its end the points

$$\begin{aligned}
&0^{n-2t+1}1^{n-2t-1}z',\\
&0^{n-2t+2}1^{n-2t-2}z',\\
&0^{n-2t+3}1^{n-2t-3}z',\\
&\dots\\
&0^{2n-4t-1}1z',\\
&10^{2n-4t-2}1z',\\
&1^20^{2n-4t-3}1z',\\
&\dots\\
&1^{n-2t}0^{n-2t-1}1z',\\
&1^{n-2t}0^{n-2t}z'.
\end{aligned}$$

We passed once through $B_t(0^n)$, and then through $B_t(1^n)$, thus again our geodesic P has at least two more jumps than P'.

So, in either case, P jumps at least $y_0 + 2\frac{t-t_0}{n-2t}$ times in total. By construction, P is a well-ending $(t+1)$-geodesic, as desired.

For (b), we proceed similarly. For the induction step, we consider the restriction g of f in the hypercube $\mathcal{H}_{n'}$, with $n' := n-2(n-2t)$, which assigns to each x'' in $\mathcal{H}_{n'}$ the value $g(x'') = f(0^{n-2t}1^{n-2t}x'')$. Then $t' := t-(n-2t)$ is valid for n', and the induction hypothesis yields a geodesic $\tilde{P}$ in $\mathcal{H}_{n'}$ that jumps at least $y_0 + 2\frac{t-t_0}{n-2t} - 2$ times in g. As above, we turn $\tilde{P}$ into a path P' in $\mathcal{H}_n$.

Now, if P' ends in a point with less than t entries equal to 0 or less than t entries equal to 1, then it is not difficult to change the beginning and the ending of P' in a way that the obtained path starts and ends in points with at least t 0's and at least t 1's, and still jumps at least as often in f as P' does. So assume P' starts and ends in points with at least t 0's and at least t 1's. Now, depending on the colour of the last point $0^{n-2t}1^{n-2t}z$ on P', we extend P' to a geodesic P by first going to $B_t(0^n)$ and then to $B_t(1^n)$, or first going to $B_t(1^n)$ and then to $B_t(0^n)$, but in either case ending in $1^{n-2t}0^{n-2t}z$. This gives two more jumps, as desired. $\square$

5. Final remarks

We considered the memoryless case of the binary-valued consensus problem, previously studied by Dolev and Rajsbaum [5]. A measure of the minimum instability of consensus functions that are representative is studied. Namely, the value of the parameter $inst(n,t)$ is presented and bounds for its value are derived, in general and for specific cases. The problem of determining the precise value of the parameter $inst(n,t)$ for arbitrary values of n and t seems quite challenging.

A conjecture that $inst(n,t) = 2t + 1$ for every (valid) t is presented and a few results that point towards the conjecture are proved. As there are examples that show that $inst(n,t) \le 2t + 1$, good lower bounds on $inst(n,t)$ are of interest. Some of the results we presented allow to derive better lower bounds for arbitrary t from better lower bounds for small values of t. Stronger versions of these results, as well as improvements on the lower bounds for small values of t, would help to close the gap between the best lower bound and the conjectured value.

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