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# On the structure of locally symmetric manifolds

ARIS DANIILIDIS, JÉRÔME MALICK, HRISTO SENDOV

April 3, 2014

**Abstract.** This paper studies structural properties of locally symmetric submanifolds. One of the main result states that a locally symmetric submanifold  $\mathcal{M}$  of  $\mathbf{R}^n$  admits a locally symmetric tangential parametrization in an appropriately reduced ambient space. This property has its own interest and is the key element to establish, in a follow-up paper [7], that the spectral set  $\lambda^{-1}(\mathcal{M}) := \{X \in \mathbf{S}^n : \lambda(X) \in \mathcal{M}\}$  consisting of all  $n \times n$  symmetric matrices having their eigenvalues on  $\mathcal{M}$ , is a smooth submanifold of the space of symmetric matrices  $\mathbf{S}^n$ . Here  $\lambda(X)$  is the  $n$ -dimensional ordered vector of the eigenvalues of  $X$ .

**Key words.** Locally symmetric manifold, spectral manifold, permutation, partition, symmetric matrix, eigenvalue.

**AMS Subject Classification.** *Primary* 15A18, 53B25 *Secondary* 47A75, 05A05.

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# 1 Introduction

Investigation of rotational manifolds is a classical subject in Mathematical Analysis. For example, given a graph of a smooth function  $f : \mathbf{R} \rightarrow \mathbf{R}$  one may easily come up with necessary and sufficient conditions so that when the graph of  $f$  is rotated around the  $y$ -axis, the resulting set is a smooth manifold. It is clear that this depends on the way the graph of  $f$  intersects the  $y$ -axis. For instance, if the graph of  $f$  does not touch the  $y$ -axis at all, then it is straightforward to see that the rotational set is a manifold. On the contrary, if the graph of  $f$  touches the  $y$ -axis, then a symmetry with respect to the  $y$ -axis is required, yielding the condition  $f'(0) = 0$ . In fact, provided  $f$  is smooth, it follows readily that only the portion of the graph of  $f$  that is in the non-negative orthant  $\mathbf{R}_+^2$  is important, in order to determine whether or not the rotational set is smooth. Guiding by the above intuition, our goal hereby is to investigate higher-dimensional rotational manifolds. In the sequel, we introduce the relevant notation and definitions.

Let  $\mathbf{R}_\geq^n$  stand for the closed convex cone of all vectors  $x \in \mathbf{R}^n$  with  $x_1 \geq x_2 \geq \dots \geq x_n$ . Denoting by  $\mathbf{S}^n$  the Euclidean space of  $n \times n$  symmetric matrices with inner product  $\langle X, Y \rangle = \text{tr}(XY)$ , we consider the *spectral mapping*  $\lambda$ , that is, a function from the space  $\mathbf{S}^n$  to  $\mathbf{R}_\geq^n$ , which associates to  $X \in \mathbf{S}^n$  the vector  $\lambda(X) = (\lambda_1(X), \dots, \lambda_n(X)) \in \mathbf{R}_\geq^n$  of its eigenvalues counted with multiplicities and ordered in a non-increasing way:

$$\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X).$$

Given a set  $\mathcal{M} \subset \mathbf{R}^n$  we are seeking for conditions on  $\mathcal{M}$  ensuring that the pre-image

$$\lambda^{-1}(\mathcal{M}) := \{X \in \mathbf{S}^n : \lambda(X) \in \mathcal{M}\}$$

is a smooth manifold. Naturally, we need to require that  $\mathcal{M} \cap \mathbf{R}_\geq^n \neq \emptyset$ , or equivalently, that the pre-image  $\lambda^{-1}(\mathcal{M})$  is empty. Furthermore, observe that

$$\lambda^{-1}(\mathcal{M}) = \bigcup_{x \in \mathcal{M}} \{U^\top (\text{Diag}(x))U : U \in \mathbf{O}^n\},$$

where  $\mathbf{O}^n$  is the group of  $n \times n$  orthogonal matrices and  $\text{Diag}(x)$  stands for the diagonal matrix with vector  $x$  on the diagonal. It follows that  $\lambda^{-1}(\mathcal{M})$  is invariant under the action of the orthogonal group  $\mathbf{O}^n$ . Since the norm in  $\mathbf{S}^n$  is orthogonally invariant, we have

$$\|U^\top XU - \alpha I_n\| = \|X - \alpha I_n\|$$

for any real number  $\alpha \in \mathbf{R}$  and any  $U \in \mathbf{O}^n$  where  $I_n$  denotes the identity  $n \times n$  matrix. In this sense,  $\lambda^{-1}(\mathcal{M})$  can be seen as a rotational set. In the sequel we shall refer to it as *spectral set*, in accordance with the terminology of [2, Section 5.2] as well as of the previous works [5], [10] and [11].

In a long (unpublished) note [6], we proved that if  $\mathcal{M}$  is a locally symmetric  $C^k$  submanifold of  $\mathbf{R}_\geq^n$  of dimension  $d$ , then the spectral set  $\lambda^{-1}(\mathcal{M})$  is a  $C^k$  submanifold of  $\mathbf{S}^n$ . We refer to Definition 3.6 for the terminology of locally symmetric set. Here  $k$  may take any value in the set  $\{2, 3, \dots, \infty, \omega\}$  ( $C^\omega$  stands for real-analytic). The proof turns out to be technical and almost impossible to follow. Simplifying and clarifying that proof, as well as providing the required intuition for the comprehension of the result is now carried into two parts.

The first part, represented by the current paper, deals with a structural property of  $\mathcal{M}$  stemming from the fact that it is a locally symmetric manifold. This property, established in Subsection 5.2 will be exploited in the subsequent paper [7], in order to establish that the spectral set  $\lambda^{-1}(\mathcal{M})$  is a  $C^k$  submanifold of  $\mathbf{S}^n$ , for the case  $k \in \{2, \dots, \infty, \omega\}$ . Moreover, an explicit formula for

the dimension of  $\lambda^{-1}(\mathcal{M})$  in terms of  $d$  and the characteristic partition of  $\mathcal{M}$  (see Definition 4.3) naturally arises.

Let us finally mention that the particular case of the lift  $\lambda^{-1}(\mathcal{M})$  of a locally symmetric  $C^\infty$  manifold  $\mathcal{M}$  is recovered in a recent work [4] through an indirect technique based on metric projections. This technique though, does not provide any information on the dimension of the spectral manifold  $\lambda^{-1}(\mathcal{M})$ , unless it is combined afterward with the approach adopted in the current work.

## 2 Preliminaries and notation

This section gathers several basic results about permutations that are used extensively later. In particular, after defining order relations on the group of permutations in Subsection 2.1 and the associated stratification of  $\mathbf{R}^n$  in Subsection 2.2, we introduce the subgroup of permutations that preserve balls centered at a given point.

### 2.1 Permutations and partitions

A partition  $P$  of a finite set  $N$  is a collection of non-empty, pairwise disjoint subsets of  $N$  whose union is  $N$ . The elements of a partition are sometimes called *blocks*. The partition  $\{\{i\} : i \in N\}$  is denoted by  $\text{id}_N$ . The set of all partitions of  $N$  is denoted by  $\Pi_N$ . The symbol  $\mathbf{R}^N$  denotes the set of all functions from  $N$  to  $\mathbf{R}$ . Define the set  $\mathbb{N}_n := \{1, \dots, n\}$ . When  $N = \mathbb{N}_n$ , we simply write  $\Pi_n$ ,  $\text{id}_n$ , and  $\mathbf{R}^n$ .

**Definition 2.1** (Defining an order on the partitions). Given two partitions  $P$  and  $P'$  of  $\mathbb{N}_n$  we say that  $P'$  is a *refinement* of  $P$ , written  $P \preceq P'$ , if every set in  $P$  is a (disjoint) union of sets from  $P'$ . We say that  $P'$  is a *strict refinement* of  $P$ , written  $P < P'$ , if  $P'$  is a refinement of  $P$  and a set in  $P$  is a (disjoint) union of at least two sets from  $P'$ .

Clearly, if  $P \preceq P'$  and  $P' \preceq P$ , then  $P = P'$ . Observe that this partial order is a lattice. For any partitions  $P$  and  $P'$  denote by  $P \wedge P'$  the *infimum* and by  $P \vee P'$  the *supremum* of  $P$  and  $P'$ .

**Definition 2.2** (Block-size type of a partition). Two partitions  $P$  and  $P'$  of  $\mathbb{N}_n$  are said to be of the same *block-size type*, whenever the set of cardinalities, counting repetitions, of the sets in the partitions are in a one-to-one correspondence.

Notice that if  $P$  and  $P'$  are of the same block-size type, then they are either equal or non-comparable. The following simple examples illustrate the notions introduced.

- Example 2.3.** (i) The partitions of  $\mathbb{N}_3$  that are larger than or equal to  $\{\{1, 2, 3\}\}$  are  $\{\{1, 2, 3\}\}$ ,  $\{\{1, 2\}, \{3\}\}$ ,  $\{\{1, 3\}, \{2\}\}$ ,  $\{\{2, 3\}, \{1\}\}$ , and  $\{\{1\}, \{2\}, \{3\}\}$ .
- (ii) The partitions  $\{\{1, 2, 3\}, \{4\}, \{5, 6\}, \{7\}, \{8, 9\}\}$  and  $\{\{1\}, \{2\}, \{3, 6\}, \{4, 8\}, \{5, 7, 9\}\}$  of  $\mathbb{N}_9$  have the same block-size type. They are not comparable.
- (iii) The minimum element of  $\mathbb{N}_n$  under the order relation is the partition  $\{\mathbb{N}_n\}$ . The maximum element of  $\mathbb{N}_n$  under the order relation is the partition  $\text{id}_n$ . ■

In Subsection 3.3 we will introduce a subtle refinement of the partial order relation, which will be of crucial importance for the development.

Denote by  $\Sigma^n$  the group of permutations over  $\mathbb{N}_n := \{1, \dots, n\}$ . This group has a natural action on  $\mathbf{R}^n$  defined for  $x = (x_1, \dots, x_n)$  by

$$\sigma x := (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}). \quad (2.1)$$

The group  $\Sigma^n$  has a natural action on  $\Pi_n$ . For a partition  $P = \{I_1, \dots, I_m\}$  and  $\sigma \in \Sigma^n$  define the partition

$$\sigma P := \{\{\sigma(i) : i \in I_k\} : k = 1, \dots, m\}. \quad (2.2)$$

A cycle of length  $k \in \mathbb{N}_n$  of a permutation  $\sigma \in \Sigma^n$  is a collection of  $k$  distinct elements  $i_1, \dots, i_k$  in  $\mathbb{N}_n$  such that  $\sigma(i_j) = i_{j+1 \pmod{k}}$ . We represent such a cycle by  $(i_1, \dots, i_k)$ . It is convenient to think of a cycle  $(i_1, \dots, i_k)$  of length  $k$  as a permutation on  $\mathbb{N}_n$  fixing any other element of  $\mathbb{N}_n$ . It is well-known that every permutation on  $\mathbb{N}_n$  has a cyclic decomposition: that is, every permutation  $\sigma \in \Sigma^n$  can be represented (in a unique way up to reordering) as a composition of disjoint cycles

$$\sigma = \sigma_1 \circ \dots \circ \sigma_m, \quad \text{where the } \sigma_i \text{'s are cycles.}$$

Every permutation  $\sigma \in \Sigma^n$  induces a partition  $P_\sigma$  on  $\mathbb{N}_n$ . Fix the order of the cycles in the cycle decomposition of  $\sigma$  and let  $I_k$  be the set of elements in the  $k$ -th cycle, for  $k = 1, \dots, m$ . The obtained correspondence from the set of permutations  $\Sigma^n$  to the set of partitions of  $\mathbb{N}_n$  is "onto" (but not one-to-one). Analogously, the partition induced by  $x \in \mathbf{R}^n$ , denoted by  $P_x$ , is defined by the indexes of the equal coordinates of  $x$ . More precisely, for  $i, j \in \mathbb{N}_n$  we have:

$$i, j \text{ are in the same subset of } P_x \iff x_i = x_j. \quad (2.3)$$

Notice that  $\sigma P_x = P_x$  if and only if  $\sigma x = x$ . For a vector  $x \in \mathbf{R}^n$  and a partition  $P \in \Pi_n$  define the subgroups of permutations

$$\begin{aligned} \Sigma_P^n &:= \{\sigma \in \Sigma^n : \sigma P = P\}, \\ \Sigma_x^n &:= \{\sigma \in \Sigma^n : \sigma x = x\}. \end{aligned}$$

It is easy to see that we have the following relationships:

(P1) For any  $x \in \mathbf{R}^n$ , we have  $\Sigma_x^n = \Sigma_{P_x}^n$ ;

(P2)  $\sigma \in \Sigma_P^n$  if and only if  $P_\sigma \succeq P$ ;

(P3)  $\sigma \in \Sigma_x^n$  if and only if  $P_\sigma \succeq P_x$ .

Notice that if  $P = \{I_1, \dots, I_m\}$ , then  $|\Sigma_P^n| = |I_1|! \cdots |I_m|!$ .

## 2.2 Stratification induced by the permutation group

In this section, we introduce a stratification of  $\mathbf{R}^n$  associated with the set of partitions of  $\mathbb{N}_n$ . Given a partition  $P$  of  $\mathbb{N}_n$ , define the subset  $\Delta_P$  of  $\mathbf{R}^n$  by

$$\Delta_P := \{x \in \mathbf{R}^n : P_x = P\}. \quad (2.4)$$

In other words, if  $P = \{I_1, \dots, I_m\}$ , we have the representation

$$\Delta_P = \{x \in \mathbf{R}^n : x_i = x_j \iff \exists k \in \mathbb{N}_m \text{ with } i, j \in I_k\}.$$

Obviously  $\Delta_P$  is an affine manifold, not connected in general. Note also that its orthogonal and bi-orthogonal spaces have the following expressions, respectively,

$$\Delta_P^\perp = \left\{ x \in \mathbf{R}^n : \sum_{j \in I_i} x_j = 0, \text{ for } i \in \mathbb{N}_m \right\}, \quad (2.5)$$

$$\Delta_P^{\perp\perp} = \{x \in \mathbf{R}^n : x_i = x_j \text{ for any } i, j \in I_k, k \in \mathbb{N}_m\}. \quad (2.6)$$

Note that  $\Delta_P^{\perp\perp} = \overline{\Delta_P}$ , where the latter set is the closure of  $\Delta_P$ . Thus,  $\Delta_P^{\perp}$  is a vector space of dimension  $n - m$  while  $\Delta_P^{\perp\perp}$  is a vector space of dimension  $m$ . For example,  $\Delta_{\text{id}_n}^{\perp} = \{0\}$  and  $\Delta_{\text{id}_n}^{\perp\perp} = \mathbf{R}^n$ . We show now, among other things, that  $\{\Delta_P : P \in \Pi_n\}$  is a *stratification* of  $\mathbf{R}^n$ , that is, a collection of disjoint smooth submanifolds of  $\mathbf{R}^n$  with union  $\mathbf{R}^n$  that fit together in a regular way. In this case, the submanifolds in the stratification are affine.

**Proposition 2.4** (Properties of  $\Delta_P$ ). *Let  $x \in \mathbf{R}^n$ , let  $\sigma \in \Sigma^n$ , and let  $P, P' \in \Pi_n$ . Then,*

- (1)  $x \in \Delta_P^{\perp\perp}$  if and only if  $P_x \preceq P$ ;
- (2)  $P' \preceq P$  if and only if  $\Delta_{P'} \subset \Delta_P^{\perp\perp}$ ;
- (3) The sets  $\Delta_P$  and  $\Delta_{P'}$  are disjoint for  $P \neq P'$ ;
- (4)  $\Delta_P^{\perp\perp} = \bigcup_{P' \preceq P} \Delta_{P'}$ ;
- (5)  $\Delta_P^{\perp\perp} \cap \Delta_{P'}^{\perp\perp} = \Delta_{P \wedge P'}^{\perp\perp}$ ;
- (6)  $\sigma \Delta_P = \Delta_{\sigma P}$ .

**Proof.** Assertions (1), (2), (3) and (4) are straightforward. To show assertion (5), let first  $x \in \Delta_P^{\perp\perp} \cap \Delta_{P'}^{\perp\perp}$ . Then, in view of (4), there exist  $P_1 \preceq P$  and  $P_2 \preceq P'$  such that  $x \in \Delta_{P_1} \cap \Delta_{P_2}$ . Thus, by (3),  $P_1 = P_2$  and by (2) they are both smaller than or equal to  $P \wedge P'$ . Thus,  $x \in \Delta_{P \wedge P'}^{\perp\perp}$  showing that  $\Delta_P^{\perp\perp} \cap \Delta_{P'}^{\perp\perp} \subseteq \Delta_{P \wedge P'}^{\perp\perp}$ . Let now  $x \in \Delta_{P \wedge P'}^{\perp\perp}$ . Then, for some  $P_1 \preceq P \wedge P'$  we have  $x \in \Delta_{P_1}$ . Since  $P_1 \preceq P$  and  $P_1 \preceq P'$ , the inverse inclusion follows from (4).

To prove (6) observe that  $x \in \sigma \Delta_P$  if and only if  $\sigma^{-1}x \in \Delta_P$ . The latter happens if and only if for all  $i, j \in \mathbb{N}_n$  one has  $(\sigma^{-1}x)_i = (\sigma^{-1}x)_j$  precisely when  $i, j$  belong to the same block of  $P$ . By (2.1), this is equivalent to  $x_{\sigma(i)} = x_{\sigma(j)}$  precisely when  $i, j$  belong to the same block of  $P$ . The latter is equivalent to  $x \in \Delta_{\sigma P}$ .  $\blacksquare$

In particular, it follows from Proposition 2.4 that for every  $P, P' \in \Pi_n$ , if  $\Delta_{P'} \cap \overline{\Delta_P} \neq \emptyset$ , then  $P' \preceq P$  and either  $\Delta_{P'} = \Delta_P$  or  $\Delta_{P'} \subset \overline{\Delta_P} \setminus \Delta_P$ . This yields the following result.

**Corollary 2.5** (Stratification). *The collection  $\{\Delta_P : P \in \Pi_n\}$  is an affine stratification of  $\mathbf{R}^n$ .*

**Example 2.6** (Stratification in  $\mathbf{R}^2$  and  $\mathbf{R}^3$ ). When  $n = 2$ , there are two partitions of  $\mathbb{N}_2$ :  $P_1 = \{\{1, 2\}\}$  and  $P_2 = \{\{1\}, \{2\}\}$ . Thus,  $\mathbf{R}^2 = \Delta_{P_1} \cup \Delta_{P_2}$  with the union being disjoint, where  $\Delta_{P_1} = \{(x, y) \in \mathbf{R}^2 : x = y\}$  and  $\Delta_{P_2} = \{(x, y) \in \mathbf{R}^2 : x \neq y\}$ .

When  $n = 3$ , there are five partitions of  $\mathbb{N}_3$ :  $P_1 = \{\{1, 2, 3\}\}$ ,  $P_2 = \{\{1, 2\}, \{3\}\}$ ,  $P_3 = \{\{1\}, \{2, 3\}\}$ ,  $P_4 = \{\{1, 3\}, \{2\}\}$  and  $P_5 = \{\{1\}, \{2\}, \{3\}\}$ . Thus,  $\mathbf{R}^3 = \bigcup_{k=1}^5 \Delta_{P_k}$  with the union being disjoint, where  $\Delta_{P_1} = \{(x, y, z) \in \mathbf{R}^3 : x = y = z\}$ ,  $\Delta_{P_2} = \{(x, y, z) \in \mathbf{R}^3 : x = y \neq z\}$ ,  $\Delta_{P_3} = \{(x, y, z) \in \mathbf{R}^3 : x \neq y = z\}$ ,  $\Delta_{P_4} = \{(x, y, z) \in \mathbf{R}^3 : x = z \neq y\}$ , and  $\Delta_{P_5} = \{(x, y, z) \in \mathbf{R}^3 : x \neq y \neq z \neq x\}$ . This case is illustrated on Figure 1.

The next lemma shows that the permutations in  $\Sigma_x^n$  are those preserving balls centered at  $x$ . In words, if  $\sigma \in \Sigma_x^n$ , then  $\sigma$  preserves all the balls centered at  $x$ ; and this property characterizes those permutations. Denote by  $B(x, \delta)$  the open ball centered at  $x$  with radius  $\delta$ .

**Lemma 2.7** (Local invariance and ball preservation). *For any  $x \in \mathbf{R}^n$ , we have the dichotomy:*

- (i)  $\sigma \in \Sigma_x^n \iff \forall \delta > 0 : \sigma B(x, \delta) = B(x, \delta)$ ;
- (ii)  $\sigma \notin \Sigma_x^n \iff \exists \delta > 0 : \sigma B(x, \delta) \cap B(x, \delta) = \emptyset$ .

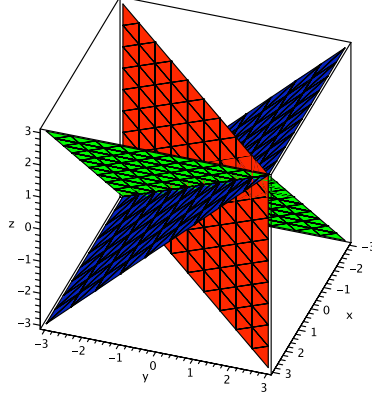


Figure 1: The stratification of  $\mathbf{R}^3$

**Proof.** Implication  $\Leftarrow$  of (i) follows by taking  $\delta \rightarrow 0$ . Implication  $\Rightarrow$  of (i) comes from the symmetry of the norm which for any  $y \in \mathbf{R}^n$  yields  $\|y - x\| = \|\sigma y - \sigma x\| = \|\sigma y - x\|$ . To prove (ii), just consider  $\delta = \|x - \sigma x\|/3$  and note that  $\delta > 0$  whenever  $\sigma \notin \Sigma_x^n$ . Now, for any  $y \in B(x, \delta)$ , we have

$$\|x - \sigma y\| \geq \| \|x - \sigma x\| - \|\sigma x - \sigma y\| \| = \|x - \sigma x\| - \|x - y\| \geq 2\delta.$$

This concludes the proof.  $\blacksquare$

The next corollary goes a bit further by saying that the preservation of only one ball, with a sufficiently small radius, also characterizes  $\Sigma_x^n$ .

**Corollary 2.8** (Invariance of one ball). *For every  $x \in \mathbf{R}^n$  there exists  $r > 0$  such that:*

$$\sigma \in \Sigma_x^n \iff \sigma B(x, r) = B(x, r) \quad \text{and} \quad \sigma \notin \Sigma_x^n \iff \sigma B(x, r) \cap B(x, r) = \emptyset.$$

**Proof.** For any  $\sigma \notin \Sigma_x^n$ , the proof of Lemma 2.7 part (ii) gives a radius, that we denote here by  $\delta_\sigma > 0$ , such that  $\sigma B(x, \delta_\sigma) \cap B(x, \delta_\sigma) = \emptyset$ . Note also that for all  $\delta \leq \delta_\sigma$ , there still holds  $\sigma B(x, \delta) \cap B(x, \delta) = \emptyset$ . Set now

$$r = \min \{ \delta_\sigma : \sigma \notin \Sigma_x^n \} > 0.$$

Thus,  $\sigma B(x, r) \cap B(x, r) = \emptyset$  for all  $\sigma \notin \Sigma_x^n$ . This yields that if a permutation preserves the ball  $B(x, r)$ , then it lies in  $\Sigma_x^n$ . The converse comes from Lemma 2.7.  $\blacksquare$

It is useful to express the orthogonal projection of a point onto a given stratum. One can state the following result.

**Lemma 2.9** (Projection onto  $\Delta_P^{\perp\perp}$ ). *For any  $P \in \Pi_n$  and  $x \in \mathbf{R}^n$  we have*

$$\text{Proj}_{\Delta_P^{\perp\perp}}(x) = \frac{1}{|\Sigma_P^n|} \sum_{\sigma \in \Sigma_P^n} \sigma x. \quad (2.7)$$

**Proof.** Letting  $P = \{I_1, \dots, I_m\}$ , it is easy to see that

$$y = \text{Proj}_{\Delta_P^{\perp\perp}}(x) \iff y_\ell = \frac{1}{|I_i|} \sum_{j \in I_i} x_j \quad \text{for all } \ell \in I_i \text{ with } i \in \mathbb{N}_m. \quad (2.8)$$

Note in passing that if the numbers  $(\sum_{j \in I_i} x_j)/|I_i|$ , for  $i \in \mathbb{N}_m$ , are distinct, then this equality also provides the projection of  $x$  onto the (non-closed) set  $\Delta_P$ .

For every  $j, \ell \in I_i$ , the coordinate  $x_j$  is repeated  $|\Sigma_P^n|/|I_i|$  times in the sum  $(\sum_{\sigma \in \Sigma_P^n} \sigma x)_\ell$ . Equation (2.8) yields the result.  $\blacksquare$

**Corollary 2.10** (Active strata around  $x$ ). *For any  $x \in \mathbf{R}^n$  there is a  $\delta > 0$  such that the ball  $B(x, \delta)$  intersects only strata  $\Delta_P$  with  $P \succeq P_x$ .*

**Proof.** Formula (2.8), applied to  $P_x$ , shows that if  $P \prec P_x$  or if  $P$  is not comparable to  $P_x$ , then  $\text{Proj}_{\Delta_P^\perp}(x) \neq x$ . Take  $\delta$  to be smaller than the distance from  $x$  to  $\Delta_P^\perp$  for any such  $P$ . This ensures that  $B(x, \delta)$  intersects only strata  $\Delta_P$  with  $P \succeq P_x$ .  $\blacksquare$

### 3 Locally symmetric manifolds

In this section we introduce and study the notion of locally symmetric manifolds. The elementary properties of the tangent and the normal spaces of such manifolds are discussed in Subsection 3.2. In Subsection 3.3, we specify the location of the manifold with respect to the stratification, see Corollary 3.17.

#### 3.1 Locally symmetric functions and manifolds

Let us start by refining the notion of symmetric function employed in previous works (see [10], [5] for example).

**Definition 3.1** (Locally symmetric function). A function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  is called *locally symmetric* around a point  $x \in \mathbf{R}^n$  if for any  $y$  close to  $x$

$$f(\sigma y) = f(y) \quad \text{for all } \sigma \in \Sigma_x^n.$$

Naturally, a vector-valued function  $g: \mathbf{R}^n \rightarrow \mathbf{R}^p$  is called *locally symmetric* around  $x$  if each component function  $g_i: \mathbf{R}^n \rightarrow \mathbf{R}$  is locally symmetric, for  $i = 1, \dots, p$ .

In view of Lemma 2.7 and its corollary, locally symmetric functions are those which are symmetric on an open ball centered at  $x$ , with respect to all permutations that preserve  $x$ . Before giving the definition of a locally symmetric manifold, let us recall the definition of a submanifold.

**Definition 3.2** (Submanifold of  $\mathbf{R}^n$ ). A nonempty set  $\mathcal{M} \subset \mathbf{R}^n$  is a  $C^k$  *submanifold* of dimension  $d$  (with  $d \in \{0, \dots, n\}$  and  $k \in \mathbb{N} \cup \{\omega\}$ ) if for every  $x \in \mathcal{M}$ , there is a neighborhood  $U \subset \mathbf{R}^n$  of  $x$  and  $C^k$  function  $\varphi: U \rightarrow \mathbf{R}^{n-d}$  with Jacobian matrix  $J\varphi(x)$  of full rank, and such that  $y \in \mathcal{M} \cap U \Leftrightarrow \varphi(y) = 0$ . The map  $\varphi$  is called *local equation* of  $\mathcal{M}$  around  $x$ .

**Remark 3.3** (Open subset). Every (nonempty) open subset of  $\mathbf{R}^n$  is trivially a  $C^k$ -submanifold of  $\mathbf{R}^n$  (for any  $k$ ) of dimension  $d = n$ .

**Definition 3.4** (Locally symmetric sets). Let  $\mathcal{M}$  be a subset of  $\mathbf{R}^n$ . The set  $\mathcal{M}$  is called *strongly locally symmetric* if  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$  and

$$\sigma \mathcal{M} = \mathcal{M}, \quad \text{for all } x \in \mathcal{M} \text{ and all } \sigma \in \Sigma_x^n.$$



The set  $\mathcal{M}$  is called *locally symmetric* if for every  $x \in \mathcal{M}$  there is a  $\delta > 0$  such that  $\mathcal{M} \cap B(x, \delta)$  is a strongly locally symmetric set. In other words, for every  $x \in \mathcal{M}$  there is a  $\delta > 0$  such that

$$\sigma(\mathcal{M} \cap B(x, \delta)) = \mathcal{M} \cap B(x, \delta) \quad \text{for all } y \in \mathcal{M} \cap B(x, \delta) \text{ and all } \sigma \in \Sigma_y^n. \quad (3.1)$$

In this case, observe that  $\mathcal{M} \cap B(x, \rho)$  for  $\rho \leq \delta$  is a strongly locally symmetric set as well (as an easy consequence of Lemma 2.7).

**Example 3.5** (Trivial examples). Obviously the whole space  $\mathbf{R}^n$  is (strongly locally) symmetric. It is also easily seen from the definition that any stratum  $\Delta_P$  is a strongly locally symmetric affine manifold. If  $x \in \Delta_P$  and the ball  $B(x, \delta)$  is small enough so that it intersects only strata  $\Delta_{P'}$  with  $P' \succeq P$ , then  $B(x, \delta)$  is strongly locally symmetric. ■

**Definition 3.6** (Locally symmetric manifold). A subset  $\mathcal{M}$  of  $\mathbf{R}^n$  is said to be a (*strongly*) *locally symmetric manifold* if it is both a connected submanifold of  $\mathbf{R}^n$  without boundary and a (strongly) locally symmetric set.

### 3.2 Structure of tangent and normal space

Denote by  $\mathbf{R}_{\geq}^n$  the closed convex cone of all vectors  $x \in \mathbf{R}^n$  with  $x_1 \geq x_2 \geq \dots \geq x_n$ . For the rest of this work, we assume the following:

*$\mathcal{M}$  is a locally symmetric,  $C^k$  submanifold of  $\mathbf{R}^n$  of dimension  $d$  such that  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$ ,*

where  $k \in \{2, \dots, \infty, \omega\}$  ( $C^\omega$  stands for real analytic).

The condition  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$  may appear weird at a first glance. But recall that our long term goal is to show that the spectral (rotational) set  $\lambda^{-1}(\mathcal{M})$  is a manifold. Since the coordinates of the eigenvalue map  $\lambda$  are ordered in a non-increasing way, the set  $\lambda^{-1}(\mathcal{M})$  is empty unless  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$ . This condition also has implications for the characteristic partition, defined in Section 4, see Theorem 4.8.

Let us now denote by  $T_{\mathcal{M}}(x)$  and  $N_{\mathcal{M}}(x)$  the tangent and normal space at  $x \in \mathcal{M}$ , respectively. In this subsection, we derive several natural properties for these two spaces, stemming from the symmetry of  $\mathcal{M}$ . The next lemma ensures that the tangent and normal spaces at  $\bar{x} \in \mathcal{M}$  inherit the local symmetry of  $\mathcal{M}$ .

**Lemma 3.7** (Local symmetry of  $T_{\mathcal{M}}(x)$ ,  $N_{\mathcal{M}}(x)$ ). *If  $x \in \mathcal{M}$ , then*

- (i)  $\sigma T_{\mathcal{M}}(x) = T_{\mathcal{M}}(x)$  for all  $\sigma \in \Sigma_x^n$ ;
- (ii)  $\sigma N_{\mathcal{M}}(x) = N_{\mathcal{M}}(x)$  for all  $\sigma \in \Sigma_x^n$ .

**Proof.** Assertion (i) follows directly from the definitions, since the elements of  $T_{\mathcal{M}}(x)$  can be viewed as the differentials at  $x$  of smooth paths on  $\mathcal{M}$  through  $x$ . Assertion (ii) stems from the fact that  $\Sigma_x^n$  is a group, as follows: for any  $w \in T_{\mathcal{M}}(x)$ ,  $v \in N_{\mathcal{M}}(x)$ , and  $\sigma \in \Sigma_x^n$  we have  $\sigma^{-1}w \in T_{\mathcal{M}}(x)$  and  $\langle \sigma v, w \rangle = \langle v, \sigma^{-1}w \rangle = 0$ , showing that  $\sigma v \in [T_{\mathcal{M}}(x)]^\perp = N_{\mathcal{M}}(x)$ . ■

Given a set  $S \subset \mathbf{R}^n$ , denote by  $\text{dist}_S(x) := \inf_{s \in S} \|x - s\|$  the distance of  $x \in \mathbf{R}^n$  to  $S$ .

**Proposition 3.8** (Local invariance of the distance). *If  $x \in \mathcal{M}$ , then*

$$\text{dist}_{(x+T_{\mathcal{M}}(x))}(y) = \text{dist}_{(x+T_{\mathcal{M}}(x))}(\sigma y) \quad \text{for any } y \in \mathbf{R}^n \text{ and } \sigma \in \Sigma_x^n.$$

**Proof.** Assume that for some  $y \in \mathbf{R}^n$  and  $\sigma \in \Sigma_x^n$  we have

$$\text{dist}_{(x+T_{\mathcal{M}}(x))}(y) < \text{dist}_{(x+T_{\mathcal{M}}(x))}(\sigma y).$$

Then, there exists  $z \in T_{\mathcal{M}}(x)$  satisfying  $\|y - (x + z)\| < \text{dist}_{(x+T_{\mathcal{M}}(x))}(\sigma y)$ , which yields (recalling  $\sigma x = x$  and the fact that the norm is symmetric)

$$\|y - (x + z)\| = \|\sigma y - (x + \sigma z)\| < \text{dist}_{(x+T_{\mathcal{M}}(x))}(\sigma y),$$

contradicting the fact that  $\sigma z \in T_{\mathcal{M}}(x)$ . The reverse inequality can be established similarly.  $\blacksquare$

Let  $\bar{\pi}_T: \mathbf{R}^n \rightarrow x + T_{\mathcal{M}}(x)$  be the projection onto the affine space  $x + T_{\mathcal{M}}(x)$ , that is,

$$\bar{\pi}_T(y) := \text{Proj}_{(x+T_{\mathcal{M}}(x))}(y), \quad (3.2)$$

and similarly, let

$$\bar{\pi}_N(y) := \text{Proj}_{(x+N_{\mathcal{M}}(x))}(y) \quad (3.3)$$

denote the projection onto the affine space  $x + N_{\mathcal{M}}(x)$ . We also introduce  $\pi_T$  and  $\pi_N$ , the projections onto the tangent and normal spaces  $T_{\mathcal{M}}(x)$  and  $N_{\mathcal{M}}(x)$ , respectively. Notice the following relationships:

$$\bar{\pi}_T(y) + \bar{\pi}_N(y) = y + x \quad \text{and} \quad \bar{\pi}_T(y) = \pi_T(y) + \pi_N(x). \quad (3.4)$$

**Corollary 3.9** (Invariance of projections). *Let  $x \in \mathcal{M}$ . Then, for all  $y \in \mathbf{R}^n$  and all  $\sigma \in \Sigma_x^n$*

$$(i) \quad \sigma \bar{\pi}_T(y) = \bar{\pi}_T(\sigma y),$$

$$(ii) \quad \sigma \bar{\pi}_N(y) = \bar{\pi}_N(\sigma y).$$

**Proof.** Let  $\bar{\pi}_T(y) = x + u$  for some  $u \in T_{\mathcal{M}}(x)$  and let  $\sigma \in \Sigma_x^n$ . By Proposition 3.8, and the symmetry of the norm we obtain

$$\text{dist}_{(x+T_{\mathcal{M}}(x))}(y) = \|y - (x + u)\| = \|\sigma y - (x + \sigma u)\| = \text{dist}_{(x+T_{\mathcal{M}}(x))}(\sigma y).$$

Since  $\sigma u \in T_{\mathcal{M}}(x)$ , we conclude  $\bar{\pi}_T(\sigma y) = x + \sigma u$  and assertion (i) follows.

For the second assertion, apply (3.4) to the point  $\sigma y \in \mathbf{R}^n$ . Then, use (i) to deduce

$$\sigma y + x = \bar{\pi}_T(\sigma y) + \bar{\pi}_N(\sigma y) = \sigma \bar{\pi}_T(y) + \bar{\pi}_N(\sigma y).$$

Applying  $\sigma^{-1}$  to this equation, recalling that  $\sigma^{-1}x = x$  and equating with (3.4) we get (ii).  $\blacksquare$

The following result relates the tangent space to the stratification.

**Proposition 3.10** (Tangential projection vs stratification). *Let  $x \in \mathcal{M}$ . Then, there exists  $\delta > 0$  such that for any  $y \in \mathcal{M} \cap B(x, \delta)$  we have*

$$y, \bar{\pi}_T(y) \in \Delta_{P'},$$

for some  $P' \succeq P_x$ .

**Proof.** Choose  $\delta > 0$  so that the ball  $B(x, \delta)$  intersects only strata  $\Delta_{P'}$  with  $P' \succeq P_x$ , see Corollary 2.10, and shrink it further to ensure (3.1) as well as that the projection  $\bar{\pi}_T$  is a one-to-one map between  $\mathcal{M} \cap B(x, \delta)$  and its range. For any  $y \in \mathcal{M} \cap B(x, \delta)$  let  $u \in T_{\mathcal{M}}(x) \cap B(0, \delta)$  be the unique element of  $T_{\mathcal{M}}(x)$  satisfying  $\bar{\pi}_T(y) = x + u$ , or in other words such that

$$\text{dist}_{(x+T_{\mathcal{M}}(x))}(y) = \|y - (x + u)\| = \min_{z \in T_{\mathcal{M}}(x)} \|(y - x) - z\|. \quad (3.5)$$

Then, for some  $P_1, P_2 \succeq P_x$  we have  $x + u \in \Delta_{P_1}$  and  $y \in \Delta_{P_2}$ . In view of Lemma 3.7 and Lemma 2.7 we deduce

$$x + \sigma_2 u = \sigma_2(x + u) \in (x + T_{\mathcal{M}}(x)) \cap B(x, \delta).$$

We shall now show that  $P_1 = P_2$ . To this end, note that for any  $\sigma_2 \in \Sigma_{P_2}^n$ , we have  $\sigma_2 x = x$  and

$$\|y - (x + \sigma_2 u)\| = \|\sigma_2 y - (\sigma_2 x + \sigma_2 u)\| = \|(y - x) - u\|.$$

It follows from (3.5) that  $\bar{\pi}_T(y) = x + \sigma_2 u$ , thus  $\sigma_2 u = u$ , which yields  $\sigma_2(x + u) = x + u$ . Hence,  $P_1 \preceq P_{\sigma_2}$ , by relationship (P2) on page 4. Since this holds for any  $\sigma_2 \in \Sigma_{P_2}^n$ , we conclude that  $P_1 \preceq P_2$ . If we assume that  $P_1 \prec P_2$ , then  $\sigma_1 y \neq y$  for  $\sigma_1$  with  $P_{\sigma_1} = P_1$  (otherwise reach a contradiction using relationship 2.1 on page 4). Since  $\sigma_1 \in \Sigma_x^n$  and condition (3.1), we have  $\sigma_1 y \in \mathcal{M} \cap B(x, \delta)$ . But  $\sigma_1 y \neq y$  yields  $\bar{\pi}_T(y) \neq \bar{\pi}_T(\sigma_1 y)$ . Thus, there exists  $v \in T_{\mathcal{M}}(x)$  with

$$\|\sigma_1 y - (x + v)\| < \|\sigma_1 y - (x + u)\| = \|y - (x + u)\|,$$

which contradicts Proposition 3.8. Thus,  $P_1 = P_2$  and  $y, x + u \in \Delta_{P_1} = \Delta_{P_2}$ .  $\blacksquare$

We end this subsection with the following important property that locates the tangent and normal spaces of  $\mathcal{M}$  at  $x$  with respect to the active stratum  $\Delta_{P_x}$ .

**Proposition 3.11** (Decomposition of  $T_{\mathcal{M}}(x)$ ,  $N_{\mathcal{M}}(x)$ ). *For any  $x \in \mathcal{M}$  we have*

$$\text{Proj}_{\Delta_{P_x}^{\perp\perp}}(T_{\mathcal{M}}(x)) = T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp},$$

which yields

$$T_{\mathcal{M}}(x) = (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \oplus (T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}). \quad (3.6)$$

Similarly,

$$N_{\mathcal{M}}(x) = (N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}) \oplus (N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}). \quad (3.7)$$

**Proof.** Lemma 2.9 and Lemma 3.7 show that for any  $u \in T_{\mathcal{M}}(x)$  we have

$$\text{Proj}_{\Delta_{P_x}^{\perp\perp}}(u) = \frac{1}{|\Sigma_{P_x}^n|} \sum_{\sigma \in \Sigma_{P_x}^n} \sigma u \in T_{\mathcal{M}}(x),$$

which yields

$$\text{Proj}_{\Delta_{P_x}^{\perp\perp}}(T_{\mathcal{M}}(x)) \subseteq T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}.$$

The opposite inclusion and decomposition (3.6) are straightforward.

Let us now prove the decomposition of  $N_{\mathcal{M}}(x)$ . For any  $u \in T_{\mathcal{M}}(x)$ , by (3.6) there are (unique) vectors  $u_{\perp} \in T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}$  and  $u_{\perp\perp} \in T_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}$  such that  $u = u_{\perp} + u_{\perp\perp}$ . Since  $\mathbf{R}^n = \Delta_{P_x}^{\perp} \oplus \Delta_{P_x}^{\perp\perp}$ , we can decompose any  $v \in N_{\mathcal{M}}(x)$  correspondingly as  $v = v_{\perp} + v_{\perp\perp}$ . Since  $u_{\perp\perp}, u_{\perp} \in T_{\mathcal{M}}(x) = N_{\mathcal{M}}(x)^{\perp}$ , we have  $\langle u_{\perp}, v \rangle = 0$  and  $\langle u_{\perp\perp}, v \rangle = 0$ . Using the fact that  $\Delta_{P_x}^{\perp}$  and  $\Delta_{P_x}^{\perp\perp}$  are orthogonal

we get  $\langle u_{\perp\perp}, v_{\perp} \rangle = 0$  (respectively,  $\langle u_{\perp}, v_{\perp\perp} \rangle = 0$ ) implying that  $\langle u_{\perp\perp}, v_{\perp\perp} \rangle = 0$  (respectively,  $\langle u_{\perp}, v_{\perp} \rangle = 0$ ), and finally  $\langle u, v_{\perp} \rangle = 0$  (respectively,  $\langle u, v_{\perp\perp} \rangle = 0$ ). Since  $u \in T_{\mathcal{M}}(x)$  has been chosen arbitrarily, we conclude  $v_{\perp} \in N_{\mathcal{M}}(x) \cap \Delta_{\bar{P}_x}^{\perp}$  and  $v_{\perp\perp} \in N_{\mathcal{M}}(x) \cap \Delta_{\bar{P}_x}^{\perp\perp}$ . In other words,  $N_{\mathcal{M}}(x)$  is equal to the (direct) sum of  $N_{\mathcal{M}}(x) \cap \Delta_{\bar{P}_x}^{\perp}$  and  $N_{\mathcal{M}}(x) \cap \Delta_{\bar{P}_x}^{\perp\perp}$ .  $\blacksquare$

The following corollary is a simple consequence of the fact that  $T_{\mathcal{M}}(x) \oplus N_{\mathcal{M}}(x) = \mathbf{R}^n$ .

**Corollary 3.12** (Decomposition of  $\Delta_{\bar{P}_x}^{\perp}, \Delta_{\bar{P}_x}^{\perp\perp}$ ). *For any  $x \in \mathcal{M}$  we have*

$$\begin{aligned}\Delta_{\bar{P}_x}^{\perp} &= (\Delta_{\bar{P}_x}^{\perp} \cap T_{\mathcal{M}}(x)) \oplus (\Delta_{\bar{P}_x}^{\perp} \cap N_{\mathcal{M}}(x)), \\ \Delta_{\bar{P}_x}^{\perp\perp} &= (\Delta_{\bar{P}_x}^{\perp\perp} \cap T_{\mathcal{M}}(x)) \oplus (\Delta_{\bar{P}_x}^{\perp\perp} \cap N_{\mathcal{M}}(x)).\end{aligned}$$

**Proof.** We shall establish the second formula, the first one can be proved analogously. Let  $y \in \Delta_{\bar{P}_x}^{\perp\perp}$  and decompose it as  $y = u + v$ , with  $u \in T_{\mathcal{M}}(x)$  and  $v \in N_{\mathcal{M}}(x)$ . By Proposition 3.11, we decompose further  $u = u_{\perp\perp} + u_{\perp}$  and  $v = v_{\perp\perp} + v_{\perp}$  with  $u_{\perp\perp}, v_{\perp\perp} \in \Delta_{\bar{P}_x}^{\perp\perp}$  and  $u_{\perp}, v_{\perp} \in \Delta_{\bar{P}_x}^{\perp}$ . Then,

$$y = (u_{\perp\perp} + v_{\perp\perp}) + (u_{\perp} + v_{\perp}) \in \Delta_{\bar{P}_x}^{\perp\perp},$$

which yields  $u_{\perp} + v_{\perp} = 0$ . Since  $\langle u_{\perp}, v_{\perp} \rangle = 0$ , we obtain  $u_{\perp} = v_{\perp} = 0$ , the assertion follows.  $\blacksquare$

The subspaces  $\Delta_{\bar{P}_x}^{\perp\perp} \cap N_{\mathcal{M}}(x)$  and  $T_{\mathcal{M}}(x) \cap \Delta_{\bar{P}_x}^{\perp}$  in the previous statements play an important role in the second part of this paper dealing with spectral manifolds.

### 3.3 Location of a locally symmetric manifold

This section deals with the structural properties of  $\mathcal{M}$  stemming from Definition 3.4.

We need the following standard technical lemma about isometries between two Riemannian manifolds. This lemma will be used as a link from local to global properties. Given a Riemannian manifold  $M$  we recall that an open neighborhood  $V$  of a point  $p \in M$  is called *normal* if every point of  $V$  can be connected to  $p$  through a unique geodesic lying entirely in  $V$ . It is well-known (see Theorem 3.7 in [8, Chapter 3] for example) that every point of a Riemannian manifold  $M$  (that is,  $M$  is at least  $C^2$ ) has a normal neighborhood. A more general version of the following lemma can be found in [9, Chapter VI], we include its proof for completeness.

**Lemma 3.13** (Determination of isometries). *Let  $M, N$  be two connected Riemannian manifolds. Let  $f_i: M \rightarrow N, i \in \{1, 2\}$  be two isometries and let  $p \in M$  be such that*

$$f_1(p) = f_2(p) \quad \text{and} \quad df_1(v) = df_2(v) \quad \text{for every } v \in T_M(p).$$

*Then,  $f_1 = f_2$ .*

**Proof.** Every isometry mapping between two Riemannian manifolds sends a geodesic into a geodesic. For any  $p \in M$  and  $v \in T_M(p)$ , we denote by  $\gamma_{v,p}$  (respectively by  $\tilde{\gamma}_{\bar{v},\bar{p}}$ ) the unique geodesic passing through  $p \in M$  with velocity  $v \in T_M(p)$  (respectively, through  $\bar{p} \in N$  with velocity  $\bar{v} \in T_N(\bar{p})$ ). Using uniqueness of the geodesics, it is easy to see that for all  $t$

$$f_1(\gamma_{v,p}(t)) = \tilde{\gamma}_{df_1(v),f_1(p)}(t) = \tilde{\gamma}_{df_2(v),f_2(p)}(t) = f_2(\gamma_{v,p}(t)). \quad (3.8)$$

Let  $V$  be a normal neighborhood of  $p$ , let  $q \in V$  and  $[0, 1] \ni t \mapsto \gamma_{v,p}(t) \in M$  be the geodesic connecting  $p$  to  $q$  and having initial velocity  $v \in T_M(p)$ . Applying (3.8) for  $t = 1$  we obtain

$f_1(q) = f_2(q)$ . Since  $q$  was arbitrarily chosen, we get  $f_1 = f_2$  on  $V$ . (Thus, since  $V$  is open, we also deduce  $df_1(v) = df_2(v)$  for every  $v \in T_M(q)$ .)

Let now  $q$  be any point in  $M$ . Since connected manifolds are also path connected, we can join  $p$  to  $q$  with a continuous path  $t \in [0, 1] \mapsto \delta(t) \in M$ . Consider the set

$$\{t \in [0, 1] : f_1(\delta(t)) = f_2(\delta(t)) \text{ and } df_1(v) = df_2(v) \text{ for every } v \in T_M(\delta(t))\}. \quad (3.9)$$

Since  $f_i: M \rightarrow N$  and  $df_i: TM \rightarrow TN$  ( $i \in \{1, 2\}$ ) are continuous maps, the above set is closed. further, since  $f_1 = f_2$  in a neighborhood of  $p$  it follows that the supremum in (3.9), denoted  $t_0$ , is strictly positive. If  $t_0 \neq 1$ , then repeating the argument for the point  $p_1 = \delta(t_0)$ , we obtain a contradiction. Thus,  $t_0 = 1$  and  $f_1(q) = f_2(q)$ . ■

The above lemma will now be used to obtain the following result which locates the locally symmetric manifold  $\mathcal{M}$  with respect to the stratification.

**Corollary 3.14** (Reduction of the ambient space). *Let  $\mathcal{M}$  be a locally symmetric manifold. If for some  $x \in \mathcal{M}$ ,  $P \in \Pi_n$ , and  $\delta > 0$  we have  $\mathcal{M} \cap B(x, \delta) \subseteq \Delta_P$ , then  $\mathcal{M} \subseteq \Delta_P^\perp$ .*

**Proof.** Fix a permutation  $\sigma$  with  $P_\sigma = P$ . Suppose first that  $\mathcal{M}$  is strongly locally symmetric. Let  $f_1: \mathcal{M} \rightarrow \mathcal{M}$  be the identity isometry on  $\mathcal{M}$  and let  $f_2: \mathcal{M} \rightarrow \mathcal{M}$  be the isometry determined by the permutation  $\sigma$ , that is,  $f_2(x) = \sigma x$  for all  $x \in \mathcal{M}$ . The assumption  $\mathcal{M} \cap B(x, \delta) \subseteq \Delta_P$  yields that the isometries  $f_1$  and  $f_2$  coincide around  $x$ , see part (6) of Proposition 2.4. Thus, by Lemma 3.13 (with  $M = N = \mathcal{M}$ ) we conclude that  $f_1$  and  $f_2$  coincide on  $\mathcal{M}$ . This shows that  $\mathcal{M} \subseteq \Delta_P^\perp$ .

In the case when  $\mathcal{M}$  is locally symmetric, assume, towards a contradiction, that there exists  $\bar{x} \in \mathcal{M} \setminus \Delta_P^\perp$ . Consider a continuous path  $t \in [0, 1] \mapsto p(t) \in \mathcal{M}$  with  $p(0) = x$  and  $p(1) = \bar{x}$ . Find  $0 = t_0 < t_1 < \dots < t_s = 1$  and  $\{\delta_i > 0 : i = 0, \dots, s\}$  such that  $\mathcal{M}_i := \mathcal{M} \cap B(p(t_i), \delta_i)$  is strongly locally symmetric, the union of all  $\mathcal{M}_i$  covers the path  $p(t)$ ,  $\mathcal{M}_{i-1} \cap \mathcal{M}_i \neq \emptyset$ , and  $\mathcal{M}_0 \subset \Delta_P$ . Let  $s'$  be the first index such that  $\mathcal{M}_{s'} \not\subseteq \Delta_P^\perp$ , clearly  $s' > 0$ . Let  $x' \in \mathcal{M}_{s'-1} \cap \mathcal{M}_{s'} \cap \Delta_P^\perp$  and note that  $x' \in \Delta_{P'}$  for some  $P' \preceq P$ . By the strong local symmetry of  $\mathcal{M}_{s'-1}$  and  $\mathcal{M}_{s'}$ , they are both invariant under the permutation  $\sigma$ . Since  $\sigma$  coincides with the identity on  $\mathcal{M}_{s'-1}$  and since  $\mathcal{M}_{s'-1} \cap \mathcal{M}_{s'}$  is an open subset of  $\mathcal{M}_{s'}$ , we see by Lemma 3.13 that  $\sigma$  coincides with the identity on  $\mathcal{M}_{s'}$ . This contradicts the fact that  $\mathcal{M}_{s'} \not\subseteq \Delta_P^\perp$ . ■

In order to strengthen Corollary 3.14 we need to introduce a new notion.

**Definition 3.15** (“Much smaller” partition). For two partitions  $P, P' \in \Pi_n$ .

- The partition  $P'$  is called *much smaller* than  $P$ , denoted  $P' \ll P$ , whenever  $P' \prec P$  and a set in  $P'$  is formed by merging at least two sets from  $P$ , one of them containing at least two elements.
- Whenever  $P' \prec P$  but  $P'$  is not much smaller than  $P$  we shall write  $P' \prec\prec P$ . In other words, if  $P' \prec P$  but  $P'$  is not much smaller than  $P$ , then every set in  $P'$  that is not in  $P$  is formed by uniting one-element sets from  $P$ .

**Example 3.16** (Smaller vs. much smaller partition). The following examples illustrate the notions of Definition 3.15. We point out that part (vii) will be used frequently.

- (i)  $\{\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}\} \ll \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}, \{7\}\}.$

- (ii) Consider  $P = \{\{1, 6, 7\}, \{2, 3\}, \{4, 5\}\}$  and  $P' = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}, \{7\}\}$ . In this case,  $P \prec P'$  but  $P$  is not much smaller than  $P'$  because only sets of length one are merged to form the sets in  $P$ . Thus,  $P \not\prec P'$ .
- (iii) If  $P'' \preceq P'$  and  $P' \prec P$ , then  $P'' \prec P$ .
- (iv) It is possible to have  $P' \prec P$  and  $P'' \prec P$  but  $P'' \not\prec P'$ , as shown by  $P = \{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$ ,  $P' = \{\{1\}, \{2, 3\}, \{4, 5\}\}$ , and  $P'' = \{\{1, 2, 3\}, \{4, 5\}\}$ .
- (v) If  $P' \prec P$  and  $P$  has at most one one-element set, then  $P' \prec P$ .
- (vi) If  $P \neq \text{id}_n$ , then  $P \prec \text{id}_n$ .
- (vii) If  $P' \preceq P$  and if  $P'$  is not much smaller than  $P$ , then either  $P' = P$  or  $P' \prec P$ .
- (viii) If  $P'' \prec P'$  and  $P' \prec P$ , then  $P'' \prec P$ . That is, the relationship ‘not much smaller’ is transitive. ■

We now describe a strengthening of Corollary 3.14. It lowers the number of strata that can intersect  $\mathcal{M}$ , hence better specifies the location of the manifold  $\mathcal{M}$ .

**Corollary 3.17** (Inactive strata). *Let  $\mathcal{M}$  be a locally symmetric manifold. If for some  $x \in \mathcal{M}$ ,  $P \in \Pi_n$  and  $\delta > 0$  we have  $\mathcal{M} \cap B(x, \delta) \subseteq \Delta_P$ , then*

$$\mathcal{M} \subseteq \Delta_P^{\perp\perp} \setminus \bigcup_{P' \prec P} \Delta_{P'}.$$

**Proof.** By Corollary 3.14, we know that  $\mathcal{M} \subseteq \Delta_P^{\perp\perp}$ . Assume, towards a contradiction, that  $\mathcal{M} \cap \Delta_{P'} \neq \emptyset$  for some  $P' \prec P$ . This implies in particular that  $P$  is not the identity partition  $\text{id}_n$ , see Example 3.16 (vi). Consider a continuous path connecting  $x$  with a point in  $\mathcal{M} \cap \Delta_{P'}$ . Let  $z$  be the first point on the path such that  $P_z \prec P$ . (Such a first point exists since whenever  $P_z \prec P$ , the points in  $\Delta_{P_z}$  are boundary points of  $\Delta_P$ .) Let  $\delta > 0$  be such that  $\mathcal{M} \cap B(z, \delta)$  is strongly locally symmetric. Let  $\bar{z} \in \mathcal{M} \cap B(z, \delta)$  be a point on the path before  $z$ . That means  $P_{\bar{z}} \prec P$  or  $P_{\bar{z}} = P$ . To summarize:

$$z \in \mathcal{M} \text{ is such that } P_z \prec P \text{ and } \bar{z} \in \mathcal{M} \cap B(z, \delta) \text{ is such that } P_{\bar{z}} \prec P \text{ or } P_{\bar{z}} = P.$$

By Definition 3.15, the fact  $P_z \prec P$  means that for some  $2 \leq \ell < k \leq n$  there is a set  $\{a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_k\}$  in  $P_z$  such that  $\{a_1, \dots, a_\ell\}$  is a set in  $P$ . Now, since  $P_{\bar{z}} \prec P$  or  $P_{\bar{z}} = P$ , the set  $\{a_1, \dots, a_\ell\}$  belongs to  $P_{\bar{z}}$  as well.

Since  $\bar{z} \in \Delta_{P_{\bar{z}}}$ , we have  $\bar{z}_{a_1} = \dots = \bar{z}_{a_\ell} =: \alpha$  and  $\bar{z}_i \neq \alpha$  for  $i \notin \{a_1, \dots, a_\ell\}$ . By the fact that  $\bar{z} \in \mathcal{M} \cap B(z, \delta)$  and the latter set is strongly locally symmetric, we deduce that

$$\sigma \bar{z} \in \mathcal{M} \subset \Delta_P^{\perp\perp} \text{ for every } \sigma \in \Sigma_z^n. \quad (3.10)$$

Let  $\sigma \in \Sigma_z^n$  be such that  $P_\sigma = P_z$  but the cycle of  $\sigma$  corresponding to the set  $\{a_1, \dots, a_\ell, a_{\ell+1}, \dots, a_k\}$  be  $(a_1, \dots, a_{\ell-1}, a_k, a_{\ell+1}, \dots, a_{k-1}, a_\ell)$ . Letting  $y := \sigma \bar{z}$ , we have  $(y_1, \dots, y_n) = (\bar{z}_{\sigma^{-1}(1)}, \dots, \bar{z}_{\sigma^{-1}(n)})$  and notice that  $y_{a_1} = \bar{z}_{\sigma^{-1}(a_1)} = \bar{z}_{a_\ell} = \alpha$ , while  $y_{a_\ell} = \bar{z}_{\sigma^{-1}(a_\ell)} \neq \alpha$ . In view of (2.6) we deduce that  $y \notin \Delta_P^{\perp\perp}$ , a contradiction. ■

## 4 The characteristic partition of $\mathcal{M}$

The location of the manifold with respect to the stratification, investigated in Subsection 3.3, leads naturally to the definition of a *characteristic* permutation associated with a locally symmetric manifold. We explain that in Subsection 4.1. On its turn the characteristic permutation induces a *canonical decomposition* of  $\mathbf{R}^n$ , see Subsection 4.2.

Anticipating the further developments, the canonical decomposition will be used to describe a reduction of the active normal space in Subsection 5.1. Finally, in Subsection 5.2 we obtain a very useful description of locally symmetric manifolds by means of a *reduced* locally symmetric local equation. This last step will be crucial in the sequel to this paper dealing with spectral manifolds.

### 4.1 Definition and basic properties of the characteristic partition $P_*$ of $\mathcal{M}$

In order to better understand the structure of the locally symmetric manifold  $\mathcal{M}$ , we exhibit the so-called *characteristic partition* of  $\mathcal{M}$ . This partition will play an important role for the reduction of the tangent/normal space of the manifold (*c.f.* Theorem 5.1), which turns out to be essential in order to obtain a symmetric tangential parametrization, see Theorem 5.4. This latter, in turn, is the key element in order to establish the smoothness of the spectral set  $\lambda^{-1}(\mathcal{M})$  in [7].

We proceed by introducing the following sets of active partitions:

$$\Delta(\mathcal{M}) := \{P \in \Pi_n : \mathcal{M} \cap \Delta_P \neq \emptyset\},$$

and

$$\Sigma_{\mathcal{M}} := \{P \in \Pi_n : \exists(x \in \mathcal{M}, \delta > 0) \text{ such that } \mathcal{M} \cap B(x, \delta) \subseteq \Delta_P\}.$$

The following result is straightforward.

**Lemma 4.1** (Maximality of  $\Sigma_{\mathcal{M}}$  in  $\Delta(\mathcal{M})$ ). *The set  $\Sigma_{\mathcal{M}}$  contains at most one partition which is maximal in  $\Delta(\mathcal{M})$ .*

**Proof.** It follows readily that  $\Delta(\mathcal{M}) \neq \emptyset$  and  $\Sigma_{\mathcal{M}} \subset \Delta(\mathcal{M})$ . Let  $P_1 \in \Delta(\mathcal{M})$  and  $P_2 \in \Sigma_{\mathcal{M}}$ . By Corollary 3.14 we deduce that  $\mathcal{M} \subseteq \Delta_{P_2}^{\perp\perp}$  and by Proposition 2.4, part (2), that  $P_1 \preceq P_2$ . This proves maximality of  $P_2$  in  $\Delta(\mathcal{M})$ . Using the above, it is easy to see that  $\Sigma_{\mathcal{M}}$  contains at most one partition.  $\blacksquare$

The next lemma is, in a sense, a converse of Corollary 3.14. It shows in particular that  $\Sigma_{\mathcal{M}} \neq \emptyset$ .

**Lemma 4.2** (Optimal reduction of the ambient space). *If  $\mathcal{M}$  is a locally symmetric manifold, then  $\Sigma_{\mathcal{M}} = \{P_*\}$  for some partition  $P_* \in \Pi_n$ . In particular, if  $\mathcal{M} \subseteq \Delta_{\bar{P}}^{\perp\perp}$ , then  $P_* \preceq \bar{P}$ .*

**Proof.** The assertion follows directly from Lemma 4.1 provided one proves that  $\Sigma_{\mathcal{M}} \neq \emptyset$ . To do so, we assume that  $\mathcal{M} \subseteq \Delta_{\bar{P}}^{\perp\perp}$  for some  $\bar{P} \in \Pi_n$  (this is always true for  $\bar{P} = \text{id}_n$ ) and we prove both that  $\Sigma_{\mathcal{M}} \neq \emptyset$  as well as the second part of the assertion. Notice that  $P \preceq \bar{P}$  for all  $P \in \Delta(\mathcal{M})$ . Let the partition  $P^\circ$  be the supremum of the nonempty set  $\Delta(\mathcal{M})$ . If  $P^\circ \in \Delta(\mathcal{M})$ , then  $P^\circ \in \Sigma_{\mathcal{M}}$  and we are done. If  $P^\circ \notin \Delta(\mathcal{M})$ , then choose any partition  $P_\circ \in \Delta(\mathcal{M})$  such that

$$\{P \in \Delta(\mathcal{M}) : P_\circ \prec P \prec P^\circ\} = \emptyset. \quad (4.1)$$

Such a partition  $P_\circ$  exists since  $\Delta(\mathcal{M})$  is a finite partially ordered set. By the definition of  $P_\circ$  there exists  $\bar{x} \in \mathcal{M} \cap \Delta_{P_\circ}$ , and by Lemma 2.7(ii) we can find  $\delta > 0$  such that  $B(\bar{x}, \delta)$  intersects only strata  $\Delta_P$  corresponding to partitions  $P \succeq P_\circ$ . If there exists  $x \in \mathcal{M} \cap B(\bar{x}, \delta)$  such that  $x \in \Delta_P$

for some partition  $P \succ P_\circ$ , then  $P \in \Delta(\mathcal{M})$  and by the definition of  $P^\circ$ , we have  $P_\circ \prec P \prec P^\circ$ , contradicting (4.1). Thus,  $\mathcal{M} \cap B(\bar{x}, \delta) \subseteq \Delta_{P_\circ}$  showing that  $P_\circ \in \Sigma_{\mathcal{M}}$ . ■

**Definition 4.3** (Characteristic partition). The partition  $P_*$  is called the *characteristic* partition of  $\mathcal{M}$ .

**Corollary 4.4** (Density of  $\mathcal{M} \cap \Delta_{P_*}$  in  $\mathcal{M}$ ). *For every  $x \in \mathcal{M}$  we have*

$$\mathcal{M} \cap \Delta_{P_*} \cap B(x, \delta) \neq \emptyset \quad \text{for every } \delta > 0.$$

**Proof.** Suppose  $x \in \mathcal{M} \cap \Delta_P$  and fix  $\delta > 0$  small enough so that  $B(x, \delta)$  intersects only strata  $\Delta_{P'}$  for  $P' \succeq P$ . Then, by Lemma 2.7, we have that the manifold  $\mathcal{M}' := \mathcal{M} \cap B(x, \delta)$  is locally symmetric. By Lemma 4.2, we obtain that  $\Sigma_{\mathcal{M}'} \neq \emptyset$ . Since  $\Sigma_{\mathcal{M}'} \subset \Sigma_{\mathcal{M}}$ , we have  $\Sigma_{\mathcal{M}'} = \Sigma_{\mathcal{M}}$ . Thus,  $\mathcal{M}' \cap B(y, \rho) \subset \Delta_{P_*}$  for some  $y \in \mathcal{M}' \subset \mathcal{M}$  and some  $\rho > 0$ , the result follows. ■

In particular, we have the following easy result.

**Corollary 4.5.** *For a locally symmetric manifold  $\mathcal{M}$  we have*

$$P_* = \text{id}_n \iff \mathcal{M} \cap \Delta_{\text{id}_n} \neq \emptyset.$$

**Proof.** The necessity is obvious, while the sufficiency follows from Lemma 4.1, since  $\text{id}_n \in \Delta(\mathcal{M})$  is the unique maximal element of  $\Pi_n$ . ■

Even though the definition of the characteristic partition  $P_*$  is local, it has global properties stemming from Corollary 3.17, that is,

$$\mathcal{M} \subseteq \Delta_{P_*}^{\perp\perp} \setminus \bigcup_{P \prec\prec P_*} \Delta_P = \Delta_{P_*} \cup \left( \bigcup_{P \prec P_*} \Delta_P \right) \subseteq \Delta_{P_*}^{\perp\perp}, \quad (4.2)$$

and  $P_*$  is the minimal partition for which (4.2) holds. The above formula determines precisely which strata can intersect  $\mathcal{M}$ . Indeed, if  $P \in \Delta(\mathcal{M})$ , then necessarily either  $P = P_*$  or  $P \prec P_*$ . Notice also that when  $P \prec P_*$ , every set in  $P$ , which is not in  $P_*$ , is obtained by merging sets of length one from  $P_*$ . Another consequence is the following relation:

$$T_{\mathcal{M}}(x) \subset \Delta_{P_*}^{\perp\perp} \quad \text{for all } x \in \mathcal{M}. \quad (4.3)$$

**Remark 4.6** (Local symmetry of active strata). Observe that for any partition  $P^\circ \in \Pi_n$ , the set

$$\Delta_{P^\circ} \cup \left( \bigcup_{P \prec P^\circ} \Delta_P \right)$$

is a locally symmetric manifold with characteristic permutation  $P^\circ$ . On the other hand, (4.2) shows that the affine space  $\Delta_P^{\perp\perp}$  is a locally symmetric manifold if and only if  $P = \text{id}_n$  or  $P = \{\mathbb{N}_n\}$ . ■

We conclude with another fact about the characteristic permutation, that stems from the assumption  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$  (see Definition 3.4). Though (4.2) describes well the strata that can intersect the manifold  $\mathcal{M}$  (which is going to be sufficient for most of our needs) we still need to say more about a slightly finer issue - a necessary condition for a stratum to intersect  $\mathcal{M} \cap \mathbf{R}_{\geq}^n$ .



**Lemma 4.7** (Active strata of  $\mathbf{R}_{\geq}^n$ ). *If  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \cap \Delta_P \neq \emptyset$ , then every set of the partition  $P$  contains consecutive integers.*

**Proof.** The lemma holds trivially for sets of cardinality one. Let us suppose, towards a contradiction, that the set  $I_\ell \in P$  contains at least two elements but does not contain consecutive numbers from  $\mathbb{N}_n$ . Let  $i, j, k \in \mathbb{N}_n$  be three indexes with  $i < j < k$ . Suppose  $i, k \in I_\ell$ . The fact  $x \in \Delta_P$  implies that  $x_i = x_k$ , while the fact that  $x \in \mathbf{R}_{\geq}^n$  implies that  $x_i \geq x_j \geq x_k$ . We obtain  $x_i = x_j = x_k$ , showing that  $j \in I_\ell$ . ■

Lemma 4.7 has consequences for the characteristic permutation  $P_*$  of  $\mathcal{M}$ .

**Theorem 4.8.** *Every set in the characteristic partition  $P_*$  contains consecutive integers.*

**Proof.** Since  $\mathcal{M} \cap \mathbf{R}_{\geq}^n \neq \emptyset$  by Definition 3.4, there is a stratum  $\Delta_P$  intersecting  $\mathcal{M} \cap \mathbf{R}_{\geq}^n$ . By Lemma 4.7, every set in  $P$  contains consecutive integers. Formula (4.2) implies that  $P$  is not much smaller than  $P_*$ , that is  $P = P_*$  or  $P \prec\sim P_*$ . Thus, if a set in  $P_*$  has more than one element, it must be an element of the partition  $P$  as well. ■

For example, according to Theorem 4.8, the partition  $\{\{1\}, \{2, 7, 4\}, \{3, 5\}, \{6\}\} \in \Pi_7$  cannot be the characteristic permutation of any locally symmetric manifold  $\mathcal{M}$  in  $\mathbf{R}^7$  that intersects  $\mathbf{R}_{\geq}^7$ .

There are further limitations imposed by the previous result. Suppose that the characteristic partition is

$$P_* = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10, 11, 12\}\} \in \Pi_{12}.$$

Consider the partition  $P = \{\{1\}, \{2\}, \{3, 4, 5\}, \{6, 8, 9\}, \{7\}, \{10, 11, 12\}\} \in \Pi_{12}$ . In comparison with Formula (4.2),  $P$  is not much smaller than  $P_*$  but the stratum  $\Delta_P$  does not intersect  $\mathcal{M} \cap \mathbf{R}_{\geq}^n$ . Thus, the set of strata that may intersect with  $\mathcal{M} \cap \mathbf{R}_{\geq}^n$  is further reduced.

## 4.2 Canonical decomposition induced by $P_*$

We explain in this subsection that the characteristic partition  $P_*$  of  $\mathcal{M}$  induces a decomposition of the space  $\mathbf{R}^n$  that will be used in the sequel to control the lift into the matrix space  $\mathbf{S}^n$ . The following example treats the particular case where  $P_*$  has at most one one-element set.

**Example 4.9.** Suppose that  $P_*$  has at most one one-element set. In other words, for every  $x \in \mathcal{M}$  at most one coordinate of the vector  $x$  is not repeated. In this case, by Example 3.16(v), every  $P$  that is smaller than  $P_*$  is much smaller than  $P_*$  and therefore (4.2) yields  $\mathcal{M} \subset \Delta_{P_*}$ . ■

Define

$$\begin{aligned} \mathbb{N}_n^1 &:= \text{the union of all sets in } P_* \text{ with exactly one element, and} \\ \mathbb{N}_n^2 &:= \text{the union of all sets in } P_* \text{ with more than one elements.} \end{aligned}$$

Clearly,  $\mathbb{N}_n$  is the disjoint union of  $\mathbb{N}_n^1$  and  $\mathbb{N}_n^2$ . (It may happen that one of the above sets is empty.)

**Definition 4.10** (Canonical split of  $\mathbf{R}^n$ ). The characteristic partition  $P_*$  of  $\mathcal{M}$  induces a *canonical split* of  $\mathbf{R}^n$  as a direct sum of the spaces  $\mathbf{R}^{\mathbb{N}_n^1}$  and  $\mathbf{R}^{\mathbb{N}_n^2}$ , as follows: any vector  $x \in \mathbf{R}^n$  is represented as

$$x = x^F \otimes x^M \tag{4.4}$$

where

- $x^F \in \mathbf{R}^{\mathbb{N}_n^1}$  is the subvector of  $x$  obtained by collecting the coordinates that have indices in one-element sets of  $P_*$ , preserving their relative order;
- $x^M \in \mathbf{R}^{\mathbb{N}_n^2}$  is the subvector of  $x$  obtained by collecting the remaining coordinates, preserving their order again.

It is readily seen that the canonical split is linear and also a reversible operation. Reversibility means that given any two vectors  $x^F \in \mathbf{R}^{\mathbb{N}_n^1}$  and  $x^M \in \mathbf{R}^{\mathbb{N}_n^2}$ , there is a unique vector  $x^F \otimes x^M \in \mathbf{R}^n$ , such that

$$(x^F \otimes x^M)^F = x^F \quad \text{and} \quad (x^F \otimes x^M)^M = x^M.$$

This operation is called *canonical product*.

**Example 4.11.** If  $P_* = \{\{1\}, \{2, 3\}, \{4\}, \{5, 6, 7\}, \{8\}\}$  and  $x \in \mathbf{R}^8$ , then  $x^F = (x_1, x_4, x_8)$  and  $x^M = (x_2, x_3, x_5, x_6, x_7)$ . Conversely, if  $x^F = (a_1, a_2, a_3)$  and  $x^M = (b_1, b_2, b_3, b_4, b_5)$ , then

$$x^F \otimes x^M = (a_1, b_1, b_2, a_2, b_3, b_4, b_5, a_3).$$

In addition, if  $x \in \mathbf{R}_{\geq}^8$ , then  $x^F \in \mathbf{R}_{\geq}^3$  and  $x^M \in \mathbf{R}_{\geq}^5$ . The converse is not true: if  $x^F \in \mathbf{R}_{\geq}^3$  and  $x^M \in \mathbf{R}_{\geq}^5$ , then in general,  $x^F \otimes x^M$  is not in  $\mathbf{R}_{\geq}^8$ . ■

In the particular case that  $P_* = \text{id}_n$ , we have  $x = x^F$  for all  $x \in \mathbf{R}^n$ .

**Definition 4.12** ( $P_*$ -decomposable partition). A partition  $P \in \Pi_n$  is called  $P_*$ -decomposable if  $P \succeq P_\circ$  for some  $P_\circ \prec\prec P_*$ .

Note that a  $P_*$ -decomposable partition  $P$  has the following property: if a set in  $P$  contains elements from  $\mathbb{N}_n^1$ , then it cannot contain elements from  $\mathbb{N}_n^2$ . According to (4.2), if  $\Delta_{P_\circ}$  intersects  $\mathcal{M}$ , then  $P_\circ$  is  $P_*$ -decomposable, moreover any  $P \succeq P_\circ$  is  $P_*$ -decomposable.

**Definition 4.13** ( $P_*$ -decomposition). For any  $P_*$ -decomposable partition  $P$  define the partitions  $P^F \in \Pi_{\mathbb{N}_n^1}$  and  $P^M \in \Pi_{\mathbb{N}_n^2}$  as follows

- $P^F$  contains those sets of  $P$  that contain only elements from  $\mathbb{N}_n^1$ ;
- $P^M$  contains the remaining sets of  $P$  (those containing only elements from  $\mathbb{N}_n^2$ ).

We have the disjoint union  $P = P^F \cup P^M$  called the  $P_*$ -decomposition of  $P$ .

For example, applying the  $P_*$ -decomposition to  $P_*$  yields  $P_*^F = \text{id}_{\mathbb{N}_n^1}$ . The following proposition is straightforward. It clarifies another particular case of the  $P_*$ -decomposition.

**Proposition 4.14.** *We have that  $P \prec\sim P_*$  if and only if  $P^F \prec \text{id}_{\mathbb{N}_n^1}$  and  $P^M = P_*^M$ .*

The  $P_*$ -decomposition is not going to be applied to partitions  $P$  that are much smaller than  $P_*$ , since these partitions may have a sets containing elements from both  $\mathbb{N}_n^1$  and  $\mathbb{N}_n^2$ . We finish this section with another straightforward statement.

**Proposition 4.15** ( $P_*$ -decomposition for active partitions). *If  $x \in \mathcal{M}$  and  $P \succeq P_x$ , then*

$$P_x^F \preceq P^F \preceq \text{id}_{\mathbb{N}_n^1} \quad \text{and} \quad P_*^M = P_x^M \preceq P^M.$$

## 5 The main results

If every (locally) symmetric submanifold  $\mathcal{M}$  would admit a (locally) symmetric local equation defined on  $\mathbf{R}^n$ , then smoothness of  $\lambda^{-1}(\mathcal{M})$  would have been easy to establish. However, unfortunately, this is not the case, as forthcoming Example 5.5 reveals.

The main result of this work asserts that every locally symmetric  $C^k$  smooth submanifold  $\mathcal{M}$  of  $\mathbf{R}^n$  admits a locally symmetric tangential parametrization defined on an appropriately reduced ambient space. This fact is essential in order to guarantee, as we show in [7], that the locally symmetric submanifold  $\mathcal{M}$  does admit a locally symmetric *reduced* local equation. This structural result will turn out to be very important for establishing the smoothness of the spectral manifold  $\lambda^{-1}(\mathcal{M})$ : indeed, the smoothness of  $\mathcal{M}$  is inherited by its locally symmetric reduced local equation, and is transferred, as we show in [7, Section 3], to an appropriately defined local equation of the spectral set  $\lambda^{-1}(\mathcal{M})$ .

### 5.1 Reduction of the normal space

In this section we fix a point  $x \in \mathcal{M}$  and reduce the relevant (active) part of the tangent and normal space with respect to the canonical split

$$\mathbf{R}^n = \mathbf{R}^{\mathbb{N}_n^1} \otimes \mathbf{R}^{\mathbb{N}_n^2} \quad (5.1)$$

induced by the characteristic partition  $P_*$  of  $\mathcal{M}$ . Consider any  $P_*$ -decomposable partition  $P \in \Pi_n$  and recall that  $P^F \in \Pi_{\mathbb{N}_n^1}$  and  $P^M \in \Pi_{\mathbb{N}_n^2}$ . These partitions define strata in  $\mathbf{R}^{\mathbb{N}_n^1}$  and  $\mathbf{R}^{\mathbb{N}_n^2}$ , respectively. For example, we define

$$\Delta_{P^F} := \{z \in \mathbf{R}^{\mathbb{N}_n^1} : P_z = P^F\} \quad (5.2)$$

and similarly  $\Delta_{P^M} \subset \mathbf{R}^{\mathbb{N}_n^2}$ . Thus, the notations  $\Delta_{P^F}^\perp$ ,  $\Delta_{P^F}^{\perp\perp}$  refer to the corresponding linear subspaces of  $\mathbf{R}^{\mathbb{N}_n^1}$ . We do the same for the stratum  $\Delta_{P^M}$  and the linear subspaces  $\Delta_{P^M}^\perp$ ,  $\Delta_{P^M}^{\perp\perp}$ . A glance at formulas (2.5) and (2.6) reveals the following relations:

$$\Delta_{P^\perp}^{\perp\perp} = \Delta_{P^F}^{\perp\perp} \otimes \Delta_{P^M}^{\perp\perp} \quad \text{and} \quad \Delta_{P^\perp}^\perp = \Delta_{P^F}^\perp \otimes \Delta_{P^M}^\perp. \quad (5.3)$$

In the sequel, we apply the canonical split (5.1) to the tangent space  $T_{\mathcal{M}}(x)$ . In view of (4.3) apply (5.3) to  $P_*$  and from the fact that  $P_*^M = P_x^M$  (see Proposition 4.14), we obtain that for every  $w \in T_{\mathcal{M}}(x)$

$$w = w^F \otimes w^M, \quad \text{where} \quad w^F \in \mathbf{R}^{\mathbb{N}_n^1} \quad \text{and} \quad w^M \in \Delta_{P_x^M}^{\perp\perp} \subset \mathbf{R}^{\mathbb{N}_n^2}. \quad (5.4)$$

Note that each coordinate of  $w^M$  is repeated at least twice.

The following theorem reveals an analogous relationship for the canonical split of the normal space  $N_{\mathcal{M}}(x)$  of  $\mathcal{M}$  at  $x$ . It is the culmination of most of the developments up to now and thus one of the main results in this work.

**Theorem 5.1** (Reduction of the normal space). *If  $x \in \mathcal{M}$  and  $v \in N_{\mathcal{M}}(x)$ , then*

$$v^F \in \Delta_{P_x^F}^{\perp\perp}. \quad (5.5)$$

**Proof.** Let us decompose  $v \in N_{\mathcal{M}}(x)$  according to Proposition 3.11, that is,  $v = v_{\perp\perp} + v_\perp$  where

$$v_{\perp\perp} \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp} \quad \text{and} \quad v_\perp \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^\perp.$$

Then,

$$v^F = v_{\perp\perp}^F + v_\perp^F \quad \text{and} \quad v^M = v_{\perp\perp}^M + v_\perp^M.$$

Since  $v_\perp \in \Delta_{P_x}^\perp$ , it follows by (5.3) that  $v_\perp^F \in \Delta_{P_x^F}^\perp$ . Since  $P_x \in \Delta(\mathcal{M})$ , by (4.2), we have  $P_x \prec P_*$ , that is,  $P_x$  is  $P_*$ -decomposable. Let  $P_x^F = \{I_1, \dots, I_m\}$  be the partition of  $\mathbb{N}_n^1$  induced by the  $P_*$ -decomposition of  $P_x$ . According to Lemma 6.1 (Appendix), we can choose  $w \in T_{\mathcal{M}}(x)$  arbitrarily close to 0, such that in the vector  $w^F \in \mathbf{R}^{\mathbb{N}_n^1}$  every subvector  $w_{I_i}^F$  has distinct coordinates for all  $i \in \mathbb{N}_m$ . Let  $\sigma^F$  be any permutation on  $\mathbb{N}_n^1$  such that  $P_{\sigma^F} \succeq P_{x^F}$ . Let  $\sigma^M$  be any permutation on  $\mathbb{N}_n^2$  such that  $P_{\sigma^M} \succeq P_{x^M}$ . In a natural way, together they define a permutation  $\sigma$  on  $\mathbb{N}_n$  such that  $\sigma \in \Sigma_x^n$ . It follows that  $(\sigma w)^F = \sigma^F w^F$ ,  $(\sigma w)^M = \sigma^M w^M = w^M$ , where the last equality holds in view of (5.4). By Lemma 3.7 part (i) we have  $\sigma w \in T_{\mathcal{M}}(x)$  allowing us to continue successively:

$$0 = \langle v_\perp, \sigma w \rangle = \langle v_\perp^F, (\sigma w)^F \rangle + \langle v_\perp^M, (\sigma w)^M \rangle = \langle v_\perp^F, \sigma^F w^F \rangle + \langle v_\perp^M, w^M \rangle.$$

This yields

$$\langle v_\perp^F, \sigma^F w^F \rangle = -\langle v_\perp^M, w^M \rangle,$$

which in view of Corollary 6.3 (applied to  $x := v_\perp^F \in \Delta_{P_x^F}^\perp$ ,  $\sigma := \sigma^F$ ,  $y := w^F$ , and  $\alpha := -\langle v_\perp^M, w^M \rangle$ ) yields  $v_\perp^F = 0$ . Recalling that  $v_{\perp\perp} \in \Delta_{P_x}^{\perp\perp}$ , we obtain  $v_{\perp\perp}^F \in \Delta_{P_x^F}^{\perp\perp}$  in view of (5.3). Thus,  $v^F = v_{\perp\perp}^F \in \Delta_{P_x^F}^{\perp\perp}$ .  $\blacksquare$

A consequence of Theorem 5.1 is the following corollary that will be needed later. Its proof can be extracted from the above proof.

**Corollary 5.2.** *If  $x \in \mathcal{M}$  and  $v_\perp \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^\perp$ , then  $v_\perp^F = 0$ .*

## 5.2 Tangential parametrization of a locally symmetric manifold

In this subsection we consider a local equation of the manifold, called *tangential parametrization*. We briefly recall some general properties of this parametrization (for any manifold  $\mathcal{M}$ ), and then we make use of Theorem 5.1 to specify it to our context.

The local inversion theorem asserts that for some  $\delta > 0$  sufficiently small the restriction of  $\bar{\pi}_T$  around  $x \in \mathcal{M}$

$$\bar{\pi}_T: \mathcal{M} \cap B(x, \delta) \rightarrow x + T_{\mathcal{M}}(x)$$

is a diffeomorphism of  $\mathcal{M} \cap B(x, \delta)$  onto its image (which is an open neighborhood of  $x$  relatively to the affine space  $x + T_{\mathcal{M}}(x)$ ). Then, there exists a smooth map

$$\phi: (x + T_{\mathcal{M}}(x)) \cap B(x, \delta) \rightarrow N_{\mathcal{M}}(x), \quad (5.6)$$

such that

$$\mathcal{M} \cap B(x, \delta) = \{y + \phi(y) \in \mathbf{R}^n : y \in (x + T_{\mathcal{M}}(x)) \cap B(x, \delta)\}. \quad (5.7)$$

In words, the function  $\phi$  measures the difference between the manifold and its tangent space. Obviously,  $\phi \equiv 0$  if  $\mathcal{M}$  is an affine manifold around  $x$ . Note that, technically, the domain of the map  $\phi$  is the open set  $\bar{\pi}_T(\mathcal{M} \cap B(x, \delta))$ , which may be a proper subset of  $(x + T_{\mathcal{M}}(x)) \cap B(x, \delta)$ . Even though we keep this in mind, it will not have any bearing on the developments in the sequel. Thus, for sake of readability we will avoid introducing more precise but also more complicated notation, for example, rectangular neighborhoods around  $x$ .

We say that the map  $\psi: (x + T_{\mathcal{M}}(x)) \cap B(x, \delta) \rightarrow \mathcal{M} \cap B(x, \delta)$  defined by

$$\psi(y) = y + \phi(y) \quad (5.8)$$

is the *tangential parametrization of  $\mathcal{M}$  around  $x$* . This function is indeed smooth, one-to-one and onto, with a full rank Jacobian matrix  $J\psi(x)$ : it is a local diffeomorphism at  $x$ , and more precisely its inverse is  $\bar{\pi}_T$ , that is, locally  $\bar{\pi}_T(\psi(y)) = y$ . The above properties of  $\psi$  hold for any manifold.

Let us return to the situation where  $\mathcal{M}$  is a locally symmetric manifold and we make the following assumption on the neighborhood.

**Assumption 5.3** (Active localization of  $\mathcal{M}$ ). Let  $\mathcal{M}$  be a locally symmetric  $C^k$ -submanifold of  $\mathbf{R}^n$  of dimension  $d$  and of characteristic partition  $P_*$ . We consider  $x \in \mathcal{M} \cap \mathbf{R}_{\geq}^n$  and we take  $\delta > 0$  small enough so that:

1.  $B(x, \delta)$  intersects only strata  $\Delta_P$  with  $P \succeq P_x$  (see Corollary 2.10);
2.  $\mathcal{M} \cap B(x, \delta)$  is a strongly locally symmetric manifold (see Equation (3.1));
3.  $\mathcal{M} \cap B(x, \delta)$  is diffeomorphic to its projection on  $x + T_{\mathcal{M}}(x)$ ; in other words, the tangential parametrization holds (see Equation (5.8)).

The first condition ensures that

$$\Delta_{P_x}^{\perp\perp} \cap B(x, \delta) = \Delta_{P_x} \cap B(x, \delta).$$

This situation enables us to specify the general properties of the tangential parametrization.

**Theorem 5.4** (Local symmetry of the tangential parametrization). *Let  $x \in \mathcal{M}$ . Then, the function  $\phi$  in the tangential parametrization satisfies*

$$\phi(x) \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}. \quad (5.9)$$

Moreover, for all  $y \in (x + T_{\mathcal{M}}(x)) \cap B(x, \delta)$  and for all  $\sigma \in \Sigma_{P_x}^n$  we have

$$\psi(\sigma y) = \sigma \psi(y) \quad (5.10)$$

and

$$\phi(\sigma y) = \sigma \phi(y) = \phi(y). \quad (5.11)$$

**Proof.** Recalling the direct decomposition of the normal space (see Proposition 3.11) we define the mappings  $\phi_{\perp\perp}(y)$  and  $\phi_{\perp}(y)$  as the projections of  $\phi(y)$  onto  $N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp}$  and  $N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp}$  respectively. Thus, (5.8) becomes

$$\psi(y) = y + \phi_{\perp\perp}(y) + \phi_{\perp}(y). \quad (5.12)$$

Splitting each term in both sides of Equation (5.12) in view of the canonical split defined in (4.4), we obtain

$$\begin{pmatrix} \psi^F(y) \\ \psi^M(y) \end{pmatrix} = \begin{pmatrix} y^F \\ y^M \end{pmatrix} + \begin{pmatrix} \phi_{\perp\perp}^F(y) \\ \phi_{\perp\perp}^M(y) \end{pmatrix} + \begin{pmatrix} \phi_{\perp}^F(y) \\ \phi_{\perp}^M(y) \end{pmatrix}.$$

We look at the second line of this vector equation. Since

$$\phi_{\perp\perp}(y) \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp\perp} \quad \text{and} \quad \phi_{\perp}(y) \in N_{\mathcal{M}}(x) \cap \Delta_{P_x}^{\perp},$$

we deduce from (5.3) that

$$\phi_{\perp\perp}^M(y) \in \Delta_{P_x^M}^{\perp\perp} \quad \text{and} \quad \phi_{\perp}^M(y) \in \Delta_{P_x^M}^{\perp}.$$

Since  $y \in x + T_{\mathcal{M}}(x)$  and  $\psi(y) \in \mathcal{M}$ , we deduce from (4.2) and (4.3) that  $y^M, \psi^M(y) \in \Delta_{P_x^M}^{\perp\perp}$  (recall that  $P_x^M = P_*^M$  by Proposition 4.15), yielding  $\phi_{\perp}^M(y) \in \Delta_{P_x^M}^{\perp\perp}$  and thus  $\phi_{\perp}^M(y) = 0$ . In addition, by Corollary 5.2, we have  $\phi_{\perp}^F(y) = 0$ . Thus,  $\phi_{\perp}(y) = 0$ , which completes the proof of (5.9).

We now show local invariance. Choose any permutation  $\sigma \in \Sigma_{P_x}^n$ . Since  $\phi(y) \in \Delta_{P_x}^{\perp\perp}$ , it follows that  $\sigma\phi(y) = \phi(y)$ . Thus,

$$\sigma\psi(y) = \sigma y + \sigma\phi(y) = \sigma y + \phi(y). \quad (5.13)$$

Since  $\mathcal{M} \cap B(x, \delta)$  is locally symmetric, we have  $\sigma\psi(y) \in \mathcal{M} \cap B(x, \delta)$ . Thus, there exists  $y_{\circ} \in (x + T_{\mathcal{M}}(x)) \cap B(x, \delta)$  such that

$$\sigma\psi(y) = \psi(y_{\circ}) = y_{\circ} + \phi(y_{\circ}). \quad (5.14)$$

Combining (5.13) with (5.14) we get

$$y_{\circ} - \sigma y = \phi(y) - \phi(y_{\circ}).$$

The left-hand side is an element of  $T_{\mathcal{M}}(x)$ , by Lemma 3.7, while the right-hand side is in  $N_{\mathcal{M}}(x)$ . Thus,  $y_{\circ} = \sigma y$  and  $\phi(y) = \phi(y_{\circ})$ , showing the local symmetry of  $\phi$  and (5.10).  $\blacksquare$

Theorem 5.4 is the culminating result of this manuscript, and should be understood in the following way:

**Every locally symmetric manifold  $\mathcal{M}$  admits  
a *reduced* locally symmetric tangential parametrization.**

This fact will allow to define, in [7], a locally symmetric reduced local equation of  $\mathcal{M}$ , and apply the so-called *transfer principle* to it, obtaining a smooth local equation of the spectral set  $\lambda^{-1}(\mathcal{M})$ , and establishing in this way that the latter is also a manifold. We recall that the transfer principle ensures, in case of a locally symmetric function  $f$  defined in  $\mathbf{R}^n$ , that the (lifted) spectral function  $f \circ \lambda$  enjoys the same degree of smoothness as  $f$ , see [1], [3], [10], [11], [12] and [13]. (A similar principle applies also to other properties, as convexity or prox-regularity, see [5].) Notice however, that the adaptation of the transfer principle in the reduced equation is not straightforward. It will be carried out in [7, Section 5].

We finish this paper with the following example, taken from [6, Example 3.8], which reveals that a locally symmetric submanifold  $\mathcal{M}$  of  $\mathbf{R}^n$  might fail to have a locally symmetric local equation defined in  $\mathbf{R}^n$ , illustrating the need the ambient space to be reduced.

**Example 5.5** (A symmetric manifold without symmetric equations). Let us consider the following symmetric (affine) submanifold of  $\mathbf{R}^2$  of dimension one:

$$\mathcal{M} = \{(x, y) \in \mathbf{R}^2 : x = y\} = \Delta((12)).$$

The associated spectral set

$$\lambda^{-1}(\mathcal{M}) = \{A \in \mathbf{S}^n : \lambda_1(A) = \lambda_2(A)\} = \{\alpha I_n : \alpha \in \mathbf{R}\}$$

is a submanifold of  $\mathbf{S}^n$  around  $I_n = \lambda^{-1}(1, 1)$ . It is interesting to observe that though  $\lambda^{-1}(\mathcal{M})$  is a (spectral) 1-dimensional submanifold of  $\mathbf{S}^n$ , this submanifold cannot be described by local equation that is a composition of  $\lambda$  with  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  a symmetric local equation of  $\mathcal{M}$  around  $(1, 1)$ . Indeed, let us assume on the contrary that such a local equation of  $\mathcal{M}$  exists, that is, there exists a smooth symmetric function  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  with surjective derivative  $\nabla\varphi(1, 1)$  which satisfies

$$\varphi(x, y) = 0 \iff x = y.$$

Consider now the two smooth paths  $c_1 : t \mapsto (t, t)$  and  $c_2 : t \mapsto (t, 2 - t)$ . Since  $\varphi \circ c_1(t) = 0$ , we infer

$$\nabla\varphi(1, 1)(1, 1) = 0. \quad (5.15)$$

On the other hand, since  $c_2'(1) = (1, -1)$  is normal to  $\mathcal{M}$  at  $(1, 1)$ , and since  $\varphi$  is symmetric, we deduce that the smooth function  $t \mapsto (\varphi \circ c_2)(t)$  has a local extremum at  $t = 1$ . Thus,

$$0 = (\varphi \circ c_2)'(1) = \nabla\varphi(1, 1)(1, -1). \quad (5.16)$$

Therefore, (5.15) and (5.16) imply that  $\nabla\varphi(1, 1) = (0, 0)$  which is a contradiction. This proves that there is no symmetric local equation  $\varphi : \mathbf{R}^2 \rightarrow \mathbf{R}$  of the symmetric manifold  $\mathcal{M}$  around  $(1, 1)$ . ■

## 6 Appendix: Lemmas needed for the proof of Theorem 5.1

This appendix contains a few results that are necessary for the proof of Theorem 5.1.

**Lemma 6.1.** *Let  $x \in \mathcal{M}$  and let  $P_x^F = \{I_1, \dots, I_m\}$  be the partition of  $\mathbb{N}_n^1$  induced by the  $P_*$ -decomposition of  $P_x$ . Then, for every  $\epsilon > 0$ , there exists a  $w \in T_{\mathcal{M}}(x) \cap B(0, \epsilon)$ , such that in the vector  $w^F \in \mathbf{R}^{\mathbb{N}_n^1}$  every subvector  $w_{I_i}^F$  has distinct coordinates for all  $i \in \mathbb{N}_m$ .*

**Proof.** By Corollary 4.4, we can choose  $y \in \mathcal{M} \cap \Delta_{P_*}$  arbitrarily close to  $x$ . Apply Proposition 3.10 to  $x$  and  $y$  to conclude that  $y, \bar{\pi}_T(y) \in \Delta_{P'}$  for some  $P' \succeq P_x$ . Necessarily, we have  $P' = P_*$ , implying that  $\bar{\pi}_T(y) \in \Delta_{P_*}$ . This shows that  $(\bar{\pi}_T(y))^F$  has distinct coordinates. In other words, there is a vector  $w \in T_{\mathcal{M}}(x)$  such that  $(\bar{\pi}_T(y))^F = (x + w)^F = x^F + w^F$  has distinct coordinates. Since  $y$  can be chosen arbitrarily close to  $x$ , we can assume that  $w$  is arbitrarily close to 0. Finally, since  $x^F \in \Delta_{P_x^F}$  and  $w^F = (x^F + w^F) - x^F$ , we conclude that  $w_{I_i}^F$  has distinct coordinates for all  $i \in \mathbb{N}_m$ . ■

Let  $y \in \mathbf{R}^n$  and consider the  $(n! + 1) \times (n + 1)$  matrix  $Y$  with first row  $(1, \dots, 1, 0) \in \mathbf{R}^{n+1}$  and consecutive rows equal to  $(\sigma y, 1)$  for each  $\sigma \in \Sigma^n$ . For example, when  $n = 2$  the matrix  $Y$  is  $3 \times 3$  and equal to

$$\begin{pmatrix} 1 & 1 & 0 \\ y_1 & y_2 & 1 \\ y_2 & y_1 & 1 \end{pmatrix}.$$

**Lemma 6.2** (Matrix of full rank). *If the numbers  $y_1, \dots, y_n$  are not all equal, then the matrix  $Y$  defined above has full rank.*

**Proof.** If  $n = 1$  the statement is trivial, so let  $n \geq 2$ . Suppose that  $(x, \alpha) \in \mathbf{R}^n \times \mathbf{R}$  is in the null space of  $Y$ . Then,  $y^\top P x + \alpha = 0$  for all permutation matrices  $P$  and  $x_1 + \dots + x_n = 0$ . Hence,  $y^\top (P - Q)x = 0$  for all permutation matrices  $P$  and  $Q$ . Without loss of generality,  $y_1 \neq y_2$ . For any distinct indices  $r$  and  $s$ , choose  $P$  and  $Q$  so that  $(P - Q)x = (x_r - x_s, x_s - x_r, 0, \dots, 0)$ . This shows that  $x_s = x_r$ . Since  $r$  and  $s$  are arbitrary, we deduce  $x = 0$  and  $\alpha = 0$ , as required. ■

The following corollary is used in the proof of Theorem 5.1.

**Corollary 6.3.** *Let  $x \in \Delta_P^\perp$  for some  $P \in \Pi_n$  and let  $P = \{I_1, \dots, I_m\}$ . Let  $y \in \mathbf{R}^n$  be such that each subvector  $y_{I_i}$  has distinct coordinates, for all  $i \in \mathbb{N}_m$ . Then, the existence of a constant  $\alpha \in \mathbf{R}$  such that*

$$\langle x, \sigma y \rangle = \alpha \quad \text{for all } \sigma \in \Sigma_P^n, \quad (6.1)$$

*is equivalent to the fact that  $x = 0$  (and thus  $\alpha = 0$ ).*

**Proof.** The sufficiency part is obvious, so we need only prove the necessity. We prove the claim by induction on  $m$ . If  $m = 1$ , then  $x \in \Delta_P^\perp$  is equivalent to  $x_1 + \cdots + x_n = 0$ . This together with (6.1) implies that the extended vector  $\bar{x} := (x, -\alpha)$  is a solution to the linear system  $Y\bar{x} = 0$ , where  $Y$  is the matrix defined above. By Lemma 6.2,  $Y$  has full column rank, which implies that  $x = 0$  and  $\alpha = 0$ . Suppose now that the result is true for  $m - 1$ , we prove it for  $m$ . For each  $\sigma \in \Sigma_P^n$  we have the natural decomposition  $\sigma = \sigma_1 \circ \cdots \circ \sigma_m$ , where each  $\sigma_j$  is a permutation on  $I_j$ —the restriction of  $\sigma$  to the set  $I_j$ ,  $j \in \mathbb{N}_m$ . Thus,

$$\langle x, \sigma y \rangle = \langle x_{I_1}, \sigma_1 y_{I_1} \rangle + \cdots + \langle x_{I_m}, \sigma_m y_{I_m} \rangle.$$

Fix a permutation  $\sigma_1$  on  $I_1$ . Since

$$\langle x_{I_2}, \sigma_2 y_{I_2} \rangle + \cdots + \langle x_{I_m}, \sigma_m y_{I_m} \rangle = \alpha - \langle x_{I_1}, \sigma_1 y_{I_1} \rangle$$

for any permutation  $\sigma_j$  on  $I_j$ ,  $j = 2, \dots, m$ , we conclude by the induction hypothesis that  $x_{I_2} = \cdots = x_{I_m} = 0$  and that  $\alpha - \langle x_{I_1}, \sigma_1 y_{I_1} \rangle = 0$ . But the permutation  $\sigma_1$  was arbitrary, so we obtain  $\langle x_{I_1}, \sigma_1 y_{I_1} \rangle = \alpha$  for all permutations  $\sigma_1$  on  $I_1$ . This, by the considerations in the base case of the induction, shows that  $x_{I_1} = 0$  and  $\alpha = 0$ . ■

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