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Contributions to local and nonlocal elliptic differential equations

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Abstract

This thesis is divided into four parts. The first part is devoted to study radial symmetry and monotonicity properties of positive solutions to fractional elliptic equations in the unit ball or in the whole space, by using the method of moving planes. In the second part, we consider the decay and symmetry properties of positive solutions for mixed integro-differential equations in the whole space. In studying the decay, we construct appropriate super and sub-solutions and then we use the method of moving planes to prove symmetry results. The third part is to investigate existence and uniqueness of weak solutions to fractional heat equations involving Radon measures. Moreover, we analyze the asymptotic behavior of the weak solution when Radon measure is the Dirac mass. In the fourth part, we study the existence of solutions to nonlinear elliptic equation which arises from Micro-Electromechanical Systems devices in the case that the elastic membranae contacts the ground plate on the boundary. We show how the boundary decay works on the existence of solutions and pull-in voltage.

Key words: Fractional Laplacian, Decay, Symmetry, Hopf's Lemma, Moving Planes, Radon measure, Dirac mass, Self-similar solution, Micro-Electromechanical Systems, Pull-in voltage, Minimal solution.

Contribuciones para ecuaciones diferenciales elípticas locales y no locales

Resumen

Esta tesis doctoral está dividida en cuatro partes. La primera parte está dedicada al estudio de la simetría radial y las propiedades de monotonicidad de soluciones positivas de ecuaciones elípticas fraccionarias en la bola unitaria o en todo el espacio, usando el método de planos móviles. En la segunda parte, se consideran propiedades de decaimiento y simetría de las soluciones positivas para ecuaciones integro-diferenciales en todo el espacio. Estudiamos el decaimiento, construyendo super y subsoluciones apropiadas y usamos el método de los planos móviles para probar las propiedades de simetría. La tercera parte es investigar la existencia y unicidad de soluciones débiles de la ecuación del calor fraccionaria, involucrando medidas de Radon. Más aún, analizamos el comportamiento asintótico de la solución débil cuando la medida de Radon es la masa de Dirac. En la cuarta parte, estudiamos la existencia de soluciones a problemas elípticos no lineales que provienen del modelamiento de dispositivos de sistemas micro-electromecánicos en el caso en que la membrana elástica entra en contacto con la placa inferior en la frontera. Mostramos, en este caso, como el decaimiento de la membrana afecta la existencia de soluciones y la tensión pull-in.

Palabras Claves: Laplaciano Fraccional, Decaimiento, Simetría, Lema de Hopf, Planos Móviles, Medida de Radon, Masa de Dirac, Solución auto-similar, Sistemas Micro-Electromecánicos, Tensión Pull-in, Solución Minimales.

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Introduction

This thesis is to study qualitative properties of positive solutions to semilinear elliptic equations involving fractional Laplacian, the weak solutions to fractional heat equations with initial measure data, and the solutions of the second order elliptic equations arising from Micro-Electromechanical Systems (MEMS).

0.1 Radial symmetry of positive solutions to equations involving the fractional Laplacian

In recent years, the study of nonlinear elliptic equations involving general integro-differential operators, especially, the fractional Laplacian, have attracted the attention of the mathematical community by great applications in physics and by important links on the theory of Lévy processes, refer to [17, 23, 24, 37, 41, 45] and reference therein. In Chapter 1, we consider symmetry and monotonicity properties of positive solutions for equations involving the fractional Laplacian. We first study the following fractional elliptic problem

$$\begin{cases} (-\Delta)^\alpha u = f(u) + g & \text{in} \quad B_1, \\ \quad u = 0 & \text{in} \quad B_1^c, \end{cases} \tag{1}$$

where B_1 denotes the open unit ball centered at the origin in $\mathbb{R}^N$ and $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|x-z|^{N+2\alpha}} dz, \tag{2}$$

$x \in B_1$. Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows.

The study of radial symmetry and monotonicity of positive solutions for nonlinear elliptic equations in bounded domains using the moving planes method based on the maximum principle was initiated with the work by Serrin [83] and Gidas, Ni and Nirenberg [50], with important subsequent advances by Berestycki and

Nirenberg [8]. We refer to the review by Pacella and Ramaswamy [77] for a more complete discussion of the method and it various applications. In [8] the maximum principle for small domain, based on the Aleksandrov-Bakelman-Pucci (ABP) estimate, was used as a tool to obtain much general results, specially avoiding regularity hypothesis on the domain. In a recent work Guillen and Schwab, [54], provided an ABP estimate for a class of fully non-linear elliptic integro-differential equations. Motivated by this work, we obtain a version of the maximum principle for small domains and we apply it with the moving planes method as in [8] to prove symmetry and monotonicity properties for positive solutions to fractional elliptic problem (1).

We consider the following hypotheses on the functions f and g:

$(F1)$ The function $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz.

(G) The function $g : B_1 \to \mathbb{R}$ is radially symmetric and decreasing in $|x|$.

Before stating our first theorem we make precise the notion of solution. We say that a continuous function $u : \mathbb{R}^N \to \mathbb{R}$ is a classical solution of equation (1) if the fractional Laplacian of u is defined at any point of B_1, according to the definition given in (2), and if u satisfies the equation and the external condition in a pointwise sense. Now we are ready for our first theorem on radial symmetry and monotonicity properties for positive solutions of (1). It states as follows:

Theorem 0.1.1 *Assume that the functions f and g satisfy $(F1)$ and (G), respectively. If u is a positive classical solution of (1), then u must be radially symmetric and strictly decreasing in $r = |x|$, for $r \in (0, 1)$.*

We devote the second part of Chapter 1 to study symmetry results for a nonlinear equation as (1), but in $\mathbb{R}^N$ and with $g \equiv 0$. For the problem in $\mathbb{R}^N$, the moving planes procedure has to start a different way because we cannot use the maximum principle for small domain. We refer to the work by Gidas, Ni and Nirenberg [51], Berestycki and Nirenberg [8], Li [62], and Li and Ni [63], where these results were studied assuming some additional hypothesis on f, allowing for decay properties of the solution u. A general result in this direction was obtained by Li [62] for the equation

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N$$

with u decaying at infinity and f satisfying the following hypothesis:

$(F2)$ there exists $s_0 > 0$, $\gamma > 0$ and $C > 0$ such that

$$\frac{f(v) - f(u)}{v - u} \leq C(u + v)^\gamma \quad \text{for all} \quad 0 < u < v < s_0. \tag{3}$$

Motivated by these results, we are interested in similar properties of positive solutions for equations involving the fractional Laplacian under assumption (F2). Here is our second main theorem.

Theorem 0.1.2 *Assume that the function f satisfies $(F1)-(F2)$ and u is a positive classical solution for the equation*

$$\begin{cases} (-\Delta)^\alpha u = f(u) \quad in \;\; \mathbb{R}^N, \\ u > 0 \;\; in \;\; \mathbb{R}^N, \;\; \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{4}$$

Assume further that there exists

$$m > \max\{\frac{2\alpha}{\gamma}, \frac{N}{\gamma+2}\} \tag{5}$$

such that u satisfies

$$u(x) = O(\frac{1}{|x|^m}), \quad as \quad |x| \to \infty, \tag{6}$$

then u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^N$.

Felmer, Quaas and Tan in [45] studied the problem (4) with $f(u) = h(u) - u$, that is,

$$\begin{cases} (-\Delta)^\alpha u + u = h(u) \quad \text{in} \quad \mathbb{R}^N, \\ u > 0 \;\; \text{in} \;\; \mathbb{R}^N, \qquad \lim_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{7}$$

They proved existence and regularity of positive solutions, and also decay and symmetry results. Precisely, it was proved that the solutions u of (7) satisfy

$$\frac{c^{-1}}{|x|^{N+2\alpha}} \le u(x) \le \frac{c}{|x|^{N+2\alpha}}, \qquad |x| \ge 1, \tag{8}$$

for some $c > 1$, when h is superlinear at 0 in the sense that

$$\lim_{s\to 0} \frac{h(s)}{s} = 0.$$

Moreover, the radial symmetry of the solutions of (7) is derived by using the moving planes method in integral form developed in [30, 64], assuming further that $h \in C^1(\mathbb{R})$, h is increasing and there exists $\tau > 0$ such that

$$\lim_{s\to 0} \frac{h'(s)}{s^\tau} = 0. \tag{9}$$

We see that Theorem 0.1.2 generalizes the symmetry result in [45], since here we do not assume f is differentiable and we do not require f to be increasing.

The third part of Chapter 1 is devoted to study the radial symmetry of non-negative solutions for the following system of nonlinear equations involving fractional Laplacians with different orders

$$\begin{cases} (-\Delta)^{\alpha_1} u = f_1(v) + g_1 & \text{in} \quad B_1, \\ (-\Delta)^{\alpha_2} v = f_2(u) + g_2 & \text{in} \quad B_1, \\ u = v = 0 & \text{in} \quad B_1^c, \end{cases} \tag{10}$$

where $\alpha_1, \alpha_2 \in (0,1)$. We have following results for system (10):

Theorem 0.1.3 *Suppose that f_1 and f_2 are locally Lipschitz continuous and increasing functions defined in $[0,\infty)$ and g_1 and g_2 satisfy (G). Assume that (u,v) are positive, classical solutions of system (10), then u and v are radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0,1)$.*

We prove the above theorems using the moving planes method in [50]. The main difficulty here comes from the fact that the fractional Laplacian is a non-local operator, and consequently maximum principle and comparison results require information on the solutions in the whole complement of the domain, not only at the boundary. To overcome this difficulty, we introduce a new truncation technique which is well adapted to be used with the method of moving planes.

0.2 Qualitative properties of positive solutions for mixed integro-differential equations

In Chapter 2, we are concerned with the decay and symmetry results of solutions to mixed integro-differential equations

$$\begin{cases} (-\Delta)_x^\alpha u + (-\Delta)_y u + u = f(u), \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M, \\ u > 0 \ \text{ in } \mathbb{R}^N \times \mathbb{R}^M, \qquad \lim_{|(x,y)| \to +\infty} u(x,y) = 0, \end{cases} \tag{11}$$

where $N \geq 1$, $M \geq 1$, the operator $(-\Delta)_y$ denotes the usual Laplacian with respect to y, while $(-\Delta)_x^\alpha$ denotes the fractional Laplacian of exponent $\alpha \in (0,1)$ with respect to x, i.e.

$$(-\Delta)_x^\alpha u(x,y) = \int_{\mathbb{R}^N} \frac{u(x,y) - u(z,y)}{|x-z|^{N+2\alpha}} dz, \tag{12}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$.

When $N = 0$, equation (11) becomes the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + u = f(u) & \text{in} \quad \mathbb{R}^M, \\ u > 0 \ \text{ in } \ \mathbb{R}^M, & \lim_{|y|\to+\infty} u(y) = 0. \end{cases} \tag{13}$$

It was the seminal work by Gidas, Ni and Nirenberg [51] that settled the decay and symmetry properties for (13) when the non-linearity is merely Lipschitz continuous and superlinear at the zero in the sense that

$$f(s) = O(s^p) \quad \text{as } s \to 0, \tag{14}$$

for some $p > 1$ and $M \geq 3$. They proved that the solutions of (13) are radially symmetric and they satisfy the precise decay estimate

$$\lim_{|y|\to+\infty} u(y)e^{|y|}|y|^{\frac{M-1}{2}} = c, \tag{15}$$

for certain constant $c > 0$. On the other hand, when $M = 0$, equation (11) reduces to the fractional nonlinear Schrödinger equation (7) which has been studied by Felmer, Quaas and Tan [45], especially the decay as (8) and radial symmetry results.

Both operators, the Laplacian and the fractional Laplacian, are particular cases of a general class of elliptic operators connected to backward stochastic differential equations associated to Brownian and Levy-Itô processes, see for example Barles, Buckdahn and Pardoux [2], Benth, Karlsen and Reikvam [6] and Pham [79]. Recently, Barles, Chasseigne, Ciomaga and Imbert in [3, 4] and Ciomaga in [38] considered the existence and regularity of solutions for equations involving mixed integro-differential operators belonging to the general class of backward stochastic differential equations mentioned above. A particular case of elliptic integro-differential operator of mixed type is the one considering the laplacian in some of the variables and the fractional laplacian in the others, modeling diffusion sensible to the direction, such as the operator in equation (11).

In view of the known results on decay and symmetry for solutions of equations (13) and (7) just described above, it is interesting to ask if these results still hold for solutions of the equation of mixed type (11), where the elliptic operator represents diffusion depending on the direction in space. Regarding the asymptotic decay of solution at infinity, the question is interesting since a proper mix of the two variables should be obtained for the decay estimates. The natural way to estimate the decay is through the construction of super and sub-solutions involving the fundamental solution of the elliptic operator. Moreover, the solution of (11) cannot be radially symmetric, so this property cannot be used to estimate the decay. On the other hand, regarding symmetry results, we may still have symmetry in x and y, but the moving planes method would require an adequate version of the Hopf's

Lemma, that we prove here.

Our first theorem in Chapter 2 concerns the decay of solutions for (11) with general nonlinearity and it states as follows.

Theorem 0.2.1 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 1$ and the non-linearity $f : (0,+\infty) \to \mathbb{R}$ is a continuous function satisfying*

$$-\infty < B := \liminf_{v \to 0^+} \frac{f(v)}{v} \leq A := \limsup_{v \to 0^+} \frac{f(v)}{v} < 1. \tag{16}$$

Let u be a positive classical solution of problem (11), then for any $\epsilon > 0$ small, there exists $C_\epsilon > 1$ such that for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,

$$C_\epsilon^{-1}(1+|x|)^{-N-2\alpha} e^{-\theta_2|y|} \leq u(x,y) \leq C_\epsilon (1+|x|)^{-N-2\alpha} e^{-\theta_1|y|}, \tag{17}$$

where

$$\theta_1 = \sqrt{1-A} - \epsilon \quad \text{and} \quad \theta_2 = \sqrt{1-B} + \epsilon. \tag{18}$$

When we compare estimate (17) with (15) for $N = 0$, we first observe that in ours an exponential decay is obtained, but with a constant C_ϵ depending on ϵ, which is a parameter controlling the rate of exponential decay. This is more clear when $A = B = 0$. On the other hand we are making much more general assumptions on f, in particular, we are not making any assumption on the radial symmetry of the solution, which is crucial in proving (15). We do not know of a decay estimate better than

$$C_\epsilon^{-1} e^{-\theta_2|y|} \leq u(y) \leq C_\epsilon e^{-\theta_1|y|}, \qquad y \in \mathbb{R}^M, \tag{19}$$

for solutions of (13) under assumption (16) for f, and where radial symmetry of the solutions is not available, like in a case where f may depend on y. On the other hand, when $M = 0$, we recover (8) from (17). For the proof of the decay estimate (17) we construct suitable super and sub-solutions and we use comparison principle with a version of Hopf's lemma.

When we assume further hypothesis we can get sharper estimates for the decay of the solutions of equation (11). Precisely, we have the following result:

Theorem 0.2.2 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 5$ and the non-linearity $f : (0,+\infty) \to \mathbb{R}$ is a non-negative function satisfying (14). Let u be a positive classical solution of (11), then there exists a constant $c > 1$ such that for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,*

$$\frac{1}{c}\rho(x,y) \leq u(x,y) \leq c\rho(x,y)(1+|y|)^{\frac{1}{2}}, \tag{20}$$

where the function ρ is defined as

$$\rho(x,y) = \min\{\frac{1}{(1+|x|)^{N+2\alpha}}\,,\, e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}\,,\, \frac{e^{-|y|}|y|^{1-\frac{M}{2}}}{(1+|x|)^{N+2\alpha}}\}. \tag{21}$$

We notice that this theorem gives the expected exponential decay for positive solutions, as suggested by (15), assuming the dimension of the space satisfies $M \geq 5$. Moreover, it gives the expected polynomial correction for the lower bound with a gap in the power for the upper bound. This theorem is proved under the assumption (14) on the non-linearity, constructing super and sub solutions devised upon the fundamental solution of $(-\Delta)_x^\alpha + (-\Delta)_y + id$. In our argument, a crucial role is played by the estimate already obtained in Theorem 0.2.1. Since the fundamental solution of $(-\Delta)_x^\alpha + (-\Delta)_y + id$ has $\mathbb{R}^N \times \{0\}$ as singular set, we cannot use the method in [51] in order to derive our estimate. Moreover, some other arguments in [51] cannot be used either because the solutions of (11) are not radial, since the differential operator is not radially invariant and there are no solutions depending only on one of the x or y variables, as can be seen from (20),

Even though solutions of (11) are not radially symmetric, we can prove partial symmetry in each of the variables x and y and this is the content of our third theorem.

Theorem 0.2.3 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 1$ and the function $f : (0,+\infty) \to \mathbb{R}$ is locally Lipschitz and satisfies (16). We more assume that f satisfies*

(F) there exist $u_0 > 0$, $\gamma > \frac{N}{N+M} \cdot \frac{2\alpha}{N+2\alpha}$ and $\bar{c} > 0$ such that

$$\frac{f(v)-f(u)}{v-u} \leq \bar{c}v^\gamma \quad \text{for all} \quad 0 < u < v < u_0. \tag{22}$$

Then every positive classical solution u of equation (11) satisfies

$$u(x,y) = u(r,s)$$

and $u(r,s)$ is strictly decreasing in r and s, where $r = |x|$ and $s = |y|$.

When $N = 0$, we see that assumption (F) implies $\gamma > 0$ and (22) coincides with the assumption considered in [62]. When $M = 0$, assumption (F) implies that $\gamma > \frac{2\alpha}{N+2\alpha}$ and it coincides with the assumption considered in [46], when the solutions is assumed to decay as a power $N + 2\alpha$ at infinity. We remark that the operator $(-\Delta)_x^\alpha + (-\Delta)_y$ is a combination of two operators with different differential orders in $x-$variable and $y-$variable, and this produced a combined polynomial-exponential decay and does not allow for radial symmetry, but only partial symmetry as stated in Theorem 0.2.3.

The proof of Theorem 0.2.3 is based on the moving planes method as developed in [46, 62]. In these arguments, the strong maximum principle plays a crucial role and it is available for the Laplacian and for the fractional Laplacian. However, in the case of our mixed integro-differential operator some difficulties arise and we overcome them with a version of the Hopf's Lemma.

0.3 Fractional heat equations with subcritical absorption with initial data measure

Chapter 3 is devoted to study weak solutions to fractional heat equations

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t,u) = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{23}$$

where $\alpha \in (0,1)$, $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $Q_\infty = (0,\infty) \times \mathbb{R}^N$ with $N \geq 2$, ν belongs to the space $\mathfrak{M}^b(\mathbb{R}^N)$ of bounded Radon measures in $\mathbb{R}^N$.

In a pioneering work, Brezis and Friedman [12] have studied semilinear the heat equation with measure as initial data

$$\begin{aligned} \partial_t u - \Delta u + u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{24}$$

where $k > 0$ and δ_0 is the Dirac mass at the origin. They proved that if $1 < p < (N+2)/N$, then for every $k > 0$ there exists a unique solution u_k to (24). When $p \geq (N+2)/N$, problem (24) has no solution and even more, they proved that no nontrivial solution of the above equation vanishing on $\mathbb{R}^N \setminus \{0\}$ at $t = 0$ exists. When $1 < p < 1 + \frac{2}{N}$, Brezis, Peletier and Terman used a dynamical system technique in [13] to prove the existence of a *very singular solution* u_s to

$$\partial_t u - \Delta u + u^p = 0 \quad \text{in} \quad Q_\infty, \tag{25}$$

vanishing at $t = 0$ on $\mathbb{R}^N \setminus \{0\}$. This function u_s is self-similar, i.e. expressed under the form

$$u_s(t,x) = t^{-\frac{1}{p-1}} f\left(\frac{|x|}{\sqrt{t}}\right), \tag{26}$$

and f is uniquely determined by the following conditions

$$\begin{aligned} &f'' + \left(\tfrac{N-1}{\eta} + \tfrac{1}{2}\eta\right) f' + \tfrac{1}{p-1} f - f^p = 0 \quad \text{on} \quad \mathbb{R}_+ \\ &f > 0 \quad \text{and} \quad f \text{ is smooth on } \mathbb{R}_+ \\ &f'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \eta^{\frac{2}{p-1}} f(\eta) = 0. \end{aligned} \tag{27}$$

Furthermore, it satisfies

$$f(\eta) = c_1 e^{-\eta^2} \eta^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\} \quad \text{as } \eta \to \infty$$

for some $c_1 > 0$. Later on, Kamin and Peletier in [58] proved that the sequence of weak solutions u_k converges to the very singular solution u_s as $k \to \infty$. After that, Marcus and Véron in [70] studied the equation in the framework of the *initial trace* theory. They pointed out the role of the very singular solution of (25) in the study of the singular set of the initial trace, showing in particular that it is the unique positive solution of (25) satisfying

$$\lim_{t\to 0} \int_{B_\epsilon} u(t,x)dx = \infty \qquad \forall \epsilon > 0,\ B_\epsilon = B_\epsilon(0), \tag{28}$$

and

$$\lim_{t\to 0} \int_{K} u(t,x)dx = 0 \qquad \forall K \subset \mathbb{R}^N \setminus \{0\},\ K \text{ compact}. \tag{29}$$

If one replaces u^p by $t^\beta u^p$ with $p \in (1, 1 + \frac{2(1+\beta)}{N})$, these results were extended by Marcus and Véron ($\beta \geq 0$) in [70] and then Al Sayed and Véron ($\beta > -1$) in [82]. The initial data problem with measure and general absorption term

$$\begin{aligned} \partial_t u - \Delta u + h(t,x,u) = 0 \quad &\text{in} \quad (0,\mathrm{T}) \times \Omega, \\ u = 0 \quad &\text{in} \quad (0,T) \times \partial\Omega, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \Omega, \end{aligned} \tag{30}$$

in a bounded domain Ω is a domain in $\mathbb{R}^N$, has been studied by Marcus and Véron in [70] in the framework of the initial trace theory. They proved that the following general integrability condition on h

$$\begin{aligned} &0 \leq | h(t,x,r) | \leq \tilde{h}(t) f(|r|) \qquad \forall (x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \\ &\textstyle\int_0^T \tilde{h}(t) f(\sigma t^{\frac{N}{2}}) t^{-\frac{N}{2}} dt < \infty \qquad \forall \sigma > 0 \\ &\text{either } \tilde{h}(t) = t^\alpha \text{ with } \alpha \geq 0 \text{ or } f \text{ is convex}, \end{aligned} \tag{31}$$

in order the problem has a unique solution for any bounded measure. In the particular case with $h(t,x,r) = t^\beta |u|^{p-1} u$, is fulfilled if $1 < p < 1 + \frac{2(1+\beta)}{N}$ and $\beta > -1$, and the very singular solution exists in this range of values.

Motivated by a growing number of applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, (see e.g. [20, 21, 26, 27, 31, 37, 44, 46]). Recently, in [32] we obtained the existence and uniqueness of weak solution to

semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + f(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{32}$$

when ν is Radon measure and f satisfies a subcritical integrability condition.

One purpose of Chapter 3 is to study the existence and uniqueness of weak solutions to semilinear fractional heat equation (23) in a measure framework. We first make precise the notion of weak solution of (23) that we will use in this chapter.

Definition 0.3.1 *We say that u is a weak solution of (23), if for any $T > 0$, $u \in L^1(Q_T)$, $h(t,u) \in L^1(Q_T)$ and*

$$\begin{aligned} &\int_{Q_T} (u(t,x)[-\partial_t \xi(t,x) + (-\Delta)^\alpha \xi(t,x)] + h(t,u)\xi(t,x))\, dxdt \\ &= \int_{\mathbb{R}^N} \xi(0,x) d\nu - \int \mathbb{R}^N \xi(T,x) u(T,x) dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}, \end{aligned} \tag{33}$$

where $Q_T = (0,T) \times \mathbb{R}^N$ and $\mathbb{Y}_{\alpha,T}$ is a space of functions $\xi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying

(i) $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t \xi\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \xi\|_{L^\infty(Q_T)} < +\infty$;

(ii) for $t \in (0,T)$, there exist $M > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, $\|(-\Delta)^\alpha_\epsilon \xi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le M$.

Before stating our main theorems, we introduce the subcritical integrability condition for the nonlinearity h, that is,

(H) *(i)* The function $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and for any $t \in (0,\infty)$, $h(t,0) = 0$ and $h(t,r_1) \ge h(t,r_2)$ if $r_1 \ge r_2$.

(ii) There exist $\beta > -1$ and a continuous, nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t,r)| \le t^\beta g(|r|) \qquad \forall (t,r) \in (0,\infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s) s^{-1-p^*_\beta} ds < +\infty, \tag{34}$$

where

$$p^*_\beta = 1 + \frac{2\alpha(1+\beta)}{N}. \tag{35}$$

We denote by $H_\alpha : (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}_+$ the heat kernel for $(-\Delta)^\alpha$ in $(0,\infty)\times\mathbb{R}^N$, by $\mathbb{H}_\alpha[\nu]$ the associated heat potential of $\nu\in\mathfrak{M}^b(\mathbb{R}^N)$, defined by

$$\mathbb{H}_\alpha[\nu](t,x) = \int_{\mathbb{R}^N} H_\alpha(t,x,y)d\nu(y)$$

and by $\mathcal{H}_\alpha[\mu]$ the Duhamel operator defined for $(t,x)\in Q_T$ and any $\mu\in L^1(Q_T)$ by

$$\mathcal{H}_\alpha[\mu](t,x) = \int_0^t \mathbb{H}_\alpha[\mu(s,.)](t-s,x)ds = \int_0^t\int_{\mathbb{R}^N} H_\alpha(t-s,x,y)\mu(s,y)dyds.$$

Now we state our first theorem as follows.

Theorem 0.3.1 *Assume that $\nu\in\mathfrak{M}^b(\mathbb{R}^N)$ and the function h satisfies (H). Then problem (23) admits a unique weak solution u_ν such that*

$$\mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,\mathbb{H}_\alpha[\nu_+])] \le u_\nu \le \mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,-\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \tag{36}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν. Furthermore,

(i) if ν is nonnegative, so is u_ν;

(ii) the mapping: $\nu\mapsto u_\nu$ is increasing and stable in the sense that if $\{\nu_n\}$ is a sequence of positive bounded Radon measures converging to ν in the weak sense of measures, then $\{u_{\nu_n}\}$ converges to u_ν locally uniformly in Q_∞.

According to Theorem 0.3.1, there exists a unique positive weak solution u_k to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{37}$$

where $\beta>-1$, $k>0$ and $p\in(0,p^*_\beta)$. We observe that $u_k\to\infty$ in $(0,\infty)\times\mathbb{R}^N$ as $k\to\infty$ for $p\in(0,1]$, see Proposition 3.4.2 for details. Our next interest of Chapter 3 is to study the limit of u_k as $k\to\infty$ for $p\in(1,p^*_\beta)$, which exists since $\{u_k\}_k$ are an increasing sequence of functions, bounded by $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$, and we set

$$u_\infty = \lim_{k\to\infty} u_k \quad \text{in} \quad Q_\infty. \tag{38}$$

Actually, u_∞ and $\{u_k\}_k$ are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \tag{39}$$

see Proposition 3.4.3 for details.

Definition 0.3.2 *(i) A solution u of (39) is called a self-similar solution if*

$$u(t,x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x) \qquad (t,x) \in Q_\infty.$$

(ii) A solution u of (39) is called a very singular solution if it vanishes on $\mathbb{R}^N \setminus \{0\}$ at $t = 0$ and

$$\lim_{t \to 0^+} \frac{u(t,0)}{\Gamma_\alpha(t,0)} = +\infty,$$

where $\Gamma_\alpha := \mathbb{H}_\alpha[\delta_0]$ is the fundamental solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \delta_0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{40}$$

We remark that for $p \in (1, p^*_\beta)$, a self-similar solution u of (39) is also a very singular solution, since

$$\lim_{t \to 0^+} \Gamma_\alpha(t,0) t^{\frac{N}{2\alpha}} = c_2, \tag{41}$$

for some $c_2 > 0$. For any self-similar solution u of (39), $v(\eta) := u(1, t^{-\frac{1}{2\alpha}} x)$ with $\eta = t^{-\frac{1}{2\alpha}} x$ is a solution of *the self-similar equation*

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \eta - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{42}$$

Since $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}$ is a constant nonzero solution of (42), the function

$$U_p(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}} \qquad t > 0 \tag{43}$$

is a flat self-similar solution of (39). It is actually the maximal solution of the ODE $y' + t^\beta y^p = 0$ defined on $\mathbb{R}_+$. Our next goal in Chapter 3 is to study non-flat self-similar solutions of (39).

Theorem 0.3.2 *Assume that $\beta > -1$, u_∞ is defined by (38) and*

$$p^{**}_\beta < p < p^*_\beta,$$

where $p^{**}_\beta = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$. Then u_∞ is a very singular self-similar solution of (39) in Q_∞. Moreover, there exists $c_3 > 1$ such that

$$\frac{c_3^{-1}}{1+|x|^{N+2\alpha}} \le u_\infty(1,x) \le \frac{c_3 \ln(2+|x|)}{1+|x|^{N+2\alpha}} \qquad x \in \mathbb{R}^N. \tag{44}$$

When $p^{**}_\beta < p < p^*_\beta$ with $\beta > -1$, we observe that u_∞ and U_p are self-similar solutions of (39) and u_∞ is non-flat. Now we are ready to consider the uniqueness

of non-flat self-similar solution of (39) with decay at infinity, precisely, we study the uniqueness of self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \\ \lim_{|x|\to\infty} u(1,x) = 0. \end{aligned} \tag{45}$$

We remark if u is self-similar, then the assumption $\lim_{|x|\to\infty} u(1,x) = 0$ is equivalent to $\lim_{|x|\to\infty} u(t,x) = 0$ for any $t > 0$. Finally, we state the properties of u_∞ when $1 < p \le p_\beta^{**}$ as follows.

Theorem 0.3.3 *(i) Assume $1 < p < p_\beta^{**}$ and u_∞ is defined by (38). Then $u_\infty = U_p$, where U_p is given by (43).*

(ii) Assume $p = p_\beta^{**}$ and u_∞ is defined by (38). Then u_∞ is a self-similar solution of (39) such that

$$u_\infty(t,x) \ge \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} \qquad (t,x) \in (0,1) \times \mathbb{R}^N, \tag{46}$$

for some $c_4 > 0$.

We note that Theorem 0.3.3 indicates that there is no self-similar solution of (39) with initial data $u(0,\cdot) = 0$ in $\mathbb{R}^N \setminus \{0\}$, since u_∞ is the least self-similar solution. In Theorem 0.3.3 part (ii), we do not know if the self-similar solution is flat or not. From the above theorems, we have the following result.

Theorem 0.3.4 *(i) Assume $p_\beta^{**} < p < p_\beta^*$. Then problem (42) admits a minimal positive solution v_∞ satisfying*

$$\lim_{|\eta|\to\infty} |\eta|^{\frac{2\alpha(1+\beta)}{p-1}} v_\infty(\eta) = 0. \tag{47}$$

Furthermore,

$$\frac{c_3^{-1}}{1 + |\eta|^{N+2\alpha}} \le v_\infty(\eta) \le \frac{c_3 \ln(2 + |\eta|)}{1 + |\eta|^{N+2\alpha}} \qquad \forall \eta \in \mathbb{R}^N \tag{48}$$

(ii) Assume $1 < p < p_\beta^{**}$. Then problem (42) admits no positive solution satisfying (47).

The question of uniqueness of the very singular solution in the case $p_\beta^{**} < p < p_\beta^*$ remains an open problem.

0.4 On semi-linear elliptic equation arising from Micro-Electromechanical Systems with contacting elastic membrane

In Chapter 4, we are concerned with the existence of solutions to the nonlinear elliptic problem

$$\begin{cases} -\Delta u = \frac{\lambda}{(a-u)^2} & \text{in} \quad \Omega, \\ 0 < u < a & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{49}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$ and the function $a : \bar{\Omega} \to [0, 1]$ satisfies $a(x) \geq \kappa \text{dist}(x, \partial\Omega)^\gamma$ for some $\kappa > 0$ and $\gamma \in (0, 1)$. This equation arises from Micro-Electromechanical Systems devices in the case that the elastic membranae contacts the ground plate on the boundary.

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. They are successfully utilized in components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches, chemical sensors, etc. In MEMS devices, a key component is called the electrostatic actuation, which is based on an electrostatic-controlled tunable, it is a simple idealized electrostatic device. The upper part of this electrostatic device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage λ is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when λ is increased beyond a certain critical value λ^*−known as pull-in voltage−the steady state of the elastic membrane is lost, and proceeds to touchdown or snap through at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless deflection of the membrane, was derived and analyzed in [49, 55, 56, 57, 65, 79, 95] and reference therein. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless deflection u of the membrane on a bounded domain Ω in $\mathbb{R}^2$ is found to satisfy the equation

$$-\Delta u = \frac{\lambda}{(1-u)^2} \qquad \text{in} \quad \Omega, \tag{50}$$

with Dirichlet boundary conditions. Here the term on the right hand side of equation (50) is the Coulomb force. Later on, Ghoussoub and Guo in [49, 55] studied

the nonlinear elliptic problem

$$-\Delta u = \frac{\lambda f(x)}{(1-u)^2} \qquad \text{in} \quad \Omega \tag{51}$$

with the Dirichlet boundary condition, which models a simple electrostatic MEMS device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at 1. Here Ω is a bounded domain of $\mathbb{R}^N$ and the function $f \geq 0$ represents the permittivity profile and $\lambda > 0$ is a constant which is increasing with respect to the applied voltage. It is known that for any given suitable f, there exists a critical value λ^* (pull-in voltage) such that if $\lambda \in (0, \lambda^*)$, problem (51) is solvable, while for $\lambda > \lambda^*$, no solution for (51) exists.

In an effort to achieve better MEMS design, the membrane can be technologically fabricated into non-flat shape like the surface of a semi-ball, which contacts the ground plate along the boundary. In Chapter 4, we study how the shape of the membranes effects on the pull-in voltage. In what follows, we assume that Ω is a C^2 bounded domain in $\mathbb{R}^N$, with $N \geq 1$, the function $a : \bar{\Omega} \to [0, 1]$ is in the class of $C^\gamma(\Omega) \cap C(\bar{\Omega})$ and satisfies

$$a(x) \geq \kappa \rho(x)^\gamma, \quad \forall\, x \in \Omega \tag{52}$$

for some $\kappa > 0$ and $\gamma \in (0, 1)$, where $\rho(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$. Our purpose of Chapter 4 is to consider the solutions to semilinear elliptic equation

$$\begin{cases} -\Delta u = \frac{\lambda}{(a-u)^2} & \text{in} \quad \Omega, \\ 0 < u < a & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{53}$$

where parameter $\lambda > 0$ characterizes the relative strength of the electrostatic and mechanical forces in the system. Equation (53) models a closed MEMS device, where the elastic membrane contacts the ground plate on the boundary. The function a is initially state of the elastic membrane. The solution u of (53) shows the steady state of deformation for the membrane when we applied voltage to this device. To this problem, we have the following existence results.

Theorem 0.4.1 *Assume that $a \in C^\gamma(\Omega) \cap C(\bar{\Omega})$ satisfies (52) with $\gamma \in (0, \frac{2}{3}]$ and $\kappa > 0$, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\kappa, \gamma) > 0$ such that*

(i) for $\lambda \in (0, \lambda^*)$, problem (53) admits a minimal solution u_λ and the mapping: $\lambda \mapsto u_\lambda$ is increasing;

(ii) for $\lambda > \lambda^*$, there is no solution for (53);

(iii) moreover, if there exists $c_0 \geq \kappa$ such that

$$a(x) \leq c_0 \rho(x)^\gamma, \qquad x \in \Omega, \tag{54}$$

then there exists $\lambda_* = \lambda_*(\kappa, \gamma) \in (0, \lambda^*]$ *such that for* $\lambda \in (0, \lambda_*)$, $u_\lambda \in H_0^1(\Omega)$ *and*

$$\text{for} \quad \gamma \neq \frac{1}{2}, \qquad \frac{1}{c}\rho(x)^{\min\{1,2-2\gamma\}} \leq u_\lambda(x) \leq c\rho(x)^{\min\{1,2-2\gamma\}}, \qquad \forall x \in \Omega$$

$$\text{for} \quad \gamma = \frac{1}{2}, \qquad \frac{1}{c}\rho(x)\ln\frac{1}{\rho(x)} \leq u_\lambda(x) \leq c\rho(x)\ln\frac{1}{\rho(x)}, \qquad \forall x \in A_{\frac{1}{2}},$$

where $c \geq 1$ *and* $A_{\frac{1}{2}} = \{x \in \Omega : \ \rho(x) < \frac{1}{2}\}$.

We remark that the membrane contacts the ground plate on the boundary with decay rate ρ^γ, $\gamma \in (0, \frac{2}{3}]$, there still has a positive finite pull-in voltage λ^*. The decay of a plays an important role to study the decay and the regularity of the minimal solution and the estimate of λ^* and λ_*. The decay rate of function a determines completely non-existence of pull-in voltage when $\gamma > \frac{2}{3}$. Precisely, we have the following non-existence result.

Theorem 0.4.2 *Assume that* $a \in C(\bar{\Omega})$ *is positive and satisfies (54) with* $\gamma \in (\frac{2}{3}, 1)$ *and* $c_0 > 0$. *Then problem (53) admits no nonnegative solution for any* $\lambda > 0$.

From Theorem 0.4.1, we observe that the mapping $\lambda \mapsto u_\lambda$ is increasing and uniformly bounded by function a, then the limit

$$u_{\lambda^*} := \lim_{\lambda \to \lambda^*} u_\lambda \quad \text{in} \ \ \bar{\Omega}, \tag{55}$$

is well-defined, where u_λ is the minimal solution of (53) with $\lambda \in (0, \lambda^*)$. Our final purpose in Chapter 4 is to prove that u_{λ^*} is a solution of (53) in a weak sense. The extremal solution u_{λ^*} always is found in the weak sense and then it could be improved the regularity up to the classical sense when $1 \leq N \leq 7$. Before stating this result, we introduce the definition of weak solution.

Definition 0.4.1 *A function* u *is a weak solution of (53) if* $0 \leq u \leq a$ *and*

$$\int_\Omega u(-\Delta)\xi dx = \int_\Omega \frac{\lambda\xi}{(a-u)^2}dx, \quad \forall \xi \in C_c^2(\Omega),$$

where $C_c^2(\Omega)$ *is the space of all* C^2 *functions with compact support in* Ω.

A solution (or weak solution) u *of (53) is stable (resp. semi-stable) if*

$$\int_\Omega |\nabla\xi|^2 dx > \int_\Omega \frac{2\lambda\xi^2}{(a-u)^3}dx, \quad (\text{resp. } \geq), \quad \forall \xi \in C_c^2(\Omega) \setminus \{0\}.$$

Theorem 0.4.3 *Assume that* $\lambda \in (0, \lambda^*)$, *the function* a *satisfies (52) and (54) with* $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$, u_λ *is the minimal solution of (53) and* u_{λ^*} *is given by (55). Then*

(i) u_{λ^*} *is a weak solution of (53) and* $u_{\lambda^*} \in W_0^{1,\frac{N}{N-\beta}}(\Omega)$ *for any* $\beta \in (0, \gamma)$.

(ii) when $1 \leq N \leq 7$, $c_0 = \kappa$ *and* $\Omega = B_1(0)$, u_{λ^*} *is a classical solution of (53).*

(iii) u_λ *is a stable solution of (53) when* $\lambda \in (0, \lambda^*)$ *and* u_{λ^*} *is a semi-stable weak solution of (53).*

Chapter 1

Radial symmetry of positive solutions to equations involving the fractional Laplacian

Abstract: in this chapter[1], we study radial symmetry and monotonicity properties for positive solution of elliptic equations involving the fractional Laplacian.

1.1 Introduction

The purpose of this chapter is to study symmetry and monotonicity properties of positive solutions for equations involving the fractional Laplacian through the use of moving planes arguments. The first part of this chapter is devoted to the following semi-linear Dirichlet problem

$$\begin{cases} (-\Delta)^\alpha u = f(u) + g, & \text{in} \quad B_1, \\ u = 0, & \text{in} \quad B_1^c, \end{cases} \tag{1.1}$$

where B_1 denotes the open unit ball centered at the origin in $\mathbb{R}^N$ and $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} dy, \tag{1.2}$$

[1]This chapter is based on the paper: P. Felmer and Y. Wang, *Radial symmetry of positive solutions to equations involving the fractional Laplacian,* Communications in Contemporary Mathematics, Vol. 16, No. 01, (2014).

$x \in B_1$. Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows.

During the last years, non-linear equations involving general integro-differential operators, especially, fractional Laplacian, have been studied by many authors. Caffarelli and Silvestre [22] gave a formulation of the fractional Laplacian through Dirichlet-Neumann maps. Various regularity issues for fractional elliptic equations has been studied by Cabré and Sire [17], Caffarelli and Silvestre [23], Capella, Dávila, Dupaigne and Sire [24], Ros-Oton and Serra [81] and Silvestre [87]. Existence and related results were studied by Cabré and Tan [37], Dipierro, Palatucci and Valdinoci [41], Felmer, Quaas and Tan [45], and Servadei and Valdinoci [86]. Great attention has also been devoted to symmetry results for equations involving the fractional Laplacian in $\mathbb{R}^N$, such as in the work by Li [64] and Chen, Li and Ou [29, 30], where the method of moving planes in integral form has been developed to treat various equations and systems, see also Ma and Chen [67]. On the other hand, using the local formulation of Caffarelli and Silvestre, Cabré and Sire [18] applied the sliding method to obtain symmetry results for nonlinear equations with fractional laplacian and Sire and Valdinoci [88] studied symmetry properties for a boundary reaction problem via a geometric inequality. Finally, in [45] the authors used the method of moving planes in integral form to prove symmetry results for

$$(-\Delta)^{\alpha} u + u = h(u) \quad \text{in} \quad \mathbb{R}^{\mathrm{N}}, \tag{1.3}$$

taking advantage of the representation formula for u given by

$$u(x) = \mathcal{K} * h(u)(x), \quad x \in \mathbb{R}^N,$$

where the kernel $\mathcal{K}$, associated to the linear part of the equation, plays a key role in the arguments. This approach is not possible to be used for problem (1.1), since a similar representation formula is not available in general.

The study of radial symmetry and monotonicity of positive solutions for non-linear elliptic equations in bounded domains using the moving planes method based on the Maximum Principle was initiated with the work by Serrin [83] and Gidas, Ni and Nirenberg [50], with important subsequent advances by Berestycki and Nirenberg [8]. We refer to the review by Pacella and Ramaswamy [77] for a more complete discussion of the method and it various applications. In [8] the Maximum Principle for small domain, based on the Aleksandrov-Bakelman-Pucci (ABP) estimate, was used as a tool to obtain much general results, specially avoiding regularity hypothesis on the domain. In a recent article Guillen and Schwab, [54], provided an ABP estimate for a class of fully non-linear elliptic integro-differential equations. Motivated by this work, we obtain a version of the Maximum Principle for small domain and we apply it with the moving planes method as in [8] to prove symmetry and monotonicity properties for positive solutions to problem (1.1) in the ball and in more general domains.

We consider the following hypotheses on the functions f and g:

$(F1)$ The function $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz.

(G) The function $g : B_1 \to \mathbb{R}$ is radially symmetric and decreasing in $|x|$.

Before stating our first theorem we make precise the notion of solution that we use in this chapter. We say that a continuous function $u : \mathbb{R}^N \to \mathbb{R}$ is a classical solution of equation (1.1) if the fractional Laplacian of u is defined at any point of B_1, according to the definition given in (1.2), and if u satisfies the equation and the external condition in a pointwise sense. This notion of solution is extended in a natural way to the other equations considered in this chapter.

Now we are ready for our first theorem on radial symmetry and monotonicity properties for positive solutions of equation (1.1). It states as follows:

Theorem 1.1.1 *Assume that the functions f and g satisfy $(F1)$ and (G), respectively. If u is a positive classical solution of (1.1), then u must be radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0, 1)$.*

The proof of Theorem 1.1.1 is given in Section §1.3, where we prove a more general symmetry and monotonicity result for equation (1.1) on a general domain Ω, which is convex and symmetric in one direction, see Theorem 1.3.1.

We devote the second part of this chapter to study symmetry results for a non-linear equation as (1.1), but in $\mathbb{R}^N$ and with $g \equiv 0$. For the problem in $\mathbb{R}^N$, the moving planes procedure has to start a different way because we cannot use the Maximum Principle for small domain. We refer to the work by Gidas, Ni and Nirenberg [51], Berestycki and Nirenberg [8], Li [62], and Li and Ni [63], where these results were studied assuming some additional hypothesis on f, allowing for decay properties of the solution u. A general result in this direction was obtained by Li [62] for the equation

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N,$$

with u decaying at infinity and f satisfying the following hypothesis:

$(F2)$ There exists $s_0 > 0$, $\gamma > 0$ and $C > 0$ such that

$$\frac{f(v) - f(u)}{v - u} \leq C(u + v)^\gamma \quad \text{for all} \quad 0 < u < v < s_0. \tag{1.4}$$

Motivated by these results, we are interested in similar properties of positive solutions for equations involving the fractional Laplacian under assumption (F2). Here is our second main theorem.

Theorem 1.1.2 *Assume that the function f satisfies $(F1)-(F2)$ and u is a positive classical solution for the equation*

$$\begin{cases} (-\Delta)^{\alpha} u = f(u) \quad in \;\; \mathbb{R}^N, \\ u > 0 \;\; in \;\; \mathbb{R}^N, \;\; \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.5}$$

Assume further that there exists

$$m > \max\{\frac{2\alpha}{\gamma}, \frac{N}{\gamma+2}\} \tag{1.6}$$

such that u satisfies

$$u(x) = O(\frac{1}{|x|^m}), \quad as \quad |x| \to \infty, \tag{1.7}$$

then u is radially symmetric and strictly decreasing about some point in $\mathbb{R}^N$.

In [45], Felmer, Quaas and Tan studied symmetry of positive solutions using the integral form of the moving planes method, assuming that the function f is such that $h(\xi) \equiv f(\xi) + \xi$ is super-linear, with sub-critical growth at infinity and

(H) $\qquad h \in C^1(\mathbb{R})$, increasing and there exists $\tau > 0$ such that

$$\lim_{v\to 0} \frac{h'(v)}{v^{\tau}} = 0.$$

We see that Theorem 1.1.2 generalizes Theorem 1.3 in [45], since here we do not assume f is differentiable and we do not require h to be increasing. In Section §1.4 we present an extension of Theorem 1.1.2 to $f(\xi) = \xi^p - \xi^q$, with $0 < q < 1 < p$, that is not covered by the results in [45] either, see Theorem 1.4.1. This non-linearity was studied by Valdebenito in [92], where decay and symmetry results were obtained using local extension as in Caffarelli and Silvestre [22] and regular moving planes.

For the particular case $f(u) = u^p$, for some $p > 1$, we see that (H) is not satisfied, but that (F2) does hold. Thus, if we knew the solution of (1.5) satisfies decay assumption (1.7) in this setting, we would have symmetry results in these cases. See [51] and [62] for the proof of decay properties in the case of the Laplacian.

The third part of this chapter is devoted to investigate the radial symmetry of non-negative solutions for the following system of non-linear equations involving fractional Laplacians with different orders,

$$\begin{cases} (-\Delta)^{\alpha_1} u = f_1(v) + g_1, & \text{in} \quad B_1, \\ (-\Delta)^{\alpha_2} v = f_2(u) + g_2, & \text{in} \quad B_1, \\ u = v = 0, & \text{in} \quad B_1^c, \end{cases} \tag{1.8}$$

where $\alpha_1, \alpha_2 \in (0,1)$. We have following results for system (1.8):

Theorem 1.1.3 *Suppose that f_1 and f_2 are locally Lipschitz continuous and increasing functions defined in $[0,\infty)$ and g_1 and g_2 satisfy (G). Assume that (u,v) are positive, classical solutions of system (1.8), then u and v are radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0,1)$.*

We prove our theorems using the moving planes method. The main difficulty comes from the fact that the fractional Laplacian is a non-local operator, and consequently Maximum Principle and Comparison Results require information on the solutions in the whole complement of the domain, not only at the boundary. To overcome this difficulty, we introduce a new truncation technique which is well adapted to be used with the method of moving planes.

1.2 Preliminaries

A key tool in the use of the moving planes method is the Maximum Principle for small domain, which is a consequence of the ABP estimate. In [54], Guillen and Schwab showed an ABP estimate (see Theorem 9.1) for general integro-differential operators. In this section we recall this estimate in the case of the fractional Laplacian in any open and bounded domain. Then we obtain the Maximum Principle for small domains.

We start with the ABP estimate for the fractional Laplacian, which is stated as follows:

Proposition 1.2.1 *Let Ω be a bounded, connected open subset of $\mathbb{R}^N$. Suppose that $h : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N)$ is a classical solution of*

$$\begin{cases} \Delta^\alpha w(x) \leq h(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.9}$$

Then there exists a positive constant C, depending on N and α, such that

$$-\inf_\Omega w \leq C d^\alpha \|h^+\|^{1-\alpha}_{L^\infty(\Omega)} \|h^+\|^\alpha_{L^N(\Omega)}, \tag{1.10}$$

where $d = diam(\Omega)$ is the diameter of Ω and $h^+(x) = \max\{h(x), 0\}$.

Here and in what follows we write $\Delta^\alpha w(x) = -(-\Delta)^\alpha w(x)$.

We have the following corollary

Corollary 1.2.1 *Under the assumptions of Proposition 1.2.1, with Ω not necessarily connected, we have*

$$-\inf_{\Omega} w \leq Cd^{\alpha}\|h^{+}\|_{L^{\infty}(\Omega)}|\Omega|^{\frac{\alpha}{N}}. \tag{1.11}$$

Proof. We let $w_0 \in L^{\infty}(\mathbb{R}^N)$ be a classical solution of

$$\begin{cases} \Delta^{\alpha} w_0(x) = \|h^{+}\|_{L^{\infty}(\Omega)}\chi_{\Omega}(x), & x \in B_d(x_0), \\ w_0(x) = 0, & x \in \mathbb{R}^N \setminus B_d(x_0), \end{cases} \tag{1.12}$$

where $x_0 \in \Omega$ and $\Omega \subset B_d(x_0)$. We observe that $B_d(x_0)$ is connected and that $w_0 \leq 0$ in all $\mathbb{R}^N$. By Comparison Principle, we see that

$$\inf_{\mathbb{R}^N} w_0 \leq \inf_{\mathbb{R}^N} w,$$

where w is the solution of (1.9). Then we use Proposition 1.2.1 to obtain that

$$-\inf_{\mathbb{R}^N} w_0 = -\inf_{B_d(x_0)} w_0 \leq C(2d)^{\alpha}\|h^{+}\|_{L^{\infty}(\Omega)}|\Omega|^{\frac{\alpha}{N}}$$

and then we conclude

$$-\inf_{\Omega} w = -\inf_{\mathbb{R}^N} w \leq Cd^{\alpha}\|h^{+}\|_{L^{\infty}(\Omega)}|\Omega|^{\frac{\alpha}{N}}. \qquad \square$$

Remark 1.2.1 *We notice that, under a possibly different constant $C > 0$, the ABP estimate for problem (1.9) with $\alpha = 1$*

$$\begin{cases} \Delta w(x) \leq h(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \partial\Omega, \end{cases}$$

is precisely (1.10) with $\alpha = 1$.

As a consequence of the ABP estimate just recalled, we have the Maximum Principle for small domain, which is stated as follows:

Proposition 1.2.2 *Let Ω be an open and bounded subset of $\mathbb{R}^N$. Suppose that $\varphi : \Omega \to \mathbb{R}$ is in $L^{\infty}(\Omega)$ and $w \in L^{\infty}(\mathbb{R}^N)$ is a classical solution of*

$$\begin{cases} \Delta^{\alpha} w(x) \leq \varphi(x)w(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.13}$$

Then there is $\delta > 0$ such that whenever $|\Omega^{-}| \leq \delta$, w has to be non-negative in Ω. Here $\Omega^{-} = \{x \in \Omega \mid w(x) < 0\}$.

Proof. By (1.13), we observe that

$$\begin{cases} \Delta^\alpha w(x) \leq \varphi(x)w(x), & x \in \Omega^-, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega^-. \end{cases}$$

Then, using Corollary 1.2.1 with $h(x) = \varphi(x)w(x)$, we obtain that

$$\|w\|_{L^\infty(\Omega^-)} = -\inf_{\Omega^-} w \quad \leq \quad Cd_0^\alpha \|(\varphi w)^+\|_{L^\infty(\Omega^-)} |\Omega^-|^{\frac{\alpha}{N}},$$

where constant $C > 0$ depends on N and α. Here $d_0 = \operatorname{diam}(\Omega^-)$. Thus

$$\|w\|_{L^\infty(\Omega^-)} \leq Cd_0^\alpha \|\varphi\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega^-)} |\Omega^-|^{\frac{\alpha}{N}}.$$

We see that, if $|\Omega^-|$ is such that $Cd_0^\alpha \|\varphi\|_{L^\infty(\Omega)} |\Omega^-|^{\alpha/N} < 1$, then we must have that

$$\|w\|_{L^\infty(\Omega^-)} = 0.$$

This implies that $|\Omega^-| = 0$ and since Ω^- is open, we have $\Omega^- = \emptyset$, completing the proof. □

1.3 Proof of Theorem 1.1.1.

In this section we provide a proof of Theorem 1.1.1 on the radial symmetry and monotonicity of positive solutions to equation (1.1) in the unit ball. For this purpose we use the moving planes method, for which we give some preliminary notation. We define

$$\Sigma_\lambda = \{x = (x_1, x') \in B_1 \mid x_1 > \lambda\}, \tag{1.14}$$

$$T_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\}, \tag{1.15}$$

$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \tag{1.16}$$

where $\lambda \in (0,1)$ and $x_\lambda = (2\lambda - x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$. For any subset A of $\mathbb{R}^N$, we write $A_\lambda = \{x_\lambda : x \in A\}$, the reflection of A with regard to T_λ.

Proof of Theorem 1.1.1. We divide the proof in three steps.

Step 1: We prove that if $\lambda \in (0,1)$ is close to 1, then $w_\lambda > 0$ in Σ_λ. For this purpose, we start proving that if $\lambda \in (0,1)$ is close to 1, then $w_\lambda \geq 0$ in Σ_λ. If we define $\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$, then we just need to prove that if $\lambda \in (0,1)$ is close to 1 then

$$\Sigma_\lambda^- = \emptyset. \tag{1.17}$$

By contradiction, we assume (1.17) is not true, that is $\Sigma_\lambda^- \neq \emptyset$. We denote

$$w_\lambda^+(x) = \begin{cases} w_\lambda(x), & x \in \Sigma_\lambda^-, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-, \end{cases} \tag{1.18}$$

$$w_\lambda^-(x) = \begin{cases} 0, & x \in \Sigma_\lambda^-, \\ w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma_\lambda^- \end{cases} \tag{1.19}$$

and we observe that $w_\lambda^+(x) = w_\lambda(x) - w_\lambda^-(x)$ for all $x \in \mathbb{R}^N$. Next we claim that for all $0 < \lambda < 1$, we have

$$(-\Delta)^\alpha w_\lambda^-(x) \leq 0, \quad \forall\, x \in \Sigma_\lambda^-. \tag{1.20}$$

By direct computation, for $x \in \Sigma_\lambda^-$, we have

$$\begin{aligned}
(-\Delta)^\alpha w_\lambda^-(x) &= \int_{\mathbb{R}^N} \frac{w_\lambda^-(x) - w_\lambda^-(z)}{|x-z|^{N+2\alpha}} dz = -\int_{\mathbb{R}^N \setminus \Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= -\int_{(B_1 \setminus (B_1)_\lambda) \cup ((B_1)_\lambda \setminus B_1)} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&\quad -\int_{(\Sigma_\lambda \setminus \Sigma_\lambda^-) \cup (\Sigma_\lambda \setminus \Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz - \int_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= -I_1 - I_2 - I_3.
\end{aligned}$$

We look at each of these integrals separately. Since $u = 0$ *in* $(B_1)_\lambda \setminus B_1$ and $u_\lambda = 0$ *in* $B_1 \setminus (B_1)_\lambda$, we have

$$\begin{aligned}
I_1 &= \int_{(B_1 \setminus (B_1)_\lambda) \cup ((B_1)_\lambda \setminus B_1)} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{(B_1)_\lambda \setminus B_1} \frac{u_\lambda(z)}{|x-z|^{N+2\alpha}} dz - \int_{B_1 \setminus (B_1)_\lambda} \frac{u(z)}{|x-z|^{N+2\alpha}} dz \\
&= \int_{(B_1)_\lambda \setminus B_1} u_\lambda(z) \left(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}}\right) dz \geq 0,
\end{aligned}$$

since $u_\lambda \geq 0$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in (B_1)_\lambda \setminus B_1$. In order to

study the sign of I_2 we first observe that $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$. Then

$$\begin{aligned} I_2 &= \int_{(\Sigma_\lambda \setminus \Sigma_\lambda^-) \cup (\Sigma_\lambda \setminus \Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz \\ &= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}} dz \\ &= \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} w_\lambda(z) \left(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}} \right) dz \geq 0, \end{aligned}$$

since $w_\lambda \geq 0$ in $\Sigma_\lambda \setminus \Sigma_\lambda^-$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in \Sigma_\lambda \setminus \Sigma_\lambda^-$. Finally, since $w_\lambda(z) < 0$ for $z \in \Sigma_\lambda^-$, we have

$$\begin{aligned} I_3 &= \int_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}} dz = \int_{\Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}} dz \\ &= -\int_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z_\lambda|^{N+2\alpha}} dz \geq 0. \end{aligned}$$

Hence, we obtain (1.20), proving the claim. Now we apply (1.20) and linearity of the fractional laplacian to obtain that, for $x \in \Sigma_\lambda^-$,

$$(-\Delta)^\alpha w_\lambda^+(x) \geq (-\Delta)^\alpha w_\lambda(x) = (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x). \tag{1.21}$$

Combining equation (1.1) with (1.21) and (1.18), for $x \in \Sigma_\lambda^-$ we have

$$\begin{aligned} (-\Delta)^\alpha w_\lambda^+(x) &\geq (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x) \\ &= f(u_\lambda(x)) + g(x_\lambda) - f(u(x)) - g(x) \\ &= \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} w_\lambda^+(x) + g(x_\lambda) - g(x). \end{aligned}$$

Let us define $\varphi(x) = -(f(u_\lambda(x)) - f(u(x)))/(u_\lambda(x) - u(x))$ for $x \in \Sigma_\lambda^-$. By assumption $(F1)$, we have that $\varphi \in L^\infty(\Sigma_\lambda^-)$. By assumption (G), we have that $g(x_\lambda) \geq g(x)$, since for all $x \in \Sigma_\lambda^-$ and $0 < \lambda < 1$, we have $|x| > |x_\lambda|$. Hence, we have

$$\Delta^\alpha w_\lambda^+(x) \leq \varphi(x) w_\lambda^+(x), \quad x \in \Sigma_\lambda^- \tag{1.22}$$

and since $w_\lambda^+ = 0$ in $(\Sigma_\lambda^-)^c$ we may apply Proposition 1.2.2. Choosing $\lambda \in (0,1)$ close enough to 1 we find that $|\Sigma_\lambda^-|$ is small and then

$$w_\lambda = w_\lambda^+ \geq 0 \quad \text{in } \Sigma_\lambda^-.$$

But this is a contradiction with our assumption so we have

$$w_\lambda \geq 0 \quad in \ \ \Sigma_\lambda.$$

In order to complete Step 1, we claim that for $0 < \lambda < 1$, if $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in Σ_λ, then $w_\lambda > 0$ in Σ_λ. Assuming the claim is true, we complete the proof, since the function u is positive in B_1 and $u = 0$ on ∂B_1, so that w_λ is positive in $\partial B_1 \cap \partial\Sigma_\lambda$ and then, by continuity $w_\lambda \neq 0$ in Σ_λ.

Now we prove the claim. Assume there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$, that is, $u_\lambda(x_0) = u(x_0)$. Then we have that

$$(-\Delta)^\alpha w_\lambda(x_0) \;=\; (-\Delta)^\alpha u_\lambda(x_0) - (-\Delta)^\alpha u(x_0) = g((x_0)_\lambda) - g(x_0).$$

Since $x_0 \in \Sigma_\lambda$, we have $|x_0| > |(x_0)_\lambda|$, then by assumption (G) we have $g((x_0)_\lambda) \geq g(x_0)$ and thus

$$(-\Delta)^\alpha w_\lambda(x_0) \geq 0. \tag{1.23}$$

On the other hand, defining $A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$, since $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and $w_\lambda(x_0) = 0$, we find

$$\begin{aligned}
(-\Delta)^\alpha w_\lambda(x_0) &= -\int_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz - \int_{\mathbb{R}^N \setminus A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz \\
&= -\int_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz - \int_{A_\lambda} \frac{w_\lambda(z_\lambda)}{|x_0 - z_\lambda|^{N+2\alpha}} dz \\
&= -\int_{A_\lambda} w_\lambda(z)\Big(\frac{1}{|x_0 - z|^{N+2\alpha}} - \frac{1}{|x_0 - z_\lambda|^{N+2\alpha}}\Big) dz.
\end{aligned}$$

Since $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in A_\lambda$, $w_\lambda(z) \geq 0$ and $w_\lambda(z) \not\equiv 0$ in A_λ, from here we get

$$(-\Delta)^\alpha w_\lambda(x_0) < 0, \tag{1.24}$$

which contradicts (1.23), completing the proof of the claim.

Step 2: We define $\lambda_0 = \inf\{\lambda \in (0,1) \mid w_\lambda > 0 \text{ in } \Sigma_\lambda\}$ and we prove that $\lambda_0 = 0$. Proceeding by contradiction, we assume that $\lambda_0 > 0$, then $w_{\lambda_0} \geq 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}. Thus, by the claim just proved above, we have $w_{\lambda_0} > 0$ in Σ_{λ_0}.

Next we claim that if $w_\lambda > 0$ in Σ_λ for $\lambda \in (0,1)$, then there exists $\epsilon \in (0, \lambda)$ such that $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon = \lambda - \epsilon$. This claim directly implies that $\lambda_0 = 0$, completing Step 2.

Now we prove the claim. Let $D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial\Sigma_\lambda) \geq \mu\}$ for $\mu > 0$ small. Since $w_\lambda > 0$ in Σ_λ and D_μ is compact, then there exists $\mu_0 > 0$ such that $w_\lambda \geq \mu_0$ in D_μ. By continuity of $w_\lambda(x)$, for $\epsilon > 0$ small enough and denoting $\lambda_\epsilon = \lambda - \epsilon$, we have that

$$w_{\lambda_\epsilon}(x) \geq 0 \text{ in } \mathrm{D}_\mu.$$

As a consequence,

$$\Sigma^-_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$$

and $|\Sigma^-_{\lambda_\epsilon}|$ is small if ϵ and μ are small. Using (1.20) and proceeding as in Step 1,

we have for all $x \in \Sigma^-_{\lambda_\epsilon}$ that

$$\begin{aligned}(-\Delta)^\alpha w^+_{\lambda_\epsilon}(x) &= (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) - (-\Delta)^\alpha w^-_{\lambda_\epsilon}(x)\\ &\geq (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x)\\ &= \varphi(x) w^+_{\lambda_\epsilon}(x) + g(x_\lambda) - g(x) \geq \varphi(x) w^+_{\lambda_\epsilon}(x),\end{aligned}$$

where $\varphi(x) = \frac{f(u_{\lambda_\epsilon}(x)) - f(u(x))}{u_{\lambda_\epsilon}(x) - u(x)}$ is bounded by assumption $(F1)$.

Since $w^+_{\lambda_\epsilon} = 0$ in $(\Sigma^-_{\lambda_\epsilon})^c$ and $|\Sigma^-_{\lambda_\epsilon}|$ is small, for ϵ and μ small, Proposition 1.2.2 implies that $w_{\lambda_\epsilon} \geq 0$ in $\Sigma_{\lambda_\epsilon}$. Thus, since $\lambda_\epsilon > 0$ and $w_{\lambda_\epsilon} \not\equiv 0$ in $\Sigma_{\lambda_\epsilon}$, as before we have $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, completing the proof of the claim.

Step 3: By Step 2, we have $\lambda_0 = 0$, which implies that $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$ and then $u(-x_1, x') = u(x_1, x')$ for $x_1 \geq 0$. Repeating this procedure in all directions we obtain radial symmetry of u.

Finally, we prove $u(r)$ is strictly decreasing in $r \in (0, 1)$. Let us consider $0 < x_1 < \widetilde{x}_1 < 1$ and let $\lambda = \frac{x_1 + \widetilde{x}_1}{2}$. Then, as proved above we have

$$w_\lambda(x) > 0 \quad \text{for} \quad x \in \Sigma_\lambda.$$

Then

$$\begin{aligned}0 < w_\lambda(\widetilde{x}_1, 0, \cdots, 0) &= u_\lambda(\widetilde{x}_1, 0, \cdots, 0) - u(\widetilde{x}_1, 0, \cdots, 0)\\ &= u(x_1, 0, \cdots, 0) - u(\widetilde{x}_1, 0, \cdots, 0),\end{aligned}$$

that is $u(x_1, 0, \cdots, 0) > u(\widetilde{x}_1, 0, \cdots, 0)$. Using the radial symmetry of u, we conclude from here the monotonicty of u. □

The proof of Theorem 1.1.1 can be applied directly to prove symmetry results for problem (1.1) in more general domains. We have the following definition

Definition 1.3.1 *We say that domain $\Omega \subset \mathbb{R}^N$ is convex in the x_1 direction:*

$$(x_1, x'), (x_1, y') \in \Omega \Rightarrow (x_1, tx' + (1-t)y') \in \Omega, \quad \forall\, t \in (0, 1).$$

Now we state the more general theorem:

Theorem 1.3.1 *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Assume further that Ω is convex in the x_1 direction and symmetric with respect to the plane $x_1 = 0$. Assume that the function f satisfies $(F1)$ and g satisfies*

$(\widetilde{G})$ *The function $g : \Omega \to \mathbb{R}$ is symmetric with respect to $x_1 = 0$ and decreasing in the x_1 direction, for $x = (x_1, x') \in \Omega$, $x_1 > 0$.*

Let u be a positive classical solution of

$$\begin{cases} (-\Delta)^\alpha u(x) = f(u(x)) + g(x), & x \in \Omega, \\ u(x) = 0, & x \in \Omega^c. \end{cases} \tag{1.25}$$

Then u is symmetric with respect to x_1 and it is strictly decreasing in the x_1 direction for $x = (x_1, x') \in \Omega$, $x_1 > 0$.

1.4 Symmetry of solutions in $\mathbb{R}^N$

In this section we study radial symmetry results for positive solution of equation (1.5) in $\mathbb{R}^N$, in particular we will provide a proof of Theorem 1.1.2. In the case of the whole space, the moving planes procedure needs to be started in a different way, because we cannot use the Maximum Principle for small domains. We use the moving plane method as for the second order equation as in the work by Li [62] (see also [77]).

In this section we use the notation introduced in (1.14)-(1.16) and we let u be a classical positive solution of (1.5). In order to prove Theorem 1.1.2 we need some preliminary lemmas.

Lemma 1.4.1 *Under the assumptions of Theorem 1.1.2, for any $\lambda \in \mathbb{R}$, we have*

$$\int_{\Sigma_\lambda} (f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ dx < +\infty.$$

Proof. By our hypothesis, for any given $\lambda \in \mathbb{R}$, we may choose $R > 1$ and some constant $c > 1$ such that

$$\frac{1}{c|x|^m} \leq u(x), u_\lambda(x) \leq \frac{c}{|x|^m} < s_0 \quad for\ all\ x \in B_R^c,$$

where s_0 is the constant in condition (F2).

If $u_\lambda(x) > u(x)$ for some $x \in \Sigma_\lambda \cap B_R^c$, we have $0 < u(x) < u_\lambda(x) < s_0$. Using (1.4) with $v = u_\lambda(x)$, then

$$\frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} \leq C(u(x) + u_\lambda(x))^\gamma \leq 2^\gamma C u_\lambda^\gamma(x),$$

then

$$\begin{aligned} (f(u_\lambda(x)) - f(u(x)))^+ (u_\lambda(x) - u(x))^+ &\leq 2^\gamma C u_\lambda^\gamma(x) [(u_\lambda(x) - u(x))^+]^2 \\ &\leq \tilde{C} u_\lambda^{\gamma+2}(x), \end{aligned}$$

for certain $\tilde{C} > 0$. We observe that, if $u_\lambda(x) \leq u(x)$ for some $x \in \Sigma_\lambda \cap B_R^c$, then inequality above is obvious. Therefore,

$$(f(u_\lambda) - f(u))^+(u_\lambda - u)^+ \leq \tilde{C} u_\lambda^{\gamma+2} \quad in \;\; \Sigma_\lambda \cap B_R^c.$$

Now we integrate in $\Sigma_\lambda \cap B_R^c$ to obtain

$$\begin{aligned} \int_{\Sigma_\lambda \cap B_R^c} (f(u_\lambda) - f(u))^+(u_\lambda - u)^+ dx &\leq \tilde{C} \int_{\Sigma_\lambda \cap B_R^c} u_\lambda^{\gamma+2}(x) dx \\ &\leq C \int_{B_R^c} |x|^{-m(\gamma+2)} dx < +\infty, \end{aligned}$$

where the last inequality holds by (1.6). Since u and u_λ are bounded and f is locally Lipschitz, we have

$$\int_{\Sigma_\lambda \cap B_R} (f(u_\lambda) - f(u))^+(u_\lambda - u)^+ dx < +\infty$$

and the proof is complete. □

It will be convenient for our analysis to define the following function

$$w(x) = \begin{cases} (u_\lambda - u)^+(x), & x \in \Sigma_\lambda, \\ (u_\lambda - u)^-(x), & x \in \Sigma_\lambda^c, \end{cases} \tag{1.26}$$

where $(u_\lambda - u)^+(x) = \max\{(u_\lambda - u)(x),\ 0\}$, $(u_\lambda - u)^-(x) = \min\{(u_\lambda - u)(x),\ 0\}$.

Lemma 1.4.2 *Under the assumptions of Theorem 1.1.2, there exists a constant $C > 0$ such that*

$$\int_{\Sigma_\lambda} (-\Delta)^\alpha (u_\lambda - u)(u_\lambda - u)^+ dx \geq C \Big(\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx \Big)^{\frac{N-2\alpha}{N}}. \tag{1.27}$$

Proof. We start observing that, given $x \in \Sigma_\lambda$, we have

$$\begin{aligned} w(x_\lambda) &= (u_\lambda - u)^-(x_\lambda) = \min\{(u_\lambda - u)(x_\lambda),\ 0\} = \min\{(u - u_\lambda)(x),\ 0\} \\ &= -\max\{(u_\lambda - u)(x),\ 0\} = -(u_\lambda - u)^+(x) = -w(x) \end{aligned}$$

and similarly $w(x) = -w(x_\lambda)$ for $x \in \Sigma_\lambda^c$ so that

$$w(x) = -w(x_\lambda) \quad \text{for} \quad x \in \mathbb{R}^N. \tag{1.28}$$

This implies

$$\int_{\mathbb{R}^N} |w|^{\frac{2N}{N-2\alpha}} dx = \int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx + \int_{\Sigma_\lambda^c} |w|^{\frac{2N}{N-2\alpha}} dx = 2 \int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx. \tag{1.29}$$

Next we see that for any $x \in \Sigma_\lambda \cap \mathrm{supp}(w)$ we have that $w(x) = (u_\lambda - u)(x)$ and

$$(-\Delta)^\alpha (u_\lambda - u)(x) \geq (-\Delta)^\alpha w(x), \quad \forall\, x \in \Sigma_\lambda \cap \mathrm{supp(w)},$$

$$\begin{aligned}
(-\Delta)^\alpha w(x) - (-\Delta)^\alpha (u_\lambda - u)(x) &= \int_{\mathbb{R}^N} \frac{(u_\lambda - u)(z) - w(z)}{|x-z|^{N+2\alpha}} dz \\
= \int_{\Sigma_\lambda \cap (\mathrm{supp}(w))^c} \frac{(u_\lambda - u)(z)}{|x-z|^{N+2\alpha}} dz &+ \int_{\Sigma_\lambda^c \cap (\mathrm{supp}(w))^c} \frac{(u_\lambda - u)(z)}{|x-z|^{N+2\alpha}} dz \\
= \int_{\Sigma_\lambda \cap (\mathrm{supp}(w))^c} (u_\lambda - u)(z) &\left(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}}\right) dz \leq 0, \qquad (1.30)
\end{aligned}$$

where we used that $u_\lambda - u \leq 0$ in $\Sigma_\lambda \cap (\mathrm{supp}(w))^c$ and $|x-z| \leq |x - z_\lambda|$ for $x, z \in \Sigma_\lambda$. From (1.30), using the equation and Lemma 1.4.1 we find that

$$\begin{aligned}
\int_{\Sigma_\lambda} (-\Delta)^\alpha w\, w dx &\leq \int_{\Sigma_\lambda} (-\Delta)^\alpha (u_\lambda - u)(u_\lambda - u)^+ dx \qquad (1.31) \\
&\leq \int_{\Sigma_\lambda} (f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ dx < \infty. \qquad (1.32)
\end{aligned}$$

From here the following integrals are finite and, taking into account (1.28), we obtain that

$$\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx &= \int_{\Sigma_\lambda} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx + \int_{\Sigma_\lambda^c} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx \\
&= 2 \int_{\Sigma_\lambda} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx. \qquad (1.33)
\end{aligned}$$

Now we can use the Sobolev embedding from $H^\alpha(\mathbb{R}^N)$ to $L^{\frac{2N}{N-2\alpha}}(\mathbb{R}^N)$ to find a constant C so that

$$\begin{aligned}
\int_{\Sigma_\lambda} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx \geq C\left(\int_{\mathbb{R}^N} |w|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}} \\
&= C\left(2 \int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx\right)^{\frac{N-2\alpha}{N}}. \qquad (1.34)
\end{aligned}$$

On the other hand, from (1.28) and (1.31) we find that

$$\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{2}} w|^2 dx &= \int_{\mathbb{R}^N} (-\Delta)^\alpha w \cdot w dx = 2 \int_{\Sigma_\lambda} (-\Delta)^\alpha w \cdot w dx \\
&\leq 2 \int_{\Sigma_\lambda} (-\Delta)^\alpha (u_\lambda - u)(u_\lambda - u)^+ dx. \qquad (1.35)
\end{aligned}$$

From (1.34) and (1.35) the proof of the lemma is completed. □

Now we are ready to complete the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. We divide the proof into three steps.

Step 1: We show that $\lambda_0 := \sup\{\lambda \mid u_\lambda \leq u \ in \ \Sigma_\lambda\}$ is finite. Using $(u_\lambda - u)^+$ as a test function in the equation for u and u_λ, using (1.4) and Hölder inequality, for λ big (negative), we find that

$$\begin{aligned}\int_{\Sigma_\lambda} (-\Delta)^\alpha (u_\lambda - u)(u_\lambda - u)^+ dx &= \int_{\Sigma_\lambda} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx \\ &\leq \int_{\Sigma_\lambda} [\frac{f(u_\lambda) - f(u)}{u_\lambda - u}]^+ [(u_\lambda - u)^+]^2 dx \\ &\leq C \int_{\Sigma_\lambda} u_\lambda^\gamma w^2 dx \leq \bar{C} \int_{\Sigma_\lambda} |x_\lambda|^{-m\gamma} w^2 dx \\ &\leq \bar{C} (\int_{\Sigma_\lambda} |x_\lambda|^{-\frac{Nm\gamma}{2\alpha}} dx)^{\frac{2\alpha}{N}} (\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}}.\end{aligned}$$

By Lemma 1.4.2, there exists a constant $C > 0$ such that

$$(\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}} \leq C(\int_{\Sigma_\lambda} |x_\lambda|^{-\frac{Nm\gamma}{2\alpha}} dx)^{\frac{2\alpha}{N}} (\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}},$$

but we have

$$\int_{\Sigma_\lambda} |x_\lambda|^{-\frac{Nm\gamma}{2\alpha}} dx \leq \int_{\Sigma_\lambda^c} |x|^{-\frac{Nm\gamma}{2\alpha}} dx \leq \int_{B^c_{|\lambda|}} |x|^{-\frac{Nm\gamma}{2\alpha}} dx = c|\lambda|^{\frac{N}{2\alpha}(2\alpha - m\gamma)},$$

so that, using (1.6), we can choose $R > 0$ big enough such that $CR^{2\alpha - m\gamma} \leq \frac{1}{2}$, then we obtain

$$\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx = 0, \quad \forall \ \lambda < -R.$$

Thus $w = 0$ in Σ_λ and then $u_\lambda \leq u$ in Σ_λ, for all $\lambda < -R$, concluding that $\lambda_0 \geq -R$. On the other hand, since u decays at infinity, then there exists λ_1 such that $u(x) < u_{\lambda_1}(x)$ for some $x \in \Sigma_{\lambda_1}$. Hence λ_0 is finite.

Step 2: We prove that $u \equiv u_{\lambda_0}$ in Σ_{λ_0}. Assuming the contrary, we have $u \neq u_{\lambda_0}$ and $u \geq u_{\lambda_0}$ in Σ_{λ_0}. Assume next that there exists $x_0 \in \Sigma_{\lambda_0}$ such that $u_{\lambda_0}(x_0) = u(x_0)$, then we have

$$(-\Delta)^\alpha u_{\lambda_0}(x_0) - (-\Delta)^\alpha u(x_0) = f(u_{\lambda_0}(x_0)) - f(u(x_0)) = 0. \tag{1.36}$$

On the other hand,

$$
\begin{aligned}
&(-\Delta)^\alpha u_{\lambda_0}(x_0) - (-\Delta)^\alpha u(x_0) = -\int_{\mathbb{R}^N} \frac{u_{\lambda_0}(y) - u(y)}{|x_0 - y|^{N+2\alpha}} dy \\
= \quad & -\int_{\Sigma_{\lambda_0}} (u_{\lambda_0}(y) - u(y))(\frac{1}{|x_0 - y|^{N+2\alpha}} - \frac{1}{|x_0 - y_{\lambda_0}|^{N+2\alpha}}) dy > 0,
\end{aligned}
$$

which contradicts (1.36). As a sequence, $u > u_{\lambda_0}$ in Σ_{λ_0}.

To complete Step 2, we only need to prove that $u \geq u_\lambda$ in Σ_λ continues to hold when $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, where $\varepsilon > 0$ small. Let us consider then $\varepsilon > 0$, to be chosen later, and take $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. Let $P = (\lambda, 0)$ and $B(P, R)$ be the ball centered at P and with radius $R > 1$ to be chosen later. Define $\tilde{B} = \Sigma_\lambda \cap B(P, R)$ and let us consider $(u_\lambda - u)^+$ test function in the equation for u and u_λ in Σ_λ, then from Lemma 1.4.2 we find

$$
(\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}} \quad \leq \quad C \int_{\Sigma_\lambda} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx. \tag{1.37}
$$

We estimate the integral on the right. Since f is locally Lipschitz, using Hölder inequality, we have

$$
\begin{aligned}
&\int_{\tilde{B}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx \leq C \int_{\tilde{B}} |w|^2 \chi_{\mathrm{supp}(u_\lambda - u)^+} dx \\
= \quad & C|\tilde{B} \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N}} (\int_{\tilde{B}} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}}.
\end{aligned} \tag{1.38}
$$

On the other hand, for the integral over $\Sigma_\lambda \setminus \tilde{B}$, we assume R and R_0 are such that $\Sigma_\lambda \setminus \tilde{B} \subset B^c(P, R) \subset B^c_{R_0}(0)$, proceeding as in Step 1, we have

$$
\begin{aligned}
\int_{\Sigma_\lambda \setminus \tilde{B}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dx \quad &\leq \quad C \int_{\Sigma_\lambda \setminus \tilde{B}} u_\lambda^\gamma w^2 dx \\
&\leq \quad C(\int_{\Sigma_\lambda \setminus \tilde{B}} |x_\lambda|^{-\frac{Nm\gamma}{2\alpha}} dx)^{\frac{2\alpha}{N}} (\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}} \\
&\leq \quad C{R_0}^{2\alpha - m\gamma} (\int_{\Sigma_\lambda} |w|^{\frac{2N}{N-2\alpha}} dx)^{\frac{N-2\alpha}{N}}.
\end{aligned} \tag{1.39}
$$

Now we choose R_0 such that $C{R_0}^{2\alpha - m\gamma} < 1/2$, then choose R so that $\Sigma_\lambda \setminus \tilde{B} \subset B^c(P, R) \subset B^c_{R_0}(0)$ and then choose $\varepsilon > 0$ so that $C|\tilde{B} \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N}} < 1/2$. With this choice of the parameters, from (1.37), (1.38) and (1.39) it follows that $w = 0$ in Σ_λ, which is a contradiction, completing Step 2.

Step 3: By translation, we may say that $\lambda_0 = 0$. An repeating the argument from the other side, we find that u is symmetric about x_1-axis. Using the same argument in any arbitrary direction, we finally conclude that u is radially symmetric.

Finally, we prove that $u(r)$ is strictly decreasing in $r > 0$, by using the same arguments as in the case of a ball. This completes the proof. $\square$

At the end of this section we want to give a theorem on radial symmetry of solutions for equation (1.5) in a case where f is only locally Lipschitz in $(0, \infty)$, see [40] and [39] for the case of the Laplacian. In precise terms we have

Theorem 1.4.1 *Let u be a positive classical solution of*

$$\begin{cases} (-\Delta)^\alpha u = u^p - u^q \quad in \;\; \mathbb{R}^N, \\ u > 0 \;\; in \;\; \mathbb{R}^N, \;\; \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.40}$$

satisfying

$$u(x) = O(|x|^{-\frac{N+2\alpha}{q}}) \qquad as \; |x| \to \infty, \tag{1.41}$$

where $\alpha \in (0,1)$, $N \geq 2$ and $0 < q < 1 < p$. Then u is radially symmetric and strictly decreasing about some point.

Proof. We denote $f(u) = u^p - u^q$ for $u > 0$, and consider $\gamma > 0$ and s_0 small enough, then for all u, v satisfying $0 < u < v < s_0$, we have

$$\frac{f(v) - f(u)}{v - u} < 0 \leq C(u + v)^\gamma,$$

for some constant $C > 0$, so that (F2) holds. We also observe that for a positive classical solution u of (1.40), $u \geq c$ in any bounded domain Ω, for a constant $c > 0$ depending on Ω and then, in (1.38) we may use Lipschitz continuity of f in the bounded interval $[c, \sup u]$. We set $m = \frac{N+2\alpha}{q}$ and γ may be chosen so that (1.6) holds. The proof of Theorem 1.4.1 goes in the same way as that of Theorem 1.1.2. $\square$

Remark 1.4.1 *In a work by Valdebenito [92], the estimate (1.41) is obtained by using super solutions and Theorem 1.4.1 is proved using the local extension of equation (1.40) as given by Caffarelli and Silvestre in [22] and then using a regular moving planes argument as developed for elliptic equations with non-linear boundary conditions by Terracini [91].*

1.5 Symmetry results for system

The aim of this section is to prove Theorem 1.1.3 by the moving planes method applied to a system of equations in the unit ball B_1. Let Σ_λ and T_λ be defined as in Section §1.3. For $x = (x_1, x') \in \mathbb{R}^N$ and $\lambda \in (0,1)$ we let $x_\lambda = (2\lambda - x_1, x')$,

$$u_\lambda(x) = u(x_\lambda), \qquad w_{\lambda,u}(x) = u_\lambda(x) - u(x),$$

$$v_\lambda(x) = v(x_\lambda), \quad \text{and} \quad w_{\lambda,v}(x) = v_\lambda(x) - v(x).$$

Proof of Theorem 1.1.3. We will split this proof into three steps.

Step 1: We start the moving planes proving that if λ is close to 1, then $w_{\lambda,u}$ and $w_{\lambda,v}$ are positive in Σ_λ. For that purpose we define

$$\Sigma^-_{\lambda,u} = \{x \in \Sigma_\lambda \mid w_{\lambda,u}(x) < 0\} \quad \text{and} \quad \Sigma^-_{\lambda,v} = \{x \in \Sigma_\lambda \mid w_{\lambda,v}(x) < 0\}.$$

We show next that $\Sigma^-_{\lambda,u}$ is empty for λ close to 1. Assume, by contradiction, that $\Sigma^-_{\lambda,u}$ is not empty and define

$$w^+_{\lambda,u}(x) = \begin{cases} w_{\lambda,u}(x), & x \in \Sigma^-_{\lambda,u}, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma^-_{\lambda,u} \end{cases} \tag{1.42}$$

and

$$w^-_{\lambda,u}(x) = \begin{cases} 0, & x \in \Sigma^-_{\lambda,u}, \\ w_{\lambda,u}(x), & x \in \mathbb{R}^N \setminus \Sigma^-_{\lambda,u}. \end{cases} \tag{1.43}$$

Using the arguments given in Step 1 of the proof of Theorem 1.1.1, we get

$$(-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) \geq (-\Delta)^{\alpha_1} w_{\lambda,u}(x) \quad \text{and} \quad (-\Delta)^{\alpha_1} w^-_{\lambda,u}(x) \leq 0, \tag{1.44}$$

for all $x \in \Sigma^-_{\lambda,u}$. From here, using equation (1.8), for $x \in \Sigma^-_{\lambda,u}$ we have

$$\begin{aligned} (-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) &\geq (-\Delta)^{\alpha_1} u_\lambda(x) - (-\Delta)^{\alpha_1} u(x) \\ &= f_1(v_\lambda(x)) + g_1(x_\lambda) - f_1(v(x)) - g_1(x) \\ &= \varphi_v(x) w_{\lambda,v}(x) + g_1(x_\lambda) - g_1(x) \\ &\geq \varphi_v(x) w_{\lambda,v}(x), \end{aligned} \tag{1.45}$$

where $\varphi_v(x) = (f_1(v_\lambda(x)) - f_1(v(x)))/(v_\lambda(x) - v(x))$ and where we used that g_1 is radially symmetric and decreasing, with $|x| > |x_\lambda|$. We further observe that, since f_1 is locally Lipschitz continuous, we have that $\varphi_v(\cdot) \in L^\infty(\Sigma^-_{\lambda,u})$. Now we consider (1.45) together with $w^+_{\lambda,u} = 0$ in $(\Sigma^-_{\lambda,u})^c$ and $w^+_{\lambda,u} < 0$ in $\Sigma^-_{\lambda,u}$, to use Proposition 1.2.1 to find a constant $C > 0$, depending on N and α only, such that

$$\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} \leq C \|(-\varphi_v w_{\lambda,v})^+\|^{1-\alpha_1}_{L^\infty(\Sigma^-_{\lambda,u})} \|(-\varphi_v w_{\lambda,v})^+\|^{\alpha_1}_{L^N(\Sigma^-_{\lambda,u})} \tag{1.46}$$

We observe that $diam(\Sigma^-_{\lambda,u}) \leq 1$. Since f_1 is increasing, we have

$$-\varphi_v w_{\lambda,v} = f_1(v) - f_1(v_\lambda) \leq 0 \ \ in \ (\Sigma^-_{\lambda,v})^c \quad \text{and} \tag{1.47}$$

$$-\varphi_v w_{\lambda,v} = f_1(v) - f_1(v_\lambda) > 0 \ \ in \ \Sigma^-_{\lambda,v}. \tag{1.48}$$

Denoting $\Sigma_\lambda^- = \Sigma_{\lambda,u}^- \cap \Sigma_{\lambda,v}^-$, from (1.46), (1.47) and (1.48), we obtain

$$\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C\|(-\varphi_v w_{\lambda,v})^+\|_{L^\infty(\Sigma_\lambda^-)}|\Sigma_\lambda^-|^{\frac{\alpha_1}{N}}, \tag{1.49}$$

Similar to (1.42) and (1.43), we define

$$w_{\lambda,v}^+(x) = \begin{cases} w_{\lambda,v}(x), & x \in \Sigma_{\lambda,v}^-, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma_{\lambda,v}^- \end{cases}$$

and

$$w_{\lambda,v}^-(x) = \begin{cases} 0, & x \in \Sigma_{\lambda,v}^-, \\ w_{\lambda,v}(x), & x \in \mathbb{R}^N \setminus \Sigma_{\lambda,v}^-. \end{cases}$$

With this definition (1.49) becomes

$$\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C\|w_{\lambda,v}^+\|_{L^\infty(\Sigma_\lambda^-)}|\Sigma_\lambda^-|^{\frac{\alpha_1}{N}}, \tag{1.50}$$

where we used that φ_v is bounded and we have changed the constant C, if necessary. At this point we observe that if $w_{\lambda,v}^+ = 0$ then $w_{\lambda,u}^+ = 0$ providing a contradiction. Thus we have that $\Sigma_{\lambda,v}^- \neq \emptyset$ and we may argue in a completely analogous way to obtain

$$\|w_{\lambda,v}^+\|_{L^\infty(\Sigma_{\lambda,v}^-)} \leq C\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_\lambda^-)}|\Sigma_\lambda^-|^{\frac{\alpha_2}{N}}, \tag{1.51}$$

that combined with (1.50) yields

$$\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} \leq C^2|\Sigma_\lambda^-|^{\frac{\alpha_1+\alpha_2}{N}}\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)},$$

and

$$\|w_{\lambda,v}^+\|_{L^\infty(\Sigma_{\lambda,v}^-)} \leq C^2|\Sigma_\lambda^-|^{\frac{\alpha_1+\alpha_2}{N}}\|w_{\lambda,v}^+\|_{L^\infty(\Sigma_{\lambda,v}^-)}.$$

Now we just take λ close enough to 1 so that $C^2|\Sigma_\lambda^-|^{\frac{\alpha_1+\alpha_2}{N}} < 1$ and we conclude that $\|w_{\lambda,u}^+\|_{L^\infty(\Sigma_{\lambda,u}^-)} = \|w_{\lambda,v}^+\|_{L^\infty(\Sigma_{\lambda,v}^-)} = 0$, so $|\Sigma_{\lambda,u}^-| = |\Sigma_{\lambda,v}^-| = 0$ and since $\Sigma_{\lambda,u}^-$ and $\Sigma_{\lambda,v}^-$ are open we have that $\Sigma_{\lambda,u}^-, \Sigma_{\lambda,v}^- = \emptyset$, which is a contradiction.

Thus we have that $w_{\lambda,u} \geq 0$ in Σ_λ when λ is close enough to 1. Similarly, we obtain $w_{\lambda,v} \geq 0$ in Σ_λ for λ close to 1. In order to complete Step 1 we will prove a bit more general statement that will be useful later, that is, given $0 < \lambda < 1$, if $w_{\lambda,u} \geq 0, w_{\lambda,v} \geq 0$, $w_{\lambda,u} \not\equiv 0$ and $w_{\lambda,v} \not\equiv 0$ in Σ_λ, then $w_{\lambda,u} > 0$ and $w_{\lambda,v} > 0$ in Σ_λ. For proving this property suppose there exists $x_0 \in \Sigma_\lambda$ such that

$$w_{\lambda,u}(x_0) = 0. \tag{1.52}$$

On one hand, by using similar arguments yielding (1.24) we find that

$$(-\Delta)^{\alpha_1} w_{\lambda,u}(x_0) < 0. \tag{1.53}$$

On the other hand, by our assumption we have that $w_{\lambda,v}(x_0) = v_\lambda(x_0) - v(x_0) \geq 0$ and since $|x_0| > |(x_0)_\lambda|$, from the monotonicity hypothesis on f_1 and g_1, we obtain

$$f_1(v_\lambda(x_0)) \geq f_1(v(x_0)), \qquad g_1((x_0)_\lambda) \geq g_1(x_0).$$

Thus, using (1.8), we find

$$(-\Delta)^{\alpha_1} w_{\lambda,u}(x_0) \;\; = \;\; f_1(v_\lambda(x_0)) + g_1((x_0)_\lambda) - f_1(v(x_0)) - g_1(x_0) \geq 0,$$

which is impossible with (1.53). This completes Step 1.

Step 2: We prove that $\lambda_0 = 0$, where

$$\lambda_0 = \inf\{\lambda \in (0,1) \mid w_{\lambda,u} \ , \ w_{\lambda,v} > 0 \ \text{ in } \ \Sigma_\lambda\}.$$

If not, that is, if $\lambda_0 > 0$ we have that $w_{\lambda_0,u}, w_{\lambda_0,v} \geq 0$ and $w_{\lambda_0,u}, w_{\lambda_0,v} \not\equiv 0$ in Σ_{λ_0}. If we use the property we just proved above, we may assume that $w_{\lambda_0,u} > 0$ and $w_{\lambda_0,v} > 0$ in Σ_{λ_0}. In what follows we argue that the plane can be moved to left, that is, that there exists $\epsilon \in (0,\lambda)$ such that $w_{\lambda_\epsilon,u} > 0$ and $w_{\lambda_\epsilon,v} > 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon = \lambda_0 - \epsilon$, providing a contradiction with the definition of λ_0.

Let us consider the set $D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial\Sigma_\lambda) \geq \mu\}$ for $\mu > 0$ small. Since $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ and D_μ is compact, then there exists $\mu_0 > 0$ such that $w_{\lambda,u}, w_{\lambda,v} \geq \mu_0$ in D_μ. By continuity of $w_{\lambda,u}(x)$ and $w_{\lambda,v}(x)$, for $\epsilon > 0$ small enough, we have that

$$w_{\lambda_\epsilon,u}, \ w_{\lambda_\epsilon,v} \geq 0 \ \text{ in } \ \mathrm{D}_\mu$$

and, as a consequence, $\Sigma^-_{\lambda_\epsilon,u}, \Sigma^-_{\lambda_\epsilon,v} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$, and $|\Sigma^-_{\lambda_\epsilon,u}|$ and $|\Sigma^-_{\lambda_\epsilon,v}|$ are small if ϵ and μ are small.

Since f_1 and f_2 are locally Lipschitz continuous and increasing, g_1 and g_2 are radially symmetric and decreasing, we may repeat the arguments given in Step 1 to obtain

$$\|w^+_{\lambda_\epsilon,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})} \leq C^2 |\Sigma^-_{\lambda_\epsilon}|^{\frac{\alpha_1+\alpha_2}{N}} \|w^+_{\lambda_\epsilon,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})}$$

and

$$\|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})} \leq C^2 |\Sigma^-_{\lambda_\epsilon}|^{\frac{\alpha_1+\alpha_2}{N}} \|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})}$$

where $\Sigma^-_{\lambda_\epsilon} = \Sigma^-_{\lambda_\epsilon,u} \cap \Sigma^-_{\lambda_\epsilon,v}$. Now we may choose ϵ and μ small such that $C^2|\Sigma^-_{\lambda_\epsilon}|^{\frac{\alpha_1+\alpha_2}{N}} < 1$, then we obtain $\|w^+_{\lambda_\epsilon,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})} = \|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})} = 0$. From here we argue as in Step 1 to obtain that $w_{\lambda_\epsilon,u}$ and $w_{\lambda_\epsilon,v}$ are positive in $\Sigma_{\lambda_\epsilon}$, completing Step 2.

Finally, we obtain that u and v are radially symmetric and strictly decreasing respect to $r = |x|$ for $r \in (0,1)$ in the same way in Step 3 in the proof of Theorem

1.1.1. □

1.6 The case of a non-local operator with non homogeneous kernel.

The main purpose of this section is to discuss radial symmetry for a problem with a non-local operator $\mathcal{L}$ of fractional order, but with a non-homogeneous kernel. The operator is defined as follows:

$$\mathcal{L}u(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y))K_\mu(x-y)dy, \tag{1.54}$$

where the kernel K_μ satisfies that

$$K_\mu(x) = \begin{cases} \frac{1}{|x|^{N+2\alpha_1}}, & |x| < 1, \\ \frac{\mu}{|x|^{N+2\alpha_2}}, & |x| \geq 1 \end{cases} \tag{1.55}$$

with $\mu \in [0,1]$ and $\alpha_1, \alpha_2 \in (0,1)$. Being more precise, we consider the equation

$$\begin{cases} \mathcal{L}u(x) = f(u(x)) + g(x), & x \in B_1, \\ u(x) = 0, & x \in B_1^c, \end{cases} \tag{1.56}$$

and our theorem states as follows.

Theorem 1.6.1 *Assume that the function f satisfies $(F1)$ and g satisfies (G). If u is a positive classical solution of (1.56), then u must be radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0,1)$.*

The idea for Theorem 1.6.1 is to take advantage of the fact that the non-local operator $\mathcal{L}$ differs from the fractional Laplacian by a zero order operator. Using this idea, we obtain a Maximum Principle for domains with small volume through the ABP-estimate given Proposition 1.2.1 and we are able to use the moving planes method as in the case of the fractional Laplacian. We prove first

Proposition 1.6.1 *Let Σ_λ and Σ_λ^- be defined as in the Section §1.3. Suppose that $\varphi \in L^\infty(\Sigma_\lambda)$ and that $w_\lambda \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ is a solution of*

$$\begin{cases} -\mathcal{L}w_\lambda(x) \leq \varphi(x)w_\lambda(x), & x \in \Sigma_\lambda, \\ w_\lambda(x) \geq 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda, \end{cases} \tag{1.57}$$

where $\mathcal{L}$ was defined in (1.54). Then, if $|\Sigma_\lambda^-|$ is small enough, w_λ is non-negative

in Σ_λ*, that is,*

$$w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.$$

Proof. We define $w_\lambda^+(x)$ as in (1.18), then we have

$$\begin{aligned}
\mathcal{L}w_\lambda^+(x) &= \int_{B_1(x)} \frac{w_\lambda^+(x) - w_\lambda^+(z)}{|x-z|^{N+2\alpha_1}} dz + \mu \int_{\mathbb{R}^N \setminus B_1(x)} \frac{w_\lambda^+(x) - w_\lambda^+(z)}{|x-z|^{N+2\alpha_2}} dz \\
&= (-\Delta)^{\alpha_1} w_\lambda^+(x) \\
&\quad + \int_{\mathbb{R}^N \setminus B_1(x)} (w_\lambda^+(x) - w_\lambda^+(z))(\frac{\mu}{|x-z|^{N+2\alpha_2}} - \frac{1}{|x-z|^{N+2\alpha_1}}) dz \\
&\leq (-\Delta)^{\alpha_1} w_\lambda^+(x) + 2C_0 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)}, \quad x \in \Sigma_\lambda^-,
\end{aligned}$$

where $C_0 = \int_{\mathbb{R}^N \setminus B_1} |\frac{\mu}{|y|^{N+2\alpha_2}} - \frac{1}{|y|^{N+2\alpha_1}}| dy$. Thus we have

$$\Delta^{\alpha_1} w_\lambda^+(x) \leq -\mathcal{L}w_\lambda^+(x) + 2C_0 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)}, \quad x \in \Sigma_\lambda^-. \tag{1.58}$$

Since K_μ is radially symmetric and decreasing in $|x|$, we may repeat the arguments used to prove (1.20) to get

$$\mathcal{L}w_\lambda^-(x) \leq 0, \quad \forall\, x \in \Sigma_\lambda^-, \tag{1.59}$$

where $0 < \lambda < 1$ and w_λ^- was defined in (1.19). Using (1.58), the linearity of $\mathcal{L}$, (1.59) and equation (1.57), for all $x \in \Sigma_\lambda^-$, we have

$$\begin{aligned}
\Delta^{\alpha_1} w_\lambda^+(x) &\leq -\mathcal{L}w_\lambda(x) + \mathcal{L}w_\lambda^-(x) + 2C_0 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} \\
&\leq -\mathcal{L}w_\lambda(x) + 2C_0 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} \\
&\leq \varphi(x) w_\lambda(x) + 2C_0 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} \leq C_1 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)},
\end{aligned} \tag{1.60}$$

where $C_1 = \|\varphi\|_{L^\infty(\Sigma_\lambda)} + 2C_0$ and we notice that $w_\lambda = w_\lambda^+$ in Σ_λ^-. Hence, we have

$$\begin{cases}
\Delta^{\alpha_1} w_\lambda^+(x) \leq C_1 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)}, & x \in \Sigma_\lambda^-, \\
w_\lambda^+(x) = 0, & x \in \mathbb{R}^N \setminus \Sigma_\lambda^-.
\end{cases} \tag{1.61}$$

Then, using Proposition 1.2.1 with $h(x) = C_1 \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)}$, we obtain a constant $C > 0$ such that

$$\|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} = -\inf_{\Sigma_\lambda^-} w_\lambda^+ \leq C d^{\alpha_1} \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} |\Sigma_\lambda^-|^{\frac{\alpha_1}{N}},$$

where $d = diam(\Sigma_\lambda^-)$. If $|\Sigma_\lambda^-|$ is small enough we conclude that $\|w_\lambda\|_{L^\infty(\Sigma_\lambda^-)} = \|w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} = 0$, from where we complete the proof. □

Now we provide a proof for Theorem 1.6.1.

Proof of Theorem 1.6.1. The proof of this theorem goes like the one for Theorem 1.1.1 where we use Proposition 1.6.1 instead of Proposition 1.2.1 and $\mathcal{L}$ instead of $(-\Delta)^\alpha$. The only place where there is a difference is in the following property: for $0 < \lambda < 1$, if $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in Σ_λ, then $w_\lambda > 0$ in Σ_λ.

For $\mu \in (0, 1]$, since K_μ is radially symmetric and strictly decreasing, the proof of the property is similar to that given in Theorem 1.1.1. So we only need to prove it in case $\mu = 0$ so the kernel K_0 vanishes outside the unit ball B_1. Let us assume that $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in Σ_λ and, by contradiction, let us assume $\Sigma_0 = \{x \in \Sigma_\lambda \mid w_\lambda(x) = 0\} \neq \emptyset$. By our assumptions on w_λ we have that $\Sigma_\lambda \setminus \Sigma_0 = \{x \in \Sigma_\lambda \mid w_\lambda(x) > 0\}$ is open and nonempty. Let us consider $x_0 \in \Sigma_0$ such that

$$dist(x_0, \Sigma_\lambda \setminus \Sigma_0) \leq 1/2, \tag{1.62}$$

and observe that $(\Sigma_\lambda \setminus \Sigma_0) \cap B_1(x_0)$ is nonempty. Using (1.56) we have

$$\begin{aligned} \mathcal{L}w_\lambda(x_0) &= \mathcal{L}u_\lambda(x_0) - \mathcal{L}u(x_0) \\ &= f(u_\lambda(x_0)) - f(u(x_0)) + g((x_0)_\lambda) - g(x_0) \\ &= g((x_0)_\lambda) - g(x_0) \geq 0, \end{aligned} \tag{1.63}$$

where the last inequality holds by monotonicity assumption on g and since $|x_0| > |(x_0)_\lambda|$. On the other hand, denoting by $A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$, since $w_\lambda(x_0) = 0$ and $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$, we have

$$\begin{aligned} \mathcal{L}w_\lambda(x_0) &= -\int_{A_\lambda} w_\lambda(z)K_0(x_0 - z)dz - \int_{\mathbb{R}^N \setminus A_\lambda} w_\lambda(z)K_0(x_0 - z)dz \\ &= -\int_{A_\lambda} w_\lambda(z)K_0(x_0 - z)dz - \int_{A_\lambda} w_\lambda(z_\lambda)K_0(x_0 - z_\lambda)dz \\ &= -\int_{A_\lambda} w_\lambda(z)(K_0(x_0 - z) - K_0(x_0 - z_\lambda))dz. \end{aligned}$$

Since $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in A_\lambda$, by definition of K_0, Σ_λ and Σ_0, we have that

$$K_0(x_0 - z) > K_0(x_0 - z_\lambda) \quad \text{and} \quad w_\lambda(z) > 0 \quad \text{for} \quad z \in (\Sigma_\lambda \setminus \Sigma_0) \cap B_1(x_0),$$

and we also have that $w_\lambda(z) \geq 0$ and $K_0(x_0 - z) \geq K_0(x_0 - z_\lambda)$ for all $z \in A_\lambda$, so that

$$\mathcal{L}w_\lambda(x_0) < 0,$$

contradicting (1.63). Hence Σ_0 is empty and then $w_\lambda > 0$ in Σ_λ, completing the proof of the theorem. □

Remark 1.6.1 *The theorem we just proved can be extended to more general non-*

homogeneous kernels in the following class

$$K(x) = \begin{cases} |x|^{-N-2\alpha}, & x \in B_r, \\ \theta(x), & x \in B_r^c, \end{cases} \tag{1.64}$$

here $\alpha \in (0,1)$, $r > 0$ *and the function* $\theta : B_r^c \to \mathbb{R}$ *satisfies that*

(C) $\theta \in L^1(B_r^c)$ *is nonnegative, radially symmetric and such that the kernel* K *is decreasing.*

Chapter 2

Qualitative properties of positive solutions for mixed integro-differential equations

Abstract: in this chapter[1], we consider the decay and symmetry properties of solutions to mixed integro-differential equations

$$\begin{cases} (-\Delta)_x^\alpha u + (-\Delta)_y u + u = f(u) & \text{in } \mathbb{R}^N \times \mathbb{R}^M, \\ u > 0 \ \text{ in } \mathbb{R}^N \times \mathbb{R}^M, & \lim_{|(x,y)|\to+\infty} u(x,y) = 0, \end{cases} \tag{2.1}$$

where $N \geq 1$, $M \geq 1$, the operator $(-\Delta)_y$ is the laplacian with respect to y, $(-\Delta)_x^\alpha$ is the fractional laplacian of exponent $\alpha \in (0,1)$ with respect to x. In studying the decay, we construct appropriate super and sub solutions and then we use the moving planes method to prove the symmetry properties.

2.1 Introduction

The study of qualitative properties of positive solutions to semi-linear elliptic equations in $\mathbb{R}^N$ has been the concern of numerous authors along the last several decades. The asymptotic behavior of the solution at infinity, the actual rate of decay and symmetry properties have been the most studied qualitative properties for these equations. It was the seminal work by Gidas, Ni and Nirenberg [51] that

[1]This chapter is based on the paper: P. Felmer and Y. Wang, *Qualitative properties of positive solutions for mixed integro-differential equations,* submitted.

settled these two main qualitative properties for the semi-linear elliptic equation

$$\begin{cases} -\Delta u + u = f(u) & \text{in} \quad \mathbb{R}^M, \\ u > 0 \ \text{ in } \ \mathbb{R}^M, & \lim_{|y|\to+\infty} u(y) = 0, \end{cases} \tag{2.2}$$

when the non-linearity is merely Lipschitz continuous, super-linear at the zero, in the sense that

$$f(s) = O(s^p) \quad \text{as } s \to 0, \tag{2.3}$$

for some $p > 1$, and $M \geq 3$. Gidas, Ni and Nirenberg proved that the solutions of (2.2) are radially symmetric and they satisfy the precise decay estimate

$$\lim_{|y|\to+\infty} u(y)e^{|y|}|y|^{\frac{M-1}{2}} = c, \tag{2.4}$$

for certain constant $c > 0$. After this work, many authors extended the results in various directions, generalizing the non-linearity, the elliptic operator or the hypotheses on the solutions. Out of the very many contributions in this direction we mention here only a few: Berestycki and Lions [7], Berestycki and Nirenberg [8], Brock [15], Busca and Felmer [16], Cortázar, Elgueta and Felmer [40], Da Lio and Sirakov [42], Dolbeault and Felmer [43], Gui [53], Kwong [60], Li and Ni [63] and Pacella and Ramaswamy [77].

Recently, much attention has been given to the study of elliptic equations of fractional order. In this direction, Felmer, Quaas and Tan in [45] studied the problem

$$\begin{cases} (-\Delta)^\alpha u + u = f(u) & \text{in} \quad \mathbb{R}^N, \\ u > 0 \ \text{ in } \ \mathbb{R}^N, & \lim_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{2.5}$$

They proved existence and regularity of positive solutions, and also decay and symmetry results. Precisely, it was proved that the solutions u of (2.5) satisfy

$$\frac{c^{-1}}{|x|^{N+2\alpha}} \leq u(x) \leq \frac{c}{|x|^{N+2\alpha}}, \qquad |x| \geq 1, \tag{2.6}$$

for some $c > 1$, when f is superlinear at 0 in the sense that

$$\lim_{s\to 0} \frac{f(s)}{s} = 0.$$

The radial symmetry of the solutions of (2.5) is derived by using the moving planes method in integral form developed in [30, 64], assuming further that $f \in C^1(\mathbb{R})$, it is increasing and there exists $\tau > 0$ such that

$$\lim_{s\to 0} \frac{f'(s)}{s^\tau} = 0. \tag{2.7}$$

This symmetry result was generalized by the authors in [46], using an appropriate truncation argument together with the moving planes method with ideas developed in [62]. We refer to some other papers with more discussions on qualitative properties of solutions to fractional elliptic problems as Cabré and Sire [18], Caffarelli and Silvestre [22], Chen, Li and Ou [30], Barles, Chasseigne, Ciomaga and Imbert [38], Dipierro Palatucci, Valdinoci [41], Li [64], Quaas and Xia [80], Ros-Oton and Serra [81] and Sire and Valdinoci [88].

Both operators, the laplacian and the fractional laplacian, are particular cases of a general class of elliptic operators connected to backward stochastic differential equations associated to Brownian and Levy-Itô processes, see for example Barles, Buckdahn and Pardoux [2], Benth, Karlsen and Reikvam [6] and Pham [79]. Recently, Barles, Chasseigne, Ciomaga and Imbert in [3, 4] and Ciomaga in [38] considered the existence and regularity of solutions for equations involving mixed integro-differential operators belonging to the general class of backward stochastic differential equations mentioned above. A particular case of elliptic integro-differential operator of mixed type is the one considering the laplacian in some of the variables and the fractional laplacian in the others, modeling diffusion sensible to the direction. In view of (2.2) and (2.5) we may write similarly

$$\begin{cases} (-\Delta)_x^\alpha u + (-\Delta)_y u + u = f(u), \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M, \\ u > 0 \ \text{ in } \mathbb{R}^N \times \mathbb{R}^M, \qquad \lim_{|(x,y)|\to+\infty} u(x,y) = 0, \end{cases} \tag{2.8}$$

where $N \geq 1$, $M \geq 1$. The operator $(-\Delta)_y$ denotes the usual laplacian with respect to y, while $(-\Delta)_x^\alpha$ denotes the fractional laplacian of exponent $\alpha \in (0,1)$ with respect to x, i.e.

$$(-\Delta)_x^\alpha u(x,y) = \int_{\mathbb{R}^N} \frac{u(x,y) - u(z,y)}{|x-z|^{N+2\alpha}} dz, \tag{2.9}$$

for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. Here the integral is understood in the principal value sense.

In view of the known results on decay and symmetry for solutions of equations (2.2) and (2.5) just described above, it is interesting to ask if these results still hold for solutions of the equation of mixed type (2.8), where the elliptic operator represents diffusion depending on the direction in space. Regarding the asymptotic decay of solution at infinity, the question is interesting since a proper mix of the two variables should be obtained for the decay estimates. The natural way to estimate the decay is through the construction of super and sub solutions involving the fundamental solution of the elliptic operator, which in this case is singular in $\mathbb{R}^N \times \{0\}$. Moreover, the solution of (2.8) cannot be radially symmetric, so this property cannot be used to estimate the decay. On the other hand, regarding radial symmetry, we may still have symmetry in x and y, but the moving planes method

would require an adequate version of the Hopf's Lemma, that we prove here.

Our first theorem in this chapter concerns the decay of solutions for (2.8) with general nonlinearity and it states as follows.

Theorem 2.1.1 *Let $\alpha \in (0,1)$, $N, M \in \mathbb{N}$, $N \geq 1$ and $M \geq 1$ and let us assume that the function $f : (0, +\infty) \to \mathbb{R}$ is continuous and it satisfies*

$$-\infty < B := \liminf_{v \to 0^+} \frac{f(v)}{v} \leq A := \limsup_{v \to 0^+} \frac{f(v)}{v} < 1. \tag{2.10}$$

Let u be a positive classical solution of problem (2.8), then for any $\epsilon > 0$ small, there exists $C_\epsilon > 1$ such that for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$,

$$C_\epsilon^{-1}(1 + |x|)^{-N-2\alpha} e^{-\theta_2 |y|} \leq u(x, y) \leq C_\epsilon (1 + |x|)^{-N-2\alpha} e^{-\theta_1 |y|}, \tag{2.11}$$

where

$$\theta_1 = \sqrt{1 - A} - \epsilon \quad \text{and} \quad \theta_2 = \sqrt{1 - B} + \epsilon. \tag{2.12}$$

When we compare estimate (2.11) with (2.4) for $N = 0$, we first observe that in ours an exponential decay is obtained, but with a constant C_ϵ depending on ϵ, which is a parameter controlling the rate of exponential decay. This is more clear when $A = B = 0$. On the other hand we are making much more general assumptions on f and, in particular, we are not making any assumption on the radial symmetry of the solution, which is crucial in proving (2.4). We do not know of a decay estimate better than

$$C_\epsilon^{-1} e^{-\theta_2 |y|} \leq u(y) \leq C_\epsilon e^{-\theta_1 |y|}, \qquad y \in \mathbb{R}^M, \tag{2.13}$$

for solutions of (2.2) under assumption (2.10) for f, and where radial symmetry of the solutions is not available, like in a case where f may depend on y. On the other hand, when $M = 0$, we recover (2.6) from (2.11). For the proof of the decay estimate (2.11) we construct suitable super and sub solutions and we use comparison principle with a version of Hopf's lemma.

When we assume further hypothesis we can get sharper estimates for the decay of the solutions of equation (2.8). Precisely, we have the following result:

Theorem 2.1.2 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 5$ and the non-linearity $f : (0, +\infty) \to \mathbb{R}$ is non-negative and it satisfies (2.3). Let u be a positive classical solution of (2.8), then there exists a constant $c > 1$ such that for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$,*

$$\frac{1}{c} \rho(x, y) \leq u(x, y) \leq c \rho(x, y)(1 + |y|)^{\frac{1}{2}}, \tag{2.14}$$

where the function ρ is defined as

$$\rho(x,y) = \min\{\frac{1}{(1+|x|)^{N+2\alpha}}\,,\, e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}\,,\, \frac{e^{-|y|}|y|^{1-\frac{M}{2}}}{(1+|x|)^{N+2\alpha}}\}. \tag{2.15}$$

We notice that this theorem gives the expected exponential decay for positive solutions, as suggested by (2.4), assuming the dimension of the space satisfies $M \geq 5$. Moreover, it gives the expected polynomial correction for the lower bound with a gap in the power for the upper bound. This theorem is proved under the assumption (2.3) on the non-linearity, constructing super and sub solutions devised upon the fundamental solution of $(-\Delta)_x^\alpha + (-\Delta)_y + id$. In our argument, a crucial role is played by the estimate already obtained in Theorem 2.1.1. Since the fundamental solution of $(-\Delta)_x^\alpha + (-\Delta)_y + id$ has $\mathbb{R}^N \times \{0\}$ as singular set, we cannot use the method in [51] in order to derive our estimate. Moreover, some other arguments in [51] cannot be used either because the solutions of (2.8) are not radial, since the differential operator is not radially invariant and there are no solutions depending only on one of the x or y variables, as can be seen from (2.14),

Even though solutions of (2.8) are not radially symmetric, we can prove partial symmetry in each of the variables x and y and this is the content of our third theorem.

Theorem 2.1.3 *Assume that $\alpha \in (0,1)$, $N \geq 1$, $M \geq 1$ and the function $f : (0,+\infty) \to \mathbb{R}$ is locally Lipschitz and it satisfies (2.10). Moreover, we assume that f also satisfies*

(F) *there exist $u_0 > 0$, $\gamma > \frac{N}{N+M} \cdot \frac{2\alpha}{N+2\alpha}$ and $\bar{c} > 0$ such that*

$$\frac{f(v)-f(u)}{v-u} \leq \bar{c}v^\gamma \quad \text{for all} \quad 0 < u < v < u_0. \tag{2.16}$$

Then, every positive classical solution u of equation (2.8) satisfies

$$u(x,y) = u(r,s)$$

and $u(r,s)$ is strictly decreasing in r and s, where $r = |x|$ and $s = |y|$.

When $N = 0$, we see that assumption (F) implies $\gamma > 0$ and (2.16) coincides with the assumption considered in [62]. When $M = 0$, assumption (F) implies that $\gamma > \frac{2\alpha}{N+2\alpha}$ and it coincides with the assumption considered in [46], when the solutions is assumed to decay as a power $N + 2\alpha$ at infinity. We remark that the operator $(-\Delta)_x^\alpha + (-\Delta)_y$ is a combination of two operators with different differential orders in $x-$variable and $y-$variable, and this produced a combined polynomial-exponential decay and does not allow for radial symmetry, but only partial symmetry as stated in Theorem 2.1.3.

The proof of Theorem 2.1.3 is based on the moving planes method as developed in [46, 62]. In these arguments, the strong maximum principle plays a crucial role and it is available for the laplacian and for the fractional laplacian. However, in the case of our mixed integro-differential operator some difficulties arise and we overcome them with a version of the Hopf's Lemma.

2.2 Preliminaries

This section is devoted to study the Strong Maximum Principle for mixed integro-differential operators as in equation (2.8). To this end, we prove first a suitable form of the Hopf's Lemma.

However, before to go to this, we recall some basic properties of the Sobolev embeddings. If we denote the Sobolev spaces

$$H(\mathbb{R}^{N+M}) = \{w \in L^2(\mathbb{R}^{N+M}) | \int_{\mathbb{R}^M} \int_{\mathbb{R}^N} (|\xi_1|^{2\alpha} + |\xi_2|^2 + 1)|\hat{w}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 < \infty\}$$

and

$$H^\alpha(\mathbb{R}^{N+M}) = \{w \in L^2(\mathbb{R}^{N+M}) \mid \int_{\mathbb{R}^{N+M}} (|\xi|^{2\alpha} + 1)|\hat{w}(\xi)|^2 d\xi < \infty\},$$

with norms

$$\|w\|_H = (\int_{\mathbb{R}^M} \int_{\mathbb{R}^N} (|\xi_1|^{2\alpha} + |\xi_2|^2 + 1)|\hat{w}(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2)^{\frac{1}{2}}$$

and

$$\|w\|_{H^\alpha} = (\int_{\mathbb{R}^{N+M}} (|\xi|^{2\alpha} + 1)|\hat{w}(\xi)|^2 d\xi)^{\frac{1}{2}},$$

respectively, then it is not difficult to see that the following proposition holds.

Proposition 2.2.1 *For $\alpha \in (0, 1)$, we have that*

$$H(\mathbb{R}^{N+M}) \subset H^\alpha(\mathbb{R}^{N+M}) \subset L^p(\mathbb{R}^{N+M}),$$

where the first inclusion is continuous and the second inclusion is continuous if $1 \leq p \leq \frac{2(N+M)}{N+M-2\alpha}$. Moreover,

$$H(\mathbb{R}^{N+M}) \subset L^p_{loc}(\mathbb{R}^{N+M})$$

is compact if $1 \leq p < \frac{2(N+M)}{N+M-2\alpha}$.

We devote the rest of this section to prove the Strong Maximum Principle in our context and to this end, we start with versions of the Maximum Principle and

the Hopf's Lemma. In what follows, given Ω an open subset in $\mathbb{R}^N \times \mathbb{R}^M$, we define its closed cylindrical extension in the direction x as

$$\tilde{\Omega} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M : \exists\, x' \in \mathbb{R}^N \text{ s.t. } (x', y) \in \bar{\Omega}\}.$$

Given a function h defined in an appropriate domain, we consider the mixed integro-differential operator

$$\mathcal{L}w(x, y) = (-\Delta)_x^\alpha w(x, y) + (-\Delta)_y w(x, y) + h(x, y)w(x, y).$$

Lemma 2.2.1 *Assume that Ω is an open domain of $\mathbb{R}^N \times \mathbb{R}^M$ and the function $h : \Omega \to \mathbb{R}$ satisfies $h \geq 0$ in Ω. If the function $w \in C(\bar{\Omega}) \cap L^\infty(\tilde{\Omega})$ satisfies*

$$\begin{cases} \mathcal{L}w \geq 0 \quad \text{in} \quad \Omega, \quad w \geq 0 \quad \text{in} \quad \tilde{\Omega} \setminus \Omega, \\ \liminf_{(x,y)\in\Omega, |(x,y)|\to\infty} w(x, y) \geq 0 \end{cases} \tag{2.17}$$

then $w \geq 0$ in $\tilde{\Omega}$.

Proof. If not, we may assume that there exists some $(x_0, y_0) \in \Omega$ such that

$$w(x_0, y_0) = \min_{(x,y)\in\tilde{\Omega}} w(x, y) < 0.$$

Then

$$(-\Delta)_x^\alpha w(x_0, y_0) = \int_{\mathbb{R}^N} \frac{w(x_0, y_0) - w(z, y_0)}{|x_0 - z|^{N+2\alpha}} dz < 0$$

and

$$(-\Delta)_y w(x_0, y_0) \leq 0$$

and then, since h is non-negative we have $\mathcal{L}w(x_0, y_0) < 0$, which contradicts (2.17), completing the proof. □

It what follows we prove a version of the Hopf's Lemma and for this purpose we need to give some conditions to the boundary of the domain where the function is defined. We say that the domain $\Omega \subset \mathbb{R}^N \times \mathbb{R}^M$ satisfies *interior cylinder condition at* $(x_0, y_0) \in \partial\Omega$ if there exist $r > 0$ and $\tilde{y} \in \mathbb{R}^M$ such that $O_r = B_r^N(x_0) \times B_r^M(\tilde{y})$ satisfies

$$O_r \subset \Omega \quad \text{and} \quad (x_0, y_0) \in \partial O_r, \tag{2.18}$$

where $B_r^N(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$ and $B_r^M(\tilde{y}) = \{y \in \mathbb{R}^M : |y - \tilde{y}| < r\}$ and, obviously $|\tilde{y} - y_0| = r$. We define also

$$D = \{(x, y) \in O_r : |x - x_0| < \frac{r}{2},\ |y - \tilde{y}| > \frac{r}{2}\}. \tag{2.19}$$

Lemma 2.2.2 *[Hopf's Lemma] Let Ω be an open set satisfying interior cylinder condition at $(x_0, y_0) \in \partial\Omega$. Assume that $h \in L^\infty(D)$ and $w \in C(\bar{\Omega}) \cap Ł^\infty(\tilde{\Omega})$ satisfies*

$$\mathcal{L}w \geq 0 \quad \text{in } \Omega$$

and

$$0 = w(x_0, y_0) < w(x, y), \quad \forall (x, y) \in \Omega.$$

Further assume that for $r > 0$ be given in (2.19) and for any $(x, y) \in D$ we have

$$\int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{w(z, y)}{|x - z|^{N+2\alpha}} dz \geq 0. \tag{2.20}$$

Then

$$\limsup_{s \to 0^+} \frac{w(x_0, y_0) - w(x_0, y_0 + s\tilde{y})}{s} < 0, \tag{2.21}$$

moreover, if the limit exists, then

$$\frac{\partial w}{\partial n}(x_0, y_0) < 0, \tag{2.22}$$

where n is the unit exterior normal vector of Ω at the point (x_0, y_0).

Proof. Let us define

$$\varphi_M(y) = e^{-\beta|y-\tilde{y}|^2} - e^{-\beta r^2}, \quad y \in \bar{B}_r^M(\tilde{y}), \tag{2.23}$$

where $\beta > 0$ will be chosen later. By direct computation, we have that

$$-\Delta\varphi_M(y) = (2M\beta - 4\beta^2|y - \tilde{y}|^2)e^{-\beta|y-\tilde{y}|^2}. \tag{2.24}$$

Next we consider the function

$$v(x, y) = \varphi_N(x)\varphi_M(y), \quad (x, y) \in \tilde{O}_r,$$

where φ_N is the first eigenfunction of Dirichlet problem

$$\begin{cases} (-\Delta)^\alpha \varphi_N(x) = \lambda_1 \varphi_N(x), & x \in B_{r/2}^N(x_0), \\ \varphi_N(x) = 0, & x \in \mathbb{R}^N \setminus B_{r/2}^N(x_0), \end{cases} \tag{2.25}$$

where φ_N is positive and bounded in $B_{r/2}^N(x_0)$ and the first eigenvalue λ_1, is positive, see Propositions 9 and 4 in [86] and [85], respectively.

For $(x,y) \in D$, by (2.24) and (2.25), we obtain that

$$\begin{aligned}
\mathcal{L}v(x,y) &= \varphi_M(y)(-\Delta)^\alpha \varphi_N(x) + \varphi_N(x)(-\Delta\varphi_M(y)) + h(x,y)\varphi_N(x)\varphi_M(y) \\
&= \varphi_N(x)[\lambda_1\varphi_M(y) + (2M\beta - 4\beta^2|y-\tilde{y}|^2)e^{-\beta|y-\tilde{y}|^2} + h(x,y)\varphi_M(y)] \\
&\leq \varphi_N(x)e^{-\beta|y-\tilde{y}|^2}(\lambda_1 + 2M\beta - \beta^2 r^2 + \|h\|_{L^\infty(D)}),
\end{aligned}$$

where the last inequality holds by the fact that $0 \leq \varphi_M(y) < e^{-\beta|y-\tilde{y}|^2}$ and $|y-\tilde{y}| > r/2$ in D. Let us choose $\beta > 0$ big enough such that

$$\mathcal{L}v \leq 0 \quad \text{in } D. \tag{2.26}$$

On the other hand, since $\varphi_N(x) = 0$ for $|x-x_0| \geq r/2$ and $\varphi_M(y) = 0$ for $|y-\tilde{y}| = r$, it is obvious that $v = 0$ in $A_1 \cup A_2$ where $A_1 = \{(x,y) \in \tilde{D} : |x-x_0| \geq r/2\}$ and $A_2 = \{(x,y) \in \bar{D} : |y-\tilde{y}| = r\}$. If we define the set $A_3 := \{(x,y) \in \bar{D} : |y-\tilde{y}| = r/2\}$, we see that $\tilde{D} \setminus D = A_1 \cup A_2 \cup A_3$. We also observe that v is a bounded function in $\tilde{O}_r$.

Next we prove (2.21) assuming $h \geq 0$. Defining

$$W(x,y) = \begin{cases} w(x,y), & (x,y) \in \bar{O}_r, \\ 0, & (x,y) \in \tilde{O}_r \setminus \bar{O}_r \end{cases} \tag{2.27}$$

and using (2.20), we have that for any $(x,y) \in D$,

$$\mathcal{L}W(x,y) = \mathcal{L}w(x,y) + \int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{w(z,y)}{|x-z|^{N+2\alpha}} dz \geq 0.$$

Combining with (2.26), we have that, for every $\epsilon > 0$

$$\mathcal{L}(W - \epsilon v) \geq 0 \quad \text{in } D. \tag{2.28}$$

Since v is bounded in $\tilde{O}_r$, the set A_3 is a compact subset of O_r and $w > 0$ in O_r, then there exists $\epsilon > 0$ small such that

$$W = w \geq \epsilon v \quad \text{in } A_3.$$

Since $v = 0$ in $A_1 \cup A_2$, $w \geq 0$ in $\bar{O}_r$ and (2.27), we have $W \geq 0 = \epsilon v$ in $A_1 \cup A_2$. Consequently,

$$W - \epsilon v \geq 0 \quad \text{in } \tilde{D} \setminus D.$$

Then we can use Lemma 2.2.1, recalling that $h \geq 0$ to obtain that

$$W - \epsilon v \geq 0 \quad \text{in } D.$$

In view of the definition of W, since $D \subset \bar{O}_r$, we find that $w - \epsilon v \geq 0$ in D and

noticing that $w(x_0, y_0) = v(x_0, y_0) = 0$ we obtain that

$$\frac{w(x_0, y_0) - w(x_0, y_0 + s\tilde{y})}{s} \leq \epsilon \frac{v(x_0, y_0) - v(x_0, y_0 + s\tilde{y})}{s},$$

for all $s \in (0, r/2)$. Thus, we have

$$\begin{aligned}
\limsup_{s \to 0^+} \frac{w(x_0, y_0) - w(x_0, y_0 + s\tilde{y})}{s} &\leq \epsilon \lim_{s \to 0^+} \frac{v(x_0, y_0) - v(x_0, y_0 + s\tilde{y})}{s} \\
&= \epsilon \varphi_N(x_0) \lim_{s \to 0^+} \frac{\varphi_M(y_0) - \varphi_M(y_0 + s\tilde{y})}{s} \\
&= -2\epsilon\beta r^2 e^{-\beta r^2} \varphi_N(x_0) \\
&< 0,
\end{aligned}$$

completing the proof of (2.21).

The case for general h can be done simply by replacing h by h^+. In fact, since $w > 0$ in Ω, we have

$$(-\Delta)_x^\alpha w(x, y) + (-\Delta)_y w(x, y) + h^+(x, y) w(x, y) \geq 0, \quad (x, y) \in \Omega$$

and similarly we obtain that

$$(-\Delta)_x^\alpha v(x, y) + (-\Delta)_y v(x, y) + h^+(x, y) v(x, y) \leq 0, \quad (x, y) \in D,$$

so we may proceed as before to get (2.21) and the proof is complete. □

In order to state the Strong Maximum Principle to be used in our moving planes procedure, it is convenient to consider property (P):

(P) We say that a function $w : \tilde{\Omega} \to \mathbb{R}$ satisfies property (P) if whenever $(x_0, y_0) \in \Omega$ such that

$$0 = w(x_0, y_0) = \inf_{(x,y) \in \Omega} w(x, y),$$

then

$$w(x, y_0) \equiv 0, \quad \forall x \in \mathbb{R}^N.$$

The following lemma is in preparation of the strong maximum principle.

Lemma 2.2.3 *Let Ω be an open set in $\mathbb{R}^N \times \mathbb{R}^M$ and w have property (P). We denote*

$$\Omega_0 = \{(x, y) \in \Omega : w(x, y) = \inf_{\Omega} w = 0\}. \tag{2.29}$$

If $\emptyset \neq \Omega_0 \subsetneqq \Omega$, then $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at any point $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$.

Proof. Since $\emptyset \neq \Omega_0 \subsetneqq \Omega$, we have that $\emptyset \neq \partial\Omega_0 \cap \Omega \subset \partial(\Omega \setminus \Omega_0)$. For any $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$, let us denote $r = \frac{1}{4}dist((x_0, y_0), \partial\Omega)$ and let $\tilde{y} \in \mathbb{R}^M$ such that $(x_0, \tilde{y}) \in \Omega \setminus \Omega_0$ and $|\tilde{y} - y_0| = r$. Since w has property (P), then $w = 0$ in $\tilde{\Omega}_0$, where $\tilde{\Omega}_0$ is the extension of Ω_0 in x-direction and as $\Omega \setminus \Omega_0$ is open, we have that $B_r^N(x_0) \times B_r^M(\tilde{y}) \subset \Omega \setminus \Omega_0$. Therefore, $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$.

□

Theorem 2.2.1 *[Strong Maximum Principle] Let Ω be an open set of $\mathbb{R}^N \times \mathbb{R}^M$, the function $h \in L^\infty_{loc}(\Omega)$ and $w \in C(\bar{\Omega}) \cap L^\infty(\tilde{\Omega})$ has the property (P) satisfying*

$$\mathcal{L}w \geq 0 \quad \text{in } \Omega \quad \text{and} \quad w \geq 0 \quad \text{in } \Omega. \tag{2.30}$$

Assume that $\Omega_0 \neq \emptyset$ defined by (2.29) and there exists some $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$ such that (2.20) holds in corresponding D.

Then w must be 0 in $\tilde{\Omega}$.

Proof. Assume that $\Omega_0 \neq \Omega$. By Lemma 2.2.3, $\Omega \setminus \Omega_0$ satisfies interior cylinder condition at $(x_0, y_0) \in \partial\Omega_0 \cap \Omega$ and then $w(x_0, y_0) = 0$ by $w \in C(\bar{\Omega})$ and the definition of Ω_0. Furthermore, we observe that $\bar{D}$ is compact in Ω and then $h \in L^\infty(\bar{D})$. Using Lemma 2.2.2, we obtain (2.21), which is impossible by the fact of $w(x_0, y_0) = \inf_\Omega w = 0$. Therefore, $\Omega_0 = \Omega$, i.e. $w \equiv 0$ in Ω. Since w has property (P), then $w \equiv 0$ in $\tilde{\Omega}$. □

2.3 Decay estimate

2.3.1 Proof of Theorem 2.1.1

In this subsection, we prove Theorem 2.1.1 on decay estimates for positive classical solutions of equation (2.8). The main work is to construct appropriate super and sub solutions and then the decay estimate is derived by Lemma 2.2.1.

Before proving Theorem 2.1.1, we introduce some computations gathered in the next proposition. For $\alpha \in (0, 1)$ and $\mu > 0$, we define the function $\psi_\mu : \mathbb{R}^N \to \mathbb{R}$ as follows

$$\psi_\mu(x) = \begin{cases} \mu^{-N-2\alpha}, & |x| < \mu, \\ |x|^{-N-2\alpha}, & |x| \geq \mu. \end{cases} \tag{2.31}$$

Proposition 2.3.1 *For any $\mu > 0$, there exists $R_0 > 3\mu$ and $c > 0$, independent of μ, such that*

$$-c\mu^{-2\alpha}\psi_\mu(x) \leq (-\Delta)^\alpha \psi_\mu(x) \leq -c^{-1}\mu^{-2\alpha}\psi_\mu(x), \quad x \in B^c_{R_0}. \tag{2.32}$$

Proof. We consider along the proof that $\mu > 0$ and $x \in \mathbb{R}^N$ satisfies $|x| > 3\mu$. We define

$$A(\mu, x, z) = \frac{\psi_\mu(x+z) + \psi_\mu(x-z) - 2\psi_\mu(x)}{|z|^{N+2\alpha}}, \qquad z \in \mathbb{R}^N$$

and we observe that

$$(-\Delta)^\alpha \psi_\mu(x) = -\frac{1}{2}\int_{\mathbb{R}^N} A(\mu, x, z)dz. \tag{2.33}$$

Now we compute the integral above by decomposing the domain in various pieces. First we consider the integral over $B_{\frac{|x|}{3}}(0)$. We observe that $|x \pm z| \geq \mu$ for all $z \in B_{\frac{|x|}{3}}(0)$, then by (2.31) we obtain

$$\begin{aligned}
& |\int_{B_{\frac{|x|}{3}}(0)} A(\mu, x, z)dz| \\
= \; & |\int_{B_{\frac{|x|}{3}}(0)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} - 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}} dz| \\
= \; & |x|^{-N-4\alpha} |\int_{B_{\frac{1}{3}}(0)} \frac{|z+e_x|^{-N-2\alpha} + |z-e_x|^{-N-2\alpha} - 2}{|z|^{N+2\alpha}} dz| \\
\leq \; & c_1 |x|^{-N-4\alpha} \int_{B_{\frac{1}{3}}(0)} \frac{|z|^2}{|z|^{N+2\alpha}} dz \leq c_2 |x|^{-N-4\alpha},
\end{aligned} \tag{2.34}$$

where $e_x = \frac{x}{|x|}$ and $c_1, c_2 > 0$ are independent of μ. Next we consider the integral over $B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)$. We observe that for all $z \in B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)$ we have $|x+z| \geq |x-z| \geq \mu$ and then we obtain

$$\begin{aligned}
& \int_{B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)} A(\mu, x, z)dz \\
= \; & \int_{B_{\frac{|x|}{3}}(x) \setminus B_\mu(x)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} - 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}} dz \\
= \; & |x|^{-N-4\alpha} \int_{B_{\frac{1}{3}}(e_x) \setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z+e_x|^{-N-2\alpha} + |z-e_x|^{-N-2\alpha} - 2}{|z|^{N+2\alpha}} dz \\
\leq \; & c_3 |x|^{-N-4\alpha} \int_{B_{\frac{1}{3}}(e_x) \setminus B_{\frac{\mu}{|x|}}(e_x)} |z-e_x|^{-N-2\alpha} dz \leq c_4 \mu^{-2\alpha} |x|^{-N-2\alpha},
\end{aligned}$$

where the first inequality holds since $|z+e_x| \geq |z-e_x|$ for $z \in B_{\frac{1}{3}}(e_x) \setminus B_{\frac{\mu}{|x|}}(e_x)$

and $|z| \geq \frac{2}{3}$ for $z \in B_{\frac{1}{3}}(e_x)$. For the inequality on the other side, we obtain

$$\begin{aligned}
&\int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} A(\mu,x,z)dz \\
= \;& |x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z+e_x|^{-N-2\alpha}+|z-e_x|^{-N-2\alpha}-2}{|z|^{N+2\alpha}}dz \\
\geq \;& |x|^{-N-4\alpha}\Big(\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} \frac{|z-e_x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz - \int_{B_{\frac{1}{3}}(e_x)} \frac{2}{|z|^{N+2\alpha}}dz\Big) \\
\geq \;& c_5|x|^{-N-4\alpha}\int_{B_{\frac{1}{3}}(e_x)\setminus B_{\frac{\mu}{|x|}}(e_x)} |z-e_x|^{-N-2\alpha}dz - c_6|x|^{-N-4\alpha} \\
\geq \;& c_7\mu^{-2\alpha}|x|^{-N-2\alpha} - c_8|x|^{-N-4\alpha},
\end{aligned}$$

where the second inequality holds by $|z| \leq \frac{4}{3}$ for $z \in B_{\frac{1}{3}}(e_x)$. Consequently,

$$c_7\mu^{-2\alpha}|x|^{-N-2\alpha} - c_8|x|^{-N-4\alpha} \leq \int_{B_{\frac{|x|}{3}}(x)\setminus B_\mu(x)} A(\mu,x,z)dz \leq c_4\mu^{-2\alpha}|x|^{-N-2\alpha}, \tag{2.35}$$

where the constants $c_4, c_7, c_8 > 0$ are independent of μ. The estimate for the integral over $B_{\frac{|x|}{3}}(-x) \setminus B_\mu(-x)$ is similar.

Next we consider the integral over $B_\mu(x)$. We observe that, for $z \in B_\mu(x)$ we have since $|x+z| > \mu > |x-z|$ and $|z| \geq |x| - \mu \geq \frac{2|x|}{3}$, thus

$$\begin{aligned}
&\int_{B_\mu(x)} A(\mu,x,z)dz = \int_{B_\mu(x)} \frac{|x+z|^{-N-2\alpha}+\mu^{-N-2\alpha}-2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz \\
&\leq 2\int_{B_\mu(x)} \frac{\mu^{-N-2\alpha}}{|z|^{N+2\alpha}}dz \leq c_9\mu^{-2\alpha}(|x|-\mu)^{-N-2\alpha} \leq c_{10}\mu^{-2\alpha}|x|^{-N-2\alpha}
\end{aligned}$$

and, for the other inequality

$$\begin{aligned}
\int_{B_\mu(x)} A(\mu,x,z)dz \;&\geq\; \int_{B_\mu(x)} \frac{-2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}}dz \\
&\geq\; -c_{11}\mu^N|x|^{-N-2\alpha}(|x|-\mu)^{-N-2\alpha} \geq -c_{12}|x|^{-N-4\alpha},
\end{aligned}$$

where c_9, c_{10}, c_{11} and c_{12} are positive constant independent of μ. Therefore,

$$-c_{12}|x|^{-N-4\alpha} \leq \int_{B_\mu(x)} A(\mu,x,z)dz \leq c_{10}\mu^{-2\alpha}|x|^{-N-2\alpha}. \tag{2.36}$$

The integral over $B_\mu(-x)$ is exactly the same. Finally, we consider the complementary integral over $D(x) = \mathbb{R}^N \setminus (B_{\frac{|x|}{3}}(0) \cup B_{\frac{|x|}{3}}(x) \cup B_{\frac{|x|}{3}}(-x))$. For $|x| > 3\mu$ and $z \in D(x)$, we have that $|x \pm z| \geq \frac{|x|}{3}$, thus

$$\begin{aligned} |\int_{D(x)} A(\mu, x, z)dz| &\leq \int_{D(x)} \frac{|x+z|^{-N-2\alpha} + |x-z|^{-N-2\alpha} + 2|x|^{-N-2\alpha}}{|z|^{N+2\alpha}} dz \\ &\leq c_{13}|x|^{-N-2\alpha} \int_{\mathbb{R}^N \setminus B_{\frac{|x|}{3}}(0)} \frac{1}{|z|^{N+2\alpha}} dz \\ &\leq c_{14}|x|^{-N-4\alpha}, \end{aligned} \tag{2.37}$$

where $c_{13} > 0$ and $c_{14} > 0$ are independent of μ. Therefore, by (2.34)-(2.37), there exist $c_{15}, c_{16} > 1$ independent of μ such that

$$\begin{aligned} &c_{15}^{-1}\mu^{-2\alpha}|x|^{-N-2\alpha} - c_{15}|x|^{-N-4\alpha} \leq \int_{\mathbb{R}^N} A(\mu, x, z)dz \\ &\leq c_{16}\mu^{-2\alpha}|x|^{-N-2\alpha} + c_{16}|x|^{-N-4\alpha} \leq c_{15}\mu^{-2\alpha}|x|^{-N-2\alpha}, \end{aligned}$$

where we used that $|x| > 3\mu$. Choosing $R_0 > 3\mu$ such that $c_{15}^{-1}\mu^{-2\alpha} - c_{15}|x|^{-2\alpha} \geq \frac{1}{2}c_{15}^{-1}\mu^{-2\alpha}$ for $|x| \geq R_0$, together with (2.33), we obtain (2.32). □

In what follows we provide a proof of our first theorem on the decay of the positive solutions of our equation.

Proof of Theorem 2.1.1. By definition of A and B in (2.10), for any $\epsilon > 0$, there exits $\delta_\epsilon > 0$ such that

$$(B - \epsilon^2)t \leq f(t) \leq (A + \epsilon^2)t, \quad \forall\, t \in (0, \delta_\epsilon). \tag{2.38}$$

Since u is a positive solution of (2.8) vanishing at infinity, there exists $R_\epsilon > 0$ such that $0 < u(x, y) < \delta_\epsilon$ for any $(x, y) \in B^c_{R_\epsilon}$. Therefore,

$$(-\Delta)^\alpha_x u + (-\Delta)_y u + (1 - A - \epsilon^2)u \leq 0 \quad \text{in } B^c_{R_\epsilon} \tag{2.39}$$

and

$$(-\Delta)^\alpha_x u + (-\Delta)_y u + (1 - B + \epsilon^2)u \geq 0 \quad \text{in } B^c_{R_\epsilon}. \tag{2.40}$$

Next we define the function $\phi_\nu : \mathbb{R}^M \to \mathbb{R}$ as $\phi_\nu(y) = e^{-\nu|y|}$, where $\nu > 0$ and we find that for $y \in \mathbb{R}^M \setminus \{0\}$,

$$-\Delta\phi_\nu(y) = \nu\left(\frac{M-1}{|y|} - \nu\right)\phi_\nu(y). \tag{2.41}$$

Step 1. *There exists $C(\epsilon) > 1$ such that*

$$u(x, y) \leq C(\epsilon)e^{-\theta_1|y|}, \qquad (x, y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{2.42}$$

To prove (2.42) we let $U_1(x,y) = \phi_{\theta_1}(y)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$ and then, by (2.41), we have

$$\begin{aligned}
&(-\Delta)_x^\alpha U_1 + (-\Delta)_y U_1 + (1 - A - \epsilon^2) U_1 \\
&= \left[\theta_1 \left(\frac{M-1}{|y|} - \theta_1\right) + 1 - A - \epsilon^2\right] U_1 \geq 0,
\end{aligned} \tag{2.43}$$

if $\epsilon \leq \sqrt{1-A}$. By definition of U_1 and ϕ_{θ_1} we have that $U_1 = 1$ in $\mathbb{R}^N \times \{0\}$ and $U_1 \geq e^{-\theta_1 R_\epsilon}$ in $\bar{B}_{R_\epsilon}$ and, since u is bounded, there exists $\rho_1 > 0$ depending on ϵ, such that

$$W_1 = \rho_1 U_1 - u \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon} \cup (\mathbb{R}^N \times \{0\}).$$

Combining (2.39) with (2.43), we obtain

$$(-\Delta)_x^\alpha W_1 + (-\Delta)_y W_1 + (1 - A - \epsilon^2) W_1 \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon}^c \cap (\mathbb{R}^N \times \{0\})^c.$$

By Lemma 2.2.1, this implies that $W_1 \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^M$ and then

$$u(x,y) \leq \rho_1 U_1(x,y) = \rho_1 \phi_{\theta_1}(y) = \rho_1 e^{-\theta_1 |y|}, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{2.44}$$

Step 2. *There exists $C(\epsilon) > 1$ such that*

$$u(x,y) \leq C(\epsilon)|x|^{-N-2\alpha}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{2.45}$$

Let c and R_0 be as in Proposition 2.3.1 $\mu = (c/(2\epsilon\sqrt{(1-A)} - 2\epsilon^2))^{\frac{1}{2\alpha}}$ and consider the function $U_2(x,y) = \psi_\mu(x)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. Then, by (2.32), we have for all $(x,y) \in (B_{R_0}^N(0))^c \times \mathbb{R}^M$ that

$$\begin{aligned}
&(-\Delta)_x^\alpha U_2 + (-\Delta)_y U_2 + (1 - A - \epsilon^2) U_2 \\
&\geq (-c\mu^{-2\alpha} + 1 - A - \epsilon^2) U_2 \geq 0
\end{aligned} \tag{2.46}$$

for $0 < \epsilon < \sqrt{1-A}$. Let us denote $W_2 = \rho_2 U_2 - u$, where $\rho_2 > 0$ is such that

$$W_2 \geq \rho_2 (R_0 + R_\epsilon)^{-N-2\alpha} - u \geq 0 \quad \text{in } \bar{B}_{R_\epsilon} \cup (\overline{B_{R_0}^N(0)} \times \mathbb{R}^M).$$

Combining (2.39) with (2.46), we obtain that

$$(-\Delta)_x^\alpha W_2 + (-\Delta)_y W_2 + (1 - A - \epsilon^2) W_2 \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon}^c \cap (\overline{B_{R_0}^N(0)} \times \mathbb{R}^M)^c.$$

By Lemma 2.2.1, we have that $W_2 = \rho_2 U_2 - u \geq 0$ in $R^N \times \mathbb{R}^M$ and then, for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$,

$$u(x,y) \leq \rho_2 U_2(x,y) = \rho_2 \psi_\mu(x) \leq \rho_2 |x|^{-N-2\alpha}.$$

Step 3. *There exists $C(\epsilon) > 1$ such that*

$$u(x,y) \leq C(\epsilon)|x|^{-N-2\alpha}e^{-\theta_1|y|}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M. \tag{2.47}$$

Let us consider the function $V(x,y) = \psi_\mu(x)\phi_{\theta_1}(y)$, for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, with μ as defined above. From (2.32) and (2.41), we have that

$$\begin{aligned} &(-\Delta)_x^\alpha V + (-\Delta)_y V + (1 - A - \epsilon^2)V \\ &\geq \left[-c\mu^{-2\alpha} + \theta_1\left(\frac{M-1}{|y|} - \theta_1\right) + 1 - A - \epsilon^2\right] V \geq 0, \end{aligned} \tag{2.48}$$

for $(x,y) \in (B_{R_0}^N(0))^c \times (\mathbb{R}^M \setminus \{0\})$ and assuming that $0 < \epsilon < \sqrt{1-A}$. Since u, V are bounded in $\bar{B}_{R_\epsilon}$ and V is positive, there is $\bar{\rho}_1 > 0$ large such that

$$\bar{\rho}_1 V - u \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon}.$$

By (2.42) and (2.44), we may choose $\bar{\rho}_2 > 0$ such that

$$\begin{aligned} \bar{\rho}_2 V - u &\geq \bar{\rho}_2 R_0^{-N-2\alpha}\phi_{\theta_1}(y) - u \geq 0 \quad \text{in} \quad \overline{B_{R_0}^N(0)} \times \mathbb{R}^M \quad \text{and} \\ \bar{\rho}_2 V - u &\geq \bar{\rho}_2 \psi_\mu(x) - u \geq 0 \quad \text{in} \quad \mathbb{R}^N \times \{0\}. \end{aligned}$$

Taking $\bar{\rho} = \max\{\bar{\rho}_1, \bar{\rho}_2\}$, defining $W = \bar{\rho}V - u$ and combining (2.39) with (2.48), we have that

$$W \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon} \cup (\overline{B_{R_0}^N(0)} \times \mathbb{R}^M) \cup (\mathbb{R}^N \times \{0\}) \quad \text{and}$$

$$(-\Delta)_x^\alpha W + (-\Delta)_y W + (1 - A - \epsilon^2)W \geq 0 \quad \text{in} \quad \bar{B}_{R_\epsilon}^c \cap ((B_{R_0}^N(0))^c \times (\mathbb{R}^M \setminus \{0\})).$$

Then, by Lemma 2.2.1, we have that $\bar{\rho}V - u \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^M$. Thus, there exists $C(\epsilon) > 1$ such that

$$u(x,y) \leq C(\epsilon)\psi_\mu(x)\phi_{\theta_1}(y) \leq C(\epsilon)|x|^{-N-2\alpha}e^{-\theta_1|y|}, \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M.$$

Step 4. *There exists $C_1(\epsilon) > 0$ and $R > 0$ such that*

$$u(x,y) \geq C_1(\epsilon)e^{-\theta_2|y|}, \qquad (x,y) \in \overline{B_R^N(0)} \times \mathbb{R}^M. \tag{2.49}$$

Let R_0 be as in Proposition 2.3.1 and let $R > R_0$ such that $\lambda_1 < \epsilon^2$, where λ_1 is the first eigenvalue of the fractional Dirichlet problem (2.25) with $x_0 = 0$ and $r = 4R$. Let φ_N be the first eigenfunction of (2.25) and define $V_1(x,y) = \varphi_N(x)\phi_{\theta_2}(y)$ for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$. From (2.25) and (2.41), for $(x,y) \in B_{2R}^N(0) \times (B_{R_1}^M(0))^c$ with

$R_1 = \frac{M-1}{\epsilon}$, we have

$$\begin{aligned}
&(-\Delta)_x^\alpha V_1 + (-\Delta)_y V_1 + (1 - B + \epsilon^2) V_1 \\
&= \left[\lambda_1 + \theta_2 \left(\frac{M-1}{|y|} - \theta_2\right) + 1 - B + \epsilon^2\right] V_1 \\
&\leq [\epsilon^2 + \theta_2(\epsilon - \theta_2) + 1 - B + \epsilon^2] V_1 \leq 0,
\end{aligned} \tag{2.50}$$

if $\epsilon < \sqrt{1-B}$. Let us define $w_1 = u - r_1 V_1$, where $r_1 > 0$ is such that

$$w_1 \geq 0 \quad \text{in } \overline{B}_{R_\epsilon} \cup (\overline{B_{2R}^N(0) \times B_{R_1}^M(0)})$$

and observe that $w_1 \geq 0$ in $(B_{2R}^N(0))^c \times \mathbb{R}^M$ since $V_1 = 0$. Combining (2.40) with (2.50), we obtain that

$$(-\Delta)_x^\alpha w_1 + (-\Delta)_y w_1 + (1 - B + \epsilon^2) w_1 \geq 0 \quad \text{in} \quad (B_{2R}^N(0) \times (B_{R_1}^M(0))^c) \cap B_{R_\epsilon}^c$$

and then, by Lemma 2.2.1, we have that

$$w_1 = u - r_1 V_1 \geq 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^M.$$

Since φ_N is classical solution of (2.25) with $r = 4R$ and $x_0 = 0$ then $\varphi_N(x)$ is positive in $\overline{B_R^N(0)} \subset \mathbb{R}^N$, we can finally choose $C_1(\epsilon) > 0$ such that

$$u(x,y) \geq r_1 \varphi_N(x) \phi_{\theta_2}(y) \geq C_1(\epsilon) e^{-\theta_2 |y|}, \quad \forall (x,y) \in \overline{B_R^N(0)} \times \mathbb{R}^M. \tag{2.51}$$

Step 5. *There exists $C_1(\epsilon) > 0$ such that, for R and R_1 as in Step 4,*

$$u(x,y) \geq C_1 |x|^{-N-2\alpha}, \qquad (x,y) \in (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}. \tag{2.52}$$

To prove this, we define $V_2(x,y) = \psi_\mu(x) \eta_M(y)$ for $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$, where η_M is the solution of

$$\begin{cases} -\Delta \eta_M(y) = \bar{\lambda}_1 \eta_M(y), & y \in B_{R_2}^M(0), \\ \eta_M(y) = 0, & y \in (B_{R_2}^M(0))^c, \end{cases} \tag{2.53}$$

with $R_2 > R_1$ such that $\bar{\lambda}_1 < \epsilon^2$. Here $\mu = [c(1 - B + 2\epsilon^2)]^{\frac{-1}{2\alpha}}$ with c as in Proposition 2.3.1 and ψ_μ defined in (2.31). By (2.32) and (2.53), for $(x,y) \in ((B_R^N(0))^c \times \mathbb{R}^M) \cap (\mathbb{R}^N \times B_{R_2}^M(0))$, we have that

$$\begin{aligned}
&(-\Delta)_x^\alpha V_2 + (-\Delta)_y V_2 + (1 - B + \epsilon^2) V_2 \\
&\leq (-c^{-1} \mu^{-2\alpha} + \bar{\lambda}_1 + 1 - B + \epsilon^2) V_2 = 0.
\end{aligned} \tag{2.54}$$

Let $w_2 = u - r_2 V_2$, with $r_2 > 0$ such that

$$w_2 \geq 0 \quad \text{in } \bar{B}_{R_\epsilon} \cup \overline{(B_R^N(0)} \times \mathbb{R}^M) \cup (\mathbb{R}^N \times (B_{R_2}^M(0))^c).$$

Combining (2.40) with (2.54), we obtain that

$$(-\Delta)_x^\alpha w_2 + (-\Delta)_y w_2 + (1 - B + \epsilon^2) w_2 \geq 0$$

in $B_{R_\epsilon}^c \cap ((B_R^N(0))^c \times \mathbb{R}^M) \cap (\mathbb{R}^N \times B_{R_2}^M(0))$. By Lemma 2.2.1, we have then

$$w_2 = u - r_2 V_2 \geq 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^M.$$

Since η_M is positive in $\overline{B_{R_1}^M(0)} \subset B_{R_2}^M(0)$, there exists $C_1(\epsilon) > 0$ such that for any $(x, y) \in (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}$, we have that

$$u(x, y) \geq r_2 \psi_\mu(x) \eta_M(y) \geq C_1(\epsilon) |x|^{-N-2\alpha}.$$

Step 6. *There exist $C_1(\epsilon) > 0$ such that, for R as in Step 4,*

$$u(x, y) \geq C_1(\epsilon) |x|^{-N-2\alpha} e^{-\theta_2 |y|}, \qquad (x, y) \in (B_R^N(0))^c \times \mathbb{R}^M. \tag{2.55}$$

To prove this we let $\tilde{V}(x, y) = \psi_\mu(x) \phi_{\theta_2}(y)$, for $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$ with μ as defined above. Using (2.32) and (2.41), for $(x, y) \in (B_R^N(0))^c \times (B_{R_1}^M(0))^c$ with $R_1 = \frac{M-1}{\epsilon}$, we have that

$$\begin{aligned}
&(-\Delta)_x^\alpha \tilde{V} + (-\Delta)_y \tilde{V} + (1 - B + \epsilon^2) \tilde{V} \\
&\leq \left[-c^{-1} \mu^{-2\alpha} + \theta_2 \left(\frac{M-1}{|y|} - \theta_2 \right) + 1 - B + \epsilon^2 \right] \tilde{V} \\
&\leq [\theta_2(\epsilon - \theta_2) + 1 - B + \epsilon^2] \tilde{V} \leq 0,
\end{aligned} \tag{2.56}$$

if $0 < \epsilon < \sqrt{1 - B}$. Since u is positive and V is bounded in $\overline{B}_{R_\epsilon}$, we can choose $\tilde{r}_1 > 0$ such that

$$u - \tilde{r}_1 V \geq 0 \quad \text{in } \overline{B}_{R_\epsilon}.$$

Since ψ_μ is bounded in $\overline{B_R^N(0)}$, using (2.51), there exists $\tilde{r}_2 > 0$ such that

$$u - \tilde{r}_2 V \geq u - \tilde{r}_2 c_1 e^{-\theta_2 |y|} \geq 0 \quad \text{in } \overline{B_R^N(0)} \times \mathbb{R}^M,$$

and by (2.52), there exists $\tilde{r}_3 > 0$ such that

$$u - \tilde{r}_3 V \geq u - \tilde{r}_3 |x|^{-N-2\alpha} \geq 0 \quad \text{in } (B_R^N(0))^c \times \overline{B_{R_1}^M(0)}.$$

Taking $\tilde{r} = \min\{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3\}$ and combining (2.40) with (2.56), we obtain that

$$\begin{aligned} w = u - \tilde{r}V \geq 0 \quad &\text{in} \quad \bar{B}_{R_\epsilon} \cup (\overline{B_R^N(0)} \times \mathbb{R}^M) \cup ((B_R^N(0))^c \times \overline{B_{R_1}^M(0)}) \quad \text{and} \\ (-\Delta)_x^\alpha w + (-\Delta)_y w + (1 - B + \epsilon) w \geq 0 \quad &\text{in} \quad \bar{B}_{R_\epsilon}^c \cap ((B_R^N(0))^c \times (B_{R_1}^M(0))^c). \end{aligned}$$

Thus Lemma 2.2.1, we have that $w \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^M$ and then (2.55) holds.

Finally, Step 1 − Step 6 completes the proof. □

2.3.2 Proof of Theorem 2.1.2

This subsection is devoted to prove Theorem 2.1.2. Our proof is based on the fundamental solution of the mixed integro-differential operator. We first study the fundamental solution $\mathcal{K}$ for

$$(-\Delta)_x^\alpha u + (-\Delta)_y u + u = 0 \quad \text{in} \quad \mathbb{R}^N \times (\mathbb{R}^M \setminus \{0\}),$$

which can be characterized by

$$\mathcal{K}(x, y) = \int_0^\infty e^{-t} \mathcal{H}(x, y, t) dt, \tag{2.57}$$

where

$$\mathcal{H}(x, y, t) = \int_{\mathbb{R}^M} \int_{\mathbb{R}^N} e^{-2\pi i (x,y)\cdot(\xi_1,\xi_2) - t(|\xi_1|^{2\alpha} + |\xi_2|^2)} d\xi_1 d\xi_2. \tag{2.58}$$

In fact, for $\phi \in \mathcal{S}$, we have that

$$\begin{aligned} \langle \mathcal{K}, \phi \rangle &= \textstyle\int_{\mathbb{R}^{N+M}} \int_0^\infty \int_{\mathbb{R}^{N+M}} e^{-2\pi i (x,y)\cdot(\xi_1,\xi_2) - t(|\xi_1|^{2\alpha} + |\xi_2|^2 + 1)} \phi(x, y) d\xi_1 d\xi_2 dt dx dy \\ &= \textstyle\int_{\mathbb{R}^{N+M}} \left[\int_0^\infty e^{-t(|\xi_1|^{2\alpha} + |\xi_2|^2 + 1)} dt \int_{\mathbb{R}^{N+M}} e^{-2\pi i (x,y)\cdot(\xi_1,\xi_2)} \phi(x, y) dx dy \right] d\xi_1 d\xi_2 \\ &= \textstyle\int_{\mathbb{R}^{N+M}} \left[\frac{1}{|\xi_1|^{2\alpha} + |\xi_2|^2 + 1} \int_{\mathbb{R}^{N+M}} e^{-2\pi i (x,y)\cdot(\xi_1,\xi_2)} \phi(x, y) dx dy \right] d\xi_1 d\xi_2 \\ &= \left\langle \frac{1}{|\xi_1|^{2\alpha} + |\xi_2|^2 + 1}, \mathcal{F}\phi \right\rangle. \end{aligned}$$

Next we want to find some properties of $\mathcal{H}$. To this end, we consider

$$\mathcal{H}_\alpha(x, t) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi_1 - t|\xi_1|^{2\alpha}} d\xi_1 \quad \text{and} \quad \mathcal{H}_1(y, t) = \int_{\mathbb{R}^M} e^{-2\pi i y\cdot\xi_2 - t|\xi_2|^2} d\xi_2.$$

It is well known that the function $\mathcal{H}_\alpha$ has the following properties:

$$\mathcal{H}_\alpha(x, t) = t^{-\frac{N}{2\alpha}} \mathcal{H}_\alpha(t^{-\frac{1}{2\alpha}} x, 1) \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{N+2\alpha} \mathcal{H}_\alpha(x, 1) = C,$$

where $C > 0$, which imply that there exists $c_1 > 0$ and $c_2 >$ such that

$$c_1 \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\} \leq \mathcal{H}_\alpha(x,t) \leq c_2 \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\}, \tag{2.59}$$

see [61, 45]. By the definition of $\mathcal{H}$, we have that

$$\mathcal{H}(x,y,t) = \mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t). \tag{2.60}$$

Since we have

$$\mathcal{H}_1(y,t) = (4\pi t)^{-\frac{M}{2}} e^{-\frac{|y|^2}{4t}}, \tag{2.61}$$

see [61], together with (2.57)-(2.60), for $|y| > 2$,

$$\begin{aligned}
\mathcal{K}(x,y) &= \int_0^\infty e^{-t}\mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t)dt \\
&\geq c_1 \int_0^\infty e^{-t} \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt \\
&\geq c_1 \int_{\frac{|y|}{2}}^{\frac{|y|}{2}+1} e^{-t} \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt \\
&\geq c_3 \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}, |x|^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\},
\end{aligned}$$

for some $c_3 > 0$. On the other hand, since for $n \geq 3$ we have

$$\int_0^\infty e^{-t}(4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}} dt \leq c_4 e^{-|y|}|y|^{2-n}(1+|y|)^{\frac{n-3}{2}}$$

with $c_4 > 0$ (see [61]), for $M \geq 5$ we have that

$$\begin{aligned}
&\mathcal{K}(x,y) = \int_0^\infty e^{-t}\mathcal{H}_\alpha(x,t)\mathcal{H}_1(y,t)dt \\
\leq\ & c_2 \int_0^\infty e^{-t} \min\{t^{-\frac{N}{2\alpha}}, t|x|^{-N-2\alpha}\}(4\pi t)^{-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt \\
\leq\ & c_5 \min\{\int_0^\infty e^{-t}(4\pi t)^{-\frac{N}{2\alpha}-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt, |x|^{-N-2\alpha}\int_0^\infty e^{-t}(4\pi t)^{1-\frac{M}{2}} e^{-\frac{|y|^2}{4t}} dt\} \\
\leq\ & c_6 \min\{e^{-|y|}|y|^{2-\frac{N}{\alpha}-M}(1+|y|)^{\frac{N}{2\alpha}+\frac{M}{2}-\frac{3}{2}}, |x|^{-N-2\alpha}e^{-|y|}|y|^{4-M}(1+|y|)^{\frac{M-5}{2}}\}
\end{aligned}$$

Therefore, for $N \geq 1$ and $M \geq 5$, there exist $c_8 > c_7 > 0$ such that

$$c_7\rho(x,y) \leq \mathcal{K}(x,y) \leq c_8\rho(x,y)|y|^{\frac{1}{2}}, \quad (x,y) \in \mathbb{R}^N \times (B_2^M(0))^c, \tag{2.62}$$

where $\rho(x,y)$ is defined in (2.15). In what follows, we construct super and sub-solutions to obtain the decay estimate given in Theorem 2.1.2.

Proof of Theorem 2.1.2. By the estimate in Theorem 2.1.1, we observe that,

for constants $c_{10} > c_9 > 0$ such that

$$c_9(1+|x|)^{-N-2\alpha} \le u(x,y) \le c_{10}(1+|x|)^{-N-2\alpha}, \quad (x,y) \in \mathbb{R}^N \times B_2^M(0),$$

so we only need to prove (2.14) holds for $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$.

Step 1: *Lower bound.* Let $\tilde{u} = \mathcal{K} * \chi_{B_1^N(0)\times B_1^M(0)}$, where $\chi_{B_1^N(0)\times B_1^M(0)}$ is the characteristic function of $B_1^N(0) \times B_1^M(0)$. By (2.62), we have that

$$\tilde{u}(x,y) \ge c_{11} \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}, (1+|x|)^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\}, \tag{2.63}$$

for all $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$, where $c_{11} > 0$. By definition of $\tilde{u}$, we have

$$(-\Delta)_x^\alpha \tilde{u} + (-\Delta)_y \tilde{u} + \tilde{u} = 0 \quad \text{in} \quad \mathbb{R}^N \times (\mathbb{R}^M \setminus \{0\}) \setminus (B_1^N(0) \times B_1^M(0))$$

and, by (2.62) and Theorem 2.1.1, there exists $c_{12} > 0$ such that $u \ge c_{11}\tilde{u}$ in $\mathbb{R}^N \times \{y \in \mathbb{R}^M : |y| = 2\}$. Since f is nonnegative, we use the Comparison Principle to obtain that, for any $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$

$$u(x,y) \ge c_{11}\tilde{u}(x,y) \ge c_{12} \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}}, (1+|x|)^{-N-2\alpha}e^{-|y|}|y|^{1-\frac{M}{2}}\}.$$

Step 2: *Upper bound.* For $y \in \mathbb{R}^M$ with $|y| \ge 2$, there exists $1 \le i \le M$ such that $|y_i| > 1$, we may assume that $y_1 > 1$. Let $\bar{u}(x,y) = \mathcal{K}(x,y)(1-|y_1|^{-1})$, then by direct computation

$$\begin{aligned}(-\Delta)_y \bar{u} &= (1-|y_1|^{-1})(-\Delta)_y \mathcal{K} - 2y_1^{-2}\partial_{y_1}\mathcal{K} + 2\mathcal{K}y_1^{-3} \\ &\ge (-\Delta)_y \mathcal{K}(1-|y_1|^{-1}) + 2\mathcal{K}y_1^{-3},\end{aligned}$$

where the last inequality holds since $y_1 > 0$ and $\partial_{y_1}\mathcal{K} < 0$. Therefore, by (2.62), we have that for $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$,

$$\begin{aligned} & (-\Delta)_x^\alpha \bar{u}(x,y) + (-\Delta)_y \bar{u}(x,y) + \bar{u}(x,y) \\ \ge\ & [(-\Delta)_x^\alpha \mathcal{K} + (-\Delta)_y \mathcal{K} + \mathcal{K}](1-|y_1|^{-1}) + 2\mathcal{K}(x,y)y_1^{-3} \ge 2\mathcal{K}(x,y)|y|^{-3} \\ \ge\ & 2c_8 \min\{e^{-|y|}|y|^{-\frac{N}{2\alpha}-\frac{M}{2}-3}, |x|^{-N-2\alpha}e^{-|y|}|y|^{-\frac{M}{2}-2}\}. \end{aligned} \tag{2.64}$$

Since $f(u) = O(u^p)$ near $u = 0$ for some $p > 1$, by Theorem 2.1.1 with $\epsilon = \frac{p-1}{4p}$, we have that

$$(-\Delta)_x^\alpha u + (-\Delta)_y u + u = f(u) \le c_{13}(1+|x|)^{-(N+2\alpha)p}e^{-\frac{3p+1}{4}|y|},$$

where $c_{13} > 0$. We notice that $\frac{3p+1}{4} > 1$. By definition of $\bar{u}$, (2.62) and Theorem 2.1.1 with $\epsilon = \frac{p-1}{4p}$, there exists $c_{14} > 0$ such that $u \le c_{14}\bar{u}$ in $\mathbb{R}^N \times \{y \in \mathbb{R}^M : |y| =$

2}. By Comparison Principle, we have that

$$\begin{aligned} u(x,y) &\leq c_{14}\bar{u}(x,y) \leq c_{14}\mathcal{K}(x,y) \\ &\leq c_{15}\min\{e^{-|y|}|y|^{\frac{1}{2}-\frac{N}{2\alpha}-\frac{M}{2}}, (1+|x|)^{-N-2\alpha}e^{-|y|}|y|^{\frac{3}{2}-\frac{M}{2}}\} \end{aligned}$$

for all $(x,y) \in \mathbb{R}^N \times (B_2^M(0))^c$ and some $c_{15} > 0$. This complete the proof.

□

2.4 Symmetry results

In this section, we prove Theorem 2.1.3 by moving planes method. Let u be a classical positive solution of (2.8) and consider first the y-direction. Let

$$\Sigma_\lambda^{y_1} = \{(x, y_1, y') \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{M-1} \mid y_1 > \lambda\},$$

$$T_\lambda^{y_1} = \{(x, y_1, y') \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{M-1} \mid y_1 = \lambda\}$$

and $u_\lambda(x, y_1, y') = u(x, 2\lambda - y_1, y')$ for $\lambda \in \mathbb{R}$. We introduce a preliminary inequality which plays a crucial role in the procedure of moving planes.

Lemma 2.4.1 *Under the assumptions of Theorem 2.1.3, for any $\lambda \in \mathbb{R}$, there exists $c_1 > 0$, independent of λ, such that*

$$\begin{aligned} & c_1\Big(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy\Big)^{\frac{N+M-2\alpha}{N+M}} \\ \leq & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha(u_\lambda - u) + (-\Delta)_y(u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy < \infty. \end{aligned}$$

Proof. First we show that the integrals are finite. We observe that u_λ satisfies the same equation (2.8) as u in $\Sigma_\lambda^{y_1}$. Taking $(u_\lambda - u)^+$ as test function in the equations for u and u_λ, subtracting and integrating in $\Sigma_\lambda^{y_1}$, we find

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha(u_\lambda - u) + (-\Delta)_y(u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy \\ = & \int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy. \end{aligned} \tag{2.65}$$

Now we only need to prove that

$$\int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy < +\infty. \tag{2.66}$$

In fact, for any given $\lambda \in \mathbb{R}$, using (2.11), we choose $R > 1$ such that

$$0 < u_\lambda(x,y) \le C_\epsilon (1+|x|)^{-N-2\alpha} e^{-\theta_1 |y_\lambda|} < s_0, \quad \forall (x,y) \in B_R^c,$$

where $y_\lambda = (2\lambda - y_1, y')$ for $y = (y_1, y') \in \mathbb{R}^M$ and s_0 is from (F).

If $u_\lambda(x,y) > u(x,y)$ for some $(x,y) \in \Sigma_\lambda^{y_1} \cap B_R^c$, we have

$$0 < u(x,y) < u_\lambda(x,y) < s_0.$$

Using (2.16) with $v = u_\lambda(x,y)$, then

$$\frac{f(u_\lambda(x,y)) - f(u(x,y))}{u_\lambda(x,y) - u(x,y)} \le \bar{c} u_\lambda^\gamma(x,y),$$

then

$$(f(u_\lambda(x,y)) - f(u(x,y)))^+ (u_\lambda(x,y) - u(x,y))^+ \le \bar{c} u_\lambda^{\gamma+2}(x,y).$$

The inequality above is obvious if $u_\lambda(x,y) \le u(x,y)$ for some $(x,y) \in \Sigma_\lambda^{y_1} \cap B_R^c$. Then

$$(f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ \le \bar{c} u_\lambda^{\gamma+2} \quad \text{in} \ \ \Sigma_\lambda^{y_1} \cap B_R^c.$$

Therefore,

$$\begin{aligned}
& \int_{\Sigma_\lambda^{y_1} \cap B_R^c} (f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ dxdy \\
\le \ & \bar{c} \int_{\Sigma_\lambda^{y_1} \cap B_R^c} u_\lambda^{\gamma+2}(x,y) dxdy \\
\le \ & \bar{c} C_\epsilon \int_{\Sigma_\lambda^{y_1}} (1+|x|)^{-(N+2\alpha)(\gamma+2)} e^{-(\gamma+2)\theta_1 |y_\lambda|} dxdy \\
\le \ & \bar{c} C_\epsilon \int_{\mathbb{R}^N} (1+|x|)^{-(N+2\alpha)(\gamma+2)} dx \int_{\mathbb{R}^M} e^{-(\gamma+2)\theta_1 |y|} dy < +\infty,
\end{aligned}$$

where the last inequality holds by $\gamma > \frac{2\alpha N}{(N+M)(N+2\alpha)}$. Since u and u_λ are bounded and f is locally Lipschitz, we have

$$\int_{\Sigma_\lambda^{y_1} \cap B_R} (f(u_\lambda) - f(u))^+ (u_\lambda - u)^+ dxdy < +\infty.$$

Therefore, (2.66) holds. Together with (2.65), we have the second inequality in the result.

Next we show that the first inequality holds in Lemma 2.4.1. Let us denote

$$w(x,y)=\begin{cases}(u_\lambda-u)^+(x,y), & (x,y)\in\Sigma_\lambda^{y_1},\\ (u_\lambda-u)^-(x,y), & (x,y)\in(\Sigma_\lambda^{y_1})^c\end{cases}\tag{2.67}$$

and

$$\mathrm{supp}(w)=\overline{\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^M\mid w(x,y)\neq 0\}},$$

where $(u_\lambda-u)^+(x,y)=\max\{(u_\lambda-u)(x,y),\ 0\}$, $(u_\lambda-u)^-(x,y)=\min\{(u_\lambda-u)(x,y),\ 0\}$. We observe that $w(x,y_1,y')=-w(x,2\lambda-y_1,y')$ for $(x,y_1,y')\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^{M-1}$ and

$$w=u_\lambda-u\quad\text{in}\ \ \mathrm{supp}(w).\tag{2.68}$$

It is obvious that for $(x,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$, $\{z\in\mathbb{R}^N|\ (z,y)\in(\Sigma_\lambda^{y_1})^c\}=\emptyset$ and

$$\begin{aligned}\mathbb{R}^N\ =\ &\{z\in\mathbb{R}^N|\ (z,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)\}\cup\\ &\{z\in\mathbb{R}^N|\ (z,y)\in\Sigma_\lambda^{y_1}\cap(\mathrm{supp}(w))^c\}\cup\{z\in\mathbb{R}^N|\ (z,y)\in(\Sigma_\lambda^{y_1})^c\}.\end{aligned}$$

Combining with (2.68), then for $(x,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$,

$$\begin{aligned}&(-\Delta)_x^\alpha w(x,y)-(-\Delta)_x^\alpha(u_\lambda-u)(x,y)=\int_{\mathbb{R}^N}\frac{(u_\lambda-u)(z,y)-w(z,y)}{|x-z|^{N+2\alpha}}dz\\ &=\int_{\{z\in\mathbb{R}^N:(z,y)\in\Sigma_\lambda^{y_1}\cap(\mathrm{supp}(w))^c\}}\frac{(u_\lambda-u)(z,y)}{|x-z|^{N+2\alpha}}dz\le 0,\end{aligned}\tag{2.69}$$

where the last inequality holds by $u_\lambda-u\le 0$ in $\Sigma_\lambda^{y_1}\cap(\mathrm{supp}(w))^c$. On one hand, from (2.69) and $w=(u_\lambda-u)^+>0$ in $\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$, we have that

$$\int_{\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)}(-\Delta)_x^\alpha w\,w\,dxdy\le\int_{\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)}(-\Delta)_x^\alpha(u_\lambda-u)(u_\lambda-u)^+dxdy.\tag{2.70}$$

On the other hand, we know that $w(x,y)=(u_\lambda-u)(x,y)$ and $(-\Delta)_y w(x,y)=(-\Delta)_y(u_\lambda-u)(x,y)$ for $(x,y)\in\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)$. Together with (2.70), then

$$\begin{aligned}&\int_{\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)}[(-\Delta)_x^\alpha w+(-\Delta)_y w+w]\,w\,dxdy\\ \le\ &\int_{\Sigma_\lambda^{y_1}\cap\mathrm{supp}(w)}[(-\Delta)_x^\alpha(u_\lambda-u)+(-\Delta)_y(u_\lambda-u)+(u_\lambda-u)](u_\lambda-u)^+dxdy\end{aligned}$$

and then by the fact of $w = (u_\lambda - u)^+ = 0$ in $\Sigma_\lambda^{y_1} \cap (\mathrm{supp}(w))^c$, we have that

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w]\, w dxdy \\ \leq & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+. \end{aligned} \tag{2.71}$$

By the definition of w, we have that

$$\int_{\mathbb{R}^{N+M}} |w|^2 dxdy = 2 \int_{\Sigma_\lambda^{y_1}} |w|^2 dxdy,$$

$$\int_{\mathbb{R}^{N+M}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy = 2 \int_{\Sigma_\lambda^{y_1}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy,$$

$$\int_{\mathbb{R}^{N+M}} (-\Delta)_y w\, w dxdy = 2 \int_{\Sigma_\lambda^{y_1}} (-\Delta)_y w\, w dxdy,$$

$$\int_{\mathbb{R}^{N+M}} (-\Delta)_x^\alpha w\, w dxdy = 2 \int_{\Sigma_\lambda^{y_1}} (-\Delta)_x^\alpha w\, w dxdy,$$

then, together with Proposition 2.2.1, we obtain that

$$\begin{aligned} & \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w]\, w dxdy \\ = & \frac{1}{2} \int_{\mathbb{R}^{N+M}} [(-\Delta)_x^\alpha w + (-\Delta)_y w + w]\, w dxdy \\ \geq & c_3 (\int_{\mathbb{R}^{N+M}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\ = & c_3 (2 \int_{\Sigma_\lambda^{y_1}} |w|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}}, \end{aligned} \tag{2.72}$$

for some $c_3 > 0$. Combining (2.71) with (2.72), by $w = (u_\lambda - u)^+$ in $\Sigma_\lambda^{y_1}$, we get the first inequality in Lemma 2.4.1. The proof is complete. □

Lemma 2.4.2 *Under the assumptions of Theorem 2.1.3, for any $\lambda \in \mathbb{R}$, there exists $c_4 > 0$ independent of λ such that*

$$\begin{aligned} & c_4 (\int_{\Sigma_\lambda^{x_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\ \leq & \int_{\Sigma_\lambda^{x_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy < \infty, \end{aligned}$$

where $\Sigma_\lambda^{x_1} = \{(x_1, x', y) \in \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^M \mid x_1 > \lambda\}$.

Proof. The proof proceeds similarly to the proof of Lemma 2.4.1, the only difference is to show (2.69) with $(x,y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)$. It is obvious that

$$\begin{aligned}\mathbb{R}^N &= \{z \in \mathbb{R}^N |\ (z,y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)\} \cup \\ &\quad \{z \in \mathbb{R}^N |\ (z,y) \in \Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c\} \cup \\ &\quad \{z \in \mathbb{R}^N |\ (z,y) \in (\Sigma_\lambda^{x_1})^c \cap (\mathrm{supp}(w))^c\} \cup \\ &\quad \{z \in \mathbb{R}^N |\ (z,y) \in (\Sigma_\lambda^{x_1})^c \cap \mathrm{supp}(w)\}\end{aligned}$$

and $w = u_\lambda - u$ in $\mathrm{supp}(w)$, then for $(x,y) \in \Sigma_\lambda^{x_1} \cap \mathrm{supp}(w)$,

$$\begin{aligned}&(-\Delta)_x^\alpha w(x,y) - (-\Delta)_x^\alpha (u_\lambda - u)(x,y) = \int_{\mathbb{R}^N} \frac{(u_\lambda - u)(z,y) - w(z,y)}{|x-z|^{N+2\alpha}} dz \\ = &\int_{\{z \in \mathbb{R}^N |\ (z,y) \in \Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c\}} \left(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}}\right)(u_\lambda - u)(z,y) dz \\ \le &\ 0,\end{aligned}$$

where $z_\lambda = (2\lambda - z_1, z')$ for $z = (z_1, z') \in \mathbb{R}^N$ and the last inequality holds by $u_\lambda - u \le 0$ in $\Sigma_\lambda^{x_1} \cap (\mathrm{supp}(w))^c$. □

Theorem 2.4.1 *Under the assumptions of Theorem 2.1.3, for $x \in \mathbb{R}^N$, we have*

$$u(x,y) = u(x,|y|)$$

and u is strictly decreasing in y-direction.

Proof. We divide the proof into three steps.

Step 1: $\lambda_0 := \sup\{\lambda \mid u_\lambda \le u \text{ in } \Sigma_\lambda^{y_1}\}$ is finite. Since u decays at infinity, we observe that the set $\{\lambda \mid u_\lambda \le u \text{ in } \Sigma_\lambda^{y_1}\}$ is nonempty. Using $(u_\lambda - u)^+$ as a test function in the equation for u and u_λ, by (2.16) and Hölder inequality, for λ big (negative), we find that

$$\begin{aligned}&\int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy \\ &= \int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy \\ &= \int_{\Sigma_\lambda^{y_1}} \frac{f(u_\lambda) - f(u)}{u_\lambda - u} [(u_\lambda - u)^+]^2 dxdy \le \bar{c} \int_{\Sigma_\lambda^{y_1}} u_\lambda^\gamma [(u_\lambda - u)^+]^2 dxdy \\ &\le c_5 \int_{\Sigma_\lambda^{y_1}} (1+|x|)^{-\gamma(N+2\alpha)} e^{-\gamma\theta_1 |y_\lambda|} [(u_\lambda - u)^+]^2 dxdy \\ &\le c_5 \Big(\int_{\Sigma_\lambda^{y_1}} (1+|x|)^{-a} e^{-b|y_\lambda|} dxdy\Big)^{\frac{2\alpha}{N+M}} \Big(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy\Big)^{\frac{N+M-2\alpha}{N+M}},\end{aligned}$$

where $a = \frac{\gamma(N+2\alpha)(N+M)}{2\alpha}$ and $b = \frac{\theta_1\gamma(N+M)}{2\alpha}$. Since $\gamma > \frac{2\alpha N}{(N+2\alpha)(N+M)}$, we have that $a > N$. Then we can choose $R > 0$ such that for all $\lambda < -R$,

$$c_5(\int_{\Sigma_\lambda^{y_1}} (1+|x|)^{-a} e^{-b|y_\lambda|} dxdy)^{\frac{2\alpha}{N+M}} \leq \frac{1}{4}.$$

By Lemma 2.4.1, we obtain that

$$\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy = 0, \qquad \forall\ \lambda < -R.$$

Thus $u_\lambda \leq u$ in $\Sigma_\lambda^{y_1}$ for all $\lambda < -R$ and then conclude that $\lambda_0 \geq -R$. On the other hand, since u decays at infinity, then there exist $\lambda_1 \in \mathbb{R}$ and $(x, y) \in \Sigma_\lambda^{y_1}$ such that $u(x, y) < u_{\lambda_1}(x, y)$. Hence λ_0 is finite.

Step 2: $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$. Assuming the contrary, we have that $u \not\equiv u_{\lambda_0}$ and $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, in this case the following claim holds.

Claim 1. *If $u \not\equiv u_{\lambda_0}$ and $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then $u > u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.*

Let us assume, for the moment, that Claim 1 is true, then for any given $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, where $\epsilon > 0$ is chosen later. Let $P = (0, \cdots, \lambda, \cdots, 0) \in T_\lambda^{y_1}$ and $B(P, R)$ be the ball centered at P and with radius $R > 1$ to be chosen later. Define $B_1 = \Sigma_\lambda^{y_1} \cap B(P, R)$ and let us consider $(u_\lambda - u)^+$ test function in the equation for u and u_λ in $\Sigma_\lambda^{y_1}$, then from Lemma 2.4.1 we obtain

$$\begin{aligned}
&(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\
\leq\ & c_6 \int_{\Sigma_\lambda^{y_1}} [(-\Delta)_x^\alpha (u_\lambda - u) + (-\Delta)_y (u_\lambda - u) + (u_\lambda - u)](u_\lambda - u)^+ dxdy \\
=\ & c_6 \int_{\Sigma_\lambda^{y_1}} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy. \qquad (2.73)
\end{aligned}$$

We estimate the integral on the right. Proceeding as in Step 1, we can choose $R > 1$ big enough such that

$$c_7(\int_{\Sigma_\lambda^{y_1} \setminus B_1} (1+|x|)^{-a} e^{-b|y_\lambda|} dxdy)^{\frac{2\alpha}{N+M}} \leq \frac{1}{4}$$

for some $c_7 > 0$, where $a = \frac{\gamma(N+2\alpha)(N+M)}{2\alpha}$ and $b = \frac{\theta_1\gamma(N+M)}{2\alpha}$. Then

$$\begin{aligned}
&\int_{\Sigma_\lambda^{y_1}\setminus B_1} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy \le \bar{c}\int_{\Sigma_\lambda^{y_1}\setminus B_1} u_\lambda^\gamma |(u_\lambda - u)^+|^2 dxdy \\
&\le c_7(\int_{\Sigma_\lambda^{y_1}\setminus B_1} (1+|x|)^{-a} e^{-b|y_\lambda|} dxdy)^{\frac{2\alpha}{N+M}} (\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\
&\le \frac{1}{4}(\int_{\Sigma_\lambda^{y_1}} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}}. \qquad (2.74)
\end{aligned}$$

Now using Claim 1, we choose $\epsilon > 0$ such that $c_8|B_1 \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N+M}} < 1/4$, for some $c_8 > 0$. Since f is locally Lipschitz, using Hölder inequality, we have

$$\begin{aligned}
&\int_{B_1} (f(u_\lambda) - f(u))(u_\lambda - u)^+ dxdy \le c_{46}\int_{B_1} |(u_\lambda - u)^+|^2 \chi_{\mathrm{supp}(u_\lambda - u)^+} dxdy \\
&= c_8|B_1 \cap \mathrm{supp}(u_\lambda - u)^+|^{\frac{2\alpha}{N+M}} (\int_{B_1} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}} \\
&\le \frac{1}{4}(\int_{B_1} |(u_\lambda - u)^+|^{\frac{2(N+M)}{N+M-2\alpha}} dxdy)^{\frac{N+M-2\alpha}{N+M}}. \qquad (2.75)
\end{aligned}$$

From (2.73), (2.74) and (2.75), it follows that $(u_\lambda - u)^+ = 0$ in $\Sigma_\lambda^{y_1}$. Then $u_\lambda \le u$ in $\Sigma_\lambda^{y_1}$ for $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, which contradicts the definition of λ_0. As a consequence, we have $u \equiv u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.

In order to complete Step 2, we only need to prove Claim 1.

Proof of Claim 1. By contradiction, if there exists $(\bar{x}, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}$ such that $u(\bar{x}, \bar{y}) = u_{\lambda_0}(\bar{x}, \bar{y})$, then

$$\begin{aligned}
&(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) + (-\Delta)_y (u - u_{\lambda_0})(\bar{x}, \bar{y}) + (u - u_{\lambda_0})(\bar{x}, \bar{y}) \\
&= f(u(\bar{x}, \bar{y})) - f(u_{\lambda_0}(\bar{x}, \bar{y})) = 0.
\end{aligned}$$

Since $(u - u_{\lambda_0})(\bar{x}, \bar{y}) = \min_{\Sigma_{\lambda_0}^{y_1}} (u - u_{\lambda_0}) = 0$, we have $(-\Delta)_y (u - u_{\lambda_0})(\bar{x}, \bar{y}) \le 0$, then

$$(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) \ge 0. \qquad (2.76)$$

The other side, we observe that $\{z \in \mathbb{R}^N |\ (z, \bar{y}) \in (\Sigma_{\lambda_0}^{y_1})^c\} = \emptyset$ when $(\bar{x}, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}$. By $u(\bar{x}, \bar{y}) = u_{\lambda_0}(\bar{x}, \bar{y})$ and then

$$\begin{aligned}
(-\Delta)_x^\alpha (u - u_{\lambda_0})(\bar{x}, \bar{y}) &= -\int_{\mathbb{R}^N} \frac{(u - u_{\lambda_0})(z, \bar{y})}{|\bar{x} - z|^{N+2\alpha}} dz \\
&= -\int_{\{z \in \mathbb{R}^N |\ (z, \bar{y}) \in \Sigma_{\lambda_0}^{y_1}\}} \frac{(u - u_{\lambda_0})(z, \bar{y})}{|\bar{x} - z|^{N+2\alpha}} dz \le 0, \qquad (2.77)
\end{aligned}$$

where the last inequality holds by $u \ge u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$.

Combining (2.76) with (2.77), we obtain that $(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y})=0$ and then from (2.77), we have that

$$u(z,\bar{y})=u_{\lambda_0}(z,\bar{y}),\quad \forall z\in\mathbb{R}^N, \tag{2.78}$$

this means that $u-u_{\lambda_0}$ has property (P) and by $u\neq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$ we have

$$(\bar{x},\bar{y})\in(\Sigma_{\lambda_0}^{y_1})_0:=\{(x,y)\in\Sigma_{\lambda_0}^{y_1}\mid (u-u_{\lambda_0})(x,y)=\inf_{\Sigma_{\lambda_0}^{y_1}}(u-u_{\lambda_0})=0\}\subsetneqq\Sigma_{\lambda_0}^{y_1}.$$

Moreover, by Proposition 2.2.3 with $\Omega=\Sigma_{\lambda_0}^{y_1}$, we observe that $\Sigma_{\lambda_0}^{y_1}\setminus(\Sigma_{\lambda_0}^{y_1})_0$ satisfies interior cylinder condition at point $(x_0,y_0)\in\partial(\Sigma_{\lambda_0}^{y_1})_0\cap\Sigma_{\lambda_0}^{y_1}$. Then there exist $r>0$ small and $\tilde{y}\in\mathbb{R}^M$ such that

$$O_r:=B_r^N(x_0)\times B_r^M(\tilde{y})\subset\Sigma_{\lambda_0}^{y_1}\setminus(\Sigma_{\lambda_0}^{y_1})_0\quad\text{and}\quad(x_0,y_0)\in\partial O_r.$$

Let D be defined by (2.19). Since $u\geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then for any $(x,y)\in D$, we have

$$\int_{\mathbb{R}^N\setminus B_r^N(x_0)}\frac{(u-u_{\lambda_0})(z,y)}{|x-z|^{N+2\alpha}}dz\geq 0.$$

Finally, it is obvious that

$$(-\Delta)_x^\alpha(u-u_{\lambda_0})+(-\Delta)_y(u-u_{\lambda_0})+h(u-u_{\lambda_0})=0\quad\text{in }\Sigma_{\lambda_0}^{y_1},$$

where $h=1-\frac{f(u)-f(u_{\lambda_0})}{u-u_{\lambda_0}}\in L_{loc}^\infty(\Sigma_{\lambda_0}^{y_1})$. Then we use Theorem 2.2.1 to obtain

$$u\equiv u_{\lambda_0}\quad\text{in }\tilde{\Sigma}_{\lambda_0}^{y_1},$$

which contradicts the condition of $u\neq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{y_1}$, then we obtain the results in Claim 1.

Step 3. By translation, we may say that $\lambda_0=0$. Repeating the argument from the other side, we find that u is symmetric about y_1-axis. Using the same argument in any y-direction, we conclude that

$$u(x,y)=u(x,|y|),\quad (x,y)\in\mathbb{R}^N\times\mathbb{R}^M.$$

Finally, we prove that $u(x,|y|)$ is strictly decreasing in $|y|>0$. Indeed, for any given $y_1<\widetilde{y}_1<0$ and letting $\lambda=\frac{y_1+\widetilde{y}_1}{2}$. Then, as proved above we have

$$u>u_\lambda\quad\text{in}\quad\Sigma_\lambda^{y_1}.$$

For any given $x\in\mathbb{R}^N$, we observe that $(x,\widetilde{y}_1,0,\cdots,0)\in\Sigma_\lambda^{y_1}$, then

$$u(x,\widetilde{y}_1,0,\cdots,0)>u_\lambda(x,\widetilde{y}_1,0,\cdots,0)=u(x,y_1,0,\cdots,0).$$

Using the result of $u(x,y) = u(x,|y|)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^M$ and $|\widetilde{y}_1| < |y_1|$, we conclude monotonicity of u respect to y. This completes the proof. $\square$

Next we study the symmetry result in x-direction.

Theorem 2.4.2 *Under the assumptions of Theorem 2.1.3, for $y \in \mathbb{R}^M$, we have*

$$u(x,y) = u(|x|,y)$$

and u is strictly decreasing in x-direction.

Proof. The proof of this theorem goes like the one for Theorem 2.4.1. The only place where there is a difference is in the following property: *if $u \not\equiv u_{\lambda_0}$ and $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, then $u > u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$.* By contradiction, if there exists $(\bar{x},\bar{y}) \in \Sigma_{\lambda_0}^{x_1}$ such that $u(\bar{x},\bar{y}) = u_{\lambda_0}(\bar{x},\bar{y})$, then

$$\begin{aligned}&(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y}) + (-\Delta)_y(u-u_{\lambda_0})(\bar{x},\bar{y}) + (u-u_{\lambda_0})(\bar{x},\bar{y})\\ &= f(u(\bar{x},\bar{y})) - f(u_{\lambda_0}(\bar{x},\bar{y})) = 0.\end{aligned}$$

Since $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, we have $(u-u_{\lambda_0})(\bar{x},\bar{y}) = \min_{\Sigma_{\lambda_0}^{x_1}}(u-u_{\lambda_0}) = 0$ and $(-\Delta)_y(u-u_{\lambda_0})(\bar{x},\bar{y}) \leq 0$ and then

$$(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y}) \geq 0.$$

The other side, by direct computation, we have that

$$\begin{aligned}&(-\Delta)_x^\alpha(u-u_{\lambda_0})(\bar{x},\bar{y}) = \int_{\mathbb{R}^N} \frac{(u_{\lambda_0}-u)(z,\bar{y})}{|\bar{x}-z|^{N+2\alpha}}dz\\ &= \int_{\{z\in\mathbb{R}^N|\ (z,\bar{y})\in\Sigma_{\lambda_0}^{x_1}\}} \left(\frac{1}{|\bar{x}-z|^{N+2\alpha}} - \frac{1}{|\bar{x}-z_{\lambda_0}|^{N+2\alpha}}\right)(u_{\lambda_0}-u)(z,\bar{y})dz \leq 0,\end{aligned}$$

where $z_{\lambda_0} = (2\lambda_0 - z_1, z')$ for $z = (z_1, z') \in \mathbb{R}^N$ and the last inequality holds by $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$. Therefore,

$$u(z,\bar{y}) = u_{\lambda_0}(z,\bar{y}), \quad \forall z \in \mathbb{R}^N, \tag{2.79}$$

this means that $u - u_{\lambda_0}$ has property (P) and by $u \neq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$ we have

$$(\bar{x},\bar{y}) \in (\Sigma_{\lambda_0}^{x_1})_0 := \{(x,y) \in \Sigma_{\lambda_0}^{x_1} \mid (u-u_{\lambda_0})(x,y) = \inf_{\Sigma_{\lambda_0}^{x_1}}(u-u_{\lambda_0}) = 0\} \subsetneqq \Sigma_{\lambda_0}^{x_1}.$$

Moreover, by Proposition 2.2.3, we observe that $\Sigma_{\lambda_0}^{x_1} \setminus (\Sigma_{\lambda_0}^{x_1})_0$ satisfies interior cylinder condition at point $(x_0,y_0) \in \partial(\Sigma_{\lambda_0}^{x_1})_0 \cap \Sigma_{\lambda_0}^{x_1}$. Then there exist $r_1 > 0$ and $\tilde{y} \in \mathbb{R}^M$ such that for all $r \in (0, r_1]$,

$$O_r := B_r^N(x_0) \times B_r^M(\tilde{y}) \subset \Sigma_{\lambda_0}^{x_1} \setminus (\Sigma_{\lambda_0}^{x_1})_0 \quad \text{and} \quad (x_0,y_0) \in \partial O_r.$$

Next we show that there exists some $r \in (0, r_1]$ such that for any $(x, y) \in D$,

$$\int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{(u - u_{\lambda_0})(z, y)}{|x - z|^{N+2\alpha}} dz \geq 0, \tag{2.80}$$

where D is defined by (2.19). Indeed, since $u \not\equiv u_{\lambda_0}$ and $u \geq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, then for $(x, y) \in D \subset \Sigma_{\lambda_0}^{x_1}$, we have that

$$\int_{\mathbb{R}^N} \frac{(u - u_{\lambda_0})(z, y)}{|x - z|^{N+2\alpha}} dz > 0.$$

Let us define

$$r(x, y) = \sup\{r \in (0, r_1] : \int_{\mathbb{R}^N \setminus B_r^N(x_0)} \frac{(u - u_{\lambda_0})(z, y)}{|x - z|^{N+2\alpha}} dz \geq 0\}. \tag{2.81}$$

Let $r_m = \inf_{(x,y)\in D} r(x, y)$, it is obvious that $r_m \in [0, r_1]$. Now we prove that $r_m > 0$. By contradiction, if $r_m = 0$, then there exist a sequence $(x_n, y_n) \in D$ and $(\tilde{x}, \tilde{y}) \in \bar{D}$ such that $(x_n, y_n) \to (\tilde{x}, \tilde{y})$ and $r(x_n, y_n) \to 0$, as $n \to +\infty$. Since $r(x, y)$ is continuous, then $r(\tilde{x}, \tilde{y}) = 0$. If $(\tilde{x}, \tilde{y}) \in \bar{D} \setminus (\Sigma_{\lambda_0}^{x_1})_0$, i.e. $u(\tilde{x}, \tilde{y}) > u_{\lambda_0}(\tilde{x}, \tilde{y})$, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \frac{(u - u_{\lambda_0})(z, \tilde{y})}{|\tilde{x} - z|^{N+2\alpha}} dz \\ = & \int_{\{z\in\mathbb{R}^N |\ (z,\tilde{y})\in\Sigma_{\lambda_0}^{x_1}\}} (u - u_{\lambda_0})(z, \tilde{y})(\frac{1}{|\tilde{x} - z|^{N+2\alpha}} - \frac{1}{|\tilde{x} - z_{\lambda_0}|^{N+2\alpha}}) dz > 0. \end{aligned}$$

By the continuity of the integration and (2.81), we obtain that $r(\tilde{x}, \tilde{y}) > 0$, which is impossible.

Then $(\tilde{x}, \tilde{y}) \in \bar{D} \cap (\Sigma_{\lambda_0}^{x_1})_0$, i.e. $u(\tilde{x}, \tilde{y}) = u_{\lambda_0}(\tilde{x}, \tilde{y})$. Since the function $u - u_{\lambda_0}$ has property (P), then for any $\tilde{r} > 0$,

$$\int_{\mathbb{R}^N \setminus B_{\tilde{r}}^N(x_0)} \frac{(u - u_{\lambda_0})(z, \tilde{y})}{|\tilde{x} - z|^{N+2\alpha}} dz = 0.$$

Combining with (2.81), we obtain that $r(\tilde{x}, \tilde{y}) = r_1 > 0$, which contradicts $r(\tilde{x}, \tilde{y}) = 0$. As a consequence, we have that $0 < r_m \leq r_1$. Taking $r = r_m$, then (2.80) holds for any $(x, y) \in D$. Finally, it is obvious that

$$(-\Delta)_x^\alpha (u - u_{\lambda_0}) + (-\Delta)_y (u - u_{\lambda_0}) + h(u - u_{\lambda_0}) = 0 \quad \text{in } \Sigma_{\lambda_0}^{x_1},$$

where $h = 1 - \frac{f(u) - f(u_{\lambda_0})}{u - u_{\lambda_0}} \in L_{loc}^\infty(\Sigma_{\lambda_0}^{x_1})$. Then we use Theorem 2.2.1 to obtain that

$$u \equiv u_{\lambda_0} \quad \text{in } \tilde{\Sigma}_{\lambda_0}^{x_1},$$

which contradicts the condition of $u \neq u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$. Then $u > u_{\lambda_0}$ in $\Sigma_{\lambda_0}^{x_1}$, to complete the proof. $\square$

Chapter 3

Fractional heat equations with subcritical absorption with initial data measure

Abstract: in this chapter[1], we study existence and uniqueness of weak solutions to (F) $\partial_t u + (-\Delta)^\alpha u + h(t,u) = 0$ in $(0,\infty)\times\mathbb{R}^N$, with initial condition $u(0,\cdot) = \nu$ in $\mathbb{R}^N$, where $N \geq 2$, the operator $(-\Delta)^\alpha$ is the fractional Laplacian with $\alpha \in (0,1)$, ν is a bounded Radon measure and $h : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a subcritical integrability condition.

In particular, if $h(t,u) = t^\beta u^p$ with $\beta > -1$ and $0 < p < p^*_\beta := 1 + \frac{2\alpha(1+\beta)}{N}$, we prove that there exists a unique weak solution u_k to (F) with $\nu = k\delta_0$, where δ_0 is the Dirac mass at the origin. We obtain that $u_k \to \infty$ in $(0,\infty)\times\mathbb{R}^N$ as $k \to \infty$ for $p \in (0,1]$ and the limit of u_k exists as $k \to \infty$ when $1 < p < p^*_\beta$, we denote it by u_∞. When $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} := p^{**}_\beta < p < p^*_\beta$, u_∞ is the minimal self-similar solution of $(F)_\infty$ $\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0$ in $(0,\infty)\times\mathbb{R}^N$ with the initial condition $u(0,\cdot) = 0$ in $\mathbb{R}^N \setminus \{0\}$ and it satisfies $u_\infty(0,x) = 0$ for $x \neq 0$. While if $1 < p < p^{**}_\beta$, then $u_\infty \equiv U_p$, where U_p is the maximal solution of the differential equation $y' + t^\beta y^p = 0$ on $\mathbb{R}_+$.

3.1 Introduction

Let $h : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ be a continuous function and $Q_\infty = (0,\infty)\times\mathbb{R}^N$ with $N \geq 2$. The first object of this chapter is to consider existence and uniqueness of

[1]This chapter is based on the paper: H. Chen, L. Véron and Y. Wang, *Fractional heat equations with subcritical absorption with initial data measure,* arXiv:1401.7187.

weak solutions to fractional heat equations

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t,u) = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{3.1}$$

where ν belongs to the space $\mathfrak{M}^b(\mathbb{R}^N)$ of bounded Radon measures in $\mathbb{R}^N$ and $(-\Delta)^\alpha$ $(0 < \alpha < 1)$ is the fractional Laplacian defined by

$$(-\Delta)^\alpha u(t,x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(t,x),$$

where, for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(t,x) = \int_{\mathbb{R}^N} \frac{u(t,x) - u(t,z)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz$$

and

$$\chi_\epsilon(r) = \begin{cases} 0 & \text{if} \quad r \in [0,\epsilon], \\ 1 & \text{if} \quad r > \epsilon. \end{cases}$$

In a pioneering work, Brezis and Friedman [12] have studied semilinear the heat equation with measure as initial data

$$\begin{aligned} \partial_t u - \Delta u + u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{3.2}$$

where $k > 0$ and δ_0 is the Dirac mass at the origin. They proved that if $1 < p < (N+2)/N$, then for every $k > 0$ there exists a unique solution u_k to (3.2). When $p \geq (N+2)/N$, problem (3.2) has no solution and even more, they proved that no nontrivial solution of the above equation vanishing on $\mathbb{R}^N \setminus \{0\}$ at $t = 0$ exists. When $1 < p < 1 + \frac{2}{N}$, Brezis, Peletier and Terman used a dynamical system technique in [13] to prove the existence of a *very singular solution* u_s to

$$\partial_t u - \Delta u + u^p = 0 \quad \text{in} \quad Q_\infty, \tag{3.3}$$

vanishing at $t = 0$ on $\mathbb{R}^N \setminus \{0\}$. This function u_s is self-similar, i.e. expressed under the form

$$u_s(t,x) = t^{-\frac{1}{p-1}} f\left(\frac{|x|}{\sqrt{t}}\right), \tag{3.4}$$

and f is uniquely determined by the following conditions

$$\begin{aligned} & f'' + \left(\tfrac{N-1}{\eta} + \tfrac{1}{2}\eta\right) f' + \tfrac{1}{p-1} f - f^p = 0 \quad \text{on } \mathbb{R}_+ \\ & f > 0 \quad \text{and} \quad f \text{ is smooth on } \mathbb{R}_+ \\ & f'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \eta^{\frac{2}{p-1}} f(\eta) = 0. \end{aligned} \tag{3.5}$$

Furthermore, it satisfies

$$f(\eta) = c_1 e^{-\eta^2} \eta^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\} \quad \text{as } \eta \to \infty$$

for some $c_1 > 0$. Later on, Kamin and Peletier in [58] proved that the sequence of weak solutions u_k converges to the very singular solution u_s as $k \to \infty$. After that, Marcus and Véron in [70] studied the equation in the framework of the *initial trace* theory. They pointed out the role of the very singular solution of (3.3) in the study of the singular set of the initial trace, showing in particular that it is the unique positive solution of (3.3) satisfying

$$\lim_{t\to 0} \int_{B_\epsilon} u(t,x)dx = \infty \qquad \forall \epsilon > 0,\ B_\epsilon = B_\epsilon(0), \tag{3.6}$$

and

$$\lim_{t\to 0} \int_K u(t,x)dx = 0 \qquad \forall K \subset \mathbb{R}^N \setminus \{0\},\ K \text{ compact}. \tag{3.7}$$

If one replaces u^p by $t^\beta u^p$ with $p \in (1, 1 + \frac{2(1+\beta)}{N})$, these results were extended by Marcus and Véron ($\beta \geq 0$) in [70] and then Al Sayed and Véron ($\beta > -1$) in [82]. The initial data problem with measure and general absorption term

$$\begin{aligned} \partial_t u - \Delta u + h(t,x,u) = 0 \quad & \text{in} \quad (0,\mathrm{T}) \times \Omega, \\ u = 0 \quad & \text{in} \quad (0,T) \times \partial\Omega, \\ u(0,\cdot) = \nu \quad & \text{in} \quad \Omega, \end{aligned} \tag{3.8}$$

in a bounded domain Ω is a domain in $\mathbb{R}^N$, has been studied by Marcus and Véron in [70] in the framework of the initial trace theory. They proved that the following general integrability condition on h

$$\begin{aligned} & 0 \leq | h(t,x,r) | \leq \tilde{h}(t) f(|r|) \qquad \forall (x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \\ & \textstyle\int_0^T \tilde{h}(t) f(\sigma t^{\frac{N}{2}}) t^{-\frac{N}{2}} dt < \infty \qquad \forall \sigma > 0 \\ & \text{either } \tilde{h}(t) = t^\alpha \text{ with } \alpha \geq 0 \text{ or } f \text{ is convex,} \end{aligned} \tag{3.9}$$

in order the problem has a unique solution for any bounded measure. In the particular case with $h(t,x,r) = t^\beta |u|^{p-1} u$, is fulfilled if $1 < p < 1 + \frac{2(1+\beta)}{N}$ and

$\beta > -1$, and the very singular solution exists in this range of values.

Motivated by a growing number of applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, (see e.g. [20, 21, 26, 27, 31, 37, 44, 46]). Recently, in [32] we obtained the existence and uniqueness of weak solution to semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + f(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^c, \end{aligned} \tag{3.10}$$

when ν is Radon measure and f satisfies a subcritical integrability condition.

One purpose of this chapter is to study the existence and uniqueness of weak solutions to semilinear fractional heat equation (3.1) in a measure framework. We first make precise the notion of weak solution of (3.1) that we will use in this chapter.

Definition 3.1.1 *We say that u is a weak solution of (3.1), if for any $T > 0$, $u \in L^1(Q_T)$, $h(t,u) \in L^1(Q_T)$ and*

$$\begin{aligned} &\int_{Q_T} \left(u(t,x)[-\partial_t \xi(t,x) + (-\Delta)^\alpha \xi(t,x)] + h(t,u)\xi(t,x)\right) dxdt \\ &= \int_{\mathbb{R}^N} \xi(0,x) d\nu - \int_{\mathbb{R}^N} \xi(T,x)u(T,x)dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}, \end{aligned} \tag{3.11}$$

where $Q_T = (0,T) \times \mathbb{R}^N$ and $\mathbb{Y}_{\alpha,T}$ is a space of functions $\xi : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying

(i) $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t \xi\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \xi\|_{L^\infty(Q_T)} < +\infty$;

(ii) for $t \in (0,T)$, there exist $M > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, $\|(-\Delta)^\alpha_\epsilon \xi(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \le M$.

Before stating our main theorems, we introduce the subcritical integrability condition for the nonlinearity h, that is,

(H) *(i)* The function $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and for any $t \in (0,\infty)$, $h(t,0) = 0$ and $h(t,r_1) \ge h(t,r_2)$ if $r_1 \ge r_2$.

(ii) There exist $\beta > -1$ and a continuous, nondecreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t,r)| \le t^\beta g(|r|) \qquad \forall (t,r) \in (0,\infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s) s^{-1-p^*_\beta} ds < +\infty, \tag{3.12}$$

where

$$p_\beta^* = 1 + \frac{2\alpha(1+\beta)}{N}. \tag{3.13}$$

We denote by $H_\alpha : (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ the heat kernel for $(-\Delta)^\alpha$ in $(0,\infty) \times \mathbb{R}^N$, by $\mathbb{H}_\alpha[\nu]$ the associated heat potential of $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, defined by

$$\mathbb{H}_\alpha[\nu](t,x) = \int_{\mathbb{R}^N} H_\alpha(t,x,y)d\nu(y)$$

and by $\mathcal{H}_\alpha[\mu]$ the Duhamel operator defined for $(t,x) \in Q_T$ and any $\mu \in L^1(Q_T)$ by

$$\mathcal{H}_\alpha[\mu](t,x) = \int_0^T \mathbb{H}_\alpha[\mu(s,.)](t-s,x)ds = \int_0^t \int_{\mathbb{R}^N} H_\alpha(t-s,x,y)\mu(s,y)dyds.$$

Now we state our first theorem as follows.

Theorem 3.1.1 *Assume that $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ and the function h satisfies (H). Then problem (3.1) admits a unique weak solution u_ν such that*

$$\mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,\mathbb{H}_\alpha[\nu_+])] \le u_\nu \le \mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(.,-\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \tag{3.14}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν. Furthermore,

(i) if ν is nonnegative, so is u_ν;

(ii) the mapping: $\nu \mapsto u_\nu$ is increasing and stable in the sense that if $\{\nu_n\}$ is a sequence of positive bounded Radon measures converging to ν in the weak sense of measures, then $\{u_{\nu_n}\}$ converges to u_ν locally uniformly in Q_∞.

According to Theorem 3.1.1, there exists a unique positive weak solution u_k to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{3.15}$$

where $\beta > -1$, $k > 0$ and $p \in (0, p_\beta^*)$. We observe that $u_k \to \infty$ in $(0,\infty) \times \mathbb{R}^N$ as $k \to \infty$ for $p \in (0,1]$, see Proposition 3.4.2 for details. Our next interest of this chapter is to study the limit of u_k as $k \to \infty$ for $p \in (1, p_\beta^*)$, which exists since $\{u_k\}_k$ are an increasing sequence of functions, bounded by $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$, and we set

$$u_\infty = \lim_{k\to\infty} u_k \quad \text{in} \quad Q_\infty. \tag{3.16}$$

Actually, u_∞ and $\{u_k\}_k$ are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \tag{3.17}$$

see Proposition 3.4.3 for details.

Definition 3.1.2 *(i) A solution u of (3.17) is called a self-similar solution if*

$$u(t,x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x) \qquad (t,x) \in Q_\infty.$$

(ii) A solution u of (3.17) is called a very singular solution if it vanishes on $\mathbb{R}^N \backslash \{0\}$ at $t = 0$ and

$$\lim_{t \to 0^+} \frac{u(t,0)}{\Gamma_\alpha(t,0)} = +\infty,$$

where $\Gamma_\alpha := \mathbb{H}_\alpha[\delta_0]$ is the fundamental solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \delta_0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{3.18}$$

We remark that for $p \in (1, p^*_\beta)$, a self-similar solution u of (3.17) is also a very singular solution, since

$$\lim_{t \to 0^+} \Gamma_\alpha(t,0) t^{\frac{N}{2\alpha}} = c_2, \tag{3.19}$$

for some $c_2 > 0$. For any self-similar solution u of (3.17), $v(\eta) := u(1, t^{-\frac{1}{2\alpha}} x)$ with $\eta = t^{-\frac{1}{2\alpha}} x$ is a solution of *the self-similar equation*

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \eta - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{3.20}$$

Since $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}$ is a constant nonzero solution of (3.20), the function

$$U_p(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}} \qquad t > 0 \tag{3.21}$$

is a flat self-similar solution of (3.17). It is actually the maximal solution of the ODE $y' + t^\beta y^p = 0$ defined on $\mathbb{R}_+$. Our next goal in this chapter is to study non-flat self-similar solutions of (3.17).

Theorem 3.1.2 *Assume that $\beta > -1$, u_∞ is defined by (3.16) and*

$$p^{**}_\beta < p < p^*_\beta,$$

where $p_\beta^{**} = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$. *Then* u_∞ *is a very singular self-similar solution of (3.17) in* Q_∞. *Moreover, there exists* $c_3 > 1$ *such that*

$$\frac{c_3^{-1}}{1+|x|^{N+2\alpha}} \le u_\infty(1,x) \le \frac{c_3 \ln(2+|x|)}{1+|x|^{N+2\alpha}} \qquad x \in \mathbb{R}^N. \tag{3.22}$$

When $p_\beta^{**} < p < p_\beta^*$ with $\beta > -1$, we observe that u_∞ and U_p are self-similar solutions of (3.17) and u_∞ is non-flat. Now we are ready to consider the uniqueness of non-flat self-similar solution of (3.17) with decay at infinity, precisely, we study the uniqueness of self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \\ \lim_{|x|\to\infty} u(1,x) = 0. \end{aligned} \tag{3.23}$$

We remark if u is self-similar, then the assumption $\lim_{|x|\to\infty} u(1,x) = 0$ is equivalent to $\lim_{|x|\to\infty} u(t,x) = 0$ for any $t > 0$. Finally, we state the properties of u_∞ when $1 < p \le p_\beta^{**}$ as follows.

Theorem 3.1.3 *(i) Assume* $1 < p < p_\beta^{**}$ *and* u_∞ *is defined by (3.16). Then* $u_\infty = U_p$, *where* U_p *is given by (3.21).*

(ii) Assume $p = p_\beta^{**}$ *and* u_∞ *is defined by (3.16). Then* u_∞ *is a self-similar solution of (3.17) such that*

$$u_\infty(t,x) \ge \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}} \qquad (t,x) \in (0,1)\times\mathbb{R}^N, \tag{3.24}$$

for some $c_4 > 0$.

We note that Theorem 3.1.3 indicates that there is no self-similar solution of (3.17) with initial data $u(0,\cdot) = 0$ in $\mathbb{R}^N \setminus \{0\}$, since u_∞ is the least self-similar solution. In Theorem 3.1.3 part (ii), we do not know if the self-similar solution is flat or not. From the above theorems, we have the following result.

Theorem 3.1.4 *(i) Assume* $p_\beta^{**} < p < p_\beta^*$. *Then problem (3.20) admits a minimal positive solution* v_∞ *satisfying*

$$\lim_{|\eta|\to\infty} |\eta|^{\frac{2\alpha(1+\beta)}{p-1}} v_\infty(\eta) = 0. \tag{3.25}$$

Furthermore,

$$\frac{c_3^{-1}}{1+|\eta|^{N+2\alpha}} \le v_\infty(\eta) \le \frac{c_3 \ln(2+|\eta|)}{1+|\eta|^{N+2\alpha}} \qquad \forall \eta \in \mathbb{R}^N \tag{3.26}$$

(ii) Assume $1 < p < p_\beta^{**}$. Then problem (3.20) admits no positive solution satisfying (3.25).

The question of uniqueness of the very singular solution in the case $p_\beta^{**} < p < p_\beta^*$ remains an open problem.

3.2 Linear estimates

3.2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 3.2.1 *Let $\Theta \subset \mathbb{R}^{N+1}$ be an open domain and μ be a positive Borel measure in Θ. For $\kappa > 1$, $\kappa' = \kappa/(\kappa-1)$ and $u \in L^1_{loc}(\Theta, d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\mu)} = \inf\left\{c \in [0,\infty] : \int_E |u| d\mu \le c\left(\int_E d\mu\right)^{\frac{1}{\kappa'}},\ \forall E \subset \Theta,\ E \text{ Borel set}\right\} \tag{3.1}$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L^1_{loc}(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta,d\mu)} < \infty\}. \tag{3.2}$$

$M^\kappa(\Theta, d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|.\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm. The following property holds.

Proposition 3.2.1 *[5, 32] Assume that $1 \le q < \kappa < \infty$ and $u \in L^1_{loc}(\Theta, d\mu)$. Then there exists $c_5 > 0$ dependent of q, κ such that*

$$\int_E |u|^q d\mu \le c_5 \|u\|_{M^\kappa(\Theta,d\mu)} \left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E of Θ.

Remark 3.2.1 *If Ω is a smooth domain of $\mathbb{R}^N$, we denote by $H^\Omega_\alpha : (0,\infty) \times \Omega \times \Omega \to \mathbb{R}_+$ the heat kernel for $(-\Delta)^\alpha$ and, if $\nu \in \mathfrak{M}^b(\Omega)$, by $\mathbb{H}^\Omega_\alpha[\nu]$ the corresponding heat potential of ν defined by*

$$\mathbb{H}^\Omega_\alpha[\nu](t,x) = \int_\Omega H^\Omega_\alpha(t,x,y) d\nu(y).$$

When $\Omega = \mathbb{R}^N$, by Fourier transform, it is easy clear that

$$H_\alpha(t,x,y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x-y)\cdot\zeta - t|\zeta|^{2\alpha}} d\zeta = H_\alpha(t, x-y, 0).$$

Furthermore, $\|H_\alpha(t,.,0)\|_{L^1}$ *is independent of* t*. This implies*

$$\|\mathbb{H}_\alpha^\Omega[\nu](t,.)\|_{L^p} \leq \|\nu\|_{L^p} \qquad \forall 1 \leq p \leq \infty\,,\ \forall \nu \in L^p(\mathbb{R}^N). \tag{3.3}$$

Since $\mathbb{H}_\alpha^\Omega[\nu](t+s,.) = \mathbb{H}_\alpha^\Omega[\mathbb{H}_\alpha^\Omega[\nu](s,.)](t,.)$ *for all* $t,s>0$ *(semigroup property) and* $\nu \geq 0 \Longrightarrow \mathbb{H}_\alpha^\Omega[\nu](t,.) \geq 0$ *the semigroup* $\{\mathbb{H}_\alpha^\Omega[.](t,.)\}_{t\geq 0}$ *is sub-Markovian. Furthermore, since the operator* $(-\Delta)^\alpha$ *is symmetric in* $L^2(\mathbb{R}^N)$*, the above semigroup is analytic in* $L^p(\mathbb{R}^N)$ *for all* $1 \leq p < \infty$*: if* $1 < p < \infty$ *it follows from a general result of Sten [89]) and for* $p = 1$ *it is a consequence of regularity result from fractional powers of operators theory (see e.g. [59]). For* $1 \leq p < \infty$ *generator* A_p *of the semigroup in* $L^p(\mathbb{R}^N)$ *is the operator* $-(-\Delta)^\alpha$ *with domain*

$$D(A_p) := \{\nu \in L^p(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in L^p(\mathbb{R}^N)\}. \tag{3.4}$$

and $D(A_p)$ *is dense since it contains* $C_0^\infty(\mathbb{R}^N)$*. If* $p = \infty$*, the natural space is the space* $C_0(\mathbb{R}^N)$ *of continuous functions in* $\mathbb{R}^N$ *tending to* 0 *at infinity. The domain of the corresponding operator* A_{c_0} *is*

$$D(A_{c_0}) := \{\nu \in C_0(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in C_0(\mathbb{R}^N)\}. \tag{3.5}$$

This operator is densely defined in $C_0(\mathbb{R}^N)$*. In order to avoid confusion,* $C_c(\mathbb{R}^N)$ *(resp.* $C_c^\infty(\mathbb{R}^N)$*) denotes the space of continuous (resp.* C^∞*) functions in* $\mathbb{R}^N$ *with compact support. It is a dense subset of* $C_0(\mathbb{R}^N)$*.*

Proposition 3.2.2 *For any* $\beta > -1$ *and* $T > 0$*, there exists* $c_6 > 0$ *dependent of* N, α, β *such that for* $\nu \in \mathfrak{M}^b(\Omega)$*,*

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p_\beta^*}(Q_T^\Omega, t^\beta dxdt)} \leq c_6 \|\nu\|_{\mathfrak{M}^b(\Omega)}, \tag{3.6}$$

where p_β^* *is defined by (3.13) and* $Q_T^\Omega = (0,T) \times \Omega$*.*

In order to prove this proposition, we introduce some notations. For $\lambda > 0$ and $y \in \Omega$, let us denote

$$A_\lambda^\Omega(y) = \{(t,x) \in Q_T^\Omega : H_\alpha^\Omega(t,x,y) > \lambda\} \ \text{ and } \ m_\lambda^\Omega(y) = \int_{A_\lambda(y)} t^\beta dxdt.$$

We also set $A_\lambda^{\mathbb{R}^N} = A_\lambda$ and $m_\lambda^\Omega = m_\lambda$.

Lemma 3.2.1 *There exists* $c_7 > 0$ *such that for any* $\lambda > 1$*,*

$$A_\lambda(y) \subset (0,\ c_7 \lambda^{-\frac{2\alpha}{N}}] \times B_{c_7 \lambda^{-\frac{1}{N}}}(y), \tag{3.7}$$

where $B_r(y)$ *is the ball with radius* r *and center* y *in* $\mathbb{R}^N$*.*

Proof. We observe that $H_\alpha(t,x,y) = t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}})$, where Γ_α is the fundamental solution of (3.18). From [28], there exists $c_8 > 0$ such that

$$\Gamma_\alpha(1,z) \leq \frac{c_8}{1+|z|^{N+2\alpha}}.$$

This implies in particular

$$H_\alpha(t,x,y) \leq \frac{c_8 t^{-\frac{N}{2\alpha}}}{1+\left(t^{-\frac{1}{2\alpha}}|x-y|\right)^{N+2\alpha}}. \tag{3.8}$$

On the one hand, for $(t,x) \in A_\lambda(y)$, we have that

$$t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,0) \geq t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

which implies

$$t < \Gamma_\alpha^{\frac{2\alpha}{N}}(1,0)\lambda^{-\frac{2\alpha}{N}}. \tag{3.9}$$

On the other hand, letting $r = |x-y|$,

$$\frac{c_8 t}{t^{1+\frac{N}{2\alpha}} + r^{N+2\alpha}} \geq t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

then

$$r \leq (c_8 t\lambda^{-1})^{\frac{1}{N+2\alpha}}, \tag{3.10}$$

which, together with (3.9), implies

$$r \leq c_9\lambda^{-\frac{1}{N}},$$

for some $c_9 > 0$. □

Proof of Proposition 3.2.2. By Lemma 3.2.1, there exists $c_{10} > 0$ such that

$$m_\lambda(y) \leq c_{10}\lambda^{-1-\frac{2\alpha(1+\beta)}{N}}.$$

Clearly

$$H_\alpha^\Omega(t,x,y) \leq H_\alpha(t,x,y), \tag{3.11}$$

then for any Borel set $E \subset Q_T^\Omega$ and $y \in \Omega$, we have that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \leq \lambda\int_E t^\beta dxdt + \int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt$$

and

$$\begin{aligned}\int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt &= -\int_\lambda^{+\infty} s dm_s(y) = \lambda m_\lambda(y) + \int_\lambda^{+\infty} m_s(y)ds \\ &\le c_{10}\lambda^{-\frac{2\alpha(1+\beta)}{N}} + c_{10}\int_\lambda^{+\infty} s^{-1-\frac{2\alpha(1+\beta)}{N}} ds \\ &\le c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}},\end{aligned}$$

where $c_{11} = c_{10}\left(1 + \frac{N}{2\alpha(1+\beta)}\right)$. As a consequence, it follows

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le \lambda \int_E t^\beta dxdt + c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}}.$$

Taking $\lambda = (\int_E t^\beta dxdt)^{-\frac{N}{N+2\alpha(1+\beta)}}$, we obtain that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le (c_{11}+1)(\int_E t^\beta dxdt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}. \tag{3.12}$$

Since, by Fubini's theorem,

$$\begin{aligned}\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt &= \int_E\int_\Omega H_\alpha^\Omega(t,x,y)d|\nu(y)|t^\beta dxdt \\ &= \int_\Omega\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdtd|\nu(y)|,\end{aligned}$$

together with (3.12), it yields

$$\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt \le (c_{11}+1)\|\nu\|_{\mathfrak{M}^b(\Omega)}\left(\int_E t^\beta dxdt\right)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}.$$

Thus,

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{1+\frac{2\alpha(1+\beta)}{N}}(Q_T^\Omega, t^\beta dxdt)} \le (c_{11}+1)\|\nu\|_{\mathfrak{M}^b(\Omega)},$$

which ends the proof. □

3.2.2 The non-homogeneous problem

The following proposition is the Kato's type estimate which is essential tool to prove the uniqueness of solutions to (3.1). For $T > 0$, we denote $Q_T = (0,T)\times\mathbb{R}^N$.

Proposition 3.2.3 *Assume $\mu \in L^1(Q_T)$ and $\nu \in L^1(\mathbb{R}^N)$. Then there exists a*

unique weak solution u to the problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = \mu \quad &\text{in} \quad Q_T, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{3.13}$$

and there exists $c_{12} > 0$ such that

$$\int_{Q_T} |u| dxdt \le c_{12} \int_{Q_T} |\mu| dxdt + c_{12} \int_{\mathbb{R}^N} |\nu| dx. \tag{3.14}$$

Moreover, for any $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \ge 0$, we have that

$$\begin{aligned} &\textstyle\int_{Q_T} |u|(-\partial_t \xi + (-\Delta)^\alpha \xi) dxdt + \int_{\mathbb{R}^N} |u(T,x)| \xi(T,x) dx \\ &\qquad \le \textstyle\int_{Q_T} \xi \mathrm{sign}(u) \mu dxdt + \int_{\mathbb{R}^N} \xi(0,x) |\nu| dx \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\textstyle\int_{Q_T} u_+(-\partial_t \xi + (-\Delta)^\alpha \xi) dxdt + \int_{\mathbb{R}^N} u_+(T,x) \xi(T,x) dx \\ &\qquad \le \textstyle\int_{Q_T} \xi \mathrm{sign}_+(u) \mu dxdt + \int_{\mathbb{R}^N} \xi(0,x) \nu_+ dx. \end{aligned} \tag{3.16}$$

In order to prove Proposition 3.2.3, we introduce the following notations. We say that $u : Q_T \to \mathbb{R}$ is in $C^{\sigma,\sigma'}_{t,x}(Q_T)$ for $\sigma, \sigma' \in (0,1)$ if

$$\|u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \sup_{Q_T} \frac{|u(t,x) - u(s,y)|}{|t-s|^\sigma + |x-y|^{\sigma'}} < +\infty$$

and $u \in C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)$ if

$$\|u\|_{C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \|\partial_t u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} + \|(-\Delta)^\alpha u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} < +\infty.$$

Lemma 3.2.2 *Let $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$, $\nu \in L^\infty(\mathbb{R}^N)$ and u be a solution of problem (3.13), then there exists $\sigma \in (0,1)$ such that $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(T_0,T) \times \mathbb{R}^N$ for any $T_0 \in (0,T)$. In particular, if $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha \nu\|_{C^{1-\alpha}_x(\mathbb{R}^N)} < \infty$, then $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$.*

Proof. *Step 1.* When $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha \nu\|_{C^{1-\alpha}_x(\mathbb{R}^N)} < \infty$, it follows directly by [21, $(A.1)$] that $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$.

Step 2. When $\nu \in L^\infty(\mathbb{R}^N)$, we use [26, Theorem 6.1] to obtain that $u \in C^{\frac{\sigma}{2\alpha},\sigma}_{t,x}(Q_T)$ for some $\sigma > 0$. For any $T_0 \in (0,T)$, let $\eta : [0,T] \to [0,1]$ be a C^2 functions such that $\eta = 0$ in $[0, \frac{T_0}{4}]$ and $\eta = 1$ in $[T_0, T]$ and $v = \eta u$ in Q_T. Then we obtain that for $t \in [\frac{T_0}{4}, T]$ and $x \in \mathbb{R}^N$,

$$\partial_t v + (-\Delta)^\alpha v = \eta\mu + \eta'(t) u,$$

where $\eta\mu+\eta'(t)u \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_T)$ and $v(0,\cdot)=0$ in $\mathbb{R}^N$, Then we apply the argument in Step 1 to obtain that $v \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$. Therefore, u is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$. The proof is complete. □

Lemma 3.2.3 *(i) Let $\mu \in C^1(Q_T)\cap L^\infty(Q_T)$ and $\nu \in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then problem (3.13) admits a unique solution u and for some $\sigma\in(0,1)$, u is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0\in(0,T)$.*

(ii) Let $\mu \in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$, $\nu \in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ and u be the solution of (3.13), then $u\in L^1(Q_T)$, is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0\in(0,T)$ and for any $\xi\in\mathbb{Y}_{\alpha,T}$,

$$\begin{aligned}&\int_{Q_T} u(t,x)[-\partial_t\xi(t,x)+(-\Delta)^\alpha\xi(t,x)]dxdt\\ &\quad=\int_{Q_T}\mu(t,x)\xi(t,x)dxdt+\int_{\mathbb{R}^N}\xi(0,x)\nu dx-\int_{\mathbb{R}^N}\xi(T,x)u(T,x)dx.\end{aligned}\tag{3.17}$$

(iii) Let $\mu\in C^1(Q_T)\cap L^\infty(Q_T)$ and $\nu\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, then problem

$$\begin{aligned}-\partial_t u+(-\Delta)^\alpha u&=\mu \quad \text{in} \quad Q_T,\\ u(T,\cdot)&=\nu \quad \text{in} \quad \mathbb{R}^N\end{aligned}\tag{3.18}$$

admits a unique solution $u\in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$ for some $\sigma\in(0,1)$. Moreover, if $\mu\in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$ and $\nu\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, then $u\in\mathbb{Y}_{\alpha,T}$.

Proof. *(i)* By [26, Theorem 2.6, Theorem 6.1], there exists a unique viscosity solution $u\in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_T)$ with $\sigma>0$ to problem (3.13), and then it follows by Lemma 3.2.2 that u is $C_{t,x}^{1+\sigma',2\alpha+\sigma'}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0\in(0,T)$ and some $\sigma'\in(0,\min\{\frac{\sigma}{2\alpha},\sigma\})$. Then u is a classical solution of (3.13).

(ii) We claim that $u\in L^1(Q_T)$ and $u(t,\cdot)\in L^1(\mathbb{R}^N)$ for $t\in(0,T)$. By Duhamel formula, we have

$$\begin{aligned}\|u(t,\cdot)\|_{L^1(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N}\left(\int_0^t\int_{\mathbb{R}^N}H_\alpha(t-s,x,y)|\mu(s,y)|dyds\right)dx\\ &\qquad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}H_\alpha(t,x,y)|\nu(y)|dydx\\ &\le \|\mu\|_{L^1(Q_T)}+\|\nu\|_{L^1(\mathbb{R}^N)}\end{aligned}$$

and

$$\|u\|_{L^1(Q_T)}=\int_0^T\|u(t,\cdot)\|_{L^1(\mathbb{R}^N)}dt\le T(\|\mu\|_{L^1(Q_T)}+\|\nu\|_{L^1(\mathbb{R}^N)}).$$

In the sequel we denote by $\mathcal{H}_\alpha$ the operator of $L^1(Q_T)$ defined for all $(x,t)\in Q_T$ by

$$\mathcal{H}_\alpha[\mu](x,t)=\int_0^t \mathbb{H}_\alpha[\mu(.,s)](x,t-s)ds=\int_0^t\int_{\mathbb{R}^N} H_\alpha(t-s,x,y)\mu(s,y)dyds. \quad (3.19)$$

We claim that $\|(-\Delta)^\alpha_\epsilon u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}$ *is uniformly bounded with respect to* $\epsilon\in(0,\epsilon_0)$. Since $u(t,\cdot)\in C^{2\alpha+\sigma}_x(\mathbb{R}^N)$ for some $\sigma\in(0,\min\{2-2\alpha,1\})$, then for $x\in\mathbb{R}^N$ and $y\in B_1(0)$, $|u(x+y)+u(x-y)-2u(x)|\le\|u(t,\cdot)\|_{C^{2\alpha+\sigma}_x(\mathbb{R}^N)}|y|^{2\alpha+\sigma}$. Thus,

$$\begin{aligned}\||(-\Delta)^\alpha_\epsilon u(t,\cdot)|\|_{L^\infty(\mathbb{R}^N)} &\le \sup_{x\in\mathbb{R}^N}\Bigg[\int_{\mathbb{R}^N\setminus B_1(0)}\frac{|u(x+y)-u(x)|}{|y|^{N+2\alpha}}dy\\ &\quad+\frac{1}{2}\int_{B_1(0)\setminus B_\epsilon(0)}\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{N+2\alpha}}dy\Bigg]\\ &\le 2\|u\|_{L^1(\mathbb{R}^N)}+\int_{B_1(0)}|y|^{\sigma-N}dy\|u(t,\cdot)\|_{C^{2\alpha+\sigma}_x(\mathbb{R}^N)}.\end{aligned}$$

Next we claim that

$$\int_{Q_T}\xi(-\Delta)^\alpha_\epsilon u dxdt=\int_{Q_T}u(-\Delta)^\alpha_\epsilon\xi dxdt \qquad \forall\xi\in\mathbb{Y}_{\alpha,T}. \quad (3.20)$$

Indeed, using the fact that for any $t>0$ there holds

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)]\xi(t,x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx\\ &\qquad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,x)-u(t,z)]\xi(t,z)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx,\end{aligned}$$

then we have

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^N}\xi(t,x)(-\Delta)^\alpha_\epsilon u(t,x)dx\\ &=-\tfrac{1}{2}\textstyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left[\frac{(u(t,z)-u(t,x))\xi(t,x)}{|z-x|^{N+2\alpha}}+\frac{(u(t,x)-u(t,z))\xi(t,z)}{|z-x|^{N+2\alpha}}\right]\chi_\epsilon(|x-z|)dzdx\\ &=\tfrac{1}{2}\textstyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.\end{aligned}$$

Similarly,

$$\textstyle\int_{\mathbb{R}^N}u(t,x)(-\Delta)^\alpha_\epsilon\xi(t,x)dx=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.$$

Then (3.20) holds. Since u is $C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0\in(0,T)$ and ξ belongs to $\mathbb{Y}_{\alpha,T}$, $(-\Delta)^\alpha_\epsilon\xi(t,\cdot)\to(-\Delta)^\alpha\xi(t,\cdot)$ and $(-\Delta)^\alpha_\epsilon u(t,\cdot)\to(-\Delta)^\alpha u(t,\cdot)$ as

$\epsilon \to 0$ in $\mathbb{R}^N$ and $(-\Delta)^\alpha_\epsilon \xi(t,\cdot)$, $(-\Delta)^\alpha_\epsilon u(t,\cdot) \in L^\infty(\mathbb{R}^N)$ and $\xi(t,\cdot), u(t,\cdot) \in L^1(\mathbb{R}^N)$, then it follows by the Dominated Convergence Theorem that

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha_\epsilon u(t,x)dx = \int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha u(t,x)dx$$

and

$$\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^N} (-\Delta)^\alpha_\epsilon \xi(t,x)u(t,x)dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t,x)u(t,x)dx.$$

Combining this with (3.20), and letting $\epsilon \to 0^+$, we have that

$$\int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha u(t,x)dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t,x)u(t,x)dx,$$

integrating over $[0,T]$ and by (3.13), we conclude that (3.17) holds.

(iii) End of the proof. Let u be the weak solution of problem (3.13) and

$$w(t,x) = u(T-t,x) \qquad (t,x) \in [0,T] \times \mathbb{R}^N.$$

Then w is a solution of (3.18) and for some $\sigma \in (0,1)$, w is $C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$. On the contrary, if w is a solution of (3.18), then $u(t,x) = w(T-t,x)$ for $(t,x) \in [0,T]\times\mathbb{R}^N$ is a solution of (3.13), then the uniqueness holds since the solution of (3.13) is unique. Since $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$, then $(-\Delta)^\alpha u(t,\cdot) \in C^\sigma_x$ and then $(-\Delta)^\alpha_\epsilon u(t,\cdot)$ is bounded, which implies $u \in \mathbb{Y}_{\alpha,T}$. □

Proof of Proposition 3.2.3. *Uniqueness.* Let v be a weak solution of

$$\begin{aligned} \partial_t v + (-\Delta)^\alpha v = 0 \quad &\text{in} \quad Q_T, \\ v(0,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned} \tag{3.21}$$

We claim that $v = 0$ a.e. in Q_T.

In fact, let ω be a Borel subset of Q_T and $\eta_{\omega,n}$ be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u = \zeta_n \quad &\text{in} \quad Q_T, \\ u(T,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{3.22}$$

where $\zeta_n : \bar{Q}_T \to [0,1]$ is a function $C^1_c(Q_T)$ such that

$$\zeta_n \to \chi_\omega \quad \text{in } L^\infty(\bar{Q}_T) \quad \text{as } n \to \infty.$$

Then $\eta_{\omega,n} \in \mathbb{Y}_{\alpha,T}$ by Lemma 3.2.3, and

$$\int_{Q_T} v\zeta_n dxdt = 0.$$

Passing to the limit when $n \to \infty$, we derive

$$\int_\omega v dxdt = 0.$$

This implies $v = 0$ a.e. in Q_T.

Existence and estimate (3.15). For $\delta > 0$, we define an even convex function ϕ_δ by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2} & \text{if} \quad |t| \geq \delta, \\ \frac{t^2}{2\delta} & \text{if} \quad |t| < \delta/2. \end{cases} \tag{3.23}$$

Then for any $t, s \in \mathbb{R}$, $|\phi'_\delta(t)| \leq 1$, $\phi_\delta(t) \to |t|$ and $\phi'_\delta(t) \to \text{sign(t)}$ when $\delta \to 0^+$. Moreover,

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s - t). \tag{3.24}$$

Let $\{\mu_n\}$, $\{\nu_n\}$ be two sequences of functions in $C_0^2(Q_T)$, $C_0^2(\mathbb{R}^N)$, respectively, such that

$$\lim_{n\to\infty} \int_{Q_T} |\mu_n - \mu| dxdt = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nu_n - \nu| dx = 0.$$

We denote by u_n the corresponding solution to (3.13) where μ, ν are replaced by μ_n, ν_n, respectively. By Lemma 3.2.2 and Lemma 3.2.3(ii), $u_n \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T) \cap L^1(Q_T)$ and then we use Lemma 2.3 in [32] and Lemma 3.2.3 (ii) to obtain that for any $\delta > 0$ and $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \geq 0$,

$$\begin{aligned} &\int_{Q_T} \phi_\delta(u_n)[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} \xi(T,x)\phi_\delta(u_n(T,x)) dx \\ &\qquad = \int_{Q_T} \xi[\partial_t \phi_\delta(u_n) + (-\Delta)^\alpha \phi_\delta(u_n)] dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n) dx \\ &\qquad \leq \int_{Q_T} \xi \phi'_\delta(u_n)[\partial_t u_n + (-\Delta)^\alpha u_n] dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n) dx \\ &\qquad = \int_{Q_T} \xi \phi'_\delta(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n) dx. \end{aligned}$$

Letting $\delta \to 0^+$, we obtain

$$\begin{aligned} \int_{Q_T} |u_n|[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} \xi(T,x)|u_n(T,x)| dx \\ \leq \int_{Q_T} \xi \text{sign}(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0,x)|\nu_n| dx. \end{aligned} \tag{3.25}$$

Let η_k be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u = \varsigma_k \quad &\text{in} \quad Q_T, \\ u(T,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{3.26}$$

where $\varsigma_k : Q_T \to [0,1]$ is a C_0^2 function such that $\varsigma_k = 1$ in $(0,T)\times B_k(0)$. From the proof of Lemma 3.2.3, $\tilde{\eta}_k(t,x) := \eta_k(T-t,x)$ satisfies with $\tilde{\varsigma}_k(t,x) = \varsigma_k(T-t,x)$

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = \tilde{\varsigma}_k \quad &\text{in} \quad Q_T, \\ u(0,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned}$$

By Lemma 3.2.2, $\tilde{\eta}_k \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$ with some $\sigma \in (0,1)$ and

$$\begin{aligned} 0 \le \tilde{\eta}_k(t,x) &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1+|(s-t)^{-\frac{1}{2\alpha}}(y-x)|^{N+2\alpha}} dyds \\ &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{dz}{1+|z|^{N+2\alpha}} ds \\ &= c_{13}(T-t). \end{aligned}$$

Taking $\xi = \eta_k$ in (3.25), we derive that

$$\int_{Q_T} |u_n| \chi_{(0,T)\times B_k(0)} dxdt \le c_{13} T \int_{Q_T} |\mu_n| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx.$$

Then, letting $k \to \infty$, we have

$$\int_{Q_T} |u_n| dxdt \le c_{13} T \int_{Q_T} |\mu_n| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx. \tag{3.27}$$

Similarly,

$$\int_{Q_T} |u_n - u_m| dx \le c_{13} T \int_{Q_T} |\mu_n - \mu_m| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n - \nu_m| dx. \tag{3.28}$$

Therefore, $\{u_n\}_n$ is a Cauchy sequence in $L^1(Q_T)$ and its limit u is a weak solution of (3.13). Letting $n \to \infty$, (3.15) and (3.14) follow by (3.25) and (3.27), respectively. The proof of (3.16) is similar. □

3.3 Proof of Theorem 3.1.1

If $h(t,.)$ is monotone nondecreasing, for any $\lambda > 0$, $I + \lambda h(t,.)$ is an homeorphism of $\mathbb{R}$ and the inverse function $J_\lambda(t,.) = (I + \lambda h(t,.))^{-1}$ is a contraction. We define the Yosida approximation by

$$h_\lambda(t,.) = \frac{I - J_\lambda(t,.)}{\lambda}. \tag{3.1}$$

The function $h_\lambda(t,.)$ is monotone nondecreasing, vanishes at 0 as h does it and it is $\frac{1}{\lambda}$-Lipschitz continuous. Furthermore

$$rh_\lambda(t,r) \uparrow rh(t,r) \quad \text{as } \lambda \to 0 \qquad \forall r \in \mathbb{R}, \tag{3.2}$$

see [11, Chap 2, Prop. 2.6]. If u is a real valued function we will denote by $h \circ u$ and $h_\lambda \circ u$ respectively the functions $(t,x) \mapsto h(t,u(t,x))$ and $(t,x) \mapsto h_\lambda(t,u(t,x))$

Lemma 3.3.1 *Assume that h satisfies (H)-(i), $\lambda > 0$ and $\phi \in L^1(\mathbb{R}^N)$. Then there exists a unique solution u_ϕ of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h_\lambda \circ u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \phi \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{3.3}$$

Moreover,

$$\mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ \mathbb{H}_\alpha[\phi_+])] \le u_\phi \le \mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_T, \tag{3.4}$$

where $\phi_\pm = \max\{0, \pm\phi\}$ and

$$\|u_\phi(t,.) - u_\psi(t,.)\|_{L^1} \le \|\phi - \psi\|_{L^1} \qquad \forall 1 \le p \le \infty. \tag{3.5}$$

(i) $u_\phi \ge 0$ if $\phi \ge 0$ in Ω;

(ii) the mapping $\phi \mapsto u_\phi$ is increasing.

Proof. Existence is a consequence of the Cauchy-Lipschitz-Picard theorem (see [25, Chap 4]): we write (3.3) under the integral form $u = \mathcal{T}[u] = \mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ u]$, i.e.

$$\mathcal{T}[u](t,.) = \mathbb{H}_\alpha[\phi](t,.) - \int_0^T \mathbb{H}_\alpha[h_\lambda \circ u](t-s,.)ds \tag{3.6}$$

The space $C([0,\infty); L^1(\mathbb{R}^N))$ endowed with the norm

$$\|w\|_{C-L^1} = \sup\left\{e^{-kt}\|w(t,.)\|_{L^1} : t \ge 0\right\},$$

$(k > \lambda^{-1})$, is a Banach space. Since $u \mapsto h_\lambda(t,u)$ is $\frac{1}{\lambda}$-Lipschitz continuous, the mapping $\mathcal{T}$ is $\frac{1}{\lambda k}$-Lipschitz continuous in X_p. Thus it admits a unique fixed point u_ϕ which is an integral solution of (3.3).

$$u_\phi(t,.) = \mathbb{H}_\alpha[\phi](t,.) - \int_0^T \mathbb{H}_\alpha[h_\lambda \circ u_\phi](t-s,.)ds. \tag{3.7}$$

The semigroup $\{\mathbb{H}_\alpha[.](t,.)\}_{t\ge 0}$ is analytic in $L^1(\mathbb{R}^N)$ since generated by the fractional power of a closed operator. It follows from the classical regularity theory for

analytic semigroups as it exposed in [52, Sec 6] that that u_ϕ is a strong solution of (3.3). Since it is continuous, it is also a weak solution in the sense that

$$\begin{aligned}\int_{Q_T} (u_\phi[-\partial_t\xi + (-\Delta)^\alpha\xi] + \xi h_\lambda \circ u_\phi)\, dxdt \\ = \int_{\mathbb{R}^N} \xi(0,x)\phi(x)dx - \int_{\mathbb{R}^N} \xi(T,x)u_\phi(T,x)dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}.\end{aligned} \tag{3.8}$$

If $\phi_1, \phi_2 \in L^1(\mathbb{R}^N)$ and u_{ϕ_j} are the corresponding solutions of (3.3), it follows from the positivity of H_α that

$$(u_{\phi_2} - u_{\phi_1})_+ \leq (\mathcal{H}_\alpha[h_\lambda \circ u_{\phi_2} - h_\lambda \circ u_{\phi_1}])_+ \leq \tfrac{1}{\lambda}\mathcal{H}_\alpha[(u_{\phi_2} - u_{\phi_1})_+].$$

Therefore,

$$\|(u_{\phi_2}(t,.) - u_{\phi_1}(t,.))_+\|_{L^p} \leq \frac{1}{\lambda}\int_0^T \|(u_{\phi_2}(t-s) - u_{\phi_1}(t-s))_+\|_{L^p} ds,$$

and by Gronwall inequality

$$\|(u_{\phi_2}(t) - u_{\phi_1}(t))_+\|_{L^p} \leq e^{\frac{t}{\lambda}}\|(\phi_2 - \phi_1)_+\|_{L^p}.$$

This implies (i) and (ii). As a consequence,

$$-\mathbb{H}_\alpha[\phi_-] \leq -u_{\phi_-} \leq u_\phi \leq u_{\phi_+} \leq \mathbb{H}_\alpha[\phi_+]$$

and thus

$$h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-]) \leq h_\lambda \circ (-u_{\phi_-}) \leq h_\lambda \circ u_\phi \leq h_\lambda \circ u_{\phi_+} \leq h_\lambda \circ \mathbb{H}_\alpha[\phi_+].$$

Jointly with (3.7) it yields (3.4). □

Notation. In the sequel, if $\eta \in L^1(Q_\tau)$ and $\tau \geq T$, we denote by $\xi_{\eta,\tau}$ the solution of

$$\begin{aligned}-\partial_t\xi_\eta + (-\Delta)^\alpha\xi_\eta = \eta \qquad \text{in } Q_\tau \\ \xi_\eta(\tau,.) = 0\end{aligned} \tag{3.9}$$

If $\eta \geq 0$, then $\xi_{\eta,\tau} \geq 0$; if $\eta \in C_0^\infty(\mathbb{R}^{N+1})$, then $\eta \in \mathbb{Y}_{\alpha,\tau}$; if $\eta_n = \eta(\frac{\cdot}{n})$, where $n \in \mathbb{N}_*$ and $\eta \in C_0^\infty(\mathbb{R}^{N+1})$ is nonnegative, $0 \leq \eta \leq 1$, with value 1 on B_1 and 0 on B_2^c, then $\xi_{\eta_n,\tau} \uparrow \tau - t$ as $n \to \infty$.

In the next lemma we prove that we can replace h_λ by h.

Lemma 3.3.2 *Assume that h satisfies (H)-(i) and $\phi \in L^1(\mathbb{R}^N)$. Then there exists a unique solution $u_\phi \in C([0,\infty); L^1(\mathbb{R}^N)$ of*

$$\begin{aligned}\partial_t u + (-\Delta)^\alpha u + h \circ u = 0 \quad \text{in} \quad Q_\infty, \\ u(0,\cdot) = \phi \quad \text{in} \quad \mathbb{R}^N,\end{aligned} \tag{3.10}$$

Moreover inequality (3.5) and statements (i) and (ii) in Lemma 3.3.1 hold.

Proof. We denote by $u_{\lambda,\phi}$ the solution of (3.3).

Step 1- A priori estimate. Let $\phi \geq 0$. If we take $\xi = \xi_{\eta_n,\tau}$ in (3.8) and let $n \to \infty$, we derive

$$\int_{Q_T} (u_{\lambda,\phi} + (\tau - t)h_\lambda \circ u_{\lambda,\phi})\, dxdt + (\tau - T)\int_{\mathbb{R}^N} u_{\lambda,\phi}(T,.)dx = \tau \int_{\mathbb{R}^N} \phi(x)dx. \tag{3.11}$$

For $0 < \lambda < \lambda'$ we set $w = u_{\lambda,\phi} - u_{\lambda',\phi}$. It follows from (3.16) and inequality $h_{\lambda'} \circ u_{\lambda,\phi} \leq h_\lambda \circ u_{\lambda,\phi}$, that for any nonnegative ξ in $\mathbb{Y}_{\alpha,T}$,

$$\begin{aligned}\textstyle\int_{Q_T} \big(w_+[-\partial_t\xi + (-\Delta)^\alpha\xi] + \xi\,(h_\lambda \circ u_{\lambda,\phi} - h_\lambda \circ u_{\lambda',\phi})\,\mathrm{sign}_+(w)\big)\, dxdt \\ \leq \textstyle\int_{Q_T} w_+\,(h_{\lambda'} \circ u_{\lambda',\phi} - h_\lambda \circ u_{\lambda',\phi})\, dxdt - \int_{\mathbb{R}^N} \xi(T,x)w_+(T,x)dx,\end{aligned}$$

Since $h_\lambda(t,.)$ is nondecreasing, we derive

$$\int_{Q_T} w_+[-\partial_t\xi + (-\Delta)^\alpha\xi]dxdt \leq 0 \qquad \forall \xi \in \mathbb{Y}_{\alpha,T},\ \xi \geq 0.$$

If $\eta \in C_0^\infty(\mathbb{R}^{N+1})$ is nonnegative, then $\xi_\eta \in \mathbb{Y}_{\alpha,T}$, $\xi_\eta \geq 0$ and

$$\int_{Q_T} w_+\eta dxdt = 0.$$

This implies $u_{\lambda,\phi} \leq u_{\lambda',\phi}$.

Step 2- Truncation. We replace ϕ by $\phi_n = \inf\{\phi, n\}$ for $n \in \mathbb{N}_*$ and denote by u_{λ,ϕ_n} the corresponding solution of (3.3). By Step 1, the sequence $\{u_{\lambda,\phi_n}\}_{\lambda>0}$ is decreasing and it converges to some nonnegative u_{ϕ_n} when $\lambda \downarrow 0$. Therefore $h_\lambda \circ u_{\lambda,\phi_n} \to h \circ u_{\phi_n}$ a.e. in Q_T. It follows from (3.11) and Fatou's lemma that

$$\int_{Q_T} (u_{\phi_n} + (\tau - t)h \circ u_{\phi_n})\, dxdt + (\tau - T)\int_{\mathbb{R}^N} u_{\phi_n}(.,T)dx = \tau \int_{\mathbb{R}^N} \phi_n(x)dx. \tag{3.12}$$

Since $0 \leq u_{\lambda,\phi_n} \leq n$, then $0 \leq h_\lambda \circ u_{\lambda,\phi_n} \leq h \circ u_{\lambda,\phi_n} \leq h(n)$ by (3.5). If $E \subset Q_T$ is a Borel set,

$$\int_E h_\lambda \circ u_{\lambda,\phi_n} dxdt \leq h(n)|E|.$$

By Vitali convergence theorem $h_\lambda \circ u_{\lambda,\phi_n} \to h \circ u_{\phi_n}$ in $L^1(Q_T)$. Therefore, we can let $\lambda \to 0$ in identity (3.8) and conclude that u_{ϕ_n} is a weak solution of (3.10) with initial data ϕ_n.

Step 3- Existence with ϕ bounded. If $\phi = \phi_+ - \phi_- \in L^1(\mathbb{R}^N)$, set $\phi_{+,n} = \inf\{\phi_+, n\}$ and $\phi_{-,n} = \inf\{\phi_-, n\}$. We denote by $u_{\lambda,\phi_{+,n}}$, $u_{\phi_{+,n}}$, $u_{\lambda,-\phi_{-,n}}$ and $u_{-\phi_{-,n}}$

the corresponding solutions of (3.3) and (3.10). Then

$$\begin{gathered} u_{\lambda,-\phi_{-,n}} \leq u_{\lambda,\phi_{+,n}-\phi_{-,n}} \leq u_{\lambda,\phi_{+,n}} \\ \text{which implies} \\ h_\lambda \circ u_{\lambda,-\phi_{-,n}} \leq h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}} \leq h_\lambda \circ u_{\lambda,\phi_{+,n}}. \end{gathered} \tag{3.13}$$

Estimate (3.11) is valid under the form

$$\begin{aligned} &\int_{Q_T} \left(u_{\lambda,\phi_{+,n}} + (\tau - t) h_\lambda \circ u_{\lambda,\phi_{+,n}}\right) dxdt \\ &\qquad + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,\phi_{+,n}}(.,T)dx = \tau \int_{\mathbb{R}^N} \phi_{+,n}(x)dx. \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &\int_{Q_T} \left(u_{\lambda,-\phi_{-,n}} + (\tau - t) h_\lambda \circ u_{\lambda,-\phi_{-,n}}\right) dxdt \\ &\qquad + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,-\phi_{-,n}}(.,T)dx = -\tau \int_{\mathbb{R}^N} \phi_{-,n}(x)dx. \end{aligned} \tag{3.15}$$

Since $h_\lambda \circ u_{\lambda,\phi_{+,n}}$ and $h_\lambda \circ u_{\lambda,-\phi_{-,n}}$ are bounded in $L^1(Q_T)$ independently of λ and n, $h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}}$ endows the same property. Since

$$u_{\lambda,\phi_{+,n}-\phi_{-,n}} = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}] - \mathcal{H}_\alpha[h_\lambda \circ u_{\lambda,\phi_{+,n}-\phi_{-,n}}]$$

it follows from [52, Sec 6] that $u_{\lambda,\phi_{+,n}-\phi_{-,n}}$ remains bounded in the interpolation space $Y_1 := L^1([0,T]; D(A_1)(\mathbb{R}^N)) \cap W^{s,1}([0,T]; L^1(\mathbb{R}^N))$ for any $s \in (0,1)$ where $D(A_1)$ is defined in (3.4). Although a bounded subset K of Y_1 is not a relatively compact subset of $L^1(Q_T)$, for any ball $B \subset \mathbb{R}^N$, the set of restriction to B of functions belonging to K is relatively compact in $L^1((0,T) \times B)$. Thus, there exists a subsequence $\{\lambda_k\}$ such that $\{u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}$ converges a.e. to some function U_n. Furthermore $\{h_{\lambda_k} \circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}$ converges a.e. to $h \circ U_n$. Since the sequences $\{u_{\lambda_k,-\phi_{-,n}}\}_{\lambda_k}$, $\{u_{\lambda_k,\phi_{+,n}}\}_{\lambda_k}$, $\{h_{\lambda_k} \circ u_{\lambda_k,-\phi_{-,n}}\}_{\lambda_k}$ and $\{h_{\lambda_k} \circ u_{\lambda_k,\phi_{+,n}}\}_{\lambda_k}$ are convergent in $L^1(Q_T)$ they are uniformly integrable. Because of (3.13) the same property is shared by the two sequences $\{u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}_{\lambda_k}$ and $\{h_{\lambda_k} \circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}\}_{\lambda_k}$. Letting λ_k to 0 in the identity

$$u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}(t,.) = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}](t,.) - \int_0^T \mathbb{H}_\alpha[h_{\lambda_k} \circ u_{\lambda_k,\phi_{+,n}-\phi_{-,n}}](t-s,.)ds. \tag{3.16}$$

yields

$$U_n(t,.) = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}](t,.) - \int_0^T \mathbb{H}_\alpha[h \circ U_n](t-s,.)ds. \tag{3.17}$$

This implies that U_n is an integral solution, thus a weak solution of (3.10) with initial data $\phi_{+,n} - \phi_{-,n} = \operatorname{sgn}(\phi) \inf\{n, |\phi|\}$ and then $U_n = u_{\phi_n}$.

Step 4- Existence with $\phi \in L^1(\mathbb{R}^N)$. By Kato's inequality (3.15), we obtain that

$$\int_{Q_T} \left(|u_{\phi_k} - u_{\phi_m}|(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi |h \circ u_{\phi_k} - h \circ u_{\phi_m}|\right) dxdt$$
$$+ \int_{\mathbb{R}^N} |u_{\phi_k}(T,x) - u_{\phi_m}(T,x)|\xi(T,x)dx \leq \int_{\mathbb{R}^N} \xi(0,x)|\phi_k - \phi_m|dx,$$

for $m, k \in \mathbb{N}_*$ and $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi > 0$. Taking $\xi = \xi_{\eta_n,\tau}$ as in (3.9) and letting $n \to \infty$ yields

$$\int_{Q_T} \left(|u_{\phi_k} - u_{\phi_m}| + (\tau - t)|h \circ u_{\phi_k} - h \circ u_{\phi_m}|\right) dxdt$$
$$+ (\tau - T) \int_{\mathbb{R}^N} |u_{\phi_k}(T,.) - u_{\phi_m}(T,.)|dx \leq \tau \int_{\mathbb{R}^N} |\phi_k - \phi_m|dx. \tag{3.18}$$

Since $\{\phi_m\}$ is a Cauchy sequence in $L^1(\mathbb{R}^N)$, $\{u_{\phi_m}\}$ and $\{h \circ u_{\phi_m}\}$ are also Cauchy sequences in $C(0,T;L^1(\mathbb{R}^N))$ and $L^1(Q_T)$ respectively. Set $U = \lim_{m\to\infty} u_{\phi_m}$, then it satisfies

$$\int_{Q_T} \left(U[-\partial_t \xi + (-\Delta)^\alpha \xi] + \xi h \circ U\right) dxdt$$
$$= \int_{\mathbb{R}^N} \xi(0,x)\phi(x)dx - \int_{\mathbb{R}^N} \xi(T,x)U(T,x)dx \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}. \tag{3.19}$$

and it is also an integral solution of (3.10). Thus $u_\phi \in C([0,\infty); L^1(\mathbb{R}^N))$.

Finally, we end the proof with uniqueness which is a consequence of the inequality below

$$\int_{Q_T} \left(|U - U'| + (\tau - t)|h \circ U - h \circ U'|\right) dxdt$$
$$+ (\tau - T) \int_{\mathbb{R}^N} |U(T,.) - U'(T,.)|dx \leq \tau \int_{\mathbb{R}^N} |\phi - \phi'|dx, \tag{3.20}$$

valid for two solutions U and U' of problem (3.10) with respective initial data ϕ and ϕ', the proof of which is the same as the one of (3.18). Notice also that statement (i) and (ii) as well as inequality (3.5) follows by the above approximations. □

Remark 3.3.1 *By the same method it can be proved that for any $p \in (1,\infty)$ and $\phi \in L^p(\mathbb{R}^N)$ (resp. $\phi \in C_0(\mathbb{R}^N)$) there exists a unique solution $u_\phi \in C([0,\infty); L^p(\mathbb{R}^N))$ (resp. $u_\phi \in C([0,\infty); C_0(\mathbb{R}^N))$) solution of (3.10). Furthermore (3.5) holds.*

Proof of Theorem 3.1.1. *Existence for $\nu \geq 0$.* We consider a sequence of nonnegative functions $\{\nu_n\}_n \subset C_0^2(\mathbb{R}^N)$ such that $\nu_n \to \nu$ as $n \to \infty$ in the weak sense of bounded measures, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \zeta \nu_n dx = \int_{\mathbb{R}^N} \zeta d\nu \qquad \forall \zeta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{3.21}$$

It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)}$ is bounded independently of n and we assume that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \leq 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}$. By Lemma 3.3.1,

we denote by u_{ν_n} the corresponding solution of (3.10) initial data ν_n. Then u_n is nonnegative and satisfies that

$$0 \leq u_{\nu_n} = \mathbb{H}_\alpha[\nu_n] - \mathcal{H}_\alpha[h \circ u_{\nu_n}] \leq \mathbb{H}_\alpha[\nu_n] \quad \text{in } Q_T. \tag{3.22}$$

Jointly with (3.6) it implies

$$\|u_{\nu_n}\|_{M^{p^*_\beta}(Q_T, t^\beta dxdt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \tag{3.23}$$

We have also the following estimates from (3.8) and (3.12)

$$u_{\nu_n}(t,x) \leq \mathbb{H}_\alpha[\nu_n](t,x) \leq 2c_8 t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} \qquad \forall (t,x) \in Q_T \tag{3.24}$$

and

$$\begin{aligned} \int_{Q_T} \left(u_{\nu_n} + (\tau - t) h \circ u_{\nu_n}\right) dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\nu_n}(.,T) dx &= \tau \int_{\mathbb{R}^N} \nu_n(x) dx \\ &\leq 2\tau \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \end{aligned} \tag{3.25}$$

As in the proof of Lemma 3.3.2-Step 3, using the regularizing properties of the semigroup $\mathbb{H}_\alpha[.](t)$ (see [52, Sec 6]) infert that there exists a subsequence $\{u_{\nu_{n_k}}\}$ which converges a.e. in Q_T to some function U and $\{h \circ u_{\nu_{n_k}}\}$ converges a.e. to $h \circ U$.

For $\kappa > 0$, we denote $S_\kappa = \{(t,x) \in Q_T : |u_{n_k}(t,x)| > \kappa\}$ and $\omega(\kappa) = \int_{S_\kappa} t^\beta dxdt$. Then for any Borel set $E \subset Q_T$

$$\begin{aligned} \iint_E h \circ u_{\nu_{n_k}} dxdt &\leq \iint_{E \cap \{u_{\nu_{n_k}} \leq \kappa\}} h \circ u_{\nu_{n_k}} dxdt + \iint_{E \cap S_\kappa} h \circ u_{\nu_{n_k}} dxdt \\ &\leq g(\kappa) \iint_E t^\beta dxdt + \iint_{\{S_\kappa} t^\beta g(u_{\nu_{n_k}}) dxdt \\ &\leq g(\kappa) \iint_E t^\beta dxdt - \int_\kappa \infty g(s) d\omega(s), \end{aligned}$$

where

$$\int_\kappa^\infty g(s) d\omega(s) = \lim_{M \to \infty} \int_\kappa^M g(s) d\omega(s).$$

By (3.1) and (3.23), $\omega(s) \leq c_{14} s^{-p_\beta^*}$, thus

$$\begin{aligned}
-\int_\kappa^M g(s)d\omega(s) &= -\Big[g(s)\omega(s)\Big]_{s=\kappa}^{s=M} + \int_\kappa^M \omega(s)dg(s) \\
&\leq g(\kappa)\omega(\kappa) - g(M)\omega(M) + c_{14}\int_\kappa^M s^{-p_\beta^*}dg(s) \\
&\leq g(\kappa)\omega(\kappa) - g(M)\omega(M) + c_{14}\left(M^{-p_\beta^*}g(M) - \kappa^{-p_\beta^*}g(\kappa)\right) \\
&\qquad + \frac{c_{14}}{p_\beta^*+1}\int_\kappa^M s^{-1-p_\beta^*}g(s)ds.
\end{aligned}$$

Since $\lim_{M\to\infty} M^{-p_\beta^*}g(M) = 0$ by (3.12) and [32, Lemma 4.1] and $\omega(s) \leq c_{14}s^{-p_\beta^*}$, we derive $g(\kappa)\omega(\kappa) \leq c_{14}\kappa^{-p_\beta^*}g(\kappa)$ and then

$$-\int_\kappa^\infty g(s)d\omega(s) \leq \frac{c_{14}}{p_\beta^*+1}\int_\kappa^\infty s^{-1-p_\beta^*}g(s)ds.$$

The above quantity on the right-hand side tends to 0 when $\kappa \to \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\kappa > 0$ such that

$$\frac{c_{14}}{p_\beta^*+1}\int_\kappa^\infty s^{-1-p_\beta^*}g(s)ds \leq \frac{\epsilon}{2}$$

and there exists $\delta > 0$ such that

$$\int_E t^\beta dxdt \leq \delta \Longrightarrow g(\kappa)\int_E t^\beta dxdt \leq \frac{\epsilon}{2}.$$

This means that $\{h_{n_k} \circ u_{\nu_{n_k}}\}$ is uniformly integrable in $L^1(Q_T)$ and by Vitali convergence theorem $h_{n_k} \circ u_{\nu_{n_k}} \to h \circ U$ in $L^1(Q_T)$. Letting $n_k \to \infty$ in the identity

$$u_{\nu_{n_k}}(t,.) = \mathbb{H}_\alpha[\nu_{n_k}](t,.) - \int_0^T \mathbb{H}_\alpha[h \circ u_{\nu_{n_k}}(s,.)](t-s,.)ds$$

for some $t > 0$ such that $u_{\nu_{n_k}}(t,.) \to U(t,.)$ a.e. in $\mathbb{R}^N$ yields

$$U(t,.) = \mathbb{H}_\alpha[\nu](t,.) - \int_0^T \mathbb{H}_\alpha[h \circ U(s,.)](t-s,.)ds.$$

This is valid for almost all $t > 0$ and implies that $U \in C([0,T]; L^1(\mathbb{R}^N))$, up to a modification on a set of $t > 0$ with zero measure. Moreover

$$\int_{Q_T} \left(u_{\nu_{n_k}}(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ u_{\nu_{n_k}}\right) dxdt \\ = \int_{\mathbb{R}^N} \xi(0,x)\nu_{n_k} dx - \int_{\mathbb{R}^N} u_{\nu_{n_k}}(T,x)\xi(T,x)dx.$$

where $\xi \in \mathbb{Y}_{\alpha,T}$ is arbitrary. Thus, using the continuity of $t \mapsto U(t,.)$ in $L^1(\mathbb{R}^N)$, we derive

$$\int_{Q_T} (U(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ U)\, dxdt \\ = \int_{\mathbb{R}^N} \xi(0,x)d\nu(x) - \int_{\mathbb{R}^N} U(T,x)\xi(T,x)dx.$$

From this infers that U is a weak solution of (3.1).

Existence for general ν. For $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, a sequence $\{\nu_n\}$ in $C_0^2(\mathbb{R}^N)$ converge to ν in the weak sense of bounded measures. Because of the monotonicity of $h(t,\cdot)$,

$$-\mathbb{H}_\alpha[|\nu_n|] \le u_{-|\nu_n|} \le u_{\nu_n} \le u_{|\nu_n|} \le \mathbb{H}_\alpha[|\nu_n|].$$

Then by above analysis, the sequence $\{h \circ u_{-|\nu_n|})\}$ and $\{h \circ u_{|\nu_n|})\}$ are relatively compact in $L^1(Q_T^B)$ for any $T > 0$ and ball B and (3.23) holds for $\{u_{\nu_n}\}$. Therefore $\{u_{\nu_n}\}$ is relatively locally compact in $L^1(Q_T^B)$ and there exist some subsequence $\{u_{\nu_{n_k}}\}$ and $U \in L^1(Q_T)$ such that

$$u_{\nu_{n_k}} \to U \Longrightarrow h \circ u_{\nu_{n_k}} \to h \circ U \quad \text{as } k \to \infty \quad \text{a.e. in} \quad Q_T.$$

As in the previous case it implies that U is a weak solution of (3.1) and also an integral solution.

Uniqueness. Let u_1, u_2 be two weak solutions of (3.1) with the same initial ν and $w = u_1 - u_2$. Then

$$\partial_t w + (-\Delta)^\alpha w = h \circ u_2 - h \circ u_1 \quad \text{in} \quad Q_T.$$

Since $h \circ u_2 - h \circ u_1 \in L^1(Q_T)$, then by (3.15), for $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \ge 0$, we have that

$$\int_{Q_T} |w|[-\partial_t \xi + (-\Delta)^\alpha \xi]dxdt + \int_{\mathbb{R}^N} |w(T,x)|\xi(T,x)dxdt \\ + \int_{Q_T} (h \circ u_2 - h \circ u_1)\text{sign}(w)\xi dxdt \le 0.$$

This impies $w = 0$ by monotonicity.

Statements (i) and (ii) and inequality (3.14) follows from the fact that the same relations holds for u_{ν_n} by Lemma 3.3.2.

Stability is proved by the same approach that existence. If $\{\nu_n\}$ converges to ν in the weak sense of measures, then $\|\nu_n\|_{\mathfrak{M}^b}$ is bounded independently of n. Since the distribution function of $h \circ u_{\nu_n}$ depends only on the supremum of $\|\nu_n\|_{\mathfrak{M}^b}$, this set of functions is uniformly integrable in Q_T. This, combined with local compactness of the set $\{u_{\nu_n}\}$ in $L^1(Q_T)$, implies the convergence of a subsequence $(u_{\nu_{n_k}}, h \circ u_{\nu_{n_k}})$ to $(u_\nu, h \circ u_\nu)$ where u_ν is the solution of (3.1). Because of uniqueness, all converging subsequence have the same limit which imply the convergence of the whole sequence and stability. □

3.4 Dirac mass as initial data

In this section, we study the properties of solutions to (3.1) when $h(t,r) = t^\beta r^p$ with $\beta > -1$ and $0 < p < p^*_\beta$ and the initial data is $\nu = k\delta_0$ with $k > 0$.

Proposition 3.4.1 *Assume $0 < p < p^*_\beta$ and that u_k is the solution of (3.15), then there exists $c_{15} > 0$ such that*

$$\lim_{t \to 0^+} t^{\frac{N}{2\alpha}} u_k(t,0) = c_{15} k. \tag{3.1}$$

Proof. By (3.14) it follows that

$$u_k(t,0) \le k\mathbb{H}_\alpha[\delta_0](t,0) = k\Gamma_\alpha(t,0) \qquad t > 0. \tag{3.2}$$

We claim that there exists $c_{16} > 0$ independent of k such that

$$u_k(t,0) \ge k\Gamma_\alpha(t,0) - c_{16} k^p t^{-\frac{N}{2\alpha}p+1+\beta} \quad t \in (0,1/2). \tag{3.3}$$

Indeed, from (3.14), it infers that

$$u_k(t,0) \ge k\Gamma_\alpha(t,0) - k^p W(t,0) \qquad t \in (0,1/2),$$

where

$$W(t,x) = \int_0^T \mathbb{H}_\alpha[s^\beta(\mathbb{H}^p_\alpha[\delta_0]](t-s,x)ds \qquad (t,x) \in Q_\infty.$$

For $t \in (0, 1/4)$, there exists $c_{17}, c_{18} > 0$ such that

$$\begin{aligned}
W(t,0) &\le c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{(t-s)^{-\frac{N}{2\alpha}} s^{\beta}}{1 + ((t-s)^{-\frac{1}{2\alpha}}|y|)^{N+2\alpha}} \left(\frac{s^{-\frac{N}{2\alpha}}}{1 + (s^{-\frac{1}{2\alpha}}|y|)^{N+2\alpha}} \right)^p dyds \\
&\le c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{s^{\beta - \frac{N}{2\alpha}p} dzds}{\left(1 + \left((\frac{t-s}{s})^{\frac{1}{2\alpha}}|z|\right)^{(N+2\alpha)p}\right)(1 + |z|^{N+2\alpha})} \\
&\le c_{17} t^{\beta+1-\frac{Np}{2\alpha}} \int_0^1 \int_{\mathbb{R}^N} \frac{d\tau dZ}{\left(1 + \left(\frac{1-\tau}{\tau}\right)^{\frac{(N+2\alpha)p}{2\alpha}} |Z|^{(N+2\alpha)p}\right)(1 + |Z|^{N+2\alpha})} \\
&\le c_{18} t^{\beta+1-\frac{Np}{2\alpha}}.
\end{aligned}$$

Combining (3.19) and $-\frac{N}{2\alpha}p + 1 + \beta > -\frac{N}{2\alpha}$, we obtain that

$$\lim_{t \to 0^+} t^{\frac{N}{2\alpha}} W(t,0) = 0.$$

Therefore, (3.1) holds. □

In what follows we consider the limit of the solution $\{u_k\}$ of (3.15) as $k \to \infty$ for $p \in (0, 1]$.

Proposition 3.4.2 *Assume $0 < p \le 1$ and that u_k is the solution of (3.15), then*

$$\lim_{k \to \infty} u_k = \infty \quad \text{in} \quad Q_\infty.$$

Proof. We observe that $\mathbb{H}_\alpha[\delta_0]$ and $\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]$ are positive in $(0, \infty) \times \mathbb{R}^N$. By (3.14), for $p \in (0, 1)$ and $(t, x) \in (0, \infty) \times \mathbb{R}^N$, we have that

$$u_k \;\; \ge \;\; k\mathbb{H}_\alpha[\delta_0] - k^p W \Longrightarrow \lim_{k \to \infty} u_k = \infty.$$

For $p = 1$, it is obvious that $u_k = ku_1$ and $u_1 > 0$ in $(0, \infty) \times \mathbb{R}^N$, then

$$\lim_{k \to \infty} u_k = \infty \quad \text{in} \quad Q_\infty.$$

The proof is complete. □

Now we deal with the range $p \in (1, p^*_\beta)$.

Lemma 3.4.1 *Assume $1 < p < p^*_\beta$ and that u_k is the solution of (3.15). Then for any $k > 0$,*

$$0 \le u_k \le U_p \quad \text{in} \quad Q_\infty, \tag{3.4}$$

where U_p is given by (3.21).

Proof. Let $\{f_{n,k}\}$ be a sequence of nonnegative functions in $C_c^1(\mathbb{R}^N)$ which converges to $k\delta_0$ as $n \to \infty$. We denote by $u_{n,k}$ the corresponding solution of (3.17) with initial data by $f_{n,k}$.

We claim that

$$u_{n,k} \leq U_p \quad \text{in} \quad Q_\infty, \tag{3.5}$$

where, we recall it, U_p is the maximal solution of the ODE $y' + t^\beta y^p = 0$ on $\mathbb{R}_+$. Indeed this implies (3.4).

Step 1. We claim that

$$\lim_{|x|\to\infty} u_{n,k}(t,x) = 0 \qquad \forall t > 0. \tag{3.6}$$

From [28, 37], there exists $c_8 > 0$ such that for any $x, y \in \mathbb{R}^N$ and $t \in (0,\infty)$,

$$0 < \Gamma_\alpha(t, x-y) \leq \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}}.$$

Then for $|x| > 1$,

$$\begin{aligned} 0 \leq \mathbb{H}_\alpha[f_{n,k}](t,x) &\leq c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \frac{f_{n,k}(y)}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}} dy \\ &= c_8 \int_{\mathbb{R}^N} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \\ &= c_8 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz + \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \right), \end{aligned}$$

where $R = \frac{1}{2}|x|t^{-\frac{1}{2\alpha}}$ and $B_R = \{z \in \mathbb{R}^N : |z| < R\}$. It is obvious that

$$|x - zt^{\frac{1}{2\alpha}}| \geq |x| - |z|t^{\frac{1}{2\alpha}} \geq |x|/2 \quad \text{for all } z \in B_R.$$

Then

$$\begin{aligned} \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz &\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{B_R} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{\mathbb{R}^N} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &= c_{16} \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \end{aligned}$$

and

$$\int_{\mathbb{R}^N\setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}}dz \le \int_{\mathbb{R}^N\setminus B_R} \frac{\|f_{n,k}\|_{L^\infty(\mathbb{R}^N)}}{1+|z|^{N+2\alpha}}dz \le c_{18}R^{-2\alpha} = \frac{c_{18}t}{|x|^{2\alpha}},$$

for some $c_{18} > 0$ independent of x, t and R. Since $f_{n,k} \in C_0^1(\mathbb{R}^N)$, we have that

$$\lim_{|x|\to\infty} \sup_{|y|\ge\frac{|x|}{2}} f_{n,k}(y) = 0$$

and then for any $t > 0$, $0 \le u_{n,k}(t,x) \le \mathbb{H}_\alpha[f_{n,k}](t,x) \to 0$ as $|x| \to \infty$.

Step 2. We claim that (3.5) holds. By contradiction, if (3.5) is not verified, there exists $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^N$ such that

$$(U_p - u_{n,k})(t_0, x_0) = \min_{(t,x)\in(0,\infty)\times\mathbb{R}^N} (U_p - u_{n,k})(t,x) < 0,$$

since $U_p(t) > 0 = \lim_{|x|\to\infty} u_{n,k}(t,x)$ for any $t \in (0,\infty)$, $U_p(0) = \infty > f_{n,k}(x) = u_{n,k}(0,x)$ for $x \in \mathbb{R}^N$ and $\lim_{t\to\infty} U_p(t) = \lim_{t\to\infty} u_{n,k}(t,x) = 0$ for $x \in \mathbb{R}^N$. Then $\partial_t(U_p - u_{n,k})(t_0,x_0) = 0$. Moreover since

$$\begin{aligned}(U_p - u_{n,k})(t_0,x_0) &= \min\{U_p(t_0) - u_{n,k}(t_0,x) : x \in \mathbb{R}^N\}\\ &= U_p(t_0) - \max\{u_{n,k}(t_0,x) : x \in \mathbb{R}^N\}\end{aligned}$$

and

$$u_{n,k}(t_0,x_0) = \max\{u_{n,k}(t_0,x) : x \in \mathbb{R}^N\} \Longrightarrow (-\Delta)^\alpha u_{n,k}(t_0,x_0) \ge 0$$

and

$$0 = \partial_t(U_p - u_{n,k})(t_0,x_0) - (-\Delta)^\alpha u_{n,k}(t_0,x_0) + t_0^\beta U_p^p(t_0) - t_0^\beta u_{n,k}^p(t_0,x_0) < 0,$$

which is impossible. Thus (3.5) holds. □

Proposition 3.4.3 *(i) Assume $0 < p < p_\beta^*$ and that u_k is the solution of (3.15). Then u_k is a classical solution of (3.17).*

(ii) Assume $1 < p < p_\beta^*$ and that u_∞ is defined by (3.16). Then u_∞ is a classical solution of (3.17).

Proof. (i) Since $u_k \le k\mathbb{H}_\alpha[\delta_0]$, it infers that u_k is bounded in $(T_0,\infty) \times \mathbb{R}^N$ for $T_0 > 0$. Let $\{g_{n,k}\}$ be a sequence of nonnegative functions in $C_0^1(\mathbb{R}^N)$ which converges to $k\delta_0$ as $n \to \infty$ and $u_{n,k}$ the corresponding solution of (3.17) with initial data $g_{n,k}$. Then $\mathbb{H}_\alpha[g_{n,k}] \to k\mathbb{H}_\alpha[\delta_0]$ as $n \to \infty$ uniformly in $[T_0,\infty) \times \mathbb{R}^N$

for any $T_0 > 0$ and by the Comparison Principle, there exists $c_{19} > 1$ such that

$$0 \le u_{n,k}(t,x) \le k\mathbb{H}_\alpha[g_{n,k}] \le c_{19}k\mathbb{H}_\alpha[\delta_0] \quad \text{in} \quad [T_0,\infty) \times \mathbb{R}^N$$

and there exists $\sigma \in (0,1)$ such that $\{u_{n,k}\}$ are uniformly bounded with respect to n in $C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}((T_0,\infty) \times \mathbb{R}^N)$ with $T_0 > 0$. Therefore, by the Arzela-Ascoli theorem, $u_{n,k}$ converges to u_k in $C_{t,x}^{\frac{\sigma'}{2\alpha},\sigma'}((T_0,\infty) \times \mathbb{R}^N)$ with $\sigma' \in (0,\sigma)$ and then u_k is a viscosity solution of (3.17) in $(T_0,\infty) \times \mathbb{R}^N$. By estimate $(A.1)$ in [21], u_k is in $C_{t,x}^{1+\sigma',2\alpha+\sigma'}((T_0,\infty) \times \mathbb{R}^N)$ and u_k is a classical solution of (3.17) in $(T_0,\infty) \times \mathbb{R}^N$.

(ii) The proof is the same as part (i), just replacing $u_k \le k\mathbb{H}_\alpha[\delta_0]$ by $u_\infty \le U_p$. □

3.5 Self-similar and very singular solutions

By Theorem 3.1.1 and (3.4), we see that $\{u_k\}$ is an increasing sequence of nonnegative functions bounded from above by U_p. Then for $p \in (1, p_\beta^*)$, there exists $u_\infty = \lim_{k\to\infty} u_k$, which is a classical solution of (3.17) by Proposition 3.4.3 (ii) and satisfies

$$u_\infty \le U_p \quad \text{in} \quad Q_\infty. \tag{3.1}$$

Proposition 3.5.1 *Assume $1 < p < p_\beta^*$, then u_∞ is a self-similar solution of (3.17).*

Proof. For $\lambda > 0$, we set

$$T_\lambda[u](t,x) = \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u(\lambda^{2\alpha}t, \lambda x) \qquad (t,x) \in Q_\infty.$$

It is straightforward to verify that $T_\lambda[u_k]$ is the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 && \text{in } Q_\infty \\ u(0,.) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}-N} k\delta_0 && \text{in } \mathbb{R}^N. \end{aligned} \tag{3.2}$$

Because of uniqueness, $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$. Letting $k \to \infty$ and using the continuity of $u \mapsto T_\lambda[u]$, we have that

$$\lim_{k\to\infty} T_\lambda[u_k] = T_\lambda[u_\infty] = u_\infty$$

which implies that u_∞ is a self-similar solution (3.17). □

Let us denote

$$U_\infty(z) = u_\infty(1,z), \qquad z \in \mathbb{R}^N$$

and we observe that U_∞ is a classical solution of (3.20). It is obvious that the constant $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$ is a constant positive solution of the self-similar equation (3.20). We observe that $N < \frac{2\alpha(1+\beta)}{p-1} < N + 2\alpha$ when $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$.

We prove below this fundamental result that u_∞ is the minimal self similar solution.

Proposition 3.5.2 *Assume that* $1 < p < 1+\frac{2\alpha(1+\beta)}{N}$ *and* $\tilde{u}$ *is a positive self-similar solution of (3.23). Then* $u_\infty \leq \tilde{u}$.

Proof. For any $r > 0$, we have that

$$\begin{aligned}
\int_{B_r(0)} \tilde{u}(t,x)dx &= t^{-\frac{1+\beta}{p-1}} \int_{B_r(0)} \tilde{u}(1, t^{-\frac{1}{2\alpha}}x)dx \\
&= t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_{t^{-\frac{1}{2\alpha}}r}(0)} \tilde{u}(1,z)dz \\
&\geq t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_1(0)} \tilde{u}(1,z)dz \\
&\to +\infty \quad \text{as } t \to 0^+,
\end{aligned}$$

where last inequality holds for $t \in (0, r^{2\alpha}]$. Let $\{\epsilon_n\}$ be a sequence positive decreasing numbers converging to 0 as $n \to \infty$. For ϵ_n and $k > 0$, there exists $t_{n,k} > 0$ such that

$$\int_{B_{\epsilon_n}(0)} \tilde{u}(t_{n,k}, x)dx = k.$$

We observe that for any fixed k, $t_{n,k} \to 0$ as $n \to \infty$ since $\lim_{n\to\infty} \epsilon_n = 0$. Let $\eta_0 : \mathbb{R}^N \to [0,1]$ be a C^2 function such that $\operatorname{supp} \eta_0 \subset \bar{B}_2(0)$, $\eta_0 = 1$ in $B_1(0)$ and $\eta_n(x) = \eta_0(\epsilon_n^{-1}x)$ for $x \in \mathbb{R}^N$. Choosing $\{f_{n,k}\}$ be a sequence of C^2 functions such that

$$0 \leq f_{n,k}(x) \leq \eta_n(x)\tilde{u}(t_{n,k}, x) \qquad \forall x \in \mathbb{R}^N$$

and

$$f_{n,k} \to k\delta_0 \quad \text{as} \quad n \to \infty.$$

Let $u_{n,k}$ be the solution of (3.1) with initial data $f_{n,k}$, then

$$u_{n,k}(t,x) \leq u(t_{n,k} + t, x) \qquad \forall (t,x) \in Q_\infty$$

and by uniqueness of u_k, $\lim_{n\to\infty} u_{n,k} = u_k$, where u_k is the solution of (3.1) with initial data $k\delta_0$. Then for any k, we have $u_k \leq \tilde{u}$ in Q_∞, which implies that

$$u_\infty \leq \tilde{u} \quad \text{in} \quad Q_\infty.$$

□

3.5.1 The case $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$

We define the function w_λ by

$$w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}}w(t^{-\frac{1}{2\alpha}}|x|)\qquad (t,x)\in Q_\infty, \tag{3.3}$$

where $w(s)=\frac{\ln(e+s^2)}{1+s^{N+2\alpha}}$.

Lemma 3.5.1 *Assume $1+\frac{2\alpha(1+\beta)}{N+2\alpha}<p<1+\frac{2\alpha(1+\beta)}{N}$, then there exists $\Lambda_0>0$ such that for $\lambda\geq\Lambda_0$,*

$$\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w_\lambda^p(t,x)\geq 0\quad \forall (t,x)\in Q_\infty. \tag{3.4}$$

Proof. By direct computation, we have

$$\partial_t w_\lambda(t,x)=-\frac{\lambda(1+\beta)}{p-1}t^{-\frac{1+\beta}{p-1}-1}w(t^{-\frac{1}{2\alpha}}|x|)-\frac{\lambda}{2\alpha}t^{-\frac{1+\beta}{p-1}-\frac{1}{2\alpha}-1}|x|w'(t^{-\frac{1}{2\alpha}}|x|)$$

and

$$(-\Delta)^\alpha w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}-1}(-\Delta)^\alpha w(t^{-\frac{1}{2\alpha}}|x|),$$

which implies

$$\begin{aligned}&\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w_\lambda^p(t,x)\\&\quad=\lambda t^{-\frac{1+\beta}{p-1}-1}\left[(-\Delta)^\alpha w(s)-\tfrac{1}{2\alpha}w'(s)s-\tfrac{1+\beta}{p-1}w(s)+\lambda^{p-1}w^p(s)\right],\end{aligned} \tag{3.5}$$

where $s=|z|$ with $z=t^{-\frac{1}{2\alpha}}x$. Next, for $s>0$, we have

$$-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)=\left[\frac{N+2\alpha}{2\alpha}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}}-\frac{1+\beta}{p-1}-\frac{s^2(e+s^2)^{-1}}{\alpha\ln(e+s^2)}\right]w(s).$$

Since $\frac{N+2\alpha}{2\alpha}>\frac{1+\beta}{p-1}$, $\lim_{s\to\infty}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}}=1$ and $\lim_{s\to\infty}\frac{1}{\ln(e+s^2)}=0$, there exists $R_0>0$ and $\sigma_0>0$ such that

$$-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)\geq\sigma_0 w(s)\qquad \forall s\geq R_0. \tag{3.6}$$

For $|z| > 2$, and using the definition of the fractional Laplacian, we have

$$\begin{aligned}
-(-\Delta)^{\alpha} w(|z|) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{\ln(e+|z+\tilde{y}|^2)}{1+|z+\tilde{y}|^{N+2\alpha}} + \frac{\ln(e+|z-\tilde{y}|^2)}{1+|z-\tilde{y}|^{N+2\alpha}} - \frac{2\ln(e+|z|^2)}{1+|z|^{N+2\alpha}}\right)\frac{d\tilde{y}}{|\tilde{y}|^{N+2\alpha}} \\
&= \frac{w(|z|)}{2|z|^{2\alpha}}\int_{\mathbb{R}^N}\frac{I_z(y)}{|y|^{N+2\alpha}}dy,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
I_z(y) = &\frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z+y|^{N+2\alpha}}\frac{\ln(e+|z|^2|e_z+y|^2)}{\ln(e+|z|^2)} \\
&+ \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}}\frac{\ln(e+|z|^2|e_z-y|^2)}{\ln(e+|z|^2)} - 2
\end{aligned}$$

and $e_z = \frac{z}{|z|}$.

We claim that there exists $c_{20} > 0$ such that

$$\int_{B_{\frac{1}{2}}(-e_z)\cup B_{\frac{1}{2}}(e_z)}\frac{I_z(y)}{|y|^{N+2\alpha}}dy \leq \frac{c_{20}}{w(|z|)|z|^N}. \tag{3.8}$$

In fact, for $y \in B_{\frac{1}{2}}(-e_z)$, there exists $c_{21} > 0$ such that

$$\frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}}\frac{\ln(e+|z|^2|e_z-y|^2)}{\ln(e+|z|^2)} \leq c_{21}$$

and then

$$\begin{aligned}
\int_{B_{\frac{1}{2}}(-e_z)}\frac{I_z(y)}{|y|^{N+2\alpha}}dy &\leq \omega_N\int_0^{\frac{1}{2}}\frac{1+|z|^{N+2\alpha}}{1+(|z|r)^{N+2\alpha}}\frac{\ln(e+|z|^2r^2)}{\ln(e+|z|^2)}r^{N-1}dr + c_{22} \\
&\leq \frac{\omega_N}{w(|z|)|z|^N}\int_0^{\infty}\frac{t^{N-1}\ln(e+t^2)}{1+t^{N+2\alpha}}dt + c_{22} \\
&\leq \frac{c_{23}}{w(|z|)|z|^N},
\end{aligned}$$

where $c_{22}, c_{23} > 0$ and the last inequality holds since $w(|z|)|z|^N \to 0$ as $|z| \to \infty$. Thus,

$$\int_{B_{\frac{1}{2}}(e_z)}\frac{I_z(y)}{|y|^{N+2\alpha}}dy = \int_{B_{\frac{1}{2}}(-e_z)}\frac{I_z(y)}{|y|^{N+2\alpha}}dy \leq \frac{c_{23}}{w(|z|)|z|^N}.$$

We claim that there exists $c_{24} > 0$ such that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{24}. \tag{3.9}$$

Indeed, since the function I_z is C^2 in $\bar{B}_{\frac{1}{2}}(0)$, $I_z(0) = 0$ and $I_z(y) = I_z(-y)$, then $\nabla I_z(0) = 0$ and there exists $c_{34} > 0$ such that

$$|D^2 I_z(y)| \leq c_{25} \qquad \forall y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_z(y) \leq c_{25}|y|^2 \qquad \forall y \in B_{\frac{1}{2}}(0),$$

which implies

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{25} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \leq c_{24}.$$

We claim that there exists $c_{26} > 0$ such that

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{26}, \tag{3.10}$$

where $A = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_z) \cup B_{\frac{1}{2}}(-e_z))$. In fact, for $y \in A$, we observe that there exists $c_{27} > 0$ such that $I_z(y) \leq c_{27}$ and

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{27}}{|y|^{N+2\alpha}} \leq c_{28},$$

for some $c_{28} > 0$. Therefore, by (3.5)-(3.10), there exists $c_{29} > 0$ such that

$$(-\Delta)^\alpha w(|z|) \geq -\frac{c_{29}}{1 + |z|^{N+2\alpha}}, \qquad |z| \geq 2. \tag{3.11}$$

By (3.6) and (3.11), there exists $R_1 \geq R_0 + 2$ such that for $|z| > R_1$,

$$\begin{aligned}
(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) &\geq \sigma_0 w(|z|) - \frac{c_{29}}{1 + |z|^{N+2\alpha}} \\
&= w(|z|) \left(\sigma_0 - \frac{c_{29}}{\ln(e + |z|^2)} \right) \\
&\geq 0.
\end{aligned}$$

When $|z| \le R_1$, it is clear that there exists $c_{30} > 0$ such that

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) \ge -c_{30}.$$

Then there exists $\Lambda_0 > 0$ such that for $\lambda \ge \Lambda_0$,

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) + \lambda^{p-1} w^p(|z|) \ge 0 \qquad \forall z \in \mathbb{R}^N, \quad (3.12)$$

which, together with (3.5), implies that (3.4) holds. □

Next we prove that u_∞ is not a trivial flat solution when $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < p^*_\beta$.

Lemma 3.5.2 *Assume $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$, that w_{Λ_0} is given in (3.3) and u_∞ is given in (3.16). Then*

$$u_\infty(t,x) \le w_{\Lambda_0}(t,x) \qquad \forall (t,x) \in Q_\infty. \quad (3.13)$$

Moreover,

$$\lim_{t\to 0} u_\infty(t,\cdot) = 0 \quad \textit{uniformly on } B^c_\epsilon \quad \forall \epsilon > 0. \quad (3.14)$$

Proof. Let us denote

$$f_0(r) = \frac{k_0 \ln(e + r^2)}{1 + r^{N+2\alpha}} \quad \forall\, r \ge 0 \quad \text{and} \quad f_{n,k}(x) = kn^N f_0(n|x|) \quad \forall x \in \mathbb{R}^N,$$

where

$$k_0 = \left[\omega_N \int_0^\infty \frac{\ln(e+r^2)}{1+r^{N+2\alpha}} r^{N-1} dr\right]^{-1}.$$

Then for any $\eta \in C_c(\mathbb{R}^N)$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_{n,k}\eta dx = k \lim_{n\to\infty} \int_{\mathbb{R}^N} f_0(|x|)\eta\left(\frac{x}{n}\right) dx = k\eta(0).$$

Let $t_n = n^{-2\alpha}$ and then

$$\begin{aligned} w_{\Lambda_0}(t_n,x) &= \Lambda_0 t_n^{-\frac{1+\beta}{p-1}} \frac{\ln(e + (t_n^{-\frac{1}{2\alpha}}|x|)^2)}{1 + (t_n^{-\frac{1}{2\alpha}}|x|)^{N+2\alpha}} = \Lambda_0 n^{\frac{2\alpha(1+\beta)}{p-1}} \frac{\ln(e + (n|x|)^2)}{1 + (n|x|)^{N+2\alpha}} \\ &= \frac{\Lambda_0}{k_0} n^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) \\ &\ge \frac{\Lambda_0}{k_0} \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) = f_{n,k_{\tilde{n}}}(x), \end{aligned}$$

where $\tilde{n} \leq n$ and $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1}-N}$. We see that $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1}-N} \to \infty$ as $\tilde{n} \to \infty$, since $\frac{2\alpha(1+\beta)}{p-1} - N > 0$. Let $u_{n,k_{\tilde{n}}}$ be the solution of (3.17) with initial data $f_{n,k_{\tilde{n}}}$. By Lemma 3.5.1, $w_{\Lambda_0}(\cdot + t_n, \cdot)$ is a super-solution of (3.17) with initial data $w_{\Lambda_0}(t_n, \cdot)$, that is, for $(t,x) \in Q_\infty$,

$$\partial_t w_\lambda(t+t_n,x) + (-\Delta)^\alpha w_\lambda(t+t_n,x) + (t+t_n)^\beta w_\lambda^p(t+t_n,x) \geq 0.$$

By the Comparison Principle,

$$u_{n,k_{\tilde{n}}}(t,x) \leq w_{\Lambda_0}(t+t_n,x) \qquad \forall (t,x) \in Q_\infty,$$

for any $\tilde{n} \leq n$. Letting $n \to \infty$ infers

$$u_{k_{\tilde{n}}}(t,x) \leq w_{\Lambda_0}(t,x) \qquad \forall (t,x) \in Q_\infty, \tag{3.15}$$

where $u_{k_{\tilde{n}}}$ is the solution of (3.17) with $k_{\tilde{n}}\delta_0$ initial data. Thus (3.13) is obtained by letting $\tilde{n} \to \infty$. Finally (3.14) follows by the fact that

$$\lim_{t\to 0^+} w_{\Lambda_0}(t,x) = 0 \qquad \forall x \in \mathbb{R}^N \setminus \{0\},$$

which completes the proof. □

Lemma 3.5.3 *Assume $1 < p < p_\beta^*$, then there exists $c_{31} > 0$ such that*

$$u_\infty(t,x) \geq \frac{c_{31} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} \qquad \forall (t,x) \in (0,1) \times \mathbb{R}^N. \tag{3.16}$$

Proof. We divide the proof into two steps.

Step 1. Let $\sigma_0 = 1 + \beta - \frac{N}{2\alpha}(p-1) > 0$, $\eta(t) = 2 - t^{\sigma_0}$ for $t > 0$ and denote

$$v_\epsilon(t,x) = \epsilon \eta(t) \Gamma_\alpha(t,x),$$

where Γ_α is the fundamental solution of (3.17). In this step we prove that there exists $\epsilon_0 > 0$ such that

$$u_{k_0} \geq v_{\epsilon_0} \quad \text{in } (0,1) \times \mathbb{R}^N, \tag{3.17}$$

where $k_0 = 2\epsilon_0$ and u_{k_0} is the solution of (3.17) with initial data $k_0\delta_0$. Indeed,

$$\partial_t v_\epsilon(t,x) = \epsilon \eta'(t) \Gamma_\alpha(t,x) + \epsilon \eta(t) \partial_t \Gamma_\alpha(t,x)$$

and

$$(-\Delta)^\alpha v_\epsilon(t,x) = \epsilon \eta(t) (-\Delta)^\alpha \Gamma_\alpha(t,x).$$

Let $\Gamma_1(t^{-\frac{1}{2\alpha}}x) = \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}x)$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \le \epsilon_0$ and $(t,x) \in (0,1) \times \mathbb{R}^N$, we have that

$$\begin{aligned}
&\partial_t v_\epsilon(t,x) + (-\Delta)^\alpha v_\epsilon(t,x) + t^\beta v_\epsilon^p(t,x) \\
&\qquad = \epsilon\eta'(t)t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + \epsilon^p\eta^p(t)t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \\
&\qquad \le -\epsilon\sigma_0 t^{-\frac{N}{2\alpha}-1+\sigma_0}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + 2^p\epsilon^p t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \le 0,
\end{aligned}$$

the last inequality holds since $-\frac{N}{2\alpha} - 1 + \sigma_0 = -\frac{N}{2\alpha}p + \beta$ and Γ_1 is bounded. In particular, there holds

$$\partial_t v_{\epsilon_0}(t,x) + (-\Delta)^\alpha v_{\epsilon_0}(t,x) + t^\beta v_{\epsilon_0}^p(t,x) \le 0 \qquad \forall (t,x) \in (0,1) \times \mathbb{R}^N. \tag{3.18}$$

Let $f_n(x) = v_{\epsilon_0}(t_n, x)$ with $t_n = n^{-2\alpha}$. Since $\lim_{t\to 0^+}\eta(t) = 2$, then we have that $f_n \to 2\epsilon_0\delta_0$ as $n \to \infty$ in the weak sense of measures. There exists $N_0 > 0$ such that $t_n \in (0, \frac{1}{8})$ for $n \ge N_0$. Let w_n be the solution of (3.17) with initial data f_n, then it infers that

$$w_n(t,x) \ge v_{\epsilon_0}(t + t_n, x) \qquad (t,x) \in (0, 1 - t_n) \times \mathbb{R}^N.$$

Because u_{k_0} is uniquely defined, there holds

$$w_n \to u_{k_0} \quad \text{as } n \to \infty \qquad \text{in } (0,1) \times \mathbb{R}^N$$

and

$$\lim_{n\to\infty} v_{\epsilon_0}(t + t_n, x) = v_{\epsilon_0}(t,x) \qquad \forall (t,x) \in (0,1) \times \mathbb{R}^N,$$

which imply (3.17).

Step 2. We claim that (3.16) holds. Since

$$v_{\epsilon_0}(t,x) \ge \epsilon_0 t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) \qquad (t,x) \in (0,1) \times \mathbb{R}^N,$$

then, along with the relation $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$, we observe that for any $\lambda > 0$,

$$\begin{aligned}
u_{k_0\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}(t,x) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u_{k_0}(\lambda^{2\alpha}t, \lambda x) \\
&\ge \lambda^{\frac{2\alpha(1+\beta)}{p-1}} v_{\epsilon_0}(\lambda^{2\alpha}t, \lambda x) \\
&\ge \epsilon_0 \lambda^{\frac{2\alpha(1+\beta)}{p-1}-N} t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x).
\end{aligned}$$

Let $\varrho = \lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}$, $t_\varrho = (2\varrho)^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$ and $T_\varrho = \varrho^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$, then

$$0 < t_\varrho < T_\varrho \to 0 \quad \text{as } \varrho \to \infty.$$

For $(t,x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N$, we have that

$$u_{k_0\varrho}(t,x) \geq \epsilon_0 \varrho t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x),$$

then

$$u_\infty(t,x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \qquad \forall (t,x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N.$$

which implies (3.16)and completes the proof. □

Proof of Theorem 3.1.2. It follows from Proposition 3.5.1 and Lemma 3.5.2 that u_∞ is a nontrivial self-similar solution of (3.17) and (3.22) follows by (3.13), (3.16) and $\ln(e + |t^{-\frac{1}{2\alpha}} x|^2) \leq 2\ln(2 + |t^{-\frac{1}{2\alpha}} x|)$, which ends the proof. □

We have actually a stronger result which is a consequence of Theorem 3.1.4-(i) proved in next section:

Corollary 3.5.1 *Assume* $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$. *Then*
either

$$\tilde{u} > u_\infty \quad \text{in} \quad Q_\infty \tag{3.19}$$

or

$$\tilde{u} \equiv u_\infty \quad \text{in} \quad Q_\infty. \tag{3.20}$$

3.5.2 The case $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$

For $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$, it follows from Lemma 3.5.3 that

$$\lim_{t \to 0^+} u_\infty(t,x) = \infty \qquad \forall x \in \mathbb{R}^N. \tag{3.21}$$

Proof of Theorem 3.1.3 (i)**.** Let $f_0 \in C_c(\mathbb{R}^N)$ be a nonnegative function such that

$$\text{supp} f_0 \subset B_1(0) \quad \text{and} \quad \max_{x \in B_1(0)} f_0 = 1.$$

Denote

$$f_{n,k}(x) = k n^{\theta N} f_0(n^\theta (x - x_0)),$$

where $k \leq n^\tau$ with $\tau = \frac{1}{2}(\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha) > 0$, $\theta = \frac{\tau}{N}$ and $x_0 \in \mathbb{R}^N$. Since $f_{n,k}(x) \leq n^\tau$ for $x \in B_1(x_0)$, $f_n(x) = 0$ for $x \in B_1^c(x_0)$ and

$$v_{\epsilon_0}(t_n, x) \geq \frac{c_{39} n^{\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha}}{(2 + |x_0|)^{N+2\alpha}} \qquad \forall x \in B_1(x_0),$$

where $t_n = n^{-2\alpha}$. Then there exists $N_0 > 0$ such that for any $n \geq N_0$,

$$f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x) \qquad \forall x \in B_1(x_0).$$

Since $n^{\theta N} f_0(n^\theta(x - x_0)) \to c_{41}\delta_{x_0}$, as $n \to \infty$ in weak sense of measures, for some $c_{41} > 0$.

Let $w_{n,k}$ be the solution of (3.17) with initial data $f_{n,k}$, then

$$w_{n,k}(0, x) = f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x) \leq u_\infty(t_n, x) \qquad \forall x \in \mathbb{R}^N.$$

Therefore, by the Comparison Principle

$$w_{n,k}(t, x) \leq u_\infty(t + t_n, x) \qquad \forall (t, x) \in Q_\infty.$$

We observe that

$$\lim_{k\to\infty} [\lim_{n\to\infty} w_{n,k}(t, x)] = u_\infty(t, x - x_0) \qquad \forall (t, x) \in Q_\infty.$$

Thus, we derive that

$$u_\infty(t, x - x_0) \leq u_\infty(t, x) \qquad \forall (t, x) \in Q_\infty. \tag{3.22}$$

Then $u_\infty(t, x - x_0) = u_\infty(t, x)$ for all $(t, x) \in Q_\infty$, which implies that u_∞ is independent of x. Combining (3.1) and (3.16), implies that

$$u_\infty = \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}.$$

The proof is complete. □

In the case of $p = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$, it derive from Lemma 3.5.3 that

$$\liminf_{t\to 0^+} u_\infty(t, x) \geq \lim_{t\to 0^+} \frac{c_{40} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} = \frac{c_{40}}{|x|^{N+2\alpha}} \qquad \forall x \in \mathbb{R}^N.$$

Proof of Theorem 3.1.3 (ii)**.** We note that u_∞ is a self-similar solution of (3.17). Moreover, we derive (3.24) by (3.16), which ends the proof. □

3.5.3 The self-similar equation

In this section we prove Theorem 3.1.4.

Proof of Theorem 3.1.4 (i)**.** We set $v_\infty(\eta) = t^{\frac{1+\beta}{p-1}} u_\infty(1, \eta)$. Then relations

(3.25) and (3.26) hold from Lemmas 3.5.2 and 3.5.3. Assume $\tilde{v}$ is another positive solution of (3.20). Then $(t,x) \mapsto t^{-\frac{1+\beta}{p-1}}\tilde{v}(t^{-\frac{1}{2\alpha}}x)$ is a positive self-similar solution of (3.23). By Proposition 3.5.2 it is larger than u_∞. Thus $v_\infty \leq \tilde{v}$. Assume now that there exists $\eta_0 \in \mathbb{R}^N$ such that $v_\infty(\eta_0) = \tilde{v}(\eta_0)$. and set $w = \tilde{v} - v_\infty$. Then

$$(-\Delta)^\alpha w(\eta_0) = \lim_{\epsilon\to 0}(-\Delta)^\alpha_\epsilon w(\eta_0) = \lim_{\epsilon\to 0}\int_{B^c_\epsilon(\eta_0)} \frac{w(\eta_0) - w(\eta)}{|\eta - \eta_0|^{N+2\alpha}} d\eta < 0.$$

Since $\nabla w(\eta_0)$ we reach a contradiction. □

Proof of Theorem 3.1.4 (ii)**.** It is a consequence of the equality

$$u_\infty = U_p \Longleftrightarrow v_\infty = \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}$$

Open problem. We conjecture that in the case $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$, v_∞ is the unique positive solution of the self-similar equation satisfying (3.25). One step could be to prove that any positive solution $\tilde{v}$ satisfying (3.25) satisfies, for some $K > 1$,

$$\tilde{v} \leq K v_\infty \qquad \text{in } \mathbb{R}^N. \tag{3.23}$$

We also conjecture that v_∞ satisfies the following asymptotic behavior

$$v_\infty(\eta) = c_{N,p,\alpha,\beta}|\eta|^{-N-2\alpha} \qquad \text{as } |\eta| \to \infty. \tag{3.24}$$

Thus if any positive solution $\tilde{v}$ endows the same property, the conclusion (and the uniqueness) follows.

Chapter 4

On semi-linear elliptic equation arising from Micro-Electromechanical Systems with contacting elastic membrane

Abstract: in this chapter[1], we consider the solutions to nonlinear elliptic problem

$$\begin{cases} -\Delta u = \frac{\lambda}{(a-u)^2} & \text{in} \quad \Omega, \\ 0 < u < a & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda > 0$ and the function $a : \bar{\Omega} \to [0,1]$ satisfying $a(x) \geq \kappa \mathrm{dist}(x, \partial\Omega)^\gamma$ for some $\kappa > 0$ and $\gamma \in (0,1)$. This equation arises from Micro-Electromechanical Systems devices in the case that the elastic membranae contacts the ground plate on the boundary.

4.1 Introduction

Micro-Electromechanical Systems (MEMS) are often used to combine electronics with micro-size mechanical devices in the design of various types of microscopic machinery. They are successfully utilized in components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches, chemical sensors, etc. In MEMS devices, a key component is called the electrostatic actuation, which is based on an electrostatic-controlled

[1]This chapter is based on the paper: H. Chen, Y. Wang and F. Zhou, *On semi-linear elliptic equation arising from Micro-Electromechanical Systems with contacting elastic membrane, preprint.*

tunable, it is a simple idealized electrostatic device. The upper part of this electrostatic device consists of a thin and deformable elastic membrane that is held fixed along its boundary and which lies above a rigid grounded plate. This elastic membrane is modeled as a dielectric with a small but finite thickness. The upper surface of the membrane is coated with a negligibly thin metallic conducting film. When a voltage λ is applied to the conducting film, the thin dielectric membrane deflects towards the bottom plate, and when λ is increased beyond a certain critical value λ^*-known as pull-in voltage-the steady state of the elastic membrane is lost, and proceeds to touchdown or snap through at a finite time creating the so-called pull-in instability.

A mathematical model of the physical phenomena, leading to a partial differential equation for the dimensionless deflection of the membrane, was derived and analyzed in [49, 55, 56, 57, 65, 79, 95] and reference therein. In the damping-dominated limit, and using a narrow-gap asymptotic analysis, the dimensionless deflection u of the membrane on a bounded domain Ω in $\mathbb{R}^2$ is found to satisfy the equation

$$-\Delta u = \frac{\lambda}{(1-u)^2} \qquad \text{in} \quad \Omega \tag{4.2}$$

with the Dirichlet boundary condition. Here the term on the right hand side of equation (4.2) is the Coulomb force. Later on, Ghoussoub and Guo in [49, 55] studied the nonlinear elliptic problem

$$-\Delta u = \frac{\lambda f(x)}{(1-u)^2} \qquad \text{in} \quad \Omega \tag{4.3}$$

with the Dirichlet boundary condition, which models a simple electrostatic MEMS device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at 1. Here Ω is a bounded domain of $\mathbb{R}^N$ and the function $f \geq 0$ represents the permittivity profile and $\lambda > 0$ is a constant which is increasing with respect to the applied voltage. We know that for any given suitable f, there exists a critical value λ^* (pull-in voltage) such that if $\lambda \in (0, \lambda^*)$, problem (4.3) is solvable, while for $\lambda > \lambda^*$, no solution for (4.3) exists.

In an effort to achieve better MEMS design, the membrane can be technologically fabricated into non-flat shape like the surface of a semi-ball, which contacts the ground plate along the boundary. In this chapter, we study how the shape of the membranes effects on the existence of solutions and pull-in voltage. In what follows, we assume that Ω is a C^2 bounded domain in $\mathbb{R}^N$ with $N \geq 1$, $\rho(x) = \text{dist}(x, \partial\Omega)$ for $x \in \Omega$, the function $a : \bar{\Omega} \to [0, 1]$ is in the class of $C^\gamma(\Omega) \cap C(\bar{\Omega})$ and satisfy

$$a(x) \geq \kappa \rho(x)^\gamma, \quad \forall\, x \in \Omega \tag{4.4}$$

for some $\kappa > 0$ and $\gamma \in (0, 1)$. Our purpose of this chapter is to consider the

solutions to elliptic equation

$$\begin{cases} -\Delta u = \frac{\lambda}{(a-u)^2} & \text{in} \quad \Omega, \\ 0 < u < a & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.5}$$

where parameter $\lambda > 0$ characterizes the relative strength of the electrostatic and mechanical forces in the system. Equation (53) models a closed MEMS device, where the elastic membrane contacts the ground plate on the boundary. The function a is initially state of the elastic membrane. The solution u of (53) shows the steady state of deformation for the membrane when we applied voltage to this device. To this problem, we have the following existence results.

Theorem 4.1.1 *Assume that $a \in C^\gamma(\Omega) \cap C(\bar{\Omega})$ satisfies (4.4) with $\gamma \in (0, \frac{2}{3}]$ and $\kappa > 0$, then there exists a finite pull-in voltage $\lambda^* := \lambda^*(\kappa, \gamma) > 0$ such that*

(i) for $\lambda \in (0, \lambda^*)$, (4.5) admits a minimal solution u_λ and the mapping: $\lambda \mapsto u_\lambda$ is increasing;

(ii) for $\lambda > \lambda^*$, there is no solution for (4.5);

(iii) assume more that there exists $c_0 \geq \kappa$ such that

$$a(x) \leq c_0 \rho(x)^\gamma, \quad x \in \Omega, \tag{4.6}$$

then there exists $\lambda_* = \lambda_*(\kappa, \gamma) \in (0, \lambda^*]$ such that for $\lambda \in (0, \lambda_*)$, $u_\lambda \in H_0^1(\Omega)$ and

$$\text{for} \quad \gamma \neq \frac{1}{2}, \qquad \frac{1}{c_1}\rho(x)^{\min\{1, 2-2\gamma\}} \leq u_\lambda(x) \leq c_1 \rho(x)^{\min\{1, 2-2\gamma\}}, \qquad \forall x \in \Omega$$

$$\text{for} \quad \gamma = \frac{1}{2}, \qquad \frac{1}{c_1}\rho(x) \ln \frac{1}{\rho(x)} \leq u_\lambda(x) \leq c_1 \rho(x) \ln \frac{1}{\rho(x)}, \qquad \forall x \in A_{\frac{1}{2}},$$

where $c_1 \geq 1$ and $A_{\frac{1}{2}} = \{x \in \Omega : \ \rho(x) < \frac{1}{2}\}$.

(iv) the mappings: $\gamma \mapsto \lambda_*(\kappa, \gamma)$ and $\gamma \mapsto \lambda^*(\kappa, \gamma)$ are decreasing. Moreover, if $\Omega = B_1(0)$ and

$$a(x) = \kappa(1 - |x|)^\gamma, \qquad \forall x \in B_1(0),$$

then $\lambda_*(\kappa, \gamma), \lambda^*(\kappa, \gamma)$ have following estimates

$$\lambda_*(\kappa, \gamma) \geq \begin{cases} \frac{4\kappa^3(1-2\gamma)(N-1+2\gamma)}{27} & \text{if} \quad \gamma \in (0, \frac{1}{3}], \\ \frac{\kappa^3}{27} & \text{if} \quad \gamma \in (\frac{1}{3}, \frac{1}{2}], \\ \frac{4\kappa^3\gamma(1-\gamma)}{27} & \text{if} \quad \gamma \in (\frac{1}{2}, \frac{2}{3}] \end{cases}$$

and

$$\lambda^*(\kappa, \gamma) \leq c_2 \kappa^3 (2 - 2\gamma),$$

where $c_2 > 0$ independent of γ, κ.

We remark that the membrane contacts the ground plate on the boundary with decay rate ρ^γ, $\gamma \in (0, \frac{2}{3}]$, there still has a positive finite pull-in voltage λ^*, but the decay of a plays an important role in decay of minimal solution, the regularity of minimal solution and the estimate of λ^*. Theorem 4.1.1 shows that the membrane of the MEMS device could be designed as the surface of the unit semi-ball, that is,

$$\Omega = B_1(0) \quad \text{and} \quad a(x) = (1 - |x|^2)^{\frac{1}{2}},$$

which is equivalent to the case that $a(x) = \rho(x)^{\frac{1}{2}}$, so there exists a positive finite pull-in voltage λ^*. The decay rate of function a determined completely non-existence of pull-in voltage when $\gamma > \frac{2}{3}$. Precisely, we have following non-existence result.

Theorem 4.1.2 *Assume that $a \in C(\bar{\Omega})$ is positive and satisfies (4.6) with $\gamma \in (\frac{2}{3}, 1)$ and $c_0 > 0$. Then problem (4.5) admits no nonnegative solution for any $\lambda > 0$.*

We notice that for $\gamma \leq \frac{2}{3}$ and fixed κ, the finite pull-in voltage λ^* depends on γ, however, when $\gamma = \frac{2}{3}$, $\lambda^* > 0$ and $\lambda^* = 0$ for $\gamma > \frac{2}{3}$. Therefore, there is a gap at $\gamma = \frac{2}{3}$. Next it is challenging to study the extremal solution, i.e. when $\lambda = \lambda^*$. Especially, the decay of function a makes this issue more subtle. From Theorem 4.1.1, we observe that the mapping $\lambda \mapsto u_\lambda$ is increasing and uniformly bounded by function a, then it is well-defined that

$$u_{\lambda^*} := \lim_{\lambda \to \lambda^*} u_\lambda \quad \text{in } \bar{\Omega}, \tag{4.7}$$

where u_λ is the minimal solution of (4.5) with $\lambda \in (0, \lambda^*)$. Our final purpose in this chapter is to prove that u_{λ^*} is a solution of (4.5) in some weak sense and it is called the extremal solution. The extremal solution always is found in a weak sense and then it could be improved the regularity up to the classical sense when $1 \leq N \leq 7$. Before stating this result, we introduce the definition of weak solution.

Definition 4.1.1 *A function u is a weak solution of (4.5) if $0 \leq u \leq a$ and*

$$\int_\Omega u(-\Delta)\xi dx = \int_\Omega \frac{\lambda \xi}{(a-u)^2} dx, \quad \forall \xi \in C_c^2(\Omega),$$

where $C_c^2(\Omega)$ is the space of all C^2 functions with compact support in Ω.

A solution (or weak solution) u of (4.5) is stable (resp. semi-stable) if

$$\int_\Omega |\nabla \xi|^2 dx > \int_\Omega \frac{2\lambda \xi^2}{(a-u)^3} dx, \quad (\text{resp. } \geq) \quad \forall \xi \in C_c^2(\Omega) \setminus \{0\}.$$

Theorem 4.1.3 *Assume that $\lambda \in (0, \lambda^*)$, the function a satisfies (4.4) and(4.6) with $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$, u_λ is the minimal solution of (4.5) and u_{λ^*} is given by (4.7). Then*

(i) u_{λ^*} is a weak solution of (4.5) and $u_{\lambda^*} \in W_0^{1,\frac{N}{N-\beta}}(\Omega)$ for any $\beta \in (0, \gamma)$.

(ii) when $1 \leq N \leq 7$, $c_0 = \kappa$ and $\Omega = B_1(0)$, u_{λ^*} is a classical solution of (4.5).

(iii) u_λ is a stable solution of (4.5) with $\lambda \in (0, \lambda^*)$ and u_{λ^*} is a semi-stable weak solution of (4.5).

4.2 Existence

Denote by G_Ω the Green kernel of $-\Delta$ in $\Omega \times \Omega$ and by $\mathbb{G}_\Omega[\cdot]$ the Green operator defined as

$$\mathbb{G}_\Omega[f](x) = \int_\Omega G_\Omega(x, y) f(y) dy, \quad \forall f \in L^1(\Omega, \rho^{-1}).$$

In our analysis of the minimal solution of (4.5), the following estimates play an important role.

Lemma 4.2.1 *Let $\tau \in (0, 2)$, $A_{\frac{1}{2}} = \{x \in \Omega : \rho(x) < \frac{1}{2}\}$. For $x \in A_{\frac{1}{2}}$, denote*

$$\varrho_\tau(x) = \begin{cases} \rho(x)^{\min\{1,\tau\}} & \text{if} \quad \tau \in (0, 1) \cup (1, 2), \\ \rho(x) \ln \frac{1}{\rho(x)}, & \text{if} \quad \tau = 1 \end{cases} \tag{4.8}$$

and we make C^1 extension of ϱ into $\Omega \setminus A_{\frac{1}{2}}$ such that $\varrho_\tau > 0$ in $\Omega \setminus A_{\frac{1}{2}}$.

Then there exists $c_\tau > 1$ such that

$$\frac{1}{c_\tau} \varrho(x) \leq \mathbb{G}_\Omega[\rho^{\tau-2}](x) \leq c_\tau \varrho(x), \quad \forall x \in \Omega.$$

Proof. We assume that $\delta_1 > 0$ is such that the distance function $\rho(\cdot)$ is of class C^2 in

$$A_{\delta_1} := \{x \in \Omega : \ \rho(x) < \delta_1\}$$

and we define

$$V_\tau(x) = \begin{cases} l(x), & x \in \Omega \setminus A_{\delta_1}, \\ \rho(x)^\tau, & x \in A_{\delta_1}, \end{cases} \tag{4.9}$$

where τ is a parameter in $(0, 1)$ and the function l is positive such that V_τ is C^2 in Ω. Our aim is to estimate $-\Delta V_\tau$ near the boundary. By compactness we prove that the corresponding inequality holds in a neighborhood of any point $\bar{x} \in \partial\Omega$ and without loss of generality we may assume that $\bar{x} = 0$ and $e_N = (0, \cdots, 0, 1)$ is the unit normal vector at 0 pointing inside. We only have to consider the points

$\{x_t = te_N\}$ with $t \in (0, \delta_1)$. By geometric results,

$$\frac{\partial^2 V_\tau(te_N)}{\partial x_N^2} = \tau(\tau-1)t^{\tau-2} \quad \text{and} \quad |\frac{\partial^2 V_\tau(te_N)}{\partial x_i^2}| \le c_3, \quad i = 1, 2, \cdots, N-1,$$

where $c_3 > 0$ independent of t. Then

$$\tau(\tau-1)t^{\tau-2} - c_3 \le \Delta V_\tau(te_N) \le \tau(\tau-1)t^{\tau-2} + c_3,$$

that is,

$$\tau(\tau-1)\rho(x)^{\tau-2} - c_4 \le \Delta V_\tau(x) \le \tau(\tau-1)\rho(x)^{\tau-2} + c_4, \quad \forall x \in A_{\delta_1}, \tag{4.10}$$

where $c_4 > 0$.

For $\tau \in (0, 1)$, one has that $\tau(\tau-1) < 0$ and

$$-\Delta \mathbb{G}_\Omega[\rho^{\tau-2}] = \rho^{\tau-2},$$

then by Comparison Principle, there exists $c_5 > 1$ and $\delta_2 \in (0, \delta_1]$ such that

$$\frac{1}{c_5} V_\tau \le \mathbb{G}_\Omega[\rho^{\tau-2}] \le c_5 V_\tau \quad \text{in } A_{\delta_2}.$$

For $\tau \in (1, 2)$, take $W_t = t\mathbb{G}_\Omega[1] - V_\tau$, then from (4.10), there choosing t suitable and $c_6 \ge 1$ such that

$$c_6^{-1}\rho^{\tau-2}(x) \le -\Delta W_t(x) \le c_6 \rho^{\tau-2}(x), \quad \forall x \in A_{\delta_1}.$$

By Comparison Principle, there exists $c > 1$ that that

$$c^{-1} W_t(x) \le \mathbb{G}_\Omega[\rho^{\tau-2}](x) \le c W_t(x), \quad \forall x \in A_{\delta_1}.$$

For $\tau = 1$, we define

$$V_1(x) = \begin{cases} l(x), & x \in \Omega \setminus A_{\delta_1}, \\ \rho(x) \ln \frac{1}{\rho(x)}, & x \in A_{\delta_1}, \end{cases}$$

where $\delta_1 \in (0, \frac{1}{2}]$ and the function l is positive such that V_τ is C^2 in Ω. By directly computation, there is some $c_7 > 0$

$$\rho(x)^{-1} - c_7 \le \Delta V_1(x) \le \rho(x)^{-1} + c_7, \quad \forall x \in A_{\delta_1}.$$

Then it follows by Comparison Principle that

$$\frac{1}{c_8}\rho\ln\frac{1}{\rho} \le \mathbb{G}_\Omega[\rho^{-1}] \le c_8\rho\ln\frac{1}{\rho} \quad \text{in} \quad A_{\frac{1}{2}},$$

for some $c_8 > 1$. The proof complete. $\square$

By Lemma 4.2.1, we have following results.

Corollary 4.2.1 *For $\gamma \in (\frac{2}{3}, 1)$, we have that*

$$\lim_{x\to\partial\Omega} \mathbb{G}_\Omega[\rho^{-2\gamma}](x)\rho^{-\gamma}(x) = +\infty. \tag{4.11}$$

Proof. Take $\tau = 2-2\gamma$, then it follows by $\gamma \in (\frac{2}{3}, 1)$ that $2-2\gamma < \gamma$ and for some $c_9 > 0$,

$$\mathbb{G}_\Omega[\rho^{-2\gamma}](x) \ge c_9\rho^{2-2\gamma}(x), \quad \forall x \in \Omega,$$

which implies (4.11). $\square$

Proposition 4.2.1 *Assume that $a \in C^\gamma(\Omega) \cap C(\bar{\Omega})$ satisfies (4.4) with $\gamma \in (0, \frac{2}{3}]$, then there exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, there exists at least one solution for (4.5) and if $\lambda > \lambda^*$, there is no solution for (4.5).*

Moreover,

$$\lambda^* \le \frac{\int_\Omega a(x)dx}{\int_\Omega \frac{\mathbb{G}_\Omega[1](x)}{a(x)^2}dx}. \tag{4.12}$$

Proof. Without loss of generality, we assume

$$D(\Omega) := \sup_{x,y\in\Omega} |x-y| \le 1.$$

Existence for λ small. Let $v_0 \equiv 0$ in $\bar{\Omega}$ and

$$v_1 := \lambda\mathbb{G}_\Omega[\frac{1}{a^2}] > 0,$$

by (4.4) and Lemma 4.2.1,

$$v_1 = \lambda\mathbb{G}_\Omega[\frac{1}{a^2}] \le \lambda\mathbb{G}_\Omega[\rho^{-2\gamma}] \le \lambda\frac{c_{10}}{\kappa^2}\varrho_{2-2\gamma},$$

where $c_{10} > 0$ depending on Ω and ϱ is given by (4.8). For $\gamma \le \frac{2}{3}$, we have that $\min\{1, 2-2\gamma\} \ge \gamma$ and

$$v_1(x) \le \frac{c_{10}}{\kappa^2}\lambda\rho^\gamma(x), \quad x \in \Omega.$$

Fix any $\mu \in (0, \kappa)$, then choose λ such that

$$\frac{c_1}{\kappa^2}\lambda \leq \mu < \kappa, \tag{4.13}$$

then

$$v_2 := \lambda \mathbb{G}_\Omega[\frac{1}{(a-u_1)^2}] \geq \lambda \mathbb{G}_\Omega[\frac{1}{a^2}] = v_1$$

$$v_2 \leq \lambda \frac{1}{(\kappa-\mu)^2}\mathbb{G}_\Omega[\rho^{-2\gamma}] \leq \lambda \frac{c_{10}}{(\kappa-\mu)^2}\varrho_{2-2\gamma}.$$

Choose λ such that

$$\frac{c_{10}}{(\kappa-\mu)^2}\lambda \leq \mu \tag{4.14}$$

Combining (4.13) and (4.14), if $\lambda \leq \frac{\mu \min\{\kappa^3, (\kappa-\mu)^3\}}{c_{10}}$ then

$$v_2(x) \leq \mu\rho^\gamma(x), \quad x \in \Omega.$$

Iterating above process, we have that

$$v_n := \lambda \mathbb{G}_\Omega[\frac{1}{(a-v_{n-1})^2}] \geq v_{n-1}, \quad n \in \mathbb{N}$$

and

$$v_n(x) \leq \mu\rho^\gamma(x), \quad x \in \Omega.$$

By standard approximating procedure, $u_\lambda := \lim_{n\to\infty} v_n$ is a classical solution of (4.5) and it is normal to obtain the following assertions:

(P_λ) (i) u_λ is the minimal solution of (4.5);

(ii) if (4.5) has a super solution u for $\lambda_1 > 0$, then (4.5) admits a minimal solution u_λ for any $\lambda \in (0, \lambda_1]$;

(iii) the mapping $\lambda \mapsto u_\lambda$ is increasing.

Nonexistence for λ large. If (4.5) admits a minimal solution u_λ for $\lambda > 0$. For $\epsilon > 0$, denote

$$\Omega_\epsilon := \{x \in \Omega, \rho(x) > \epsilon\}. \tag{4.15}$$

For $\xi \in C_c^2(\Omega)$, then there exists ϵ such that

$$\text{supp}(\xi) \subset \Omega_\epsilon.$$

Then $\nabla\xi$ and $-\Delta\xi$ have compact support in Ω_ϵ. For $\lambda \in (0, \lambda^*)$, multiply (4.5) by ξ and integrate in Ω, we have that

$$\int_\Omega \frac{\lambda\xi}{(a-u_\lambda)^2}dx = \int_\Omega (-\Delta u_\lambda)\xi dx \quad = \quad \int_\Omega \nabla u_\lambda \cdot \nabla\xi dx = \int_\Omega u_\lambda(-\Delta)\xi dx. \tag{4.16}$$

Take a sequence $\{\xi_n\} \subset C_c^2(\Omega)$ such that

$$\xi_n \to \mathbb{G}_\Omega[1] \ \text{ in } \ C^2(\Omega) \quad \text{as } n \to +\infty.$$

Then it follows by (4.16) that

$$\lambda \int_\Omega \frac{\rho(x)}{(a-u_\lambda)^2}dx \le c_{11}\int_\Omega u_\lambda dx \le c_{11}\int_\Omega a(x)dx,$$

where $c_{11} > 0$. Therefore, we have that

$$\begin{aligned}\int_\Omega a(x)dx \ge \int_\Omega u_\lambda(x)dx \quad &= \quad \int_\Omega \mathbb{G}_\Omega[1](x)(-\Delta)u_\lambda(x)dx \\ &= \quad \lambda\int_\Omega \frac{\mathbb{G}_\Omega[1](x)}{[a(x)-u_\lambda(x)]^2}dx \ge \lambda\int_\Omega \frac{\mathbb{G}_\Omega[1](x)}{a^2(x)}dx,\end{aligned}$$

which implies that

$$\lambda \le \frac{\int_\Omega a(x)dx}{\int_\Omega \frac{\mathbb{G}_\Omega[1](x)}{a^2(x)}dx}.$$

Thus, the assertions in Proposition 4.2.1 follow by the existence, nonexistence result and Property (P_λ). □

Proof of Theorem 4.1.2. If there exists $\lambda > 0$ such that (4.5) admits a solution u_λ, then

$$v_1 = \lambda\mathbb{G}_\Omega[\frac{1}{a^2}] \le u_\lambda < a \quad \text{in} \quad \Omega. \tag{4.17}$$

From Corollary 4.2.1, we derive that

$$\lim_{x\to\partial\Omega} v_1(x)\rho^{-\gamma}(x) = +\infty,$$

which, together with (4.17), implies that

$$\lim_{x\to\partial\Omega} a(x)\rho^{-\gamma}(x) = +\infty,$$

then there is contradiction with (4.6). The proof ends. □

Next we do the boundary decay estimate for u_λ. First, we need following lemma.

Lemma 4.2.2 *Assume that a satisfies (4.4) and(4.6) with $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$ and u is a super solution of (4.5) with $\lambda > 0$ such that*

$$u \leq \theta a \quad \text{in} \quad \Omega, \tag{4.18}$$

for some $\theta \in (0,1)$, then (4.5) admits a minimal solution u_λ such that for some $c_{12} > 0$,

$$u_\lambda \leq c_{12}\varrho_{2-2\gamma} \quad \text{in} \quad \Omega.$$

Proof. We observe that

$$u_\lambda = \mathbb{G}_\Omega[\frac{\lambda}{(a-u_\lambda)^2}] = \lambda\kappa^{-2}\theta^{-2}\mathbb{G}_\Omega[\rho^{-2\gamma}],$$

then Lemma 4.2.1 with $\tau = 2 - 2\gamma$ implies

$$u_\lambda(x) \leq c_{12}\varrho_{2-2\gamma}(x).$$

The proof ends. □

Proposition 4.2.2 *Assume that the function a satisfies (4.4) and(4.6) with $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$. Then*
(i) for $\lambda \in (0, \lambda^*)$, there exists $c_{12} \geq 1$ such that

$$\frac{1}{c_{12}}\varrho_{2-2\gamma}(x) \leq u_\lambda(x) \leq c_{12}\rho^\gamma(x), \quad \forall x \in \Omega;$$

(ii) there exists $\lambda_* \leq \lambda^*$ such that for $\lambda \in (0, \lambda_*)$,

$$\frac{1}{c_{13}}\varrho_{2-2\gamma}(x) \leq u_\lambda(x) \leq c_{13}\varrho_{2-2\gamma}(x), \quad \forall x \in \Omega \tag{4.19}$$

for some $c_{13} \geq 1$.

Proof. *Lower bound.* For $\lambda \in (0, \lambda^*)$, the minimal solution u_λ of (4.5) could be approximated by increasing sequence $\{v_n\}$ defined by

$$v_n = \mathbb{G}_\Omega[\frac{\lambda}{(a-v_{n-1})^2}] \quad \text{and} \quad v_0 = 0.$$

By Lemma 4.2.1 with $\tau = 2 - 2\gamma$, we have that

$$u_\lambda \geq v_1 = \lambda\mathbb{G}_\Omega[\frac{1}{a^2}] \geq c_\tau\lambda\varrho_{2-2\gamma} \quad \text{in} \quad B_1(0).$$

Upper bound. The natural upper bound is $a(x) \geq \kappa(1-|x|)^\gamma$. For λ small, in the construction of v_n defined by

$$v_n = \mathbb{G}_\Omega[\frac{\lambda}{(a-v_{n-1})^2}] \quad \text{and} \quad v_0 = 0.$$

We may assume that there exists $\mu \in (0,\kappa)$ independent of n such that for any n

$$v_{n-1}(x) \leq \mu\rho(x)^\gamma, \quad \forall x \in B_1(0)$$

and then there exists $c_{14} > 0$ such that

$$v_n(x) \leq c_{14}\mu\varrho_{2-2\gamma}(x), \quad \forall x \in B_1(0),$$

where $c_{14} > 0$ independent of n. Thus, one infers that

$$u_\lambda(x) \leq c_{14}\mu\varrho_{2-2\gamma}(x), \quad \forall x \in B_1(0).$$

This means $\lambda^* > 0$.

For $\lambda' > 0$ if (4.5) admits a minimal solution $u_{\lambda'}$ such that

$$u_{\lambda'}(x) \leq c_{15}\varrho(x), \quad x \in \Omega.$$

Since the mapping $\lambda \mapsto u_\lambda$ is increasing, then there exists $\theta \in (0,\kappa)$ such that for all $\lambda \in (0,\lambda']$

$$u_\lambda(x) \leq \theta a(x), \quad x \in \Omega.$$

By Lemma 4.2.2, we have that for all $\lambda \in (0,\lambda']$.

$$u_\lambda(x) \leq c_{16}\varrho(x), \quad x \in \Omega.$$

Denote

$$\lambda_* = \sup\{\lambda > 0 : \limsup_{x\to\partial\Omega} u_\lambda(x)\varrho^{-1}(x) < +\infty\}.$$

It is obvious that $\lambda_* \leq \lambda^*$. The proof ends. □

Remark 4.2.1 *Assume that a satisfies (4.6) with $\kappa > 0$, $\gamma \in (0,\frac{2}{3}]$ and u_λ is the minimal solution of (4.5) with $\lambda \in (0,\lambda^*)$. Then u_λ satisfies (4.19) or*

$$\limsup_{\rho(x)\to 0^+} u_\lambda(x)\rho(x)^{-\gamma} \geq \kappa. \tag{4.20}$$

Proof. If (4.20) fails, there exists $\theta_1 \in (0,\kappa)$ and $\epsilon > 0$ such that

$$u_\lambda \leq \theta_1\rho^\gamma \quad \text{in} \quad B_1(0) \setminus B_{1-\epsilon}(0).$$

It infers from $u_\lambda < a$ in $B_1(0)$ that there exists $\theta_2 \in (0, \kappa)$ such that

$$u_\lambda \leq \theta_2 \rho^\gamma \quad \text{in} \quad \overline{B_{1-\epsilon}(0)},$$

Taking $\theta = \max\{\theta_1, \theta_2\} < \kappa$, we have

$$u_\lambda \leq \theta \rho^\gamma \quad \text{in} \quad B_1(0),$$

then by Lemma 4.2.2, u_λ satisfies (4.19). □

4.3 Estimates for λ^* and λ_* when $\Omega = B_1(0)$

In this section, we do the estimate for λ_* and λ^* in the case that $\Omega = B_1(0)$.

Proposition 4.3.1 *Assume that* $\Omega = B_1(0)$ *and*

$$a(x) = \kappa(1 - |x|)^\gamma, \tag{4.21}$$

where $\kappa > 0$ *and* $\gamma \in (0, \frac{2}{3}]$. *Then*

$$\lambda_*(\kappa, \gamma) \geq \begin{cases} \frac{4\kappa^3(1-2\gamma)(N-1+2\gamma)}{27} & \text{if} \quad \gamma \in (0, \frac{1}{3}], \\ \frac{\kappa^3}{27} & \text{if} \quad \gamma \in (\frac{1}{3}, \frac{1}{2}], \\ \frac{4\kappa^3\gamma(1-\gamma)}{27} & \text{if} \quad \gamma \in (\frac{1}{2}, \frac{2}{3}], \end{cases} \tag{4.22}$$

Proof. Let $w(r) = \frac{\kappa}{3}(1 - r)^{\beta_\gamma}$, where

$$\beta_\gamma = \begin{cases} 1 - 2\gamma & \text{if} \quad \gamma \in (0, \frac{1}{3}], \\ \frac{1}{2} & \text{if} \quad \gamma \in (\frac{1}{3}, \frac{1}{2}], \\ \gamma & \text{if} \quad \gamma \in (\frac{1}{2}, \frac{2}{3}], \end{cases}$$

then $\beta_\gamma \in [\gamma, 1)$ and

$$-\Delta w(|x|) = \frac{\kappa}{3}\beta_\gamma(1 - \beta_\gamma)(1 - |x|)^{\beta_\gamma - 2} + \frac{\kappa}{3}\beta_\gamma(N - 1)\frac{(1 - |x|)^{\beta_\gamma - 1}}{|x|}, \quad \forall x \in B_1(0). \tag{4.23}$$

Since $(1 - r)^{\beta_\gamma} \leq (1 - r)^\gamma$, then

$$\frac{1}{(a(x) - w(|x|))^2} = (\frac{2}{3}\kappa)^{-2}(1 - |x|)^{-2\gamma}, \quad \forall x \in B_1(0). \tag{4.24}$$

For $\gamma \in (\frac{1}{3}, \frac{2}{3}]$, we have that $\beta_\gamma - 2 \leq -2\gamma < \beta_\gamma - 1$ and (4.23) implies that

$$-\Delta w(|x|) \geq \frac{\kappa}{3}\beta_\gamma(1 - \beta_\gamma)(1 - |x|)^{\beta_\gamma - 2}, \quad \forall x \in B_1(0).$$

Then we have that

$$-\Delta w(|x|) \geq \frac{\lambda_1(\gamma)}{(a(x) - w(|x|))^2}, \quad \forall x \in B_1(0),$$

where $\lambda_1(\gamma) = \frac{4\kappa^3\beta_\gamma(1-\beta_\gamma)}{27}$ for $\gamma \in (\frac{1}{3}, \frac{2}{3}]$.

For $\gamma \in (0, \frac{1}{3}]$, we have that $-2\gamma = \beta_\gamma - 1$ and (4.23) implies that

$$-\Delta w(|x|) \geq \frac{\kappa}{3}\beta_\gamma(N - \beta_\gamma)(1 - |x|)^{\beta_\gamma - 2}, \quad x \in B_1(0).$$

Then we have that

$$-\Delta w(|x|) \geq \frac{\lambda_1(\gamma)}{(a(x) - w(|x|))^2}, \quad \forall x \in B_1(0),$$

where $\lambda_1(\gamma) = \frac{4\kappa^3(1-2\gamma)(N-1+2\gamma)}{27}$ for $\gamma \in (0, \frac{1}{3}]$.

By Lemma 4.2.2, we obtain that $\lambda_* \geq \lambda_1$. □

Next we see an upper bound for λ^* by (4.12).

Proposition 4.3.2 *Assume that $\Omega = B_1(0)$ and a satisfies (4.21) with $\kappa > 0$ and $\gamma \in (0, \frac{2}{3}]$. Then there exists $c > 0$ independent of γ, κ such that*

$$\lambda^*(\kappa, \gamma) \leq c_0\kappa^3(2 - 2\gamma).$$

Proof. From (4.12), we have to estimate $\int_{B_1(0)} a(x)dx$ and $\int_{B_1(0)} \frac{\mathbb{G}_\Omega[1](x)}{a^2(x)}dx$. Since there exists $c > 1$ such that

$$c^{-1}(1 - |x|) \leq \mathbb{G}_\Omega[1](x) \leq c(1 - |x|), \quad \forall x \in B_1(0),$$

then

$$\begin{aligned} \int_{B_1(0)} \frac{\mathbb{G}_\Omega[1](x)}{a^2(x)}dx &\geq c^{-1}\kappa^2 \int_{B_1(0)} (1 - |x|)^{1-2\gamma}dx \\ &\geq \frac{c}{(2 - 2\gamma)\kappa^2}. \end{aligned}$$

Together with

$$\int_{B_1(0)} a(x)dx \leq \kappa \int_{B_1(0)} dx,$$

it implies from (4.12) that $\lambda^*(\kappa, \gamma) \leq c_0\kappa^3(2 - 2\gamma)$. □

Lemma 4.3.1 *Assume that $\Omega = B_1(0)$, a satisfies (4.21) with $\kappa > 0$, $\gamma \in (0, \frac{2}{3}]$ and $0 < \lambda < \lambda^*(\kappa, \gamma)$. Then*

(i) the mappings: $\gamma \mapsto \lambda_*(\kappa,\gamma)$ *and* $\gamma \mapsto \lambda^*(\kappa,\gamma)$ *are decreasing for fixed* $\kappa > 0$*;*

(ii) the mapping: $\kappa \mapsto \lambda_*(\kappa,\gamma)$ *and* $\kappa \mapsto \lambda^*(\kappa,\gamma)$ *are increasing for fixed* $\gamma \in (0, \frac{2}{3}]$.

Proof. Let $0 < \gamma_2 \le \gamma_1 \le \frac{2}{3}$, for $\lambda \in (0, \min\{\lambda^*(\kappa,\gamma_1), \lambda^*(\kappa,\gamma_2)\})$, u_1, u_2 are the minimal solutions of (4.5) with $\gamma = \gamma_1$ and $\gamma = \gamma_2$ respectively. Denote $a_1(x) = \kappa(1-|x|)^{\gamma_1}$ and $a_2(x) = \kappa(1-|x|)^{\gamma_2}$, then $a_1 < a_2$ in $B_1(0)$ and for any $\lambda \in (0, \gamma^*(\kappa,\gamma_1))$,

$$-\Delta u_1 = \frac{\lambda}{(a_1-u_1)^2} \ge \frac{\lambda}{(a_2-u_1)^2}.$$

Therefore, $\lambda \in (0, \lambda^*(\kappa,\gamma_2))$, which implies that

$$\lambda^*(\kappa,\gamma_2) \ge \lambda^*(\kappa,\gamma_1).$$

It is similar to obtain the other assertions. □

4.4 Extremal solution

4.4.1 Existence of extremal solution

In this section, our aim is to investigate the extremal solution of (4.5).

Proposition 4.4.1 *Assume that* a *satisfies (4.4) and (4.6) with* $c_0 \ge \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$ *and* u_{λ^*} *is given by (4.7). Then* u_{λ^*} *is a weak solution of (4.5) with* λ^*. *Moreover, for* $\beta < \gamma$, *there exists* $c_\beta > 0$ *such that*

$$\|u_{\lambda^*}\|_{W^{1,\frac{N}{N-\beta}}(\Omega)} \le c_\beta \tag{4.25}$$

and

$$\int_\Omega \frac{\rho^{1-\beta}(x)}{(a-u_{\lambda^*})^2}dx \le c_\beta. \tag{4.26}$$

Proof. From (4.16), we have that for $\lambda \in (0, \lambda^*)$,

$$\int_\Omega u_\lambda(-\Delta)\xi dx = \int_\Omega \frac{\lambda\xi}{(a-u_\lambda)^2}dx, \qquad \xi \in C_c^2(\Omega). \tag{4.27}$$

Now take a sequence $\{\xi_n\} \subset C_c^2(\Omega)$ such that

$$\xi_n \to \mathbb{G}_\Omega[1] \ \text{ in } \ C^2_{loc}(\Omega) \quad \text{as } n \to +\infty.$$

Then it follows by (4.27) that

$$\int_\Omega \frac{\rho(x)}{(a-u_\lambda)^2}dx \le c_{17}\lambda^{-1}\int_\Omega u_\lambda dx \le c_{17}\lambda^{-1}\int_\Omega a(x)dx,$$

where $c_{17} > 0$. Again take a sequence $\{\xi_n\} \subset C_c^2(\Omega)$ such that

$$\xi_n \to \mathbb{G}_\Omega[\rho^{-1-\beta}] \ \text{ in } \ C^2_{loc}(\Omega) \quad \text{as } n \to +\infty,$$

where $\beta \in (0,\gamma)$. Since $u_\lambda < a$, we have that

$$\int_\Omega u_\lambda(-\Delta)\xi_n dx \le c_{18}\int_\Omega \rho^\alpha \rho^{-1-\beta}dx \le c_\beta,$$

where $c_\beta > 0$ satisfying $c_\beta \to +\infty$ as $\beta \to \alpha^-$. It follows by (4.27) and Lemma 4.2.1 that

$$\lambda\int_\Omega \frac{\rho^{1-\beta}(x)}{(a-u_\lambda)^2}dx \le c_{19}, \tag{4.28}$$

where $c_{19} > 0$ is independent of λ.

By [10, Theorem 2.6], for any $\beta \in (0,\gamma)$ there exists $c_{20} > 0$ such that

$$\||\nabla u_\lambda|\|_{M^{\frac{N}{N-\beta}}(\Omega)} \le c_{20}\|(a-u_\lambda)^{-2}\|_{L^1(\Omega,\ \rho^{1-\beta}dx)}. \tag{4.29}$$

Therefore,

$$\||\nabla u_\lambda|\|_{L^{\frac{N}{N-\beta}}(\Omega)} \le c_{19}\lambda^{-1}.$$

That is to say that

$$\|u_\lambda\|_{W^{1,\frac{N}{N-\beta}}(\Omega)} \le c_{19}\lambda^{-1}. \tag{4.30}$$

To prove that u_{λ^*} is a weak solution. Since the mapping $\lambda \mapsto u_\lambda$ is increasing and uniformly bounded by function a, which is in $L^1(\Omega)$, then

$$u_\lambda \to u_{\lambda^*} \ \text{ in } \ L^1(\Omega) \quad \text{as } \lambda \to \lambda^*$$

and the mapping $\lambda \mapsto \frac{\lambda}{(a-u_\lambda)^2}$ is increasing and

$$\frac{\lambda}{(a-u_\lambda)^2} \to \frac{\lambda}{(a-u_{\lambda^*})^2} \ \text{ a.e. in } \ \Omega \quad \text{as } \lambda \to \lambda^*.$$

Therefore, it follows by (4.28) that

$$\frac{\lambda}{(a-u_\lambda)^2} \to \frac{\lambda}{(a-u_{\lambda^*})^2} \ \text{ in } \ L^1(\Omega) \quad \text{as } \lambda \to \lambda^*$$

and then

$$\int_\Omega \frac{\lambda\xi}{(a-u_\lambda)^2}dx \to \frac{\lambda\xi}{(a-u_{\lambda^*})^2}dx.$$

Thus, passing the limit of (4.27), u_{λ^*} is a extremal solution of (4.5).

To prove (4.25) and (4.25). From (4.28) and the mapping $\lambda \mapsto \frac{\lambda\rho^{1-\beta}}{(a-u_\lambda)^2}$ is increasing, then there exists $c_{20} > 0$ such that

$$\int_\Omega \frac{\lambda\rho^{1-\beta}}{(a-u_\lambda)^2}dx \le c_{20}.$$

Then it follows by (4.29) that By [10, Theorem 2.6], we have (4.26). □

Lemma 4.4.1 *Assume that a satisfies (4.4) and (4.6) with $c_0 \ge \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$, λ_* is given in Theorem 4.1.1 and u_λ is the minimal solution of (4.5) with $\lambda \in (0, \lambda_*)$. Then $u_\lambda \in H_0^1(\Omega)$ and for $\lambda \in (0, \lambda_*)$, there exists $c_{21} > 0$ such that*

$$\int_\Omega |\nabla u_\lambda|^2 dx \le c_{21}\lambda \quad \text{and} \quad \int_\Omega \frac{u_\lambda}{(a-u_\lambda)^2}dx \le c_{22}.$$

Proof. By (4.19), for $\lambda \in (0, \lambda_*)$, there exists $\theta \in (0, 1)$ such that $a - u_\lambda \le \theta a$ and then

(i) for $\gamma \in (0, \frac{1}{2})$,

$$\int_\Omega \frac{u_\lambda}{(a-u_\lambda)^2}dx \le c_\tau\theta^{-2}\int_\Omega \rho^{1-2\gamma}dx < \infty;$$

(ii) for $\gamma = \frac{1}{2}$,

$$\int_\Omega \frac{u_\lambda}{(a-u_\lambda)^2}dx \le c_\tau\theta^{-2}\int_\Omega \rho^{1-2\gamma}[1 + (\ln\frac{1}{\rho})_+]dx < \infty;$$

(iii) for $\gamma \in (\frac{1}{2}, \frac{2}{3}]$,

$$\int_\Omega \frac{u_\lambda}{(a-u_\lambda)^2}dx \le \theta^{-2}\int_\Omega \rho^{2-4\gamma}dx < \infty,$$

where $2 - 4\gamma > -1$. Taking a sequence $\{\xi_n\} \subset C_c^2(\Omega)$ which converges to u_λ, then

$$\int_\Omega |\nabla u_\lambda|^2 dx = \lambda\int_\Omega \frac{u_\lambda}{(a-u_\lambda)^2}dx < +\infty.$$

The proof ends. □

Proof of Theorem 4.1.1. The existence of minimal solution for $\lambda \in (0, \lambda^*)$ and the nonexistence for $\lambda > \lambda^*$ follow by Proposition 4.2.1. And Theorem 4.1.1 (iii)

see Proposition 4.2.2. The estimates of λ_* and λ^* see Proposition 4.3.1, Proposition 4.3.2 and Lemma 4.3.1. □

4.4.2 Stability and regularity

Lemma 4.4.2 *Assume that $\lambda \in (0, \lambda^*)$, a satisfies (4.4) and (4.6) with $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$. Let u be a positive solution of (4.5) and v be a super solution of (4.5).*

If $\mu_1(\lambda, u) > 0$, then

$$u \leq v \quad \text{in} \quad \Omega.$$

If $\mu_1(\lambda, u) = 0$, then

$$u = v \quad \text{in} \quad \Omega.$$

Proof. It follows the procedure of the proof of [49, Lemma 4.1] just replaced $\frac{f}{(1-u)^2}$ by $\frac{1}{(a-u)^2}$, since $\frac{1}{(a-u)^2}$ just has the boundary singularity. □

Proposition 4.4.2 *Assume that $\lambda \in (0, \lambda^*)$, a satisfies (4.4) and (4.6) with $c_0 \geq \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$, and u_λ is the minimal solution of (4.5). Then u_λ is stable.*

Proof. Denote

$$\lambda^\sharp = \sup\{\lambda : \ u_\lambda \text{ is a stable solution of (4.5)}\}.$$

Step 1. To prove $\lambda^\sharp > 0$. It follows by [71, Theorem 1] that there exists constant $c_{22} > 0$ such that

$$\int_\Omega \xi^2 \rho^{-2} dx \leq c_{22} \int_\Omega |\nabla \xi|^2 dx, \quad \forall \xi \in C_c^2(\Omega). \tag{4.31}$$

For $\lambda < \lambda_*$, it follows by Theorem 4.1.1, there exists $\theta \in (0, 1)$ such that

$$u_\lambda \leq \theta a \quad \text{in} \quad \Omega.$$

Together with $\gamma \in (0, \frac{2}{3}]$, there exists constant such that

$$\frac{1}{(a - u_\lambda)^3} \leq c_{23} \rho^{-3\gamma} \leq c_{24} \rho^{-2} \quad \text{in} \quad \Omega,$$

where $c_{23}, c_{24} > 0$. Then for λ small enough, it follow by (4.31) that

$$\int_\Omega \frac{\lambda \xi^2}{(a - u_\lambda)^3} dx \leq \int_\Omega |\nabla \xi|^2 dx, \quad \forall \xi \in C_c^2(\Omega).$$

This means that u_λ is a stable solution of (4.5) for $\lambda > 0$ small, then $\lambda^\sharp > 0$.

Step 2. To prove $\lambda^\sharp = \lambda^*$. We prove $\lambda^\sharp = \lambda^*$ by contradiction. It is obvious that $\lambda^\sharp \le \lambda^*$ and then we may assume that $\lambda^\sharp < \lambda^*$. Choose $\lambda_1 \in (\lambda^\sharp, \lambda^*)$ and u_{λ_1} satisfies that for $\lambda \in (0, \lambda^\sharp)$, which is not empty from step 1,

$$-\Delta u_{\lambda_1} = \frac{\lambda_1}{(a-u_{\lambda_1})^2} > \frac{\lambda}{(a-u_{\lambda_1})^2}.$$

Moreover, since the mapping $\lambda \mapsto \frac{\lambda}{(a-u_\lambda)^3}$ is increasing in $L^1(\Omega, \rho^2 dx)$ and then by (4.28), $\frac{\lambda}{(a-u_\lambda)^3} \to \frac{\lambda}{(a-u_{\lambda^\sharp})^3}$ in $L^1(\Omega, \rho^2 dx)$ as $\lambda \to \lambda^\sharp$. Thus, together with u_λ is stable, we imply that $u_{\lambda^\sharp}$ is semistable. By Lemma 4.4.2, we have that $u_{\lambda^\sharp} = u_{\lambda_1}$, which is impossible. Therefore, $\lambda^\sharp = \lambda^*$. □

Proposition 4.4.3 *Assume that a satisfies (4.6) with* $\kappa > 0$, $\gamma \in (0, \frac{2}{3}]$ *and* u_{λ^*} *is given by (4.7). Then* u_{λ^*} *is semi-stable weak solution of (4.5) with* λ^*.

Proof. From (4.28) and the stability of u_λ, then

$$\int_\Omega \frac{\lambda \varphi^2}{(a-u_\lambda)^3} dx < \int_\Omega |\nabla \varphi|^2 dx$$

holds for $\varphi = \mathbb{G}_{B_1(0)}[1]$. Therefore,

$$\int_\Omega \frac{\rho^2}{(a-u_\lambda)^3} dx < c\lambda^{-1}.$$

Since the mapping $\lambda \mapsto \frac{\rho^2}{(a-u_\lambda)^3}$ is strictly increasing and bounded in $L^1(\Omega)$, Then

$$\frac{\rho^2}{(a-u_\lambda)^3} \to \frac{\rho^2}{(a-u_{\lambda^*})^3} \quad \text{as} \quad \lambda \to \lambda^* \quad \text{in} \quad L^1(\Omega)$$

and

$$\lim_{\lambda \to \lambda^*} \int_\Omega \frac{\lambda \varphi^2}{(a-u_\lambda)^3} dx = \int_\Omega \frac{\lambda^* \varphi^2}{(a-u_{\lambda^*})^3} dx$$

Therefore,

$$\int_\Omega \frac{\lambda^* \varphi^2}{(a-u_{\lambda^*})^3} dx \le \int_\Omega |\nabla \varphi|^2 dx, \quad \forall \varphi \in C_c^2(\Omega),$$

that is, u_{λ^*} is semi-stable. □

We next improve the regularity of u_{λ^*} and prove when $N \le 7$, the extremal solution u_{λ^*} is a classical solution of (4.5) with $\lambda = \lambda^*$. To this end, we need the following lemma, which is inspired by [49].

Lemma 4.4.3 *Assume that* $\lambda \in (0, \lambda^*)$, *a satisfies (4.21) with* $\kappa > 0$ *and* $\gamma \in (0, \frac{2}{3}]$. *Let* u *be a weak solution of (4.5) such that for any compact set* $K \subset \Omega$,

there exists $c_{25} > 0$ such that

$$\|\frac{1}{a-u}\|_{L^{\frac{3N}{2}}(K)} \leq c_{25}. \tag{4.32}$$

Then u is a classical solution of (4.5) and there exists $c_{26} > 0$ depending on K such that

$$\inf_{x\in K}(a(x) - u(x)) > c_{26}. \tag{4.33}$$

Proof. From (4.32), we have that

$$\frac{1}{(a-u)^2} \in L^{\frac{3N}{4}}(K)$$

and then $u \in W^{2,\frac{3N}{4}}(K)$ and by Sobolev's Theorem we can already deduce that $u \in C^{\frac{2}{3}}(K')$ with K' compact set in interior point set of K. To get more regularity, it suffices to show that $u < a$ in Ω. If not, there exists $x_0 \in \Omega$ such that $u(x_0) = a(x_0)$. Then we have that

$$\begin{aligned} |a(x) - u(x)| &= |a(x) - a(x_0)| + |u(x) - u(x_0)| \\ &\leq |u(x) - u(x_0)| \leq |x - x_0|^{\frac{2}{3}}, \end{aligned}$$

then

$$+\infty > \int_\Omega \frac{1}{(a-u)^{\frac{3N}{2}}} \geq \int_\Omega |x - x_0|^{-\frac{3N}{2}\cdot\frac{2}{3}} dx = +\infty,$$

a contradiction, which implies that we have that $a - u > 0$ in Ω. □

Proposition 4.4.4 *Assume that $1 \leq N \leq 7$, $\Omega = B_1(0)$, a satisfies (4.4) and (4.6) with $c_0 = \kappa > 0$, $\gamma \in (0, \frac{2}{3}]$ and u_{λ^*} is given by (4.7). Then u_{λ^*} is a classical solution of (4.5) with $\lambda = \lambda^*$.*

Proof. Since the mapping: $\lambda \mapsto u_\lambda$ is increasing and bounded by a, then from (4.6) and Lemma 4.4.3, we only have to improve the regularity of u_{λ^*} in any compact set of $B_1(0)$. For $\lambda \in (0, \lambda^*)$, we know that u_λ is stable, then

$$\int_\Omega \frac{\lambda\xi^2}{(a-u_\lambda)^3} dx \leq \int_\Omega |\nabla\xi|^2 dx \quad \forall \xi \in C_c^2(B_1(0)). \tag{4.34}$$

Minimal solutions u_λ is radially symmetric. Since the minimal solution u_λ could be approximated by the sequence functions

$$v_n = \lambda\mathbb{G}_{B_1(0)}[\frac{1}{(a-v_{n-1})^2}] \quad v_0 = 0.$$

It follows by radially symmetry of v_{n-1} and a that v_n is radially symmetry and then u_λ is radially symmetric. Then u_{λ^*} is radially symmetric.

We will prove that for any $r \in (0,1)$, there exists $\epsilon > 0$ depending on r such that

$$a - u_\lambda \geq \epsilon \quad \text{on} \quad \partial B_r(0).$$

Conversely, if there is r' such that

$$a - u_\lambda = 0 \quad \text{on} \quad \partial B_{r'}(0).$$

From (4.30), we have that

$$u_{\lambda^*} \in W^{1,\frac{N}{N-\beta}}(\Omega) \quad \text{for } 0 < \beta < \gamma.$$

Then there is $r_0 \in (0, r')$ and $\epsilon_0 > 0$ such that

$$a(r_0) - u_\lambda(r_0) \geq \epsilon_0 \quad \text{for} \quad \lambda \in (0, \lambda^*).$$

If not,

$$a - u_\lambda = 0 \quad \text{in} \quad B_1(0) \setminus \overline{B_{r'}}(0).$$

there is contradiction with (4.28).

Choose

$$\xi_i = \begin{cases} (a-u_\lambda)^i - \epsilon_\lambda^i & \text{in} \quad B_{r_0}(0), \\ 0 & \text{in} \quad B_1(0) \setminus B_{r_0}(0), \end{cases}$$

where $i \in (-2-\sqrt{6}, 0)$ and $\epsilon_\lambda = a - u_\lambda$ on ∂B_{r_0}. Then $\xi_i \in H_c^1(B_1(0))$. It follows by (4.34) with ξ_i, we have that

$$\begin{aligned} \int_{B_{r_0}(0)} \frac{\lambda[(a-u_\lambda)^i - \epsilon_\lambda^i]^2}{(a-u_\lambda)^3} dx &\leq \int_{B_{r_0}(0)} |\nabla((a-u_\lambda)^i)| dx \\ &= i^2 \int_{B_{r_0}(0)} (a-u_\lambda)^{2i-2} |\nabla(a-u_\lambda)|^2 dx. \quad (4.35) \end{aligned}$$

On the other hand, from (4.5), we have that

$$-\Delta(a-u) = -\Delta a - \frac{\lambda}{(a-u)^2}, \quad B_{r_0}(0). \tag{4.36}$$

Multiplying (4.36) by $\frac{i^2}{1-2i}[(a-u_\lambda)^{2i-1}-\epsilon^{2i-1}]$ and applying integration by parts yields that

$$\begin{aligned}
&\frac{i^2}{1-2i}\int_{B_{r_0}(0)}[-\Delta a-\frac{\lambda}{(a-u)^2}][(a-u_\lambda)^{2i-1}-\epsilon^{2i-1}]dx\\
&=\frac{i^2}{1-2i}\int_{B_{r_0}(0)}\nabla(a-u_\lambda)\cdot\nabla((a-u_\lambda)^{2i-1})dx\\
&=i^2\int_{B_{r_0}(0)}(a-u_\lambda)^{2i-2}|\nabla((a-u_\lambda)|^2dx,
\end{aligned}$$

together with (4.35), then we deduce that

$$\int_{B_{r_0}(0)}\frac{\lambda[(a-u_\lambda)^i-\epsilon_\lambda^i]^2}{(a-u_\lambda)^3}dx\le\frac{i^2}{1-2i}\int_{B_{r_0}(0)}[-\Delta a-\frac{\lambda}{(a-u)^2}][(a-u_\lambda)^{2i-1}-\epsilon^{2i-1}]dx,$$

thus,

$$\begin{aligned}
\lambda(2-\frac{i^2}{1-2i})\int_{B_{r_0}(0)}\frac{1}{(a-u_\lambda)^{3-2i}}dx&\le\int_{B_{r_0}(0)}\frac{4\lambda\epsilon_\lambda^i}{(a-u_\lambda)^{3-i}}dx-\int_{B_{r_0}(0)}\frac{2\lambda\epsilon_\lambda^{2i}}{(a-u_\lambda)^3}dx\\
&+\int_{B_{r_0}(0)}\frac{-\Delta a}{(a-u_\lambda)^{1-2i}}dx-\epsilon_\lambda^{2i-1}\int_{B_{r_0}(0)}\Delta adx-\lambda\int_{B_{r_0}(0)}\frac{\epsilon_\lambda^{2i-1}}{(a-u_\lambda)^2}dx.
\end{aligned}$$

Since $\Delta a\le 0$ and $\epsilon_\lambda\ge\epsilon_0$, then

$$\begin{aligned}
\lambda(2-\frac{i^2}{1-2i})\int_{B_{r_0}(0)}\frac{1}{(a-u_\lambda)^{3-2i}}dx&\le\int_{B_{r_0}(0)}[\frac{4\lambda\epsilon_\lambda^i}{(a-u_\lambda)^{3-i}}dx+\frac{-\Delta a}{(a-u_\lambda)^{1-2i}}]dx\\
&\le\int_{B_{r_0}(0)}[\frac{4\lambda^*\epsilon_0^i}{(a-u_\lambda)^{3-i}}dx+\frac{-\Delta a}{(a-u_\lambda)^{1-2i}}]dx.
\end{aligned}$$

Therefore, for $(2-\frac{i^2}{1-2i})>0$, that is, $i\in(-2-\sqrt{6},0)$, there exists $c_{27}>0$ independent of λ such that

$$\int_{B_{r_0}(0)}\frac{1}{(a-u_\lambda)^{3-2i}}dx\le c_{27}. \tag{4.37}$$

When $N\le 7$, $\frac{3N}{2}\le 3-2i$, then by Lemma 4.4.3, we have that u_λ has uniformly in $C^{2,\gamma}_{loc}(\Omega)$, then u_{λ^*} is a classical solution of (4.5) with λ^* and $a-u_{\lambda^*}>0$ in Ω. □

Proof of Theorem 4.1.3. Proposition 4.4.1 shows that u_{λ^*} is a weak solution of (4.5). The stability of u_λ of (4.5) with $\lambda\in(0,\lambda^*]$ follows by Proposition 4.4.2 and Proposition 4.4.3. When $1\le N\le 7$ and $\Omega=B_1(0)$, it from Proposition 4.4.4, u_{λ^*} is a classical solution of (4.5). □

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