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Modeling a bus through a sequence of traffic lights

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We propose a model of a bus traveling through a sequence of traffic lights, which is required to stop between the traffic signals to pick up passengers. A two dimensional model, of velocity and traveled time at each traffic light, is constructed, which shows non-trivial and chaotic behaviors for realistic city traffic parameters. We restrict the parameter values where these non-trivial and chaotic behaviors occur, by following analytically and numerically the fixed points and period 2 orbits. We define conditions where chaos may arise by determining regions in parameter space where the maximum Lyapunov exponent is positive. Chaos seems to occur as long as the ratio of the braking and accelerating capacities are greater than about ~3. © 2015 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4926669]

Traffic research is almost as old as automotive vehicles themselves; however, it is just lately that its full complexity has been recognized. Many kinds of models, going from cellular automata to systems of coupled differential equations, attest for its non-triviality. In spite of such efforts, it keeps providing interesting results. In this paper, we present and analyze the consequences of a discrete map describing the exact evolution of a bus under ideal city conditions. The buses display chaotic behavior very near (in parametric sense) of an optimal flow setting, which makes it difficult to simultaneously optimize travel time and maintain a predictable schedule. The chaotic and nontrivial dynamics are due to the finite braking and accelerating capacities of the buses. New results are related to a more complete understanding of the bus dynamics, derivation of the analytical expressions for the bounds of the nontrivial and chaotic behaviors in the bifurcation diagrams with respect to various parameters of the system, many of which can be controlled by traffic controllers. These bounds can be use to estimate the relevance of the nontrivial and chaotic dynamics of the bus trajectories. For example, we have found that chaos occurs when the ratio of the braking and accelerating capacities are greater than about ~3.

I. INTRODUCTION

The dynamics of city traffic has become an active area of research not only because of its social and economical relevance,[1] but also and because it displays many interesting features[2–11] such as complex dynamics and emergent phenomena.[12–14] This complex behavior has been studied using many different approaches, going from statistical and cellular automaton, to hydrodynamical and mean field models.[15–18] Some approaches have even included the topological complexity of the traffic networks.[19] In spite of much effort in trying to understand traffic networks, there remain many interesting problems, such as emergent phenomena,[20] chaotic behavior,[2,11] stochastic like resonances,[21] self-organization,[22] etc.

One of the main components of city traffic involves buses and their dynamics. For example, there are a number of publications that study problems related to school buses (see Refs. 23 and 24 and references therein), while others analyze environmental issues associated with bus transportation systems.[24,25] There are also a number of publications that investigate on the position of bus-stops.[26] For example, Jia *et al*.[27] and Ding *et al*.[28] use a cellular automaton model to study the impact of bus-stops on the dynamics of traffic flow; while Tang *et al*.[29] analyze the same problem using a traffic flow model. Optimization models were also used by Ibeas *et al*.[30] The interaction of the bus with traffic lights is discussed by Estrada *et al*.,[31] where a simulation optimization model is used to minimize the travel time of bus users in an urban network. There are also a number of publications that investigate the traffic light timing to optimize vehicle flow. For example, Liao and Davis[32] used Global Positioning Systems (GPS) and automated vehicle location systems on the buses in Minneapolis, to develop an adaptive signal priority strategy. Also a bus priority method for traffic light control based on two modes of operation was proposed by Koehler and Kraus.[33] Some studies combine traffic light and stop position using nonlinear map models, as was done by Mei *et al*.[34] (their interest lies in T junctions) and Hounsell *et al*.,[35] where the issue is how to tackle the challenge posed by locational error associated with GPS, where a traffic signal is located close to a bus stop in London.

Hence, the majority of the researches on buses have dealt with the problem of how traffic flow is affected by buses and vice versa (see Refs. 36–48, for examples).

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Among these researchers, there seems to be an interest on the impact of exclusive-lines on the dynamics of buses;[49–51] however, most of the cited articles are concerned with the evaluation of the single lines as viable strategies, and not on characterizing the different dynamics that can occur.

In this respect, it is interesting to mention the work by Nagatani,[52] which demonstrates the existence of chaotic behavior in the tour time of a shuttle bus moving through a traffic signal for certain passenger loading parameter and cycle time. The existence of this behavior in such a simple model must be analyzed with care, as it might have profound implications for some of these traffic flow optimization procedures. Similarly, Villalobos *et al.*[11] constructed a model for a bus moving, in a segregated manner, through a sequence of traffic lights, but having finite braking and accelerating capacities. The buses display nontrivial and chaotic behaviors very near (in parametric sense) of an optimal flow setting, which makes it difficult to simultaneously optimize travel time and maintain a predictable schedule.

Since this result may have profound implications for designing bus transport systems in cities; in this work, we propose to expand the discussion on the *Simple Bus model* presented in Ref. 11, and the discussion on the results presented in Ref. 11, derive analytical relations and restrictions for the parameters of the system, and study in detail the possible dynamics of the system. Particular emphasis will be given to understanding the root of the non-trivial and chaotic behaviors presented in this model, which we can demonstrate it is ascribable to the finite braking and accelerating capabilities of the buses. In the process, we will find the conditions that these variables need to satisfy for the existence of these non-trivial and chaotic behaviors.

We will see that these results are relevant for city parameters, so that we would expect that it should be applicable in city transport systems. Besides the intrinsic importance of characterizing the nontrivial behavior observed in this model, this research is applicable to the mass transportation system in the cities of Curitiba (Brazil), Bogotá (Colombia)—this latter one dubbed Transmilenio—, and other cities that have adopted public transportation systems based on dedicated lines. These are city traffic systems in which buses have exclusive rolling lanes (they do not interact with cars or other vehicles). Also, at times other than rush hours, they roll pretty much by themselves and interaction between buses is low, both at stations (the places where passengers get in or out of the bus) and near traffic lights. The characteristics of these systems, their comparison with other bus rapid transit systems, and its possible application as a rapid bus transport system in the United States, are discussed in Refs. 53–55.

This paper is organized as follows: in Section II, we describe the *Simple Bus model* and obtain a 2D map that gives us the state of the bus (in terms of time and speed) at one light, given the state at the previous light. Also, we explore the dynamics via numeric simulations where the Lyapunov exponent of the system is computed and represented in parameter space. Section III gives an analytical description of the dynamics, several values of the bifurcation diagrams are found, along with expressions for the fixed

point and period-2 orbits. Finally, Section IV presents a summary. There are two Appendixes, Appendix A goes into detail regarding the construction of the model, and Appendix B expands on the derivation of the analytical results presented in Section III.

II. MODEL AND DYNAMICS DESCRIPTION

Our *Simple Bus model* mimics the dynamics of one vehicle (a public transportation one) moving through a sequence of traffic lights in one dimension. The interaction with the traffic lights is modeled after what can be expected of an average driver. Let the separation between light n and $n+1$ be L_n (we set $L_n = L$, $\forall n$ in this manuscript). At time t, the nth light is green if $\sin(\omega t + \phi_n) \geq 0$, red otherwise. Here, ω_n represents the traffic's light frequency and ϕ_n its phase (we set $\omega_n = \omega$, and $\phi_n = 0 \forall n$ in this manuscript).

A bus in this sequence of traffic lights can have: (a) an acceleration a_+ until its velocity reaches the cruising speed $v_{\max}$, (b) a constant speed $v_{\max}$ with zero acceleration, or (c) a negative fixed, acceleration $-a_-$ until it stops or starts accelerating again. Hence, we can write the bus' equation of motion as

$$\frac{dv}{dt} = \begin{cases} a_+\theta(v_{\max} - v), & \text{accelerate} \\ -a_-\theta(v), & \text{brake,} \end{cases} \quad (1)$$

where θ is the Heaviside step function.

The bus is forced to make a stop for a time γ to pick and leave passengers between lights, such that the stop is located at a distance $\bar{\ell}$ from the light behind the vehicle. The bus then accelerates towards the next light.

We refer to Fig. 1 in order to describe the dynamics of the bus between two consecutive traffic lights. In Fig. 1(a), we show the speed (v) of the bus as a function of its position (x), and in Fig. 1(b) its position (x) as a function of time (t). Both figures have circled numbers that refer to intervals, or *regions*, where the dynamics are governed by the same
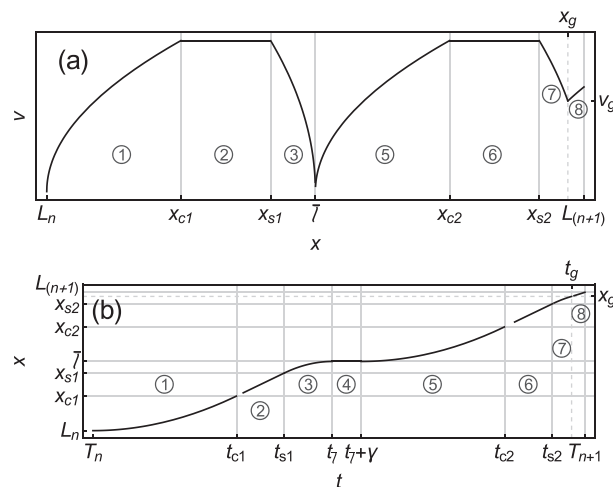

FIG. 1. Schematics for the movement of the Bus Model between two traffic lights: (a) Speed on the vertical axis and position on the horizontal, (b) position on the vertical axis and time on the horizontal. We illustrate the case where the light is red at the stop point x_{s2} and changes color before the vehicle reaches a full stop. Labels are described in Secs. II, III, and Appendix A.

dynamical equation. So, regions 1, 5, and 8 label the intervals when the bus is accelerating; regions 2 and 6 are the ones when it is traveling at constant speed; 3 and 7 identify the moments when it is braking and, finally, region 4 marks the time when the bus is standing still picking up passengers. Note that the existence of regions 7 and 8 are determined by the traffic's light color at the decision point x_{s2} (with time t_{s2}). There are three possibilities since the interaction with the traffic light is modeled after what can be expected of an average driver

(1) If the light is green at x_{s2} with t_{s2}, then the bus will continue traveling at constant speed $v_{\max}$ and will cross the light. In this case, there is no need for region 1 between light $n+1$ and $n+2$.
(2) If the light is red at x_{s2} with t_{s2}, then the bus will start braking in order to reach the traffic light with speed 0. There are two possibilities in this scenario:

   (a) The traffic light stays red, while the bus is braking. In this case, the vehicle stops at the light and waits for a green light to start accelerating, so that only region 7 is needed.
   (b) The traffic light changes to green, while the bus is braking before reaching a complete stop. In this case, the vehicle starts braking (region 7) and then accelerates again as soon as the light turns green (region 8). This is the scenario, we are illustrating in Fig. 1.

According to Fig. 1, we have that the bus always brakes at position $\bar{\ell}$ (region 3) and, may brake at the traffic light if the light is red (region 7). If the bus happens to cross the traffic light with speed $v < v_{\max}$ it will accelerate until $v_{\max}$ is reached (region 1). Since it is forced to stop at $\bar{\ell}$ it will always accelerate from this point in order to get to the following traffic light (region 5). When not accelerating or braking, a constant speed ($v_{\max}$) is maintained (regions 2 and 6). For the bus to come to a complete stop at position $\bar{\ell}$ it must start braking at x_{s1}, so that regions 3, 4, and 5 always take place.

In this manuscript, we do not contemplate the possibility that the bus gets to these two decision points (x_{s1} and x_{s2}) with speed $v < v_{\max}$. We chose to take the minimum distance that is consistent with the braking capacity ($-a_-$) of the bus, namely, $x_{s2} = x_n + L - v_{\max}^2/(2a_-)$. The same restriction applies at the bus stop, namely, $x_{s1} = x_n + \bar{\ell} - v_{\max}^2/(2a_-)$.

Let us write down some restrictions in order to keep the model as simple as possible. First, we restrict the light not to change more than once, while the bus is in regions 7 and 8 of Fig. 1, so that

$$0 < \frac{\omega}{2\pi} \leq \frac{\min(a_+, a_-)}{v_{\max}}. \tag{2}$$

Second, we expect the bus to be able to reach the cruising speed v_{max}, and stop at the stopping site $\bar{\ell}$ and at the next traffic light, hence

$$\frac{v_{\max}^2}{2a_-} < \bar{\ell} < L - \frac{v_{\max}^2}{2a_-}. \tag{3}$$

Third, to guarantee that all regions (1–8 in Fig. 1) can take place, we must enforce

$$\frac{v_{\max}^2}{2}\left(\frac{1}{a_-} + \frac{1}{a_+}\right) < \bar{\ell} < L - \frac{v_{\max}^2}{2}\left(\frac{1}{a_-} + \frac{1}{a_+}\right). \tag{4}$$

For the purpose of illustration, we will fix $\bar{\ell} = L/2$.

The 2D map

$$(t_{n+1}, v_{n+1}) = M(t_n, v_n),$$

which evolves the velocity and time at the n-th traffic light to the same variables at the $(n+1)$-th traffic light is built explicitly in Appendix A. It is valid only if Eqs. (2)–(4) hold.

It is desirable to have a normalization scheme for these variables, so that we normalize the time with the minimum travel time defined by

$$t_{\min} = T_c + \frac{v_{\max}}{2a_+a_-}(a_+ + a_-), \tag{5}$$

where $T_c = \frac{L}{v_{\max}}$. We also normalize both the speed and position, so that $u = v/v_{\max}$, $y = x/L$, and $\tau = t/t_{\min}$. We also introduce the following normalized variables: $\ell = \bar{\ell}/L$ ($0 < \ell \leq 1$), $A_+ = a_+L/v_{\max}^2$ ($A_+ \geq 0$), $A_- = a_-L/v_{\max}^2$ ($A_- \geq 0$), $\Gamma = \gamma/T_c$ ($\Gamma \geq 0$), and $\Omega = \frac{\omega t_{\min}}{2\pi}$ ($\Omega > 0$).

In terms of the normalized variables, the restrictions given by Eqs. (2) and (3), can be written as

$$0 < \Omega \leq \frac{t_{\min}}{T_c}\min(A_+, A_-) \tag{6}$$

and

$$\frac{1}{2A_-} < \ell < 1 - \frac{1}{2A_-}. \tag{7}$$

Now, the condition for all regions to exist, Eq. (4), is normalized as

$$\frac{1}{2}\left(\frac{1}{A_-} + \frac{1}{A_+}\right) < \ell < 1 - \frac{1}{2}\left(\frac{1}{A_-} + \frac{1}{A_+}\right). \tag{8}$$

Similarly, we can write

$$\frac{t_{min}}{T_c} = 1 + \left(\frac{1}{2A_-} + \frac{1}{2A_+}\right).$$

We will focus our efforts on understanding the effect that Ω and Γ (the traffic light's frequency and the time that the bus stands still at the stop point) have on the dynamics. We restrict ourselves to the case $\ell = 1/2$ without any loss of generality. As long as ℓ takes a value bounded by the inequality in Eq. (8), it has no effect on the dynamics. Unless stated otherwise, we will fix the acceleration values at $a_+ = 1\frac{m}{s^2}$ and $a_- = 5\frac{m}{s^2}$, with $L = 400$ m and $v_{\max} = 60\frac{km}{h}$, which are realistic parameters for city traffic. With these values, we have $t_{\min} \simeq 34$ s. These are reasonable values for cities like Santiago o Bogota. In Figs. 2(a) and 2(b), we show bifurcation diagrams for the speed u and time between traffic lights $\Delta\tau = (\tau_{n+1} - \tau_n)$, respectively, as a function of Ω. For now, we will take $\Gamma = 0$. All bifurcation diagrams are done by
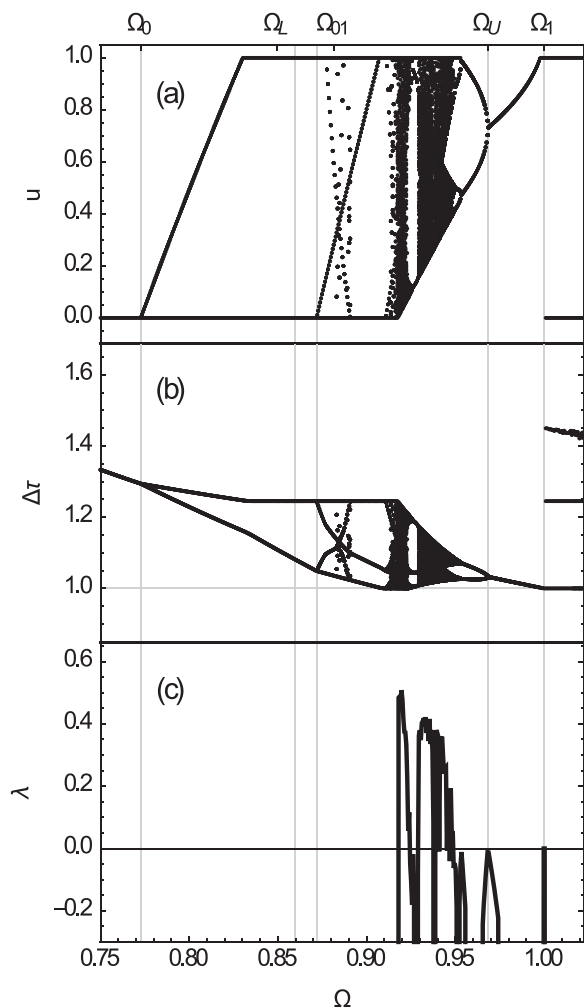
FIG. 2. Bifurcation diagrams for Ω as bifurcation parameter. Both (a) speed at the nth light and (b) time between consecutive lights are shown (b). We use $\Gamma = 0$. The vertical gray lines in (a) mark values for Ω that are used for Figs. 4(a)–4(d). Figure (c) shows the Lyapunov exponent.

plotting the iterations 900 through 1000 of the map, with initial condition $(\tau_0, u_0) = (0, 0)$. Let us define several interesting points in the bifurcation diagram. First, we observe that for $\Omega = \Omega_1 = 1$, the bus is in resonance with the traffic light crossing it with $v_{\max}$ without stopping, except at ℓ. Note that making $\Omega \gtrsim \Omega_1$ puts the bus in a situation, where it has to stop every p lights; while in the case that $\Omega \lesssim \Omega_1$, the bus may be forced to brake at every light. We will not discuss the dynamics for $\Omega > \Omega_1$ except to note that the time spent between lights grows due to the fact that the bus spends more time standing still waiting for the red light to change. For $\Omega < \Omega_1$, but close to the resonance, we note a period doubling bifurcation to chaos as we reduce Ω. We define Ω_U as the frequency where we observe the first period-doubling bifurcation. Eventually, the chaotic attractor collides with the state $(u_n, \tau_n) = (0, 0)$ producing a crisis that was characterized in Ref. 9, for a single car model. Similarly, we define Ω_L as the frequency where the lower branch of the period-2 hits the $u = 0$ threshold. At $\Omega = \Omega_{01}$, the dynamics reach a point at which the bus has to stop completely at every other traffic light. Note that Ω_U (or Ω_{01}) and Ω_L, for which we will find analytic expressions in Appendix B, provide a bound for

the existence of the nontrivial dynamics and the chaotic regime. The range of parameter where this occurs is relevant for city traffic. Furthermore, this range of Ω also contains the *Chaotic Region* (CR). For $\Omega < \Omega_L, \Omega_{01}$, we observe a point where the bus has to stop at every light and wait for it to turn green again. We call this the stop value Ω_0.

We can show that the CR is indeed chaotic, by computing the numerical maximum Lyapunov exponent as was done in Refs. 2 and 8, for a single car model. We evolve $(u_0, \tau_0) = (0, 0)$ for n iterations to make sure we had arrived to the attractor, and then we follow the actual trajectory (u_{n+m}, τ_{n+m}) and a different perturbed trajectory $(\bar{u}_{n+m}, \bar{\tau}_{n+m}) = (u_{n+m}, \tau_{n+m}) + (\delta u_m, \delta \tau_m)$ with $(\delta u_0, \delta \tau_0) = (0, 10^{-10})$, for 25 additional steps. We calculate the euclidean distance δ_m vs m between both trajectories for 10 different initial conditions, namely, trajectories starting at $n = 500 + 25 \times r$ (for $r = 0, \ldots, 9$), and fit a Lyapunov exponent as $\ln \delta_m = \ln \delta_0 + \lambda m$. Of course, due to the non-smooth nature of the map, we do not consider situations in which $\lambda \to \infty$. The result of the calculation is shown in Fig. 2(c), which clearly demonstrates that the dynamics is chaotic for the expected range of Ω.

The bifurcation diagram with respect to Ω changes as we vary Γ, not in its general shape, but in its position and width relative to Ω. Indeed, at resonance the travel time between traffic lights $t_{min} + \gamma$, should be equal to the period of the traffic light, so that

$$\Omega_1 = \frac{t_{\min}}{t_{\min} + \Gamma T_c}. \tag{9}$$

Since the resonance occurs for $\omega(t_{min} + \gamma) = 2\pi$, the bus will resonate at a decreasing $\Omega = \Omega_1$. This is expected, since it will take the bus a longer time to travel the distance between the traffic lights. The situation at Ω_U and Ω_L is a little more involved, so it will be derived in the Appendix. Meanwhile, we note that at Ω_0, the bus starts from rest and then comes to a full stop at the traffic light exactly when the traffic light turns green again, so that

$$\Omega_0 = \frac{t_{\min}}{T_c} \frac{1}{1 + \frac{3}{4}\left(\frac{1}{A_+} + \frac{1}{A_-}\right) + \Gamma}. \tag{10}$$

Similarly, at Ω_{01}, the bus starts from rest, goes through 1 traffic light with $v_{\max}$, and then comes to a full stop at the second traffic light exactly when the traffic light turns green again. Therefore

$$\Omega_{01} = \frac{t_{\min}}{T_c} \frac{1}{1 + \left(\frac{1}{A_+} + \frac{1}{A_-}\right) + \Gamma}. \tag{11}$$

We note that these frequencies decrease as Γ increases, as can be appreciated in Fig. 3(a). We also show that the width of the region where the nontrivial and chaotic dynamics changes with Γ, as displayed in Fig. 3(b), where we plot $\Delta\Omega_{U,L} = \Omega_U - \Omega_L$ (dashed) and $\Delta\Omega_{1,0} = \Omega_1 - \Omega_0$ (dotted) as a function of Γ. Clearly, the width of the chaotic and nontrivial dynamics gets reduced as we increase the bus waiting
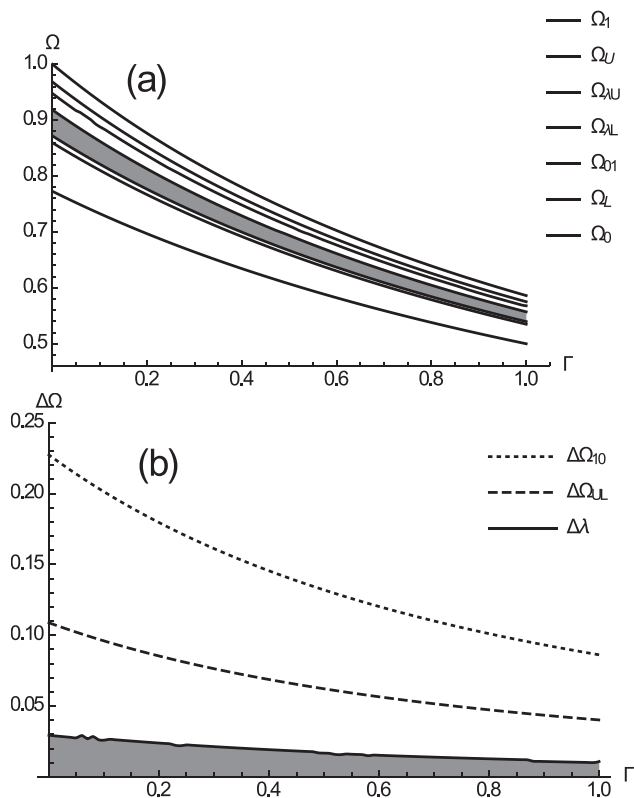
FIG. 3. (a) Ω_1, Ω_U, Ω_L, and Ω_0 as a function of Γ. The order of the curves from top to bottom is the same as the legend. $\Omega_{\lambda L}$ and $\Omega_{\lambda R}$ corresponds to the left and right boundaries of the chaotic region, which is shown at the gray area. (b) Width, $\Delta\Omega_{U,L} = \Omega_U - \Omega_L$ (dashed) and $\Delta\Omega_{1,0} = \Omega_1 - \Omega_0$ (dotted), of the region where the nontrivial dynamics occurs. The solid line shows the numerical estimation of the width of the CR.
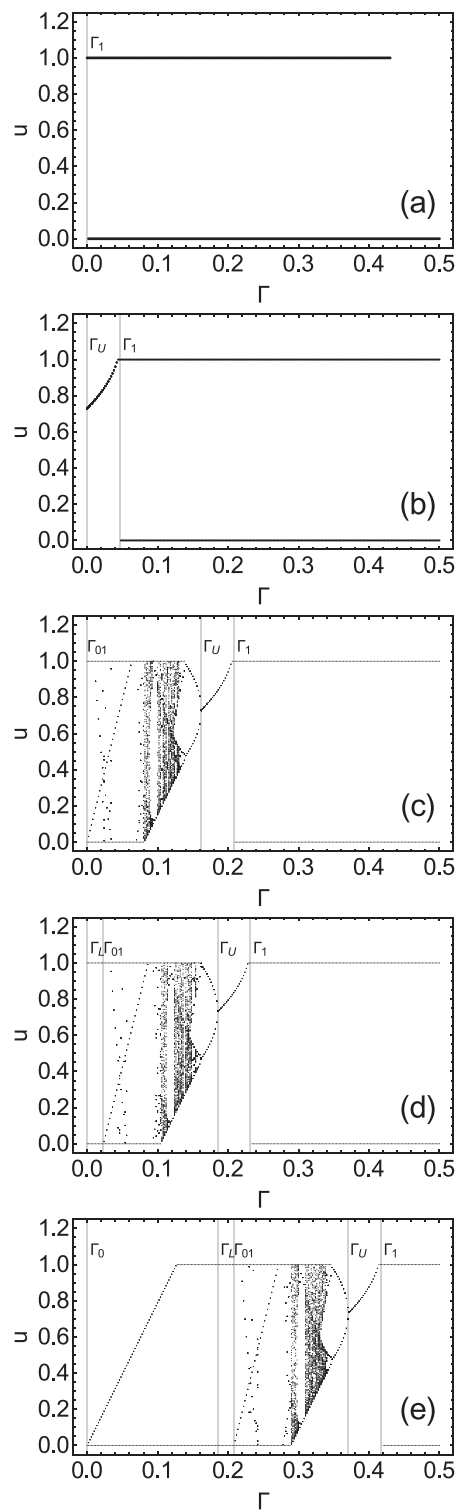

FIG. 4. Bifurcation diagrams u vs Γ for (a) $\Omega_1 = 1$, (b) $\Omega_U = 0.968354$, (c) $\Omega_{01} = 0.871795$, (d) $\Omega_L = 0.859551$, and (e) $\Omega_0 = 0.772727$. These values of Ω correspond to the vertical lines of Fig. 2.

time Γ. Hence the expected stop time at the bus stop has important consequences for the dynamics of the buses, which clearly make the system more complex. Let us note that even under the simplest possible conditions of a constant Γ, the bus dynamics are quite unpredictable for parameters that are relevant for city traffic. We can compare these results, by numerically estimating the size of the CR, defined as the range of values of Ω, for a given Γ, in which the Lyapunov exponent is positive. The width in Ω of this region is shown in Fig. 3(b) with the label $\Delta\lambda$ and plots it as solid line. This is interesting since one may argue that reducing Γ (making the stop time smaller) would reduce the average traveling time; however, according to our model this may not always be the case, as discussed below. This effect was first noticed in Ref. 11. Furthermore, the width of the nontrivial and CR change as we change Γ, for longer waiting times it gets thinner.

In Fig. 4, we show the bifurcation diagram of u as we vary Γ, for fixed Ω. As expected, the bifurcation diagram changes as we vary Ω. We use the values Ω_1, Ω_U, Ω_{01}, Ω_L, and Ω_0 that correspond to the vertical lines in Fig. 2(a). As can be seen in Fig. 4(a) at $\Gamma = 0$, the bus is synchronized with the traffic light, so we label this situation Γ_1. Increasing Γ moves the bus to a situation, where it passes many lights in green and then is forced stop for the duration of the red light. The number of consecutive green lights decreases as we increase Γ. For sufficiently large Γ (about 0.4), the bus

has to stop at every light. If we keep increasing Γ, we find that the bus enters a region of non-trivial dynamics and chaos. Decreasing Ω to Ω_U is equivalent to moving Γ_1 to the right. We see that to the left of Γ_1, we find a fixed point attractor for $u < 1$ that eventually goes through a bifurcation (at Γ_U); this is shown in Fig. 4(b). If we further decrease Ω to Ω_{01} and Ω_L of Fig. 2(a), we see that we go through a

situation in which the behavior of the bus appears to be chaotic; this is observed in Figs. 4(c) and 4(d), respectively. Finally, for Fig. 4(e), we have chosen the value of Ω_0 from Fig. 2(a). We can see there is a whole region of non-trivial dynamics and chaos, similar in appearance of chaos and non-trivial dynamics in Fig. 2. In Appendix B, we have shown how to derive equivalent definitions for Γ_1, Γ_U, Γ_L, Γ_{01}, and Γ_0. We note that as we increase Ω, the non-trivial and CR moves to lower values of Γ. This is expected, since it will take the bus a longer time to travel the distance between traffic lights, so that the bus will resonate at a decreasing Γ_1. Again, this may produce counter-intuitive results. Hence both the traffic light's frequency and the waiting time can move the system towards or away from regions, where chaos and non trivial dynamics may arise.

We now explore the effect that Ω and Γ have on the average traveling speed, which has already noted, may produce counter-intuitive results. First, we start from the initial condition $(0, 0)$ and evolve the system for 10^3 iterations to be certain it is in the attractor. Then, we take $N = 100$ iterations of the map $M(\tau, u)$ and calculate the normalized average traveling speed, $\bar{u}$, as

$$\bar{u} = \frac{N}{\tau_N},$$

where τ_N is the time it takes to travel the N traffic lights. This normalized average traveling speed is equal to one when the bus is able to cross every traffic light in green, i.e., it is in resonance at Ω_1.

Figure 5 shows $\bar{u}$ versus Ω for $\Gamma = 0$, 0.25, 0.50, and 0.75. Note that there are several resonant peaks (points where the average speed is maximum) for each value of Γ, hence reducing Γ (making the stop time smaller) does not necessarily reduce the average traveling time. We find the first resonant peak, for all values of Γ, lying over the line $\bar{u} = \Omega$. This is due to the scaling effect near resonance that these systems show.

As expected, as Γ grows, the maximum average normalized speed may go down and the minimum may be affected likewise. The maximum possible value for the normalized average speed is (at resonance)
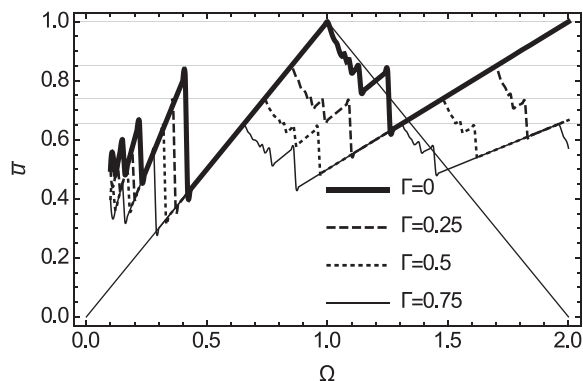
$$\bar{u}_{max} = \frac{t_{min}}{t_{min} + \Gamma T_c}. \tag{12}$$

The other effect that Γ has on the normalized average speed is to shift the resonance value of Ω. This has been shown numerically and we will discuss this effect more thoroughly in Section III, where we analyze the analytical expression for one of its resonant values. This is important mainly for two reasons: first, it is rather obvious that it will be very hard to control the stop time on a real bus system; second—as we have illustrated—this dynamical system displays chaos near the resonance value for Ω. Combining these two arguments tell us that a small change in Γ may be able to steer the system towards regions of chaotic and non trivial dynamics, even if the traffic lights have been previously synchronized.

It is clear that, from a controller point of view, one desires to keep the light frequency as close to, but below, Ω_1 as possible, because it diminishes the time spent between lights and fuel consumption (if the vehicle does not has to brake and accelerate at the lights). However, it is hard to predict in advance the waiting time Γ at each traffic light, and since the chaotic and nontrivial behaviors are so close to Ω_1, a small variation in Γ can make the dynamics unpredictable increasing both the traveling time and the bus' fuel consumption, as was already reported for a single car in Ref. 3. Therefore, for a varying loading passenger time Γ, it becomes difficult to simultaneously optimize the travel time and keep a well predicted schedule.

In Fig. 6, we show the results of calculating the maximum Lyapunov exponent of the system in the $A_+ - A_-$ diagram, for $\Gamma = 0$. In this figure, we color a point (A_+, A_-) if
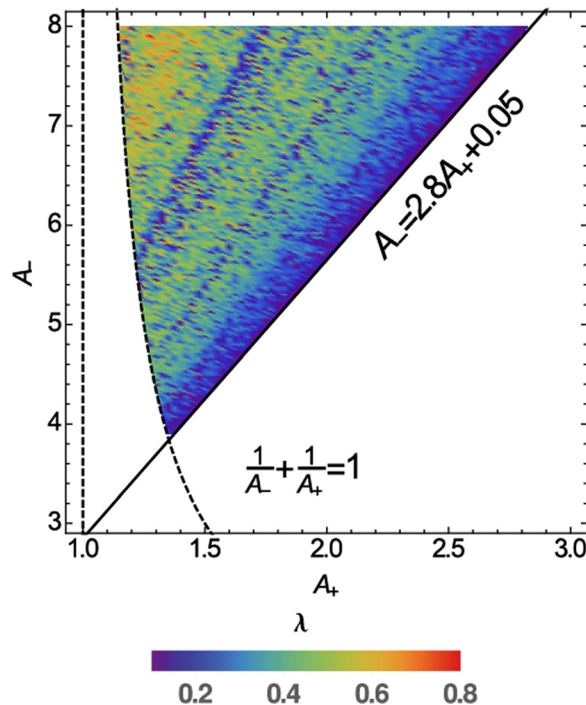

FIG. 5. Average normalized speed as a function of the light's frequency for: $\Gamma = 0$ (thick), $\Gamma = 0.25$ (dashed), $\Gamma = 0.5$ (dotted), $\Gamma = 0.75$ (thin). Peaks indicate light's frequencies for which the bus never has to break at a light, i.e., situations where the bus is completely synchronized with the green light. Horizontal lines given by Eq. (12).

FIG. 6. The $A_+ - A_-$ diagram for $\Gamma = 0$. A colored dot for a pair (A_+, A_-) marks a situation in which there exists a value of Ω with a Lyapunov exponent larger than 0.1, if the dot is white such value of Ω does not exists. The restriction left boundary $1 = A_+^{-1} + A_-^{-1}$ and a fitted lower boundary line (for $\Gamma = 0$) are shown.

we can find a value of $\Omega_L \leq \Omega \leq \Omega_U$ with a Lyapunov exponent greater than or equal to 0.1; we color it white if we cannot. We use the threshold value $\lambda_c = 0.1$ to make sure the system is chaotic, due to the numerical evaluations. We have fitted the lower boundary for the chaotic regime as

$$A_- \approx 2.8 A_+ + 0.04,$$

so that including the restriction given by Eq. (8), chaos does not appear unless $A_- > 4$. Furthermore, from the picture, and our fitted lower boundary, we can see that chaos is only possible if $a_- \gtrsim 3a_+$, and also that the width of the region of positive Lyapunov exponents gets larger as A_- grows.

In this section, we have shown several properties of our Bus model, mainly that it has rich dynamics in the form of non-trivial and chaotic behaviors, particularly for parameter values that are close to the expected values for city traffic.

III. ANALYTICAL RESULTS

We will now discuss some consequences of the analytical expressions derived in Appendix B, which define particular situations in the bifurcation diagram. We have used the following conventions as subscripts:

1 the bus resonate with the traffic lights crossing the lights at full speed. We refer to these situations as *resonant* values.

U the dynamics goes through its first period doubling bifurcation. We refer to these situations as *upper* values.

01 the point where we have an exact period 2 orbit with $u_0 = 0$ and $u_1 = 1$. We refer to these situations as *low* values.

L the point where the lower branch of the period 2 orbit hits the $u = 0$ threshold. We refer to these situations as *lower* values.

0 the bus is forced to stop completely at every traffic light. We refer to these situations as *stop* values.

These expressions, which are derived for the bifurcation diagrams with respect to A_+, A_-, Ω, and Γ, can be used to bound the different dynamical regions. Similar expressions were derived in Ref. 8 for a single car model. The procedure used to derive these relations is outlined in Appendix B.

As an example, Figs. 7(a) and 7(b) show bifurcation diagrams for the speed with A_+, A_-, as bifurcation parameters, respectively. We use the reference values, namely, $A_{+ref} = 1 \times L/v_{max}^2$, $A_{-ref} = 5 \times L/v_{max}^2$, $\Gamma = 0.5$, $\ell = 1/2$, and $\Omega = 0.71$, corresponding to the chaotic region in the bifurcation diagram for Ω, when not used as the bifurcation variable. Let us note that the restriction A_{+min} and A_{-min} imposed by Eq. (8) are represented by the vertical dashed line in Figures 7(a) and 7(b), respectively. We note that these bifurcations diagrams are consistent with the expectations provided by Figure 6, in that the chaotic region is lost as we increase A_+ with respect to A_-, and that we can obtain chaos as we increase A_- relative to A_+. This result may have important implications for the existence of chaos, and the viability of predictable bus routes in cities, specially in segregated lines, as it occurs naturally in many cities such as Santiago, Bogota, Curitiva, etc.
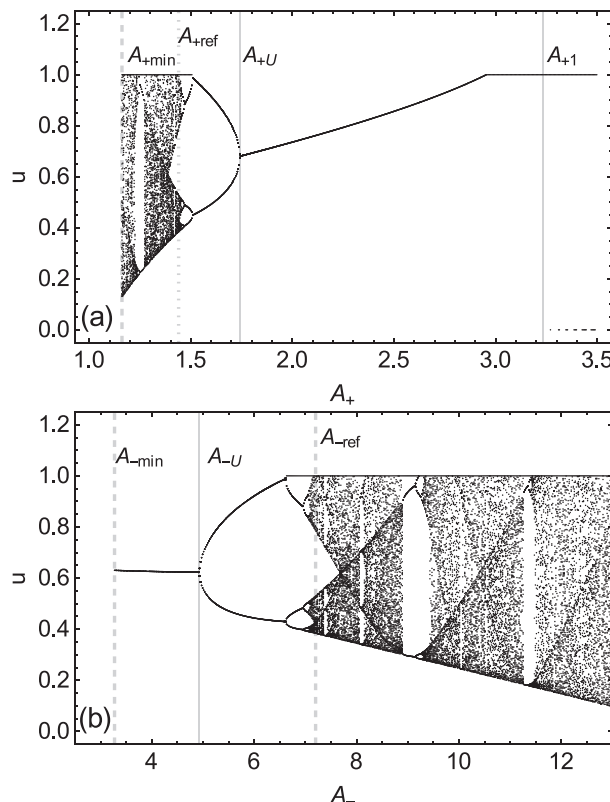

FIG. 7. Bifurcation diagrams observing the condition from Eq. (8) with (a) A_+ and (b) A_-. We use $A_{+ref} = 1 \times L/v_{max}^2$), $A_{-ref} = 5 \times L/v_{max}^2$, $\Gamma = 0.5$, $\ell = 1/2$, and $\Omega = 0.71$. The reference values A_{+ref} and A_{-ref} are shown as the vertical dotted lines.

Chaotic and non trivial dynamics regions may exist between the resonant and stop values for a given set of parameters. Looking at bifurcation diagrams, it is easy to see that for chaotic and non trivial dynamics to be present, we need

(1) $A_{+1} < A_{+U} < A_{+L}$, $A_{+01} < A_{+0}$;

(2) $A_{-1} < A_{-U} < A_{-L}$, $A_{-01} < A_{-0}$;

(3) $\Gamma_0 < \Gamma_L$, $\Gamma_{01} < \Gamma_U < \Gamma_1$; and

(4) $\Omega_0 < \Omega_L$, $\Omega_{01} < \Omega_U < \Omega_1$.

We now turn our attention to two special cases that will give us more intuition regarding the conditions for the existence of non-trivial and chaotic behaviors in the model. First, we look at what happens when we have $A_+ = A_- = A$, i.e., we force the vehicle to accelerate and brake at the same rate. In this case, we have

$$\Omega_1 = \frac{1+A}{1+A+A\Gamma},$$

$$\Omega_0 = \Omega_U = \Omega_L = \frac{1+A}{2+A+A\Gamma}.$$

Similar expressions for Γ are

$$\Gamma_1 = \frac{(1+A)(\Omega-1)}{A\Omega},$$

$$\Gamma_0 = \Gamma_U = \Gamma_L = \frac{1+A-(2+A)\Omega}{A\Omega}.$$

Hence, as we make the accelerating and braking capacities equal, the bifurcation point and the stop point become

the same; therefore, there is no period doubling and no road to chaos. Furthermore, the system's behavior becomes trivial in the sense that the transition from a non-resonant situation to a resonant one, i.e., the behavior experienced when changing Ω_0 toward Ω_1, is described by a continuous line, as can be seen in Fig. 8(a). The same can be said when the control parameter is Γ.

Let us now describe the dynamics when the braking capacity tends towards infinity, namely, $A_- \to \infty$. Taking this limit gives us

$$\Omega_1 = \Omega_U = 1,$$
$$\Omega_0 = \Omega_L = \frac{1 + 2A_+}{2 + 2A_+}.$$

Similar expressions for Γ are

$$\Gamma_1 = \Gamma_U = \frac{(1 + 2A_+)(\Omega - 1)}{2A_+\Omega},$$
$$\Gamma_0 = \Gamma_L = \frac{1 + 2A_+ - 2(1 + A_+)\Omega}{2A_+\Omega}.$$

It becomes clear that chaos and nontrivial dynamics cannot occur in the limit $A_- \to \infty$. Let's note that this is the limit that is used in a large number of cellular automaton models, in which the cars or buses are forced to have unrealistically strong braking capabilities to avoid collisions in the simulations (see, for example, Ref. 56). Hence, this restriction puts a str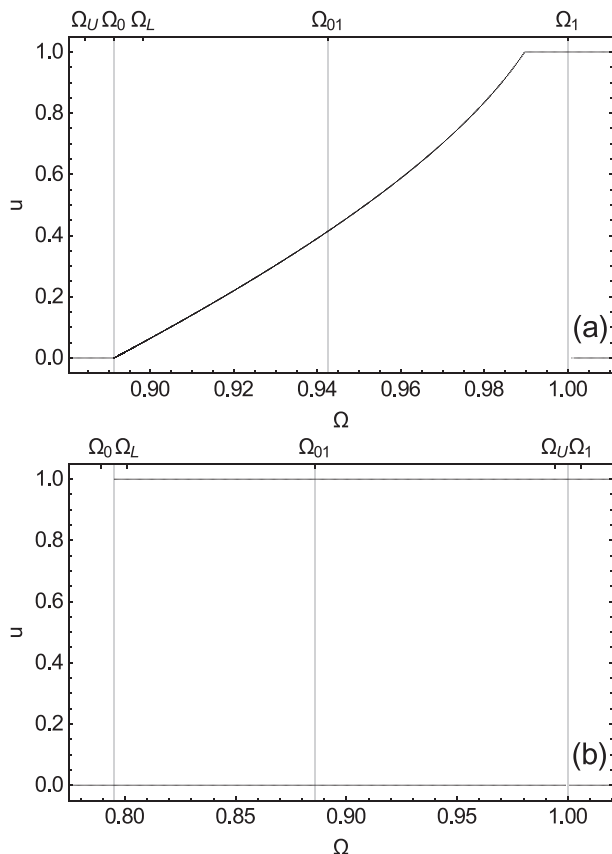ong constraint on the possibility to have a relevant part of the inherent complexity that bus systems, and city traffic, in general, should have, even in its simplest conceptual realizations. Furthermore, above we found that chaos appear for realistic accelerating and braking capacities of buses in cities, hence we expect that chaotic behavior and nontrivial dynamics are an inherent part of bus systems, and traffic in cities in general. Therefore, it is fundamental to consider the finite braking and accelerating of the buses if we pretend to understand the complexities of these transportation systems.

IV. CONCLUSIONS

Throughout this paper, we have analyzed a simple model of a bus that interacts with a sequence of traffic lights and stops in between them by using a 2D map of speed and time at the *n*-th light. The behavior of this *Simple bus model* was described in detail. The map was constructed in such a way that it gave us the speed and time at which the bus would cross the following lights given an initial speed and time. We discussed the overall behavior of the average speed as a function of the traffic light frequency (ω) and highlighted the scaling effect present.

One important conclusion is that this model displays non-trivial and chaotic behaviors for realistic city traffic parameters, and we characterized the occurrence of this behavior with the analytical results of Appendix B. These values are also used to set boundaries; in particular, we bound the chaotic region in the interval between the lower and upper frequency values. Conditions for chaos to be present where also analyzed. In particular, we found that for the non-trivial and chaotic behaviors to be present, we must have different braking and acceleration capacities (braking capacities must be larger) and finite values for both. It is interesting to note that our setting could be applicable to subway systems by getting rid of the traffic light, which is in part responsible for the chaotic behavior; therefore, the subway dynamics is more stochastic in nature defined by the time require to load and unload passengers. Here, we have considered the case of a single bus going through a sequence of traffic lights. As we allow the interaction with more buses, other interesting effects may come to play, such as synchronization or stochastic resonances,[21] produced by the interaction of the dynamics of passenger loading and unloading, with the traffic light sequence. Such problem will be addressed elsewhere.

APPENDIX A: CONSTRUCTION OF A 2D MAP FOR THE BUS MODEL: THE $M(t,v)$ MAP

The map below completely determines the bus dynamics between consecutive traffic lights, given a set of initial conditions. We use the sequence of events described in Fig. 1, but we do not assume the bus starts standing still at a red light.

FIG. 8. Bifurcation diagram for the case when (a) $A_+ = A_-$, and (b) $A_- \to \infty$. We use $\Gamma = 0$, $\ell = 1/2$, $A = A_+ = 5v_{max}^2/L$, and $A_- = 5v_{max}^2/L$.

Region 1: The bus crosses the n-th traffic light at position x_n, time t_n, with velocity v_n. It accelerates with a_+ until reaching velocity $v_{\max}$ at position $x_{c1} = x_n + (v_{\max}^2 - v_n^2)/2a_+$, at time $t_{c1} = t_n + (v_{\max} - v_n)/a_+$. Note that if $v_n = v_{\max}$, there is no need for this region.

Region 2: The bus moves at velocity $v_{\max}$, until braking in order to fully stop at $x = x_n + l$. Braking must occur at the decision position $x_{s1} = x_n + l - v_{\max}^2/2a_-$. The bus reaches the decision point at time $t_{s1} = t_{c1} + (x_{s1} - x_{c1})/v_{\max}$.

Region 3: The bus brakes with a_-. It reaches the bus stop at $x_n + l$ with speed 0, at time $t_l = t_{s1} + v_{\max}/a_-$.

Region 4: The bus loads and unloads passengers during a time γ.

Region 5: The bus accelerates with a_+. So it is like Region 1, but it now always starts from rest. At the end of the region, the bus is at position $x_{c2} = x_n + l + v_{\max}^2/2a_+$ with velocity $v_{c2} = v_{\max}$, at time $t_{c2} = t_l + \gamma + v_{\max}/a_+$.

Region 6: The bus moves at velocity $v_{\max}$ until it reaches a second decision point, where it checks the status of the upcoming traffic light. If red, it has to be able to fully stop at $x = x_n + L$. As in Region 2, the decision point is at $x_{s2} = x_n + L_n - v_{\max}^2/2a_-$, and it happens at time $t_{s2} = t_{c2} + (x_{s2} - x_{c2})/v_{\max}$. At this point, the bus has velocity $v_{s2} = v_{\max}$.

Region 7: The behavior depends on the light's status (i.e., red or green) at the decision time t_{s2}.

(A) If green, the bus does not brake. It reaches the next traffic light, which is at position $x_{n+1} = x_n + L$, with velocity $v_{n+1} = v_{\max}$ at time $t_{n+1} = t_{s2} + (v_{\max}^2/2a_-)/v_{\max} = t_{s2} + v_{\max}/2a_-$.

(B) If red, the bus starts braking. In order to decide what happens next, the time of the next green light must be calculated

$$t_g = \frac{2\pi}{\omega}\left(\left[\frac{\omega t_{s2} + \phi_n}{2\pi}\right] + 1\right) - \frac{\phi_n}{\omega},$$

where [] represents the integer part. This time must now be compared with the time, it will take the bus to fully stop, $t_t = t_{s2} + v_{\max}/a_-$.

(a) If $t_t \leq t_g$, then the bus fully stops, and stays at the traffic light at position $x_{n+1} = x_n + L_n$, with velocity $v_{n+1} = 0$, until the next green light at $t_{n+1} = t_g$.

(b) If $t_t > t_g$, then the light turns green, while the bus is braking (see **Region 8** in Fig. 1). This occurs at time t_g, at which the bus is at position $x_g = x_{s2} + v_{s2}(t_g - t_{s2}) - \frac{1}{2}a_-(t_g - t_{s2})^2$, with velocity $v_g = v_{s2} - a_-(t_g - t_{s2})$. Now, the bus starts accelerating again. Again there are two cases. In order to decide, the position $x_c = x_{s2} + (v_{\max}^2 - v_g^2)/2a_+$ at which the bus reaches velocity $v_{\max}$, must be compared with the position of the next light, $x_n + L$.

(i) If $x_c < x_n + L$, the bus reaches maximum velocity before reaching the light. Thus, it reaches the position x_c with velocity $v_{\max}$ at time $t_c = t_g + (v_{\max} - v_g)/a_+$. Then, it continues with velocity $v_{\max}$ until reaching the light at position $x_{n+1} = x_n + L$ with velocity $v_{n+1} = v_{\max}$, at time $t_{n+1} = t_c + (x_n + L - x_c)/v_{\max}$.

(ii) If $x_c > x_n + L$, the bus reaches the next light at position $x_{n+1} = x_n + L$ with non-maximum velocity $v_{n+1} = \sqrt{v_g^2 + 2a_+(x_n + L - x_g)}$, at time $t_{n+1} = t_g + (v_{n+1} - v_g)/a_+$.

These are the possible cases for the bus dynamics for the restrictions presented in the text. We have given the complete details for the construction of the map $M(t, v)$; such that we can find $(t_{n+1}, v_{n+1}) = M(t_n, v_n) \forall n \geq 1$ given (t_0, v_0).

APPENDIX B: PROCEDURES TO FIND RESONANT, STOP, AND CHAOS BOUNDING VALUES

We have already found critical values for Ω, namely

$$\Omega_1 = \frac{t_{\min}}{t_{\min} + \Gamma T_c}, \tag{B1}$$

$$\Omega_0 = \frac{t_{\min}}{T_c}\frac{1}{1 + \frac{3}{4}\left(\frac{1}{A_+} + \frac{1}{A_-}\right) + \Gamma}, \tag{B2}$$

$$\Omega_{01} = \frac{t_{\min}}{T_c}\frac{1}{1 + \left(\frac{1}{A_+} + \frac{1}{A_-}\right) + \Gamma}, \tag{B3}$$

where

$$\frac{t_{\min}}{T_c} = 1 + \left(\frac{1}{2A_-} + \frac{1}{2A_+}\right).$$

We will now derive the conditions for Ω_U and Ω_L. Ω_U is defined as the frequency in which we have the period doubling bifurcation. We will use the position and timing notation of Fig. 1. Hence, defining u_n as the normalized velocity at the nth traffic light and $u_{g,n}$ as the normalized velocity when the nth traffic light becomes green, we can compute the distance restriction for the second half of the distance between traffic lights before the nth traffic light, namely

$$\ell = \left[\frac{1}{2A_+}\right]_{n,5} + \left[\frac{1}{2} - \frac{1}{2A_+} - \frac{1}{2A_+}\right]_{n,6} + \left[\frac{1 - u_{g,n}}{2A_-}\right]_{n,7} + \left[\frac{u_n^1 - u_{g,n}}{2A_-}\right]_{n,8}.$$

Here, we have used $\ell = 1/2$. The first subscript of the brackets corresponds to the region before the given traffic light, and the second to the ranges of Fig. 1. These restrictions give

$$u_{n+1} = \begin{cases} 1 & \text{if } u_{g,n} = 1 \\ v_{g,n}\sqrt{1 + \frac{A_+}{A_-}} & \text{if } 0 < u_{g,n} < 1 \\ 0 & \text{if } u_{g,n} = 0. \end{cases} \tag{B4}$$

The first case ($u_{g,n} = 1$) takes place when the bus finds that the light is green at $x_{s2,n}$; therefore, it does not brake. The second case ($0 < u_{g,n} < 1$) takes place when the bus finds that the light is red at $x_{s2,n}$, but it changes to green (at $x_{g,n}$) before the bus reaches a full stop. The third case ($u_{g,n} = 0$) takes place when the bus finds that the light is red at $x_{s2,n}$ and it changes to green after the bus has stopped at x_{n+1}.

In terms of the time, we have that the normalized period T/T_c satisfies

$$\frac{T}{T_c} = \left[\frac{u_n - u_{g,n-1}}{A_+}\right]_{n-1,8} + \left[\frac{1 - u_n}{A_+}\right]_{n,1}$$
$$+ \left[\frac{1}{2} - \frac{1}{2A_-} - \frac{1 - u_n^2}{2A_+}\right]_{n,2} + \left[\frac{1}{A_-}\right]_{n,3}$$
$$+ \left[\Gamma \frac{t_{\min}}{T_c}\right]_{n,4} + \left[\frac{1}{A_+}\right]_{n,5}$$
$$+ \left[\frac{1}{2} - \frac{1}{2A_+} - \frac{1}{2A_-}\right]_{n,6} + \left[\frac{1 - u_{g,n}}{A_+}\right]_{n,7}. \quad (B5)$$

We can evaluate this restriction for the nth and $(n+1)$-th traffic light, using Eq. (B4), and $u_{g,n-1} = u_{g,n+1}$, to obtain $u_{g,n} \le u_{g,n+1}$. Then, using Eq. (B4), we can find the period-2 velocities at the traffic light $u_n \le u_{n+1}$, as a function of A_+, A_-, and Γ. To Find Ω_L, we follow the lower branch u_n until it hits the $v = 0$ threshold. Similarly, to find Ω_U we set $u_n = u_{n+1}$ in the period-2 orbit. The results are

$$\Omega_U = \frac{t_{\min}}{T_c} \frac{2A_-A_+(A_- + A_+)}{A_-(\Psi + \Phi) + 2A_+\Psi + 5A_+^2}, \quad (B6)$$

$$\Omega_L = \frac{t_{\min}}{T_c} \frac{A_-A_+(A_- + A_+)}{A_-\Psi + A_+\Phi + 3A_+^2}, \quad (B7)$$

where $\Psi = A_-(A_+(\Gamma + 1) + 1)$ and $\Phi = A_-A_+(\Gamma + 1)$.

It is interesting to note that these bounds in Ω are actually definitions that relate all four variables, Ω, Γ, A_-, and A_+; so that we can use the corresponding equation and solve for any of the variables in terms of the other three. For example, taking the expression for Ω_1, given by Eq. (B1), we can find

$$\Gamma_1 = \frac{(\Omega_c - 1)(A_- + A_+ + 2A_-A_+)}{2A_-A_+\Omega_c}, \quad (B8)$$

$$A_{+1} = \frac{A_-(\Omega_c - 1)}{(\Omega_c - 1)(1 + 2A_-) + 2A_-\Gamma\Omega_c}, \quad (B9)$$

$$A_{-1} = \frac{A_+(\Omega_c - 1)}{(\Omega_c - 1)(1 + 2A_+) + 2A_+\Gamma\Omega_c}, \quad (B10)$$

where $\Omega_c = \omega T_c$. The other bounds for A_+, A_-, Γ can be found directly from the respective Eqs. (B2), (B3), (B6), and (B7).

Note that all previous expressions for resonant, stop, upper, low, and lower values are independent of ℓ. This is expected since we are forcing the system to reach $u_c = 1$ before interaction with the light is possible.

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