UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE INGENIERÍA MATEMÁTICA

CONTRIBUTIONS TO ERGODIC THEORY AND TOPOLOGICAL DYNAMICS: CUBE STRUCTURES AND AUTOMORPHISMS.

TESIS PARA OPTAR AL GRADO DE DOCTOR EN CIENCIAS DE LA INGENIERÍA, MENCIÓN MODELACIÓN MATEMÁTICA
EN COTUTELA CON LA UNIVERSITÉ PARIS-EST MARNE-LA-VALLÉE

SEBASTIÁN ANDRÉS DONOSO FUENTES

PROFESOR GUÍA 1:
ALEJANDRO MAASS SEPÚLVEDA

PROFESOR GUÍA 2:
BERNARD HOST

MIEMBROS DE LA COMISIÓN:
ROMAIN DUJARDIN
RAPHAËL KRIKORIAN
SERVET MARTÍNEZ
MATHIEU MEYER
JEAN PAUL THOUVENOT

SANTIAGO DE CHILE
2015

RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE: Doctor en Ciencias de la Ingeniería, mención Modelación Matemática.
POR: Sebastián Andrés Donoso Fuentes.
FECHA: 28/05/2015
PROFESOR GUÍA: Alejandro Maass Sepúlveda.

CONTRIBUTIONS TO ERGODIC THEORY AND TOPOLOGICAL DYNAMICS: CUBE STRUCTURES AND AUTOMORPHISMS.

Esta tesis está consagrada al estudio de diferentes problemas en teoría ergódica y dinámica topológica, relacionados a “estructuras de cubos”. Consta de seis capítulos.

En la presentación general entregamos resultados generales, ligados en cierta manera a las estructuras de cubos que motivan esta tesis. Comenzamos por las estructuras de cubos introducidas en teoría ergódica por Host y Kra para probar la convergencia en L^2 de medias ergódicas múltiples. Luego presentamos su extensión a dinámica topológica, desarrollada por Host, Kra y Maass (2010), que entrega herramientas para entender la estructura topológica de sistemas dinámicos topológicos. Finalmente, mostramos las implicancias y extensiones principales derivadas de estudiar estas estructuras, motivamos los nuevos objetos introducidos en esta tesis y bosquejamos nuestras contribuciones.

En el Capítulo 1, entregamos antecedes generales en teoría ergódica y dinámica topológica, dando énfasis al estudio de ciertos factores especiales.

Desde el Capítulo 2 al Capítulo 5 desarrollamos las contribuciones de esta tesis. Cada uno está consagrado a un tópico diferente y a sus problemáticas relacionadas, tanto en teoría ergódica como en dinámica topológica. Cada uno está asociado a un artículo científico.

En el Capítulo 2 introducimos una nueva estructura de cubos para estudiar la acción de dos transformaciones S y T que conmutan, sobre un espacio métrico compacto X. En el mismo capítulo estudiamos las propiedades topológicas y dinámicas de tales estructuras y las usamos para caracterizar productos de sistemas y sus factores. También damos algunas aplicaciones, como la construcción de factores especiales. En el mismo tema, en el Capítulo 3 usamos esta nueva estructura para probar la convergencia casi segura de una media cúbica en un sistema con dos transformaciones que conmutan.

En el Capítulo 4, estudiamos el semigrupo envolvente de una clase importante de sistemas dinámicos, los nilsistemas. Usamos estructuras de cubos para mostrar relaciones entre propiedades algebraicas del semigrupo envolvente con la geometría y dinámica de un sistema. En particular, caracterizamos nilsistemas de orden 2 vía el semigrupo envolvente.

En el Capítulo 5 estudiamos grupos de automorfismos de sistemas simbólicos uno y dos dimensionales. Primero consideramos sistemas simbólicos de baja complejidad y usamos factores especiales, algunos ligados a estructuras de cubos, para estudiar el grupo de automorfismos. Nuestro resultado principal establece que en sistemas minimales de complejidad sublineal, tales grupos son generados por el shift y un conjunto finito. También, usando factores asociados a las estructuras de cubos del Capítulo 2, estudiamos el grupo de automorfismos de un sistema de embaldosados representativo.

Las referencias bibliográficas aparecen al final del documento.

Abstract

This thesis is devoted to the study of different problems in ergodic theory and topological dynamics related to “cube structures”. It consists of six chapters.

In the General Presentation we review some general results in ergodic theory and topological dynamics associated in some way to cubes structures which motivates this thesis. We start by the cube structures introduced in ergodic theory by Host and Kra (2005) to prove the convergence in L^2 of multiple ergodic averages. Then we present its extension to topological dynamics developed by Host, Kra and Maass (2010), which gives tools to understand the topological structure of topological dynamical systems. Finally we present the main implications and extensions derived of studying these structures, we motivate the new objects introduced in the thesis and sketch out our contributions.

In Chapter 1 we give a general background in ergodic theory and topological dynamics given emphasis to the treatment of special factors.

From Chapter 2 to Chapter 5 we develop the contributions of this thesis. Each one is devoted to a different topic and related questions, both in ergodic theory and topological dynamics. Each one is associated to a scientific article.

In Chapter 2 we introduce a novel cube structure to study the action of two commuting transformations S and T on a compact metric space X. In the same chapter we study the topological and dynamical properties of such structure and we use it to characterize product systems and their factors. We also provide some applications, like the construction of special factors. In the same topic, in Chapter 3 we use the new cube structure to prove the pointwise convergence of a cubic average in a system with two commuting transformations.

In Chapter 4, we study the enveloping semigroup of a very important class of dynamical systems, the nilsystems. We use cube structures to show connexions between algebraic properties of the enveloping semigroup and the geometry and dynamics of the system. In particular, we characterize nilsystems of order 2 by its enveloping semigroup.

In Chapter 5 we study automorphism groups of one-dimensional and two-dimensional symbolic spaces. First, we consider low complexity symbolic systems and use special factors, some related to the introduced cube structures, to study the group of automorphisms. Our main result states that for minimal systems with sublinear complexity such groups are spanned by the shift action and a finite set. Also, using factors associated to the cube structures introduced in Chapter 2 we study the automorphism group of a representative tiling system.

The bibliography is defer to the end of this document.

Résumé

Cette thèse est consacrée à l'étude des différents problèmes liés aux « structures des cubes », en théorie ergodique et en dynamique topologique. Elle est composée de six chapitres.

La présentation générale nous permet de présenter certains résultats généraux en théorie ergodique et dynamique topologique. Ces résultats, qui sont associés d'une certaine façon aux structures des cube, sont la motivation principale de cette thèse. Nous commençons par les structures de cube introduites en théorie ergodique par Host et Kra (2005) pour prouver la convergence dans L^2 de moyennes ergodiques multiples. Ensuite, nous présentons la notion correspondante en dynamique topologique. Cette théorie, développée par Host, Kra et Maass (2010), offre des outils pour comprendre la structure topologique des systèmes dynamiques topologiques. En dernier lieu, nous présentons les principales implications et extensions dérivées de l'étude de ces structures. Ceci nous permet de motiver les nouveaux objets introduits dans la présente thèse, afin d'expliquer l'objet de notre contribution.

Dans le Chapitre 1, nous nous attachons au contexte général en théorie ergodique et dynamique topologique, en mettant l'accent sur l'étude de certains facteurs spéciaux.

Les Chapitres 2, 3, 4 et 5 nous permettent de développer les contributions de cette thèse. Chaque chapitre est consacré à un thème particulier et aux questions qui s'y rapportent, en théorie ergodique ou en dynamique topologique, et est associé à un article scientifique.

Les structures de cube mentionnées plus haut sont toutes définies pour un espace muni d'une unique transformation. Dans le Chapitre 2, nous introduisons une nouvelle structure de cube liée à l'action de deux transformations S et T qui commutent sur un espace métrique compact X . Nous étudions les propriétés topologiques et dynamiques de cette structure et nous l'utilisons pour caractériser les systèmes qui sont des produits ou des facteurs de produits. Nous présentons également plusieurs applications, comme la construction des facteurs spéciaux.

Le Chapitre 3 utilise la nouvelle structure de cube définie dans le Chapitre 2 dans une question de théorie ergodique mesurée. Nous montrons la convergence ponctuelle d'une moyenne cubique dans un système muni deux transformations qui commutent.

Dans le Chapitre 4, nous étudions le semigroupe enveloppant d'une classe très importante des systèmes dynamiques, les nilsystèmes. Nous utilisons les structures des cubes pour montrer des liens entre propriétés algébriques du semigroupe enveloppant et les propriétés topologiques et dynamiques du système. En particulier, nous caractérisons les nilsystèmes d'ordre 2 par une propriété portant sur leur semigroupe enveloppant.

Dans le Chapitre 5, nous étudions les groupes d'automorphismes des espaces symboliques

unidimensionnels et bidimensionnels. Nous considérons en premier lieu des systèmes symboliques de faible complexité et utilisons des facteurs spéciaux, dont certains liés aux structures de cube, pour étudier le groupe de leurs automorphismes. Notre résultat principal indique que, pour un système minimal de complexité sous-linéaire, le groupe d'automorphismes est engendré par l'action du shift et un ensemble fini. Par ailleurs, en utilisant les facteurs associés aux structures de cube introduites dans le Chapitre 2, nous étudions le groupe d'automorphismes d'un système de pavages représentatif.

La bibliographie, commune à l'ensemble de la thèse, se trouve en fin document.

Acknowledgements

First, I would to thank my advisors Bernard Host and Alejandro Maass who accepted to guide me. I thank all the time they spent with me, with illuminating and encouraging discussions. I also thank the effort they put to make me able to contact different teams of research. This has clarified a lot my global vision about mathematical research.

I also thank Jean-Paul Thouvenot and Xiangdong Ye who generously accepted to be rapporteurs of this thesis. I am grateful to Romain Dujardin, Raphaël Krikorian, Servet Martínez and Mathieu Meyer for being part of the jury. It is an honor for me to have such a jury.

I am also very grateful to Bryna Kra, for her hospitality in my visit to Evanston and for all the support she has given to me. I also thank my colleague and friend Wenbo Sun with whom I look forward to continue collaborating for a long time.

I thank Fabien Durand and Samuel Petite, for make me have pleasant stays in Amiens, which have enriched a lot my mathematical background.

I would also like to thank Song Shao, Xiangdong Ye and all their students for a wonderful stay in Hefei, full of exciting mathematical discussions and very nice food.

To the graduate students at the Laboratoire d'Analyse et de Mathématiques Appliquées (LAMA): Bertrand, Marie-Noémi, Marwa, Pierre and Rania. Specially I thank Harry, Paolo and Xavier who shared with me many coffees and conversations around a beer. I am also very grateful to Sylvie Cach, who has helped me a lot in all the administrative procedures. I also thank to the graduate students at Center for Mathematical Modeling (CMM), specially to Alexis, Amaru, Héctor, Emilio, Erwin, Richard and Roberto.

I thank all people who helped me to have a very nice living in Paris. I thank all the "Chileans and friends in Paris": Ana, Antonella, Celia, Cristóbal, Daniela, Eduardo, Fabiola A., Fabiola M., Felipe, Javier, Marcy, Pablo, Sofía and Suomi. Special thanks to David, Nicolás, Mariela, Matías, Miraine and Teresa. I also thank my friends Nicolas and Jacqueline with whom I enjoyed several footings, beers and french dinners. Thanks to Valérie for her friendship and Chris for her hospitality and great dinners.

I thank my friends in Chile who have supported me during these last years, specially Constanza, Cristian, Diego, Fernando, Italo, Jimena, Karen, Omar and Sebastián. I know some barbecues and *piscolas* are waiting for me.

Me tomo la libertad de escribir este párrafo en español. Quiero agradecer especialmente a mi familia: a mi padre Javier, a mi madre Iris y a mi hermano Javier, quienes me hen dado toda la fuerza y el apoyo para seguir adelante en este proyecto de vida. De ellos estaré toda la vida agradecido.

This work was funded by the CONICYT DOCTORADO 21110300 and CMM-basal grants.

Contents

General Presentation

This thesis document presents four research articles concerning different problems on a common theme in ergodic theory and topological dynamics. It consists of six chapters and we divide it into two parts. The first one is devoted to the study of *cube structures* and its applications in ergodic theory and topological dynamics. The second part is centered on automorphism groups in symbolic dynamics. From Chapter 2 to Chapter 5 we present our research articles with their own introduction part. In this general presentation we motivate the main objects we study and explain our contributions. We start by explaining briefly the historical background and framework, and then we present our main contributions and its motivations.

Cube structures in ergodic theory

A central problem in combinatorial number theory is to understand notions of "largeness"of a subset of the integer numbers and when such a notion implies the existence of some prescribed patterns. In particular, the existence of arithmetic progressions has been a widely considered object of study. A notion of largeness that has been very well studied is the one of having positive *upper density*. The upper density of a subset of the integers S is the quantity

$$\limsup_{N\to\infty} \frac{\sharp(S\cap[0,N-1])}{N}.$$

In 1975 Szemerédi [111] proved its celebrated theorem: any subset of the integers with positive upper density contains arbitrarily long arithmetic progressions. Soon thereafter, in 1976 Furstenberg [50] proved the same result by using ergodic methods. Namely, he proved that if $(X,\mathcal{X},\mu,T)$ is a measure preserving system and $A\in\mathcal{X}$ is a set of positive measure then for every $d\in\mathbb{N}$

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(A\cap T^{-n}A\cap T^{-2n}A\cdots\cap T^{-dn}A)>0.$$

A correspondence principle allows then to translate this property into a combinatorial property of a subset of the integers. This result established a deep connection between combinatorics, number theory and ergodic theory which has been widely exploited in the last decades.

A fundamental question in ergodic theory that arise from Furstenberg's result is the convergence in L^2 of the multiple averages

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x)\cdots f_d(T^{dn} x). \tag{0.0.1}$$

The case $d=2$ was solved by Furstenberg [50]. Several works by Lesigne [86], Conze and Lesigne [24, 25, 26] and Host and Kra [65] dealt with the case $d=3$. After more than 20 years the convergence of the general case was finally solved by Host and Kra [67]. Their proof is a consequence of a deep structural theorem for measure preserving systems: they built a sequence of nested factors $(\mathcal{Z}_d)_{d\in\mathbb{N}}$ which are measurably isomorphic to inverse limits of ergodic nilsystems (translations on compact homogeneous spaces of nilpotent Lie groups). Then, they reduced the study of the multiple average by looking at $\mathcal{Z}_d$ and its orthogonal complement. Moreover, they showed that the limit of the average remains unchanged if one replaces one of the functions by its conditional expectation with respect to the $\mathcal{Z}_d$ factor. In Furstenberg terminology, this means that the factors $\mathcal{Z}_d$ are *characteristic factors* for multiple ergodic averages.

Given a probability space $(X,\mathcal{X},\mu)$ and a measure preserving transformation $T\colon X\to X$, their main idea is to build for any $d\in\mathbb{N}$ a "cube"measure $\mu^{[d]}$ in X^{2^d} and a seminorm $|||\cdot|||_d$ on the set of bounded measurable functions on X which is useful to study multiple ergodic averages. They describe the orthogonal complement of $\mathcal{Z}_d$ by the relation

$$\mathbb{E}(f|\mathcal{Z}_d)=0 \text{ if and only if } |||f|||_{d+1}=0.$$

The more remarkable (and hard) result is their structure theorem, which states that the $\mathcal{Z}_d$ factors have a very nice algebraic structure.

Theorem 0.0.1 (Host-Kra structure theorem)**.** *For any $d\in\mathbb{N}$, the factor $\mathcal{Z}_d$ is measurably isomorphic to an inverse limit of d-step nilsystems.*

Therefore, nilsystems and their inverse limits are characteristic factors for multiple averages.

The study of nilsystems as mathematical objects was considered from the 60's, but during the last years its study has been revitalized and has attracted the attention of several researchers, mainly because of its applications in additive combinatorics and number theory [49, 57, 58, 59, 60].

Moreover, the structure theorem has resulted to be very useful for the study of related convergence problems [14, 23, 44, 46, 47, 68, 71, 83], in the study of correlation sequences [13, 69, 48, 45] and even for the study of pointwise convergence problems [2, 3, 4, 22, 76, 77].

The idea of cubes was also studied in topological dynamics. In 2010 Host, Kra and Maass [70] explored the topological counterpart of the cube measures introduced in [67].

For a topological dynamical system (X,T) they introduced the space of dynamical cubes $\mathbf{Q}^{[d]}(X,T)$ and studied its properties.

Roughly speaking, the space of topological dynamical cubes $\mathbf{Q}^{[d]}(X,T)$ is a closed subset of X^{2^d} and plays the role of the support of the measure $\mu^{[d]}$ mentioned above. They showed that some properties of the space of cubes can be translated into strong dynamical properties of the system (X,T). Namely, they introduced a relation they called $\mathbf{RP}^{[d]}(X,T)$ defined in terms of cubes which allows to characterize nilrotations. Afterwards, Shao and Ye [110] proved that this relation is an equivalence one for general minimal systems and the quotient of (X,T) under this relation defines the maximal factor of (X,T) which is (topologically) isomorphic to a nilsystem. In other words, the factor $X/\mathbf{RP}^{[d]}(X,T)$ is the topological analogue of the Host-Kra factor $\mathcal{Z}_d$.

In recent years, numerous applications in ergodic theory and topological dynamics have been found for the Host-Kra-Maass topological structural theory of nilsystems. It ranges from the study of recurrence problems in topological dynamics [32, 72, 75] to, surprisingly, the study of pointwise convergence of multiple ergodic averages [76, 77] (we develop this topic later).

Objects like cube structures also appeared in the study of the convergence of averages that generalize the ones considered by Host and Kra in [67], like

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(T_1^n x) f_2(T_2^n x)\cdots f_d(T_d^n x)$$

where we are considering a probability space $(X,\mathcal{X},\mu)$ and $T_1,\dots,T_d$ are measure preserving transformations on X such that $T_i\circ T_j = T_j\circ T_i$ for every $i,j=1,\dots,d$ and $f_1,\dots,f_d$ are bounded functions.

The convergence of this average was first proved by Tao [112] using finitary methods. Soon after, Townser [113], Austin [9] and Host [64] gave other proofs for the same result using different strategies. The proof given by Towsner uses non-standard analysis and only the proofs of Austin and Host belong to ergodic theory and try to follow the ideas of structure theorems. In both Austin and Host proofs, the idea is to first find an extension of the ergodic system with convenient properties. The extension given by Host is much easier to manage so we focus our attention on that one. The main idea in Host's proof is to build an extension of X (*magic* in his terminology) such that it has a characteristic factor for the average that looks like the Cartesian product of single transformations. To build such extension and factor, cube structures are introduced, analogous to the ones in [67]. Recently, Walsh [115] proved the convergence of multiple averages for nilpotent group actions but his proof follows the original idea of Tao and does not use ergodic methods.

Contributions

Dynamical cubes in a system with two commuting transformations

Motivated by Host's construction in [64] and the topological theory of cubes of Host, Kra and Maass in [70] for one single transformation, in Chapter 2 we present our work *Dynamical cubes and a criteria for systems having product extensions* [34], joint work with Wenbo Sun, where we explore a topological counterpart of the cubes introduced in [64] to see if one can characterize interesting properties of a system with commuting transformations, as was done in [70] for one single transformation. Given a compact metric space X and two commuting homeomorphisms $S\colon X \to X$ and $T\colon X \to X$ we define the space of dynamical cubes $\mathbf{Q}_{S,T}(X)$ as

$$\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^n x, T^m x, S^n T^m x) : x \in X, n, m \in \mathbb{Z}\}} \subseteq X^4.$$

Using the space $\mathbf{Q}_{S,T}(X)$ we succeeded to characterize a simple class of systems, namely products of minimal topological dynamical systems and their factors. A product system is one of the form $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ where (Y, σ) and (W, τ) are topological dynamical systems. The condition "to complete the last coordinate of a point in $\mathbf{Q}_{S,T}(X)$ in a unique way"is equivalent to be a factor of a product system. More precisely, if $(x_0, x_1, x_2, x_3), (x_0, x_1, x_2, y_3) \in \mathbf{Q}_{S,T}(X)$ then $x_3 = y_3$. We also provide several applications of these structures in topological dynamics, like the construction of "topological magic"extensions and special factors. In what follows, further applications to the pointwise convergence of some averages and automorphisms of symbolic systems are shown.

Pointwise convergence of cubic averages

The study of the cube structure $\mathbf{Q}_{S,T}(X)$ together with new results by Huang, Shao and Ye [76] leads to prove an almost sure convergence of some cubic averages when considering two commuting transformations. This is the joint work with Wenbo Sun *A pointwise cubic average for two commuting transformations* [34].

Cubic averages are part of a plethora of non-conventional ergodic averages that has been considered since Furstenberg's work, like the multiple ergodic average 0.0.1. From all these studies it follows that the nature of the problem of pointwise convergence is completely different from the one in L^2.

Historically, in the 90's Bourgain [16] studied and proved the convergence of the average

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an} x) f_2(T^{bn} x)$$

for integers a and b and bounded functions f_1 and f_2.

Little progress has been made since Bourgain's result, mainly because the usual technique to deduce pointwise convergence uses maximal inequalities, which seems not to work for $d > 2$. Very recently, a significant step towards the general solution was done by Huang, Shao and Ye [76] who introduced a new technique to study the pointwise convergence of ergodic averages. They deeply exploited the theory of topological cubes developed in [70] and [110] to find convenient topological models for ergodic systems. Namely, they found a topological model with a uniquely ergodic space of dynamical cubes (and another structures that we do not discuss here). Then they were able to show, among other things, that multiple ergodic averages converge in a measurable distal system. They also applied this technique to deduce the pointwise convergence of cubic averages, that is, averages like

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T^i x) f_2(T^j x) f_3(T^{i+j} x)$$

or like

$$\frac{1}{N^3}\sum_{i,j,k=0}^{N-1} f_1(T^i x) f_2(T^j x) f_3(T^{i+j} x) f_4(T^k x) f_5(T^{i+k} x) f_6(T^{j+k} x) f_7(T^{i+j+k} x)$$

and their natural generalizations.

In the L^2 setting, the first convergence result of a cubic average was given by Bergelson [12] who showed the L^2 convergence of

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T^i x) f_2(T^j x) f_3(T^{i+j} x). \tag{0.0.2}$$

Host and Kra [67] generalized the L^2 convergence to higher order averages using the $\mathcal{Z}_d$ factors (which are also characteristic for this kind of averages).

When considering more transformations, one can consider different kind of averages. For example, one can consider averages like,

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^i x) f_2(T^j x) f_3(R^{i+j} x) \tag{0.0.3}$$

or like

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^i x) f_2(T^j x) f_3(S^i T^j x) \tag{0.0.4}$$

The pointwise convergence of the average 0.0.3 was proved by Assani [2]. Then, Chu and Frantzikinakis [22] proved the pointwise convergence when one consider an arbitrary number of transformations. More precisely, they proved the convergence of

$$\frac{1}{N^3}\sum_{i,j,k=0}^{N-1} f_1(T_1^i x)f_2(T_2^j x)f_3(T_3^{i+j}x)f_4(T_4^k x)f_5(T_5^{i+k}x)f_6(T_6^{j+k}x)f_7(T_7^{i+j+k}x)$$

and its natural generalizations when considering 2^d-1 transformations. In their proof, in fact no assumption of commutativity of the transformations was needed.

In the other hand, the average 0.0.4 may not converge if one does not have commutativity assumptions [81]. So averages 0.0.3 and 0.0.4 have a very different nature.

Interestingly, combining the cube structure introduced in Chapter 2, the Huang-Shao-Ye strategy and the theory developed by Host in [64] we prove the pointwise convergence of the average 0.0.4 provided that the transformations S and T commute.

Enveloping semigroups of nilsystems

Another independent application of the theory of topological cubes [70] is presented in Chapter 4, which is based on the work *Enveloping semigroups of system of order d* [39]. Given a topological dynamical system (X,T), its enveloping semigroup is the closure of the set $\{T^n : n\in\mathbb{N}\}$ in X^X in the product topology. This object was introduced by Ellis in the 60's and has proved to be a very useful tool to understand the dynamics of a system [7, 42] and properties and applications are still being found (see [56] for example). A very important feature of the enveloping semigroup is the fact that one can connect dynamical and geometrical properties of a system with algebraic properties of its enveloping semigroup and vice versa. For example, a topological dynamical system is a rotation on a compact abelian group if and only if its enveloping semigroup is an abelian group and it is distal if and only if its enveloping semigroup is a group. When the enveloping semigroup is not abelian a few results are known, in particular when the enveloping semigroup is nilpotent. This question was first studied by Glasner [53], who proved, up to some details that we do not give to simplify the discussion, that for systems who are torus extensions of equicontinuous systems the condition of having a 2-step nilpotent enveloping semigroup is equivalent to be a homogeneous space of a 2-step nilpotent Polish group. We extend this result, characterizing 2-step nilsystems through the enveloping semigroup. We introduce the notion of topologically nilpotency which is stronger than purely algebraic nilpotency and results more convenient in our context. We show that a topological dynamical system is a 2-step nilsystem if and only if its enveloping semigroup is a 2-step topologically nilpotent group. For higher orders of nilpotency the questions are more intricate and certainly require to develop new machinery. In the non abelian case, explicit computations of enveloping semigroups are rare and one can hope to succeed in this task only in very particular cases. Some examples in the literature of computations of enveloping semigroups can be found in [5, 6, 92, 99, 100, 104]. In [99, 100, 104] the authors considered particular classes of nilsystems (affine nilsystems in tori, where the dynamics is given by multiplication by particular matrices) and computed their enveloping semigroups. Using the explicit description they got they were able to deduce algebraic properties from the enveloping semigroups. Specially they deduce that such enveloping semigroups are always nilpotent groups. Using the theory of dynamical cubes introduced by Host, Kra and Maass we deduce algebraic properties of nilsystems and their inverse limits. Namely, we prove that inverse limits of d-step nilsystems have d-step nilpotent enveloping semigroups, without performing any explicit computation. These results include the previous known examples.

Automorphism groups in symbolic dynamics

The second part of this thesis is devoted to the study of automorphism groups, which is a classical topic in symbolic dynamics studied since the 70's in different contexts and that is now again under study. Even if this topic seems to be far from our previous motivation and cube structures, we arrive to them from the study of cubes. In particular, when looking for applications of our $\mathbf{Q}_{S,T}$ cubes and associated factors. Indeed, the way we propose to study automorphisms groups for tilings and other symbolic systems is by exploring in detail the fibers over these factors. We need to give a little background in symbolic dynamics. Given a finite set $\mathcal{A}$, a space shift or *subshift* over $\mathcal{A}$ is a closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ (endowed with the product topology), invariant under the *shift action* $\sigma\colon X \to X$, $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$. Subshifts are very important objects in ergodic theory and topological dynamics, see [87] for a nice survey about subshifts and their applications.

One associates to a subshift its automorphism group. An automorphism of a subshift (X,σ) is a homeomorphism $\phi\colon X \to X$ which commutes with σ (*i.e.* $\phi \circ \sigma = \sigma \circ \phi$). It is a classical result by Curtis, Hedlund and Lyndon that such maps are given by a *local map* $\widehat{\phi} : \mathcal{A}^{2\mathbf{r}+1} \to \mathcal{A}$ such that

$$\phi((x_i)_{i\in\mathbb{Z}})_n = \widehat{\phi}(x_{n-\mathbf{r}}, \ldots, x_{n+\mathbf{r}})$$

for any $n \in \mathbb{Z}$. The map $\widehat{\phi}$ is called the *sliding block code* associated to ϕ and the integer $\mathbf{r}$ is the *radius* of ϕ. We let $\mathrm{Aut}(X,\sigma)$ denote the group of automorphisms of (X,σ) and we refer to it as the *automorphism group*. The study of automorphism groups is a fundamental tool to understand the complexity of the subshifts and provides a good invariant for classifying them. Also, from a purely dynamical systems point of view, if ϕ is an automorphism of (X,σ), the topological dynamical system (X,σ,ϕ) is a nice $\mathbb{Z}^2$ action to be studied. This setting has been used to model the evolution of complex physical systems.

Automorphism groups has been widely studied in symbolic dynamics, both in the measurable and the topological setting. In ergodic theory, the group of measurable automorphisms (*i.e.* measurable functions which commute with the shift almost everywhere and preserve the measure) has been exhaustively studied for mixing systems of finite rank [43]. Orstein [94] proved that for mixing rank one systems this group consists only in the powers of the shift. Then del Junco [31] proved the same result for the rank one Chacon subshift. Finally King and Thouvenot [79] proved that for mixing systems of finite rank the group of measurable automorphisms is spanned by the powers of the transformation and a finite set. The same result was also proved by Host and Parreau [74] for some constant length substitutions.

In the topological setting, Boyle, Lind and Rudolph [17] describe the automorphism group of a positive entropy mixing shift of finite type. They showed that it is a very large object, it contains many subgroups. Recently, Hochman [62] proved similar results for multidimensional shifts of finite type with positive entropy.

Nevertheless, little was known about the automorphism group of low complexity subshifts. Here, by complexity we mean the increasing function $P_X \colon \mathbb{N} \to \mathbb{N}$ such that $P_X(n)$ is the number of non-empty cylinders of length n appearing in the subshift. We remark that the topological entropy of (X, σ) is nothing but the exponential growth rate of its complexity function. For low complexity systems, the first result in the topological setting is due to Hedlund [61], who described the automorphism group for a family of binary substitutions which includes the Thue-Morse system. He proved that $\mathrm{Aut}(X, \sigma)$ consists in powers of the shift and a flip map (a map which interchanges zeros and ones). Recently, some new results have appeared. Olli [93] proved that for Sturmian systems, $\mathrm{Aut}(X, \sigma)$ is spanned by shift and Salo and Törmä [107] proved that for constant length or primitive Pisot substitutions the group of automorphisms is spanned by σ and a finite set. In [107] it is asked whether the same result holds for any primitive substitution or more generally for linearly recurrent subshifts. In Chapter 5, we present our work *On automorphisms groups of minimal low complexity subshifts*, joint with Fabien Durand, Alejandro Maass and Samuel Petite [36]. We show, among other results that if the complexity is sublinear in a subsequence, *i.e.* if

$$\liminf_{n \in \mathbb{N}} \frac{p_X(n)}{n} < \infty$$

then $\mathrm{Aut}(X, \sigma)$ is spanned by the powers of σ and a finite set. The class of systems satisfying this condition includes primitive substitutions, linearly recurrent subshifts [39] and even some families with polynomial complexity (since we require just liminf and not limsup). We show that this behaviour is still true in a wide variety of examples and we illustrate methods to deduce such results. Our main tool is the study of classical and new relations which are preserved under the action of any automorphism. Some of those relations come from fibers associated to nilfactors, which impose severe restrictions to the group of automorphisms.

Some of the main results in [36] were independently discovered by Cyr and Kra [30] using different methods. They previously proved in [29] that for a subshift (X, σ) with subquadratic growth (*i.e.* $\liminf_{n \in \mathbb{N}} \frac{p_X(n)}{n^2} = 0$) one has that $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ is a periodic group. They came to this problem studying the Nivat conjecture, and they used a combinatorial argument for

$\mathbb{Z}^2$ subshifts by Quas and Zamboni [101] that gives conditions to have periodic directions in $\mathbb{Z}^2$ subshifts.

Finally, we come to the $\mathbf{Q}_{S,T}$ cubes and factors which provide an interesting application to study automorphism groups of tiling systems.

The study of aperiodic tiling spaces is a topic considered by many people in very different contexts: in logic they started to be studied to determine whether the plane can be covered by a set of tiles satisfying adjacency rules (the Wang tiles); in geometry they provided nice examples with interesting symmetry properties (the Penrose tilings) and in physics they appeared in material science in the 80's when studying the so called *quasicrystals*.

At the end of Chapter 5 we consider a famous tiling space, the *Robinson tiling*, which was introduced by Robinson in the 70's [105] to study undecidability problems and that has been very useful in theoretical computer science. It is also a representative element of the well studied class of hierarchical tilings. We use the theory of cubes introduced in Chapter 2 to deduce that the group of automorphisms of the minimal Robinson tiling is spanned by the shift actions. We claim that this technique can be used to prove the same kind of results for well studied families of tilings, like hierarchical tilings or others like cut and project family.

Background in ergodic theory and topological dynamics

In this chapter we give basic definitions and background in ergodic theory and topological dynamics. We refer to [117] for definitions for measure preserving systems and [7] for definitions for topological dynamical systems. We also introduce the notion of nilfactors in ergodic theory and topological dynamics which is a central object of study in this thesis. More specific definitions will be given in every particular chapter when needed.

1.1. General definitions

1.1.1. Measure preserving systems

A *measure preserving system* is a 4-tuple $(X, \mathcal{X}, \mu, G)$, where $(X, \mathcal{X}, \mu)$ is a probability space and G is a group of measurable, measure preserving transformations acting on X. When there is no confusion, we omit the σ-algebra $\mathcal{X}$ and assume without lose of generality that the probability space is standard, meaning that it is isomorphic to [0,1] endowed with the Borel σ-algebra and whose measure is a combination of the Lebesgue measure and a countable or finite set of atoms. When we consider subsets of X, we always implicitly assume that they are measurable. Similarly, functions on X are assumed to be measurable and real valued.

For any two sub σ-algebras $\mathcal{A}$ and $\mathcal{B}$ of X, let $\mathcal{A} \vee \mathcal{B}$ denote the σ-algebra generated by $\{A \cap B \colon A \in \mathcal{A}, B \in \mathcal{B}\}$. It is the smallest σ-algebra containing $\mathcal{A}$ and $\mathcal{B}$. If f is a function on $(X, \mathcal{X}, \mu)$ and $\mathcal{A}$ is a sub-algebra of $\mathcal{X}$, let $\mathbb{E}(f|\mathcal{A})$ denote the conditional expectation of f over $\mathcal{A}$.

A measure preserving system (X, μ, G) is *ergodic* if any G-invariant set of X has measure 0 or 1.

A *factor map* between the measure preserving systems (Y, ν, G) and (X, μ, G) is a measure preserving map $\pi \colon Y \to X$ such that $\pi \circ g = g \circ \pi$ for all $g \in G$. We say that (X, μ, G) is a factor of (Y, ν, G) or that (Y, ν, G) is an extension of (X, μ, G). An equivalent definition of factor maps can be formulated via sub σ-algebras (here we need to write the σ-algebra): a factor map of $(Y, \mathcal{Y}, \nu, G)$ is an invariant sub σ-algebra of $\mathcal{Y}$. The equivalence of these definitions follows from considering the σ-algebra $\pi^{-1}(\mathcal{X})$.

If π is a bi-measurable (almost everywhere defined) bijection, we say that π is an *isomorphism* and that (Y,ν,G) and (X,μ,G) are *isomorphic*.

Some tools.

Ergodic decomposition of a measure:

Let (X,μ,G) be a measure preserving system and let $\mathcal{I}$ be the σ-algebra of G-invariant sets. Let $x\to\mu_x$ be a regular version of *conditional measures* with respecto to $\mathcal{I}$. This means that the map $x\mapsto\mu_x$ is $\mathcal{I}$-measurable and

$$\mathbb{E}(f|\mathcal{I})(x)=\int f d\mu_x \quad \mu\text{-a.e. } x\in X$$

The ergodic decomposition of μ under G is $\mu=\int_X \mu_x d\mu(x)$ and μ-a.e. the system (X,μ_x,G) is ergodic.

Conditional expectation and disintegration of a measure:

Let $\pi\colon Y\to X$ be a factor map between the measure preserving systems (Y,ν,G) and (X,μ,G) and let $f\in L^2(\nu)$. The *conditional expectation* of f with respect to X is the function $\mathbb{E}(f|X)\in L^2(\mu)$ defined by the equation

$$\int_X \mathbb{E}(f|X)\cdot g d\mu=\int_Y f\cdot g\circ\pi d\nu \qquad \text{for every } g\in L^2(\mu).$$

The following result is well known (see [51], Chapter 5 for example)

Theorem 1.1.1. *Let $\pi\colon Y\to X$ be a factor map between the measure preserving systems (Y,ν,G) and (X,μ,G). There exists a unique measurable map $X\to M(Y)$, $x\mapsto\nu_x$ such that*

$$\mathbb{E}(f|X)(x)=\int f d\nu_x \tag{1.1.1}$$

for every $f\in L^1(\nu)$.

We say that $\nu=\int_X \nu_x d\mu(x)$ is the *disintegration* of ν over μ.

1.1.2. Topological dynamical systems

A *topological dynamical system* is a pair (X,G), where X is a compact metric space and G is a group of homeomorphisms of the space X into itself. We always use $d(\cdot,\cdot)$ to denote the metric in X and we let $\Delta_X\coloneqq\{(x,x)\colon x\in X\}$ denote the diagonal of $X\times X$.

Since we deal with both measure preserving systems and topological dynamical systems, we always write the measure for a measure preserving system to distinguish them.

A (topological)*factor map* between the topological dynamical systems (Y,G) and (X,G) is an onto, continuous map $\pi\colon Y\to X$ such that $\pi\circ g=g\circ\pi$ for every $g\in G$. We say that (Y,G)

is an *extension* of (X, G) or that (X, G) is a *factor* of (Y, G). When π is bijective, we say that π is an isomorphism and that (Y, G) and (X, G) are isomorphic. An equivalent definition of a (topological) factor map is given through a closed equivalence relation $\mathcal{R} \subset Y \times Y$ invariant under the diagonal $G^{\triangle} := \{(g, g) : g \in G\}$. Given such a relation one can build the quotient space $Y/\mathcal{R}$ and the canonical projection from Y onto this quotient defines a natural factor map. Conversely, for any factor map $\pi\colon Y \to X$ one can consider the invariant closed equivalence relation $\mathcal{R}_\pi = \{(y, y') \in Y \times Y : \pi(y) = \pi(y')\}$ (see [7], Chapter 1 for further details). Building factors through invariant closed equivalence relations is a very useful way to obtain interesting special factors (see section of special factors for example).

We say that (X, G) is *transitive* if there exists a point in X whose orbit $\mathcal{O}_G(x) := \{gx\colon g \in G\}$ is dense. Equivalently, (X, G) is transitive if for any two non-empty open sets $U, V \subseteq X$ there exists $g \in G$ such that $U \cap g^{-1}V \neq \emptyset$.

A system (X, G) is *weakly mixing* if the Cartesian product $X \times X$ is transitive under the action of the diagonal of G. Equivalently, (X, G) is weakly mixing if for any four non-empty open sets $A, B, C, D \subseteq X$ there exists $g \in G$ such that simultaneously $A \cap g^{-1}B \neq \emptyset$ and $C \cap g^{-1}D \neq \emptyset$.

We say that (X, G) is *minimal* if the orbit of any point is dense in X. Let (X, G) be a topological dynamical system. A point $x \in X$ is *minimal* or *almost periodic* if $(\overline{\mathcal{O}_G(x)}, G)$ is a minimal system. A system (X, G) is *pointwise almost periodic* if any $x \in X$ is an almost periodic point.

Let (X, G) be a topological dynamical system and $(x, y) \in X \times X$. We say that (x, y) is a *proximal* pair if there exists a sequence $(g_i)_{i\in\mathbb{N}}$ in G such that

$$\lim_{i\to\infty} d(g_i x, g_i y) = 0,$$

and it is a *distal* pair if it is not proximal. We let $P(X)$ denote the set of proximal pairs. A topological dynamical system (X, G) is called *distal* if (x, y) is distal whenever $x, y \in X$ are distinct. Equivalently, (X, G) is distal if $P(X) = \triangle_X$. Distal systems have a lot of interesting properties which are stated later in the document when used.

In the following two sections, we focus our attention in the case where G is the cyclic group spanned by one single transformation T.

1.2. Classical special factors

A very classical and important factor associated to a measure preserving system is the *Kronecker factor*. Thinking of σ-algebras, the Kronecker factor $\mathcal{Z}_1$ of a system (X, μ, T) is the σ-algebra spanned by the eigenfunctions of the operator $L^2(\mu) \to L^2(\mu), f \mapsto f \circ T$. It is also the smallest σ-algebra such that any invariant function of the system $(X \times X, \mu \otimes \mu, T \times T)$

is measurable with respect to $\mathcal{Z}_1 \otimes \mathcal{Z}_1$. It is well known that $\mathcal{Z}_1$ has a very nice algebraic structure: it is measurably isomorphic to a rotation on a compact abelian group, meaning that it can be represented as (Z_1, m, T) where Z_1 is a compact abelian group, T is the rotation $z \mapsto \tau z$ for a fixed $\tau \in Z_1$ and m is the Haar measure of Z_1.

The topological analogue of the Kronecker factor is the maximal equicontinuous factor. For a topological dynamical system (X, T) its maximal equicontinuous factor is the largest factor of X where the family $\{T^n : n \in \mathbb{Z}\}$ is an equicontinuous one. Similarly to the measurable case, when (X, T) is minimal the maximal equicontinuous factor is (topologically) isomorphic to a rotation (Z_1, T) where Z_1 is a compact abelian group and T is the rotation by a fixed $\tau \in Z_1$. Rotations over compact abelian groups have many good properties: for them, minimality, transitivity, ergodicity and unique ergodicity are equivalent properties.

An important feature about the maximal equicontinuous factor is that it can be built through the *regionally proximal relation* [7]. Two points $x, y \in X$ are said to be regionally proximal if for any $\delta > 0$ there exist $x', y' \in X$ and $n \in \mathbb{Z}$ such that

$$d(x, x') < \delta, \quad d(y, y') < \delta \text{ and } d(T^n x', T^n y') < \delta.$$

We let $\mathbf{RP}(X)$ denote the set of regionally proximal pairs. It is clear that $\mathbf{RP}(X)$ is a closed invariant relation on X. The non trivial fact is that is also an equivalence relation when (X, T) is minimal. Moreover, this relation characterizes being an equicontinuous system: the quotient $X/\mathbf{RP}(X)$ is the maximal equicontinuous factor of (X, T) [7].

1.3. Nilfactors

The study of nilsystems is classical in ergodic theory and topological dynamics [8, 53, 96, 116] but its relevance has grown in the last years, mainly because of its importance in the study of multiple ergodic averages [67], in the structure analysis of measurable and topological systems [67, 70] and in the analysis of the existence of certain patterns in a subset of the integers [57]. We introduce the general definitions.

1.3.1. Nilpotent groups, nilmanifolds and nilsystems

Let G be a group. For $g, h \in G$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h and for $A, B \subseteq G$ we write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups G_j, $j \geq 1$, are defined inductively by setting $G_1 = G$ and $G_{j+1} = [G_j, G]$. Let $d \geq 1$ be an integer. We say that G is d-step nilpotent if G_{d+1} is the trivial subgroup. We remark that a subgroup of a d-step nilpotent group is also d-step nilpotent, and any abelian group is 1-step nilpotent.

Let G be a d-step nilpotent Lie group and Γ a discrete cocompact subgroup of G. The compact manifold $X = G/\Gamma$ is called a *d-step nilmanifold*. The fundamental properties of nilmanifolds were established by Malcev [88]. The group G acts on X by left translations and we write this action as $(g, x) \mapsto gx$. There exists a unique probability measure invariant under the action of G, called the *Haar measure* of X.

Let $\tau \in G$ and T be the transformation $x \mapsto \tau x$. Then (X, μ, T) is called a *d-step nilsystem*. We remark that (X, μ, T) is also a topological dynamical system if we do not consider the measure. In this case we just write (X, T).

We show next some known examples of nilsystems.

Rotations:

Rotations over compact abelian groups are 1-step nilsystems.

The Heisenberg system :

Let G be the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\},$$

and consider the cocompact subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & n & m \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix} : n, m, p \in \mathbb{Z} \right\}$$

Then G/Γ is a 2-step nilmanifold. Fix an element

$$\tau = \begin{pmatrix} 1 & \tau_1 & \tau_3 \\ 0 & 1 & \tau_2 \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\{1, \tau_1, \tau_2\}$ are independent over $\mathbb{Q}$. Then system $(G/\Gamma, \tau)$ is a 2-step minimal nilsystem.

Affine nilsystems:

An important subclass of nilsystems is the class of *affine nilsystems*. Let $d \in \mathbb{N}$ and let A be a $d \times d$ integer matrix such that $(A - Id)^d = 0$ (such a matrix is called *unipotent*). Let $\vec{\alpha} \in \mathbb{T}^d$ and consider the transformation $T \colon \mathbb{T}^d \to \mathbb{T}^d$, $x \mapsto Ax + \vec{\alpha}$. Let G be the group spanned by A and all the translations of $\mathbb{T}^d$. Since A is unipotent one can check that G is a d-step nilpotent Lie group. The stabilizer of 0 is the subgroup Γ spanned by A thus we can identify $\mathbb{T}^d$ with G/Γ. The topological dynamical system $(\mathbb{T}^d, T) = (G/\Gamma, T)$ is called a *d-step affine nilsystem* and it is proved in [96] that this system is minimal if the projection of $\vec{\alpha}$ on $\mathbb{T}^d/ker(A-Id)$ defines a minimal rotation.

For example, consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\vec{\alpha} = (0,\alpha)^t$. Then the transformation $(y,x) \mapsto A(y,x)^t + \vec{\alpha}$ is nothing but the skew torus transformation $(x,y) \mapsto (x+\alpha, y+x)$.

Nilsystems, like rotations, possess very nice properties and we state some of them here. Most of them appear in the works of Auslander, Green and Hahn [8], Leibman [82, 83], Lesigne [85] and Parry [96, 97]. We refer to [63] for a nice expository of the subject.

Theorem 1.3.1. *Let (X,T) be a d-step nilsystem. Then (X,T) is a distal system.*

Moreover we have,

Theorem 1.3.2. *Let (X,μ,T) be a d-step nilsystem. The following are equivalent:*

1. *(X,μ,T) is ergodic.*
2. *(X,T) is transitive.*
3. *(X,T) is minimal.*
4. *(X,T) is uniquely ergodic, meaning that the Haar measure is the unique invariant measure.*

1.3.2. The cube measures, seminorms and Host-Kra factors

We now describe more in details the measurable cube construction of Host and Kra [67] and the topological one of Host, Kra and Maass [70]. Let $d \geq 1$ be an integer, and write $[d] = \{1,2,\ldots,d\}$. We view an element of $\{0,1\}^d$, the Euclidean cube, either as a sequence $\epsilon = (\epsilon_1,\ldots,\epsilon_d)$ of $0'$s and $1'$s; or as a subset of $[d]$. A subset ϵ corresponds to the sequence $(\epsilon_1,\ldots,\epsilon_d) \in \{0,1\}^d$ such that $i \in \epsilon$ if and only if $\epsilon_i = 1$ for $i \in [d]$. For example, $\vec{0} = (0,\ldots,0) \in \{0,1\}^d$ is the same as $\emptyset \subset [d]$ and $\vec{1} = (1,\ldots,1)$ is the same as $[d]$.

If $\vec{n} = (n_1,\ldots,n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0,1\}^d$, we define $\vec{n}\cdot\epsilon = \sum\limits_{i=1}^{n} n_i\cdot\epsilon_i = \sum\limits_{i\in\epsilon} n_i$.

If X is a set, we denote X^{2^d} by $X^{[d]}$ and we write a point $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (x_\epsilon : \epsilon \in \{0,1\}^d)$.

Let $(X,\mathcal{X},T)$ be a probability space and T an invertible measurable measure preserving transformation on X. For any $d \in \mathbb{N}$ let consider $X^{[d]}$ and let $T^{[d]}$ denote the diagonal action $T \times T \ldots \times T$ (2^d times) on $X^{[d]}$. We remark that we can naturally identify $X^{[d+1]}$ with $X^{[d]} \times X^{[d]}$.

For $d \in \mathbb{N}$, Host and Kra introduced the *cube measure* $\mu^{[d]}$ on $X^{[d]}$. These measures are defined inductively as follows. For $d = 0$, $\mu^{[0]}$ is just μ. If $\mu^{[d]}$ is already defined, then $\mu^{[d+1]}$

is the relative independent product of $(X^{[d]}, \mu^{[d]}, T^{[d]})$ with itself over the sigma algebra $\mathcal{I}_{T^{[d]}}$ of $T^{[d]}$-invariant sets. This means that if F and F' are bounded functions on $X^{[d]}$ then

$$\int_{X^{[d+1]}} F \otimes F' d\mu^{[d+1]} = \int_{X^{[d]}} \mathbb{E}(F|\mathcal{I}_{T^{[d]}})\mathbb{E}(F'|\mathcal{I}_{T^{[d]}})d\mu^{[d]}.$$

These measures are then used to build seminorms. For a function f on X one can define quantity

$$|||f|||_d := \left(\prod_{\epsilon \in \{0,1\}^d} f(x_\epsilon) d\mu^{[d]} \right)^{1/2^d}.$$

and it turns out to be a seminorm on $L^\infty(\mu)$ which is useful to control multiple averages. More precisely, one has that

$$\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x) \right\|_2 \leq \min_{1\leq j\leq d} j\, |||f_j|||$$

As mentioned before, the Host-Kra factors are defined with the relation

$$\mathbb{E}(f|\mathcal{Z}_d) = 0 \text{ if and only if } |||f|||_{d+1} = 0.$$

The connexion between multiple averages and nilsystems is the Host-Kra structure theorem:

Theorem 1.3.3. *d-step nilsystems and their inverse limits are characteristic factors for the multiple average*

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) f_2(T^{2n} x) \cdots f_d(T^{dn} x).$$

This means that one can replace any function by its conditional expectation with respect to the $\mathcal{Z}_d$ factor without affecting the limit.

1.3.3. Topological cubes and the regionally proximal relation of order d

Let (X,T) be a topological dynamical system and d an integer. We define $\mathbf{Q}^{[d]}(X,T)$ to be the closure in $X^{[d]} = X^{2^d}$ of the elements of the form

$$(T^{\vec{n}\cdot\epsilon} x : \epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0,1\}^d)$$

where $\vec{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $x \in X$.

As examples, $\mathbf{Q}^{[2]}(X,T)$ is the closure in $X^{[2]}$ of the set

$$\{(x, T^n x, T^m x, T^{n+m} x) : x \in X, n, m \in \mathbb{Z}\}$$

and $\mathbf{Q}^{[3]}(X,T)$ is the closure in $X^{[3]}$ of the set

$$\{(x, T^n x, T^m x, T^{n+m} x, T^p x, T^{n+p} x, T^{m+p} x, T^{n+m+p} x) : x \in X, n, m, p \in \mathbb{Z}\}.$$

An element in $\mathbf{Q}^{[d]}(X,T)$ is called a *cube of dimension d*. When there is no confusion, we just write $\mathbf{Q}^{[d]}(X)$ instead of $\mathbf{Q}^{[d]}(X,T)$. As mentioned before, this cube structure of a dynamical system was introduced in [70] as the topological counterpart of the theory of cube measures developed in [67].

The following structure theorem relates the notion of cubes and nilsystems. It motivates the objects introduced in Chapter 2 and is the main tool used in Chapter 4.

Theorem 1.3.4 ([70])**.** *Assume that (X,T) is a transitive topological dynamical system and let $d \geq 1$ be an integer. The following properties are equivalent:*

1. *If $\mathbf{x}, \mathbf{y} \in \mathbf{Q}^{[d+1]}(X)$ have $2^{d+1}-1$ coordinates in common, then $\mathbf{x} = \mathbf{y}$.*
2. *If $x, y \in X$ are such that $(x, y, \ldots, y) \in \mathbf{Q}^{[d+1]}(X)$, then $x = y$.*
3. *X is an inverse limit of minimal d-step nilsystems.*

We say that a minimal system (X,T) is a *system of order d* if satisfies any of the previous conditions.

The cube structure $\mathbf{Q}^{[d+1]}(X)$ also allow us to build the maximal factors of order d. Let (X,T) be a topological dynamical system and let $d \geq 1$ be an integer. A pair $(x,y) \in X \times X$ is said to be *regionally proximal of order d* if for any $\delta > 0$ there exists $x', y' \in X$ and $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ such that $d(x,x') < \delta$, $d(y,y') < \delta$ and

$$d(T^{\epsilon \cdot \vec{n}} x', T^{\epsilon \cdot \vec{n}} y') < \delta$$

for any $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \{0,1\}^d \setminus \{\vec{0}\}$.

The set of regionally proximal pairs of order d is denoted by $\mathbf{RP}^{[d]}(X,T)$ (or just $\mathbf{RP}^{[d]}(X)$ when there is no confusion), and is called *the regionally proximal relation of order d*. We remark that when $d = 1$, $\mathbf{RP}^{[1]}(X)$ is nothing but the regionally proximal relation $\mathbf{RP}(X)$.

The following theorem shows some properties of the regionally proximal relation of order d.

Theorem 1.3.5 ([70], [110])**.** *Let (X,T) be a minimal topological dynamical system and $d \in \mathbb{N}$. Then*

1. $(x,y) \in \mathbf{RP}^{[d]}(X)$ *if and only if there exists a sequence* $(\vec{n}_i)$ *in* $\mathbb{Z}^{d+1}$ *such that* $T^{\vec{n}_i \cdot \epsilon} x \to y$ *for every* $\epsilon \neq \emptyset$.

2. $\mathbf{RP}^{[d]}(X)$ *is an equivalence relation.*

3. *Let* $\pi : Y \to X$ *be a factor map between the minimal systems* (Y,T) *and* (X,T) *and* $d \in \mathbb{N}$. *Then* $\pi \times \pi(\mathbf{RP}^{[d]}(Y)) = \mathbf{RP}^{[d]}(X)$.

Furthermore the quotient of X *under* $\mathbf{RP}^{[d]}(X)$ *is the maximal* d*-step nilfactor and we denote* $X/\mathbf{RP}^{[d]}(X) = Z_d(X)$. *Particularly* $Z_1(X)$ *is the maximal equicontinuous factor. It also follows that every factor of a system of order* d *is a system of order* d.

In particular, (X,T) is a system of order d if and only if the regionally proximal relation of order d coincides with the diagonal relation.

1.4. The Enveloping semigroup

The *enveloping semigroup* (or *Ellis semigroup*) $E(X,G)$ of a topological dynamical system (X,G) is defined as the closure in X^X of the group G endowed with the product topology. This notion was introduced by Ellis and has proved to be a fundamental tool in studying topological dynamical systems. Algebraic properties of $E(X,G)$ can be translated into dynamical and geometrical properties of (X,G) and vice versa. For example, a topological dynamical system (X,G) is a rotation on a compact abelian group if and only if $E(X,G)$ is an abelian group and it is distal if and only if $E(X,G)$ is a group.

So, usually an enveloping semigroup is not a group and multiplication is not a continuous operation. In any case, for an enveloping semigroup $E(X,G)$, the applications $E(X,G) \to E(X,G)$, $p \mapsto pq$ and $p \mapsto gp$ are continuous for all $q \in E(X,G)$ and $g \in G$.

If $\pi \colon Y \to X$ is a factor map between the topological dynamical systems (Y,G) and (X,G), then π induces a unique factor map $\pi^* \colon E(Y,G) \to E(X,G)$ that satisfies $\pi^*(u)\pi(y) = \pi(uy)$ for every $u \in E(Y,G)$ and $y \in Y$.

In the following we introduce some algebraic terminology which results to have an important meaning in the enveloping semigroup. We refer to Auslander's book [7], Chapters 3 and 6 for further details.

Let (X,G) be a topological dynamical system. We say that $u \in E(X,G)$ is an *idempotent* if $u^2 = u$. By the Ellis-Nakamura Theorem, any closed subsemigroup $H \subseteq E(X,G)$ admits an idempotent. A *left ideal* $I \subseteq E(X,G)$ is a non-empty subset such that $E(X,G)I \subseteq I$. An ideal is *minimal* if it contains no proper ideals. An idempotent u is *minimal* if u belongs to some minimal ideal $I \subseteq E(X,G)$.

We summarize some results that connect algebraic properties of $E(X,G)$ with dynamical properties of (X,G). Some of those properties are useful when proving minimality of a dynamical system and we use them in Chapter 2.

Theorem 1.4.1. *Let (X,G) be a topological dynamical system and let $E(X,G)$ be its enveloping semigroup. Then*

1. *An ideal $I \subseteq E(X,G)$ is minimal if and only if (I,G) is a minimal system. Particularly, minimal ideals always exist;*
2. *An idempotent $u \in E(X,G)$ is minimal if and only if $(\overline{\mathcal{O}_G(u)},G)$ is a minimal system;*
3. *An idempotent $u \in E(X,G)$ is minimal if $vu = v$ for some $v \in E(X,G)$ implies that $uv = u$;*
4. *Let $x \in X$. Then $(\overline{\mathcal{O}_G(x)},G)$ is a minimal system if and only if there exists a minimal idempotent $u \in E(X,G)$ with $ux = x$.*

Theorem 1.4.2. *Let (X,G) be a topological dynamical system. Then*

1. *$(x,y) \in P(X)$ if and only if there exists $u \in E(X,G)$ with $ux = uy$;*
2. *Let $x \in X$ and let $u \in E(X,G)$ be an idempotent. Then $(x,ux) \in P(X)$;*
3. *Let $x \in X$. Then there exists $y \in X$ such that $(x,y) \in P(X)$ and $(\overline{\mathcal{O}_G(y)},G)$ is minimal.*
4. *If (X,G) is minimal, $(x,y) \in P(X)$ if and only if there exists $u \in E(X,G)$ a minimal idempotent such that $y = ux$.*

Proposition 1.4.3. *Let (Y,G) and (X,G) be topological dynamical systems and let $\pi\colon Y \to X$ be a factor map. If $u \in E(X,G)$ is a minimal idempotent, then there exists a minimal idempotent $v \in E(Y,G)$ such that $\pi^*(v) = u$.*

Proof. If $u \in E(X,G)$ is a minimal idempotent, let $v' \in E(X,G)$ with $\pi^*(v') = u$. Then $\pi^*(\overline{\mathcal{O}_G(v')}) = \overline{\mathcal{O}_G(u)}$. Let $J \subseteq \overline{\mathcal{O}_G(v')}$ be a minimal subsystem. Since $(\overline{\mathcal{O}_G(u)},G)$ is minimal, we have that $\pi^*(J) = \overline{\mathcal{O}_G(u)}$. Let ϕ be the restriction of π^* to J. Since u is idempotent, we have that $\phi^{-1}(u)$ is a closed subsemigroup of $E(Y,G)$. By the Ellis-Nakamura Theorem, we can find an idempotent $v \in \phi^{-1}(u)$. Since v belongs to J we have that v is a minimal idempotent.

□

Part I

Cube structures in topological dynamics

Dynamical cubes and a criteria for systems having product extensions

This chapter is based on the joint work with Wenbo Sun Dynamical Cubes and a criteria for systems having product extensions *[34]. For minimal $\mathbb{Z}^2$-topological dynamical systems, we introduce a cube structure and a variation of the regionally proximal relation for $\mathbb{Z}^2$ actions, which allow us to characterize product systems and their factors. We also introduce the concept of topological magic systems, which is the topological counterpart of measure theoretic magic systems introduced by Host in his study of multiple averages for commuting transformations. Roughly speaking, magic systems have a less intricate dynamic and we show that every minimal $\mathbb{Z}^2$ dynamical system has a magic extension. We give various applications of these structures, including the construction of some special factors in topological dynamics of $\mathbb{Z}^2$ actions.*

2.1. Introduction

We start by reviewing the motivation for characterizing cube structures for systems with a single transformation, which was first developed for ergodic measure preserving systems. To show the convergence of some multiple ergodic averages, Host and Kra [67] introduced for each $d \in \mathbb{N}$ a factor $\mathcal{Z}_d$ which characterizes the behavior of those averages. They proved that this factor can be endowed with a structure of a nilmanifold: it is measurably isomorphic to an inverse limit of ergodic rotations on nilmanifolds. To build such a structure, they introduced cube structures over the set of measurable functions of X to itself and they studied their properties. Later, Host, Kra and Maass [70] introduced these cube structures into topological dynamics. For (X,T) a minimal dynamical system and for $d \in \mathbb{N}$, they introduced the space of cubes $\mathbf{Q}^{[d+1]}(X)$ which characterizes being topologically isomorphic to an inverse limit of minimal rotations on nilmanifolds. They also defined the regionally proximal relation of order d, denoted by $\mathbf{RP}^{[d]}(X)$ which allows one to build the maximal nilfactor. They showed that $\mathbf{RP}^{[d]}(X)$ is an equivalence relation in the distal setting. Recently, Shao and Ye [110] proved that $\mathbf{RP}^{[d]}(X)$ is an equivalence relation in any minimal system and the quotient by this relation is the maximal nilfactor of order d. This theory is important in studying the structure of $\mathbb{Z}$-topological dynamical systems and recent applications of it can be found in [39], [75], [77].

Back to ergodic theory, a natural generalization of the averages considered by Host and Kra [67] are averages arise from a measurable preserving system of commuting transformations $(X, \mathcal{B}, \mu, T_1, \ldots, T_d)$. The convergence of these averages was first proved by Tao [112] with further insight given by Towsner [113], Austin [9] and Host [64]. We focus our attention on Host's proof. In order to prove the convergence of the averages, Host built an extension of X (*magic* in his terminology) with suitable properties. In this extension he found a characteristic factor that looks like the Cartesian product of single transformations. Again, to build these objects, cubes structures are introduced, analogous to the ones in [67].

2.1.1. Criteria for systems having a product extension

A *system with commuting transformations* (X, S, T) is a compact metric space X endowed with two commuting homeomorphisms S and T. The transformations S and T span a $\mathbb{Z}^2$-action, but we stress that we consider this action with a given pair of generators. Throughout Chapters 2 and 3, we always use $G \cong \mathbb{Z}^2$ to denote the group generated by S and T.

A *product system* is a system of commuting transformations of the form $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$, where σ and τ are homeomorphisms of Y and W respectively (we also say that $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ is the *product* of (Y, σ) and (W, τ)). These are the simplest systems of commuting transformations one can imagine.

We are interested in understanding how "far"a system with commuting transformations is from being a product system, and more generally, from being a factor of a product system. To address this question we need to develop a new theory of cube structures for this kind of actions which is motivated by Host's work in ergodic theory and that results in a fundamental tool.

Let (X, S, T) be a system with commuting transformations S and T. The *space of cubes* $\mathbf{Q}_{S,T}(X)$ of (X, S, T) is the closure in X^4 of the points $(x, S^n x, T^m x, S^n T^m x)$, where $x \in X$ and $n, m \in \mathbb{Z}$.

One of our main results is that this structure allows us to characterize systems with a product extension:

Theorem 2.1.1. *Let (X, S, T) be a minimal system with commuting transformations S and T. The following are equivalent:*

1. *(X, S, T) is a factor of a product system;*
2. *If $\mathbf{x}$ and $\mathbf{y} \in \mathbf{Q}_{S,T}(X)$ have three coordinates in common, then $\mathbf{x} = \mathbf{y}$;*
3. *If $(x, y, a, a) \in \mathbf{Q}_{S,T}(X)$ for some $a \in X$, then $x = y$;*

4. If $(x,b,y,b) \in \mathbf{Q}_{S,T}(X)$ for some $b \in X$, then $x = y$;

5. If $(x,y,a,a) \in \mathbf{Q}_{S,T}(X)$ and $(x,b,y,b) \in \mathbf{Q}_{S,T}(X)$ for some $a,b \in X$, then $x = y$.

Of course not any system is a factor of a product system. Nevertheless, the cube structure $\mathbf{Q}_{S,T}(X)$ also provides us a framework for studying the structure of an arbitrary system with commuting transformations. We introduce the (S,T)*-regionally proximal relation* $\mathcal{R}_{S,T}(X)$ of (X,S,T), defined as

$$\mathcal{R}_{S,T}(X) \coloneqq \{(x,y)\colon (x,y,a,a),(x,b,y,b) \in \mathbf{Q}_{S,T}(X) \text{ for some } a,b \in X\}.$$

We remark that in the case $S = T$, these definitions coincide with $\mathbf{Q}^{[2]}(X)$ and $\mathbf{RP}^{[1]}(X)$ defined in [70]. When $S \neq T$, the relation $\mathcal{R}_{S,T}(X)$ is included in the regionally proximal relation for $\mathbb{Z}^2$ actions [7] but can be different. So $\mathcal{R}_{S,T}(X)$ is a variation of $\mathbf{RP}^{[1]}(X)$ for $\mathbb{Z}^2$ actions.

In a distal system with commuting transformations, it turns out that we can further describe properties of $\mathcal{R}_{S,T}(X)$. We prove that $\mathcal{R}_{S,T}(X)$ is an equivalence relation and the quotient of X by this relation defines the *maximal factor with a product extension* (see Section 2.4 for definitions).

We also study the topological counterpart of the "magic extension"in Host's work [64]. We define the magic extension in the topological setting and show that in this setting, every minimal system with commuting transformations admits a minimal magic extension (Proposition 2.2.11). Combining this with the properties of the cube $\mathbf{Q}_{S,T}(X)$ and the relation $\mathcal{R}_{S,T}(X)$, we are able to prove Theorem 2.1.1.

We provide several applications, both in a theoretical framework and to real systems. Using the cube structure, we study some representative tiling systems. For example, we show that the $\mathcal{R}_{S,T}$ relation on the two dimensional Morse tiling system is trivial. Therefore, it follows from Theorem 2.1.1 that it has a product extension.

Another application of the cube structure is to study the properties of a system having a product system as an extension (see Section 2.5 for definitions), which include:

1. Enveloping semigroup: we show that (X,S,T) has a product extension if and only if S and T are automorphic in the enveloping semigroup.

2. Disjoint orthogonal complement: we show that if (X,S,T) is an S-T almost periodic system, then (X,S,T) is disjoint from systems with a product extension if and only if both (X,S) and (X,T) are minimal weakly mixing systems.

3. Set of return times: we show that in the distal setting, $(x,y) \in \mathcal{R}_{S,T}(X)$ if and only if the set of return time of x to any neighborhood of y is an $\mathcal{B}^*_{S,T}$ set.

4. Topological complexity: we define a relative topological complexity of a system with commuting transformations and show that in the distal setting, (X, S, T) has a product extension if and only if it has bounded topological complexity.

2.1.2. Organization of the Chapter

In Section 2.2, we formally define the cube structure, the (S, T)-regionally proximal relation and the magic extension in the setting of systems with commuting transformations. We prove that every minimal system with commuting transformations has a minimal magic extension, and then we use this to give a criteria for systems having a product extension (Theorem 2.1.1). We also present properties of the relation $\mathcal{R}_{S,T}(X)$ in an arbitrary system with commuting transformations and discuss some connections with equicontinuity and related notions.

In Section 2.3, we compute the $\mathcal{R}_{S,T}(X)$ relation for some tiling systems and provide some applications.

In Section 2.4, we study further properties of the $\mathcal{R}_{S,T}(X)$ relation in the distal case.

In Section 2.5, we study various properties of systems with product extensions, which includes the study of its enveloping semigroup, disjoint orthogonal complement, set of return times, and topological complexity.

2.2. Cube structures and general properties

2.2.1. Cube structures and the (S, T)-regionally proximal relation

Definition 2.2.1. For a system (X, S, T) with commuting transformations S and T, let $\mathcal{F}_{S,T}$ denote the subgroup of G^4 generated by $\mathrm{id} \times S \times \mathrm{id} \times S$ and $\mathrm{id} \times \mathrm{id} \times T \times T$ (recall that G is the group spanned by S and T). Write $G^{\Delta} \coloneqq \{g \times g \times g \times g \in G^4 \colon g \in G\}$. Let $\mathcal{G}_{S,T}$ denote the subgroup of G^4 generated by $\mathcal{F}_{S,T}$ and G^{Δ}.

The main structure studied in this chapter is a notion of cubes for a system with commuting transformations:

Definition 2.2.2. Let (X, S, T) be a system with commuting transformations S and T. We define

$$\begin{aligned}
&\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^n x, T^m x, S^n T^m x) \colon x \in X, n, m \in \mathbb{Z}\}};\\
&\mathbf{Q}_{S}(X) = \pi_0 \times \pi_1(\mathbf{Q}_{S,T}(X)) = \overline{\{(x, S^n x) \in X \colon x \in X, n \in \mathbb{Z}\}};\\
&\mathbf{Q}_{T}(X) = \pi_0 \times \pi_2(\mathbf{Q}_{S,T}(X)) = \overline{\{(x, T^n x) \in X \colon x \in X, n \in \mathbb{Z}\}};\\
&\mathbf{K}^{x_0}_{S,T} = \overline{\{(S^n x_0, T^m x_0, S^n T^m x_0) \in X^3 \colon n, m \in \mathbb{Z}\}} \text{ for all } x_0 \in X,
\end{aligned}$$

where $\pi_i\colon X^4 \to X$ is the projection to the i-th coordinate in X^4 for $i = 0, 1, 2, 3$.

We start with some basic properties of $\mathbf{Q}_{S,T}(X)$. The following proposition follows immediately from the definitions:

Proposition 2.2.3. *Let (X, S, T) be a minimal system with commuting transformations S and T. Then,*

1. *$(x, x, x, x) \in \mathbf{Q}_{S,T}(X)$ for every $x \in X$;*
2. *$\mathbf{Q}_{S,T}(X)$ is invariant under $\mathcal{G}_{S,T}$;*
3. *(Symmetries) if $(x_0, x_1, x_2, x_3) \in \mathbf{Q}_{S,T}(X)$, then $(x_2, x_3, x_0, x_1), (x_1, x_0, x_3, x_2) \in \mathbf{Q}_{S,T}(X)$ and $(x_0, x_2, x_1, x_3) \in \mathbf{Q}_{T,S}(X)$;*
4. *(Projection) if $(x_0, x_1, x_2, x_3) \in \mathbf{Q}_{S,T}(X)$, then (x_0, x_1), $(x_2, x_3) \in \mathbf{Q}_S(X)$ and (x_0, x_2), $(x_1, x_3) \in \mathbf{Q}_T(X)$;*
5. *If $(x_0, x_1) \in \mathbf{Q}_S(X)$, then $(x_0, x_1, x_0, x_1) \in \mathbf{Q}_{S,T}(X)$; If $(x_0, x_1) \in \mathbf{Q}_T(X)$, then $(x_0, x_0, x_1, x_1) \in \mathbf{Q}_{S,T}(X)$;*
6. *(Symmetry) $(x, y) \in \mathbf{Q}_R(X)$ if and only if $(y, x) \in \mathbf{Q}_R(X)$ for all $x, y \in X$, where R is either S or is T.*

Remark 2.2.4. We remark that when $S = T$ one has an additional symmetry, namely $(x_0, x_1, x_2, x_3) \in \mathbf{Q}_{S,T}(X)$ if and only if $(x_0, x_2, x_1, x_3) \in \mathbf{Q}_{S,T}(X)$.

It is easy to see that $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is a topological dynamical system. Moreover, we have:

Proposition 2.2.5. *Let (X, S, T) be a minimal system with commuting transformations S and T. Then $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is a minimal system. Particularly, taking R to be either S or T, $\mathbf{Q}_R(X)$ is minimal under the action generated by $\mathrm{id} \times R$ and $g \times g$ for $g \in G$.*

Proof. We use results on the enveloping semigroups given in the Background Chapter.

The proof is similar to the one given in page 46 of [55] for some similar diagonal actions. Let $E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ be the enveloping semigroup of $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$. For $i = 0, 1, 2, 3$, let $\pi_i\colon \mathbf{Q}_{S,T}(X) \to X$ be the projection onto the i-th coordinate and let $\pi_i^*\colon E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T}) \to E(X, G)$ be the induced factor map.

Let $u \in E(\mathbf{Q}_{S,T}(X), G^{\Delta})$ denote a minimal idempotent. We show that u is also a minimal idempotent in $E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$. By Theorem 1.4.1, it suffices to show that if $v \in E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ with $vu = v$, then $uv = u$. Projecting onto the corresponding coordinates, we deduce that $\pi_i^*(vu) = \pi_i^*(v)\pi_i^*(u) = \pi_i^*(v)$ for $i = 0, 1, 2, 3$. It is clear that the projection of a minimal idempotent to $E(\mathbf{Q}_{S,T}(X), G^{\Delta})$ is a minimal idempotent in $E(X, G)$. Since

$\pi_i^*(v)\pi_i^*(u) = \pi_i^*(v)$, by Theorem 1.4.1 we deduce that $\pi_i^*(u)\pi_i^*(v) = \pi_i^*(u)$ for $i = 0, 1, 2, 3$. Since the projections onto the coordinates determine an element of $E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$, we have that $uv = u$. Thus we conclude that u is a minimal idempotent in $E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$.

For any $x \in X$, (x, x, x, x) is a minimal point under G^{Δ}. So there exists a minimal idempotent $u \in E(\mathbf{Q}_{S,T}(X), G^{\Delta})$ such that $u(x, x, x, x) = (x, x, x, x)$. Since u is also a minimal idempotent in $E(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$, the point (x, x, x, x) is minimal under $\mathcal{G}_{S,T}$. Since the orbit closure of (x, x, x, x) under $\mathcal{G}_{S,T}$ is $\mathbf{Q}_{S,T}(X)$, we have that $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is a minimal system.

The fact that $\mathbf{Q}_R(X)$ is minimal follows immediately by taking projections. $\square$

We remark that $\mathbf{K}^{x_0}_{S,T}$ is invariant under $\widehat{S} \coloneqq S \times \mathrm{id} \times S$ and $\widehat{T} \coloneqq \mathrm{id} \times T \times T$. We let $\mathcal{F}^{x_0}_{S,T}$ denote the action spanned by $\widehat{S}$ and $\widehat{T}$. We note that $(\mathbf{K}^{x_0}_{S,T}, \mathcal{F}^{x_0}_{S,T})$ is not necessarily minimal, even if X is minimal (the minimality of $\mathbf{K}^{x_0}_{S,T}$ implies the minimality of $\overline{\mathcal{O}_S(x_0)}$ under S and the minimality of $\overline{\mathcal{O}_T(x_0)}$ under T, which does not always hold). See the examples in Section 2.3.

The following lemma follows from the definitions:

Lemma 2.2.6. *Let $\pi \colon Y \to X$ be a factor map between two minimal systems (Y, S, T) and (X, S, T) with commuting transformations S and T. Then $\pi \times \pi \times \pi \times \pi(\mathbf{Q}_{S,T}(Y)) = \mathbf{Q}_{S,T}(X)$. Therefore, $\pi \times \pi(\mathbf{Q}_S(Y)) = \mathbf{Q}_S(X)$ and $\pi \times \pi(\mathbf{Q}_T(Y)) = \mathbf{Q}_T(X)$.*

Associated to the cube structure, we define a relation in X as was done in [70] with cubes associated to a $\mathbb{Z}$-system. This is the main relation we study in this work:

Definition 2.2.7. Let (X, S, T) be a minimal system with commuting transformations S and T. We define

$$\begin{aligned}
\mathcal{R}_S(X) &= \{(x, y) \in X \times X \colon (x, y, a, a) \in \mathbf{Q}_{S,T}(X) \text{ for some } a \in X\};\\
\mathcal{R}_T(X) &= \{(x, y) \in X \times X \colon (x, b, y, b) \in \mathbf{Q}_{S,T}(X) \text{ for some } b \in X\};\\
\mathcal{R}_{S,T}(X) &= \mathcal{R}_S(X) \cap \mathcal{R}_T(X).
\end{aligned}$$

It then follows from (3) of Proposition 2.2.3 that $\mathcal{R}_S(X), \mathcal{R}_T(X), \mathcal{R}_{S,T}(X)$ are symmetric relations, *i.e.* $(x, y) \in A$ if and only if $(y, x) \in A$ for all $x, y \in X$, where A is $\mathcal{R}_S(X), \mathcal{R}_T(X)$ or $\mathcal{R}_{S,T}(X)$. It is worth noting that in the case $S = T$, $\mathcal{R}_{S,T}(X)$ is the regionally proximal relation $\mathbf{RP}^{[1]}(X)$ defined in [70].

Using these definitions, our main Theorem 2.1.1 can be rephrased as (we postpone the proof to Section 2.2.4):

Theorem. *Let (X, S, T) be a minimal system with commuting transformations S and T. The following are equivalent:*

1. *(X, S, T) is a factor of a product system;*

2. *If* $\mathbf{x}$ *and* $\mathbf{y} \in \mathbf{Q}_{S,T}(X)$ *have three coordinates in common, then* $\mathbf{x} = \mathbf{y}$*;*

3. $\mathcal{R}_S(X) = \Delta_X$*;*

4. $\mathcal{R}_T(X) = \Delta_X$*;*

5. $\mathcal{R}_{S,T}(X) = \Delta_X$*.*

Remark 2.2.8. In the case where $(X, S, T) = (Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ is exactly a product system, we have that

$$\mathbf{Q}_{S,T}(X) = \{((y_1, w_1), (y_2, w_1), (y_1, w_2), (y_2, w_2))\colon \ y_1, y_2 \in Y,\ w_1, w_2 \in W\}.$$

In this case, $\mathcal{R}_{S,T}(X) = \Delta_X$ holds for trivial reasons. Suppose that $((y_1, w_1), (y_2, w_2)) \in \mathcal{R}_{S,T}(X)$ for some (y_1, w_1), $(y_2, w_2) \in X$. Since $((y_1, w_1), (y_2, w_2)) \in \mathcal{R}_S(X)$, there exists $a \in X$ such that $((y_1, w_1), (y_2, w_2), a, a) \in \mathbf{Q}_{S,T}(X)$. Therefore $w_2 = w_1$ and $(y_1, w_2) = a = (y_2, w_2)$, which implies that $y_1 = y_2$. Thus $\mathcal{R}_{S,T}(X) = \Delta_X$.

2.2.2. Magic systems

We construct an extension of a system with commuting transformations which behaves like a product system for use in the sequel. Following the terminology introduced in [64] in the ergodic setting, we introduce the notion of a magic system in the topological setting:

Definition 2.2.9. A minimal system (X, S, T) with commuting transformations S and T is called a *magic system* if $\mathcal{R}_S(X) \cap \mathcal{R}_T(X) = \mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$.

We remark that the inclusion in one direction always holds:

Lemma 2.2.10. *Let* (X, S, T) *be a system with commuting transformations* S *and* T*. Then* $\mathcal{R}_S(X) \cap \mathcal{R}_T(X) \subseteq \mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$*.*

Proof. Suppose $(x, y) \in \mathcal{R}_S(X) \cap \mathcal{R}_T(X)$. Then in particular $(x, y) \in \mathcal{R}_S(X)$. So there exists $a \in X$ such that $(x, y, a, a) \in \mathbf{Q}_{S,T}(X)$. Taking the projections onto the first two coordinates, we have that $(x, y) \in \mathbf{Q}_S(X)$. Similarly, $(x, y) \in \mathbf{Q}_T(X)$, and so $\mathcal{R}_S(X) \cap \mathcal{R}_T(X) \subseteq \mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$. □

In general, not every system with commuting transformations is magic. In fact, $\mathcal{R}_S(X) \cap \mathcal{R}_T(X)$ and $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$ may be very different. For example, let $(\mathbb{T} = \mathbb{R}/\mathbb{Z}, T)$ be a rotation on the circle given by $Tx = x + \alpha \mod 1$ for all $x \in \mathbb{T}$, where α is an irrational number. Then $\mathbf{Q}_T(\mathbb{T}) \cap \mathbf{Q}_T(\mathbb{T}) = \mathbb{T} \times \mathbb{T}$. But $\mathcal{R}_T(\mathbb{T}) \cap \mathcal{R}_T(\mathbb{T}) = \{(x, x) \in \mathbb{T}^2 \colon x \in \mathbb{T}\}$ (here we take $S = T$). However, we can always regard a minimal system with commuting transformations as a factor of a magic system:

Proposition 2.2.11 (Magic extension). *Let (X,S,T) be a minimal system with commuting transformations S and T. Then (X,S,T) admits a minimal magic extension, meaning it has an extension which is a minimal magic system.*

Proof. We use some results of Section 4 of [54], where Glasner studied the so called prolongation relation and its relation with closed orbits to propose a topological analogue of the ergodic decomposition. By Lemmas 4.1 and 4.5 in [54], we can find a point $x_0 \in X$ such that $\mathbf{Q}_S[x_0] \coloneqq \{x \in X \colon (x_0,x) \in \mathbf{Q}_S(X)\}$ and $\mathbf{Q}_T[x_0] \coloneqq \{x \in X \colon (x_0,x) \in \mathbf{Q}_T(X)\}$ coincide with $\overline{\mathcal{O}_S(x_0)}$ and $\overline{\mathcal{O}_T(x_0)}$ respectively (moreover, the set of such points is a G_δ set).

Let Y be a minimal subsystem of the system $(\mathbf{K}_{S,T}^{x_0}, \widehat{S}, \widehat{T})$, where $\widehat{S} = S \times \mathrm{id} \times S$, $\widehat{T} = \mathrm{id} \times T \times T$. Since the projection onto the last coordinate defines a factor map from $(Y,\widehat{S},\widehat{T})$ to (X,S,T), there exists a minimal point of Y of the form $\vec{z} = (z_1,z_2,x_0)$. Hence, Y is the orbit closure of (z_1,z_2,x_0) under $\widehat{S}$ and $\widehat{T}$. We claim that $(Y,\widehat{S},\widehat{T})$ is a magic extension of (X,S,T).

It suffices to show that for any $\vec{x} = (x_1,x_2,x_3), \vec{y} = (y_1,y_2,y_3) \in Y$, $(\vec{x},\vec{y}) \in \mathbf{Q}_{\widehat{S}}(Y) \cap \mathbf{Q}_{\widehat{T}}(Y)$ implies that $(\vec{x},\vec{y}) \in \mathcal{R}_{\widehat{S}}(Y) \cap \mathcal{R}_{\widehat{T}}(Y)$. Since $(\vec{x},\vec{y}) \in \mathbf{Q}_{\widehat{S}}(Y)$ and the second coordinate of Y is invariant under $\widehat{S}$, we get that $x_2 = y_2$. Similarly, $(\vec{x},\vec{y}) \in \mathbf{Q}_{\widehat{T}}(Y)$ implies that $x_1 = y_1$.

We recall that $d(\cdot,\cdot)$ is a metric in X defining its topology. Let $\epsilon > 0$. Since $(\vec{x},\vec{y}) \in \mathbf{Q}_{\widehat{S}}(Y)$, there exists $\vec{x'} = (x_1',x_2',x_3') \in Y$ and $n_0 \in \mathbb{Z}$ such that $d(x_i,x_i') < \epsilon$ for $i = 1,2,3$ and that $d(S^{n_0}x_1',x_1) < \epsilon$, $d(S^{n_0}x_3',y_3) < \epsilon$. Let $0 < \delta < \epsilon$ be such that if $x,y \in X$ and $d(x,y) < \delta$, then $d(S^{n_0}x,S^{n_0}y) < \epsilon$.

Since $\vec{x'} \in Y$, there exist $n,m \in \mathbb{Z}$ such that $d(x_1',S^n z_1)$, $d(x_2',T^m z_2)$, $d(x_3',S^nT^m x_0) < \delta$. Then $d(S^{n_0}x_1',S^{n_0+n}z_1)$, $d(S^{n_0}x_3',S^{n_0+n}T^m x_0) < \epsilon$.

Let $0 < \delta' < \delta$ be such that if $x,y \in X$ and $d(x,y) < \delta'$, then $d(S^n x,S^n y) < \delta$. Since $\vec{z} \in \mathbf{K}_{S,T}^{x_0}$, we have that $z_1 \in \mathbf{Q}_T[x_0]$. By assumption, there exists $m_0 \in \mathbb{Z}$ such that $d(T^{m_0}x_0,z_1) < \delta'$. Then $d(S^nT^{m_0}x_0,S^n z_1) < \delta$ and $d(S^{n+n_0}T^{m_0}x_0,S^{n+n_0}z_1) < \epsilon$.

Denote $\vec{z'} = (S^n z_1, T^m z_2, S^nT^m x_0) \in Y$. Then the distance between

$$(\vec{z'}, \widehat{S}^{n_0}\vec{z'}, \widehat{T}^{m_0-m}\vec{z'}, \widehat{S}^{n_0}\widehat{T}^{m_0-m}\vec{z'})$$

and the corresponding coordinates of $w = (\vec{x},\vec{y},\vec{u},\vec{u})$ is smaller than $C\epsilon$ for some uniform constant $C > 0$, where $\vec{u} = (x_1,a,x_1)$ for some $a \in X$ (the existence of a follows by passing to a subsequence). We conclude that $(\vec{x},\vec{y}) \in \mathcal{R}_{\widehat{S}}(Y)$. Similarly $(\vec{x},\vec{y}) \in \mathcal{R}_{\widehat{T}}(Y)$. □

Moreover, if (X,S,T) is a system with commuting transformations S and T and $(Y,\widehat{S},\widehat{T})$ is the magic extension described in Proposition 2.2.11, we have:

Corollary 2.2.12. *If $((x_1,x_2,x_3),(x_1,x_2,y_3)) \in \mathbf{Q}_{\widehat{S}}(Y)$, then $((x_1,x_2,x_3),(x_1,x_2,y_3)) \in \mathcal{R}_{\widehat{S}}(Y)$.*

The following lemma is proved implicitly in Proposition 2.2.11. We state it here for use in the sequel:

Lemma 2.2.13. *Let (X,S,T) be a minimal system with commuting transformations S and T. Let $(Y,\widehat{S},\widehat{T})$ be the magic extension given by Proposition 2.2.11 and let $\vec{x}=(x_1,x_2,x_3)$, $\vec{y}=(y_1,y_2,y_3)$ be points in Y. For R being either S or T, if $(\vec{x},\vec{y})\in\mathcal{R}_{\widehat{R}}(Y)$ then $x_1=y_1$, $x_2=y_2$ and $(x_3,y_3)\in\mathcal{R}_R(X)$.*

2.2.3. Partially distal systems

We recall that a topological dynamical system (X,G) is distal if $x\neq y$ implies that

$$\inf_{g\in G} d(gx,gy)>0.$$

We introduce a definition of partial distality, which can be viewed as a generalization of distality, and is the main ingredient in the proof of Theorem 2.1.1.

Let (X,S,T) be a minimal system with commuting transformations S and T. For R being either S or T, let $P_R(X)$ be the set of proximal pairs under R.

Definition 2.2.14. Let (X,S,T) be a minimal system with commuting transformations S and T. We say that (X,S,T) is *partially distal* if $\mathbf{Q}_S(X)\cap P_T(X)=\mathbf{Q}_T(X)\cap P_S(X)=\Delta_X$.

We remark that when $S=T$, partial distality coincides with distality. If $\mathbf{Q}_S(X)$ is an equivalence relation on X, then the system (X,S,T) being partially distal implies that the quotient map $X\to X/\mathbf{Q}_S(X)$ is a distal extension between the systems (X,T) and $(X/\mathbf{Q}_S(X),T)$.

The following lemma allows us to lift a minimal idempotent in $E(X,G)$ to a minimal idempotent in $E(X^4,\mathcal{F}_{S,T})$. Recall that taking R to be either S or T, if $u\in E(X,R)$ is an idempotent, then $(x,ux)\in P_R(X)$ for all $x\in X$ (Theorem 1.4.2).

Lemma 2.2.15. *Let (X,S,T) be a minimal system with commuting transformations S and T, and let $u\in E(X,G)$ be a minimal idempotent. Then there exists a minimal idempotent $\widehat{u}\in E(X^4,\mathcal{F}_{S,T})$ of the form $\widehat{u}=(e,u_S,u_T,u)$, where $u_S\in E(X,S)$ and $u_T\in E(X,T)$ are minimal idempotents. Moreover, if (X,S,T) is partially distal, we have that $u_Su=u_Tu=u$.*

Proof. For $i=0,1,2,3$, let π_i be the projection from X^4 onto the i-th coordinate and let π_i^* be the induced factor map in the enveloping semigroups. Hence $\pi_1^*\colon E(X^4,\mathcal{F}_{S,T})\to E(X,S)$, $\pi_2^*\colon E(X^4,\mathcal{F}_{S,T})\to E(X,T)$, and $\pi_3^*\colon E(X^4,\mathcal{F}_{S,T})\to E(X,G)$ are factor maps. By Proposition 1.4.3, we can find a minimal idempotent $\widehat{u}\in E(X^4,\mathcal{F}_{S,T})$ such that $\pi_3^*(\widehat{u})=u$. Since the projection of a minimal idempotent is a minimal idempotent, $\widehat{u}$ can be written in the form $\widehat{u}=(e,u_S,u_T,u)$, where $u_S\in E(X,S)$ and $u_T\in E(X,T)$ are minimal idempotents.

Now suppose that (X, S, T) is partially distal. Let $u \in E(X, G)$ and $\widehat{u} = (e, u_S, u_T, u) \in (X^4, \mathcal{F}_{S,T})$ be minimal idempotents in the corresponding enveloping semigroups. Note that $(ux, u_Sux, u_Tux, uux) = (ux, u_Sux, u_Tux, ux) \in \mathbf{Q}_{S,T}(X)$ for all $x \in X$. So we have that $(ux, u_Sux) \in P_S(X) \cap \mathbf{Q}_T(X)$ and $(ux, u_Tux) \in P_T(X) \cap \mathbf{Q}_S(X)$. Thus $u_Sux = u_Tux = ux$ for all $x \in X$ since X is partially distal. This finishes the proof. $\square$

Corollary 2.2.16. *Let (X, S, T) be a partially distal system with commuting transformations S and T. Then for every $x \in X$, the system $(\mathbf{K}^x_{S,T}, \widehat{S} = S \times \mathrm{id} \times S, \widehat{T} = \mathrm{id} \times T \times T)$ with commuting transformations $\widehat{S}$ and $\widehat{T}$ is a minimal system. Moreover, $(\mathbf{K}^x_{S,T}, \widehat{S}, \widehat{T})$ is a magic extension of (X, S, T).*

Proof. Since (X, S, T) is a minimal system, there exists a minimal idempotent $u \in E(X, G)$ such that $ux = x$. By Lemma 2.2.15, there exists a minimal idempotent $\widehat{u} \in E(X^4, \mathcal{F}_{S,T})$ such that $\widehat{u}(x, x, x) = (x, x, x)$, which implies that (x, x, x) is a minimal point of $\mathbf{K}^x_{S,T}$. The proof that $(\mathbf{K}^x_{S,T}, \widehat{S}, \widehat{T})$ is a magic extension is similar to Proposition 2.2.11.

$\square$

Corollary 2.2.17. *Let (X, S, T) be a partially distal system. Then (X, S) and (X, T) are pointwise almost periodic.*

Proof. By Lemma 2.2.15, for any $x \in X$, we can find minimal idempotents $u_S \in E(X, S)$ and $u_T \in E(X, T)$ such that $u_Sx = u_Tx = x$. This is equivalent to being pointwise almost periodic. $\square$

2.2.4. Proof of Theorem 2.1.1

Before completing the proof of Theorem 2.1.1, we start with some lemmas:

Lemma 2.2.18. *For any minimal system (X, S, T) with commuting transformations S and T, $\mathbf{Q}_S(X) \cap P_T(X) \subseteq \mathcal{R}_S(X)$.*

Proof. Suppose $(x, y) \in \mathbf{Q}_S(X) \cap P_T(X)$. Since $(x, y) \in P_T(X)$, there exists a sequence $(m_i)_{i \in \mathbb{N}}$ in $\mathbb{Z}$ such that $d(T^{m_i}x, T^{m_i}y) \to 0$. We can assume that $T^{m_i}x$ and $T^{m_i}y$ converge to $a \in X$. Since $(x, y) \in \mathbf{Q}_S(X)$, we have that $(x, y, x, y) \in \mathbf{Q}_{S,T}(X)$ and therefore $(x, y, T^{m_i}x, T^{m_i}y) \to (x, y, a, a) \in \mathbf{Q}_{S,T}(X)$. We conclude that $(x, y) \in \mathcal{R}_S(X)$. $\square$

Lemma 2.2.19. *Let (X, S, T) be a minimal system with commuting transformations S and T such that $\mathcal{R}_S(X) = \Delta_X$. Then for every $x \in X$, $(\mathbf{K}^x_{S,T}, \widehat{S}, \widehat{T})$ is a minimal system. Particularly, for every $x \in X$ we have that $(\overline{\mathcal{O}_S(x)}, S)$ and $(\overline{\mathcal{O}_T(x)}, T)$ are minimal systems.*

Proof. Since $\mathcal{R}_S(X) = \Delta_X$, by Lemma 2.2.18, we deduce that $\mathbf{Q}_S(X) \cap P_T(X) = \Delta_X$. For any $x \in X$, let $u \in E(X, G)$ be a minimal idempotent with $ux = x$ and let $(e, u_S, u_T, u) \in$

$E(X^4, \mathcal{F}_{S,T})$ be a lift given by Lemma 2.2.15. Then $(x, u_S x, u_T x, ux) = (x, u_S x, u_T x, x) \in \mathbf{Q}_{S,T}(X)$. Projecting to the last two coordinates, we get that $(u_T x, x) \in \mathbf{Q}_S(X)$. On the other hand, $(u_T x, x) \in P_T(X)$ as $u_T \in E(X,T)$ is an idempotent. Since $\mathbf{Q}_S(X) \cap P_T(X) = \Delta_X$, we deduce that $x = u_T x$ and thus $(x, u_S x, u_T x, ux) = (x, u_S x, x, x)$. Since $\mathcal{R}_S(X) = \Delta_X$, we have that $(u_S x, u_T x, ux) = (x, x, x)$ and this point is minimal.

The second statement follows by projecting $\mathbf{K}^x_{S,T}$ onto the two first coordinates. $\square$

Lemma 2.2.20. *Let (X, S, T) be a minimal system with commuting transformations S and T. If $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X) = \Delta_X$, then $\mathcal{R}_S(X) = \Delta_X$.*

Proof. We remark that if $(x, a, b, x) \in \mathbf{Q}_{S,T}(X)$, then (x, a) and (x, b) belong to $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X)$. Consequently, if $(x, a, b, x) \in \mathbf{Q}_{S,T}(X)$, then $a = b = x$. Now let $(x, y) \in \mathcal{R}_S(X)$ and let $a \in X$ such that $(x, y, a, a) \in \mathbf{Q}_{S,T}(X)$. By minimality we can take two sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $S^{n_i}T^{m_i}a \to x$. We can assume that $S^{n_i}y \to y'$ and $T^{m_i}a \to a'$, and thus $(x, S^{n_i}y, T^{m_i}a, S^{n_i}T^{m_i}a) \to (x, y', a', x) \in \mathbf{Q}_{S,T}(X)$. We deduce that $y' = a' = x$ and particularly $T^{m_i}a \to x$. Hence $(x, y, T^{m_i}a, T^{m_i}a) \to (x, y, x, x) \in \mathbf{Q}_{S,T}(X)$ and therefore $x = y$. $\square$

We are now ready to prove Theorem 2.1.1:

Proof of Theorem 2.1.1.

(1) $\Rightarrow$ (2). Let $\pi \colon Y \times W \to X$ be a factor map between the minimal systems $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ and (X, S, T). Let (x_0, x_1, x_2, x_3) and $(x_0, x_1, x_2, x_3') \in \mathbf{Q}_{S,T}(X)$. It suffices to show that $x_3 = x_3'$. Since $\pi^4(\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W)) = \mathbf{Q}_{S,T}(X)$, there exist $((y_0, w_0), (y_1, w_0), (y_0, w_1), (y_1, w_1))$ and $((y_0', w_0'), (y_1', w_0'), (y_0', w_1'), (y_1', w_1'))$ in $\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W)$ such that $\pi(y_0, w_0) = x_0 = \pi(y_0', w_0')$, $\pi(y_1, w_0) = x_1 = \pi(y_1', w_0')$, $\pi(y_0, w_1) = x_2 = \pi(y_0', w_1')$, $\pi(y_1, w_1) = x_3$ and $\pi(y_1', w_1') = x_3'$.

Let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be sequences in $\mathbb{Z}$ such that $\sigma^{n_i}y_0 \to y_1$ and $\tau^{m_i}w_0 \to w_1$. We can assume that $\sigma^{n_i}y_0' \to y_1''$ and $\tau^{m_i}w_0' \to w_1''$ so that $((y_0', w_0'), (y_1'', w_0'), (y_0', w_1''), (y_1'', w_1'')) \in \mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W)$. Since $\pi(y_0, w_0) = \pi(y_0', w_0')$, we have that

$$\pi^4((y_0', w_0'), (y_1'', w_0'), (y_0', w_1''), (y_1'', w_1'')) = (x_0, x_1, x_2, x_3).$$

Particularly, $\pi(y_1', w_0') = \pi(y_1'', w_0')$ and $\pi(y_0', w_1') = \pi(y_0', w_1'')$. By minimality of (Y, σ) and (W, τ), we deduce that $\pi(y_1', w) = \pi(y_1'', w)$ and $\pi(y, w_1') = \pi(y, w_1'')$ for every $y \in Y$ and for every $w \in W$. Hence $x_3 = \pi(y_1'', w_1'') = \pi(y_1'', w_1') = \pi(y_1', w_1') = x_3'$.

(2) $\Rightarrow$ (3). Let $(x, y) \in \mathcal{R}_S(X)$ and let $a \in X$ such that $(x, y, a, a) \in \mathbf{Q}_{S,T}(X)$. We remark that this implies that $(x, a) \in \mathbf{Q}_T(X)$ and then $(x, x, a, a) \in \mathbf{Q}_{S,T}(X)$. Since (x, x, a, a) and (x, y, a, a) belong to $\mathbf{Q}_{S,T}(X)$, we have that $x = y$.

(3) $\Rightarrow$ (1). By Lemma 2.2.19, for every $x_0 \in X$, we can build a minimal magic system $(\mathbf{K}^{x_0}_{S,T}, \widehat{S}, \widehat{T})$ which is an extension of (X, S, T) whose factor map is the projection onto the

last coordinate. We remark that if $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$ are such that $(\vec{x},\vec{y})\in\mathcal{R}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$, then by Lemma 2.2.13, $x_1=y_1$, $x_2=y_2$ and $(x_3,y_3)\in\mathcal{R}_S(X)$. Hence, if $\mathcal{R}_S(X)$ coincides with the diagonal, so does $\mathcal{R}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$.

Let $\phi\colon \mathbf{K}_{S,T}^{x_0}\to\overline{\mathcal{O}_S(x_0)}\times\overline{\mathcal{O}_T(x_0)}$ be the projection onto the first two coordinates. Then ϕ is a factor map between the minimal systems $(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T})$ and $(\overline{\mathcal{O}_S(x_0)}\times\overline{\mathcal{O}_T(x_0)},S\times\mathrm{id},\mathrm{id}\times T)$ with commuting transformations. We remark that the latter is a product system.

We claim that the triviality of the relation $\mathcal{R}_S(X)$ implies that ϕ is actually an isomorphism. It suffices to show that $(a,b,c),(a,b,d)\in\mathbf{K}_{S,T}^{x_0}$ implies that $c=d$. By minimality, we can find a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $S^{n_i}a\to x_0$. Since $\mathcal{R}_S(X)=\Delta_X$, we have that $\lim S^{n_i}c=b=\lim S^{n_i}d$. So $\lim\widehat{S}^{n_i}(a,b,c)=\lim\widehat{S}^{n_i}(a,b,d)$ and hence $((a,b,c),(a,b,d))\in P_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$. Since $\mathcal{R}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$ is the diagonal, by Lemma 2.2.19 applied to the system $(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T})$ we have that every point in $\mathbf{K}_{S,T}^{x_0}$ has a minimal $\widehat{S}$-orbit. This implies that (a,b,c) and (a,b,d) are in the same $\widehat{S}$-minimal orbit closure and hence they belong to $\mathbf{Q}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$. By Proposition 2.2.11, since they have the same first two coordinates, we deduce that $((a,b,c),(a,b,d))\in\mathcal{R}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0})$, which is trivial. We conclude that $(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T})$ is a product system and thus (X,S,T) has a product extension.

(2) $\Rightarrow$ (4) is similar to (2) $\Rightarrow$ (3); (4) $\Rightarrow$ (1) is similar to (3) $\Rightarrow$ (1); (3) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1). By Proposition 2.2.11, we have a magic extension $(Y,\widehat{S},\widehat{T})$ of (X,S,T) with $Y\subseteq\mathbf{K}_{S,T}^{x_0}$ for some $x_0\in X$. The magic extension satisfies $\mathbf{Q}_{\widehat{S}}(Y)\cap\mathbf{Q}_{\widehat{T}}(Y)=\mathcal{R}_{\widehat{S}}(Y)\cap\mathcal{R}_{\widehat{T}}(Y)$. Since $\mathcal{R}_S(X)\cap\mathcal{R}_T(X)$ is the diagonal, by Lemma 2.2.13, we have that $\mathcal{R}_{\widehat{S}}(Y)\cap\mathcal{R}_{\widehat{T}}(Y)=\mathbf{Q}_{\widehat{S}}(Y)\cap\mathbf{Q}_{\widehat{T}}(Y)$ is also the diagonal. By Lemma 2.2.20, we have that $\mathcal{R}_{\widehat{S}}(Y)$ coincides with the diagonal relation. Therefore, $(Y,\widehat{S},\widehat{T})$ satisfies property (3) and we have proved above that this implies that $(Y,\widehat{S},\widehat{T})$ (and consequently (X,S,T)) has a product extension. This finishes the proof. $\square$

We remark that if (X,S,T) has a product extension, then Theorem 2.1.1 gives us an explicit (or algorithmic) way to build such an extension. In fact, we have:

Proposition 2.2.21. *Let (X,S,T) be a minimal system with commuting transformations S and T. The following are equivalent:*

1. *(X,S,T) has a product extension;*

2. *There exists $x\in X$ such that the last coordinate of $\mathbf{K}_{S,T}^{x}$ is a function of the first two coordinates. In this case, $(\mathbf{K}_{S,T}^{x},\widehat{S},\widehat{T})$ is a product system;*

3. *For any $x\in X$, the last coordinate of $\mathbf{K}_{S,T}^{x}$ is a function of the first two coordinates. In this case, $(\mathbf{K}_{S,T}^{x},\widehat{S},\widehat{T})$ is a product system.*

Proof. (1) $\Rightarrow$ (3). By Theorem 2.1.1, when (X,S,T) has a product extension, then the last coordinate of $\mathbf{Q}_{S,T}(X)$ is a function of the first three ones, which implies (3).

(3) $\Rightarrow$ (2). Is obvious.

(2) $\Rightarrow$ (1). Let $Y \subseteq \mathbf{K}_{S,T}^{x}$ be a minimal subsystem and let $(x_1, x_2, x_3) \in Y$. We remark that $(Y, \widehat{S}, \widehat{T})$ is an extension of (X, S, T) and that the last coordinate of Y is a function of the first two coordinates. Hence, the factor map $(x_1', x_2', x_3') \to (x_1', x_2')$ is an isomorphism between $(Y, \widehat{S}, \widehat{T})$ and $(\overline{\mathcal{O}_S(x_1)} \times \overline{\mathcal{O}_T(x_2)}, S \times \mathrm{id}, \mathrm{id} \times T)$, which is a product system.

$\square$

We can also give a criterion to determine when a minimal system (X, S, T) with commuting transformations S and T is actually a product system:

Proposition 2.2.22. *Let (X, S, T) be a minimal system with commuting transformations S and T. Then (X, S, T) is a product system if and only if $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X) = \Delta_X$.*

Proof. Suppose that $(X, S, T) = (Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ is a product system and (y_1, w_1), $(y_2, w_2) \in \mathbf{Q}_{\sigma\times\mathrm{id}}(Y \times W) \cap \mathbf{Q}_{\mathrm{id}\times\tau}(Y \times W)$. Then $((y_1, w_1), (y_2, w_2)) \in \mathbf{Q}_{\mathrm{id}\times\tau}(Y \times W)$ implies that $y_1 = y_2$, and $((y_1, w_1), (y_2, w_2)) \in \mathbf{Q}_{\sigma\times\mathrm{id}}(Y \times W)$ implies that $w_1 = w_2$. Therefore, $\mathbf{Q}_S(Y \times W) \cap \mathbf{Q}_T(Y \times W) = \Delta_{Y\times W}$.

Conversely, suppose that $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X) = \Delta_X$. By Lemma 2.2.20, Theorem 2.1.1 and Proposition 2.2.21, we have that for any $x_0 \in X$, $(\mathbf{K}_{S,T}^{x_0}, \widehat{S}, \widehat{T})$ is a product extension of (X, S, T). We claim that these systems are actually isomorphic. Recall that the factor map $\pi\colon \mathbf{K}_{S,T}^{x_0} \to X$ is the projection onto the last coordinate. It suffices to show that $(x_1, x_2) = (x_1', x_2')$ for all $(x_1, x_2, x), (x_1', x_2', x) \in \mathbf{K}_{S,T}^{x_0}$. Let $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ be sequences in $\mathbb{Z}$ such that $S^{n_i}T^{m_i}x \to x_0$. We can assume that $S^{n_i}x_1 \to a_1$, $S^{n_i}x_1' \to a_1'$, $T^{m_i}x_2 \to b_1$ and $T^{m_i}x_2' \to b_1'$. Therefore, (x_0, a_1, b_1, x_0) and (x_0, a_1', b_1', x_0) belong to $\mathbf{Q}_{S,T}(X)$. Since $\mathbf{Q}_S(X) \cap \mathbf{Q}_T(X) = \Delta_X$, we have that $a_1 = b_1 = a_1' = b_1' = x_0$. We can assume that $S^{n_i}x \to x'$ and thus $(x_0, S^{n_i}x_1, x_2, S^{n_i}x) \to (x_0, x_0, x_2, x')$, $(x_0, S^{n_i}x_1', x_2', S^{n_i}x) \to (x_0, x_0, x_2', x')$. Moreover, these points belong to $\mathbf{Q}_{S,T}(X)$. Since $\mathcal{R}_S(X)$ is the diagonal, we conclude that $x_2 = x' = x_2'$. Similarly, $x_1 = x_1'$ and the proof is finished.

$\square$

2.2.5. Equicontinuity and product extensions

Let (X, S, T) be a system with commuting transformations S and T. Let suppose that (X, S, T) has a product extension. In this section we show that one can always find a product extension where the factor map satisfies some kind of equicontinuity conditions.

We recall the definition of equicontinuity:

Definition 2.2.23. Let (X, G) be a topological dynamical system, where G is an arbitrary group action. We say that (X, G) is *equicontinuous* if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ for $x, y \in X$, then $d(gx, gy) < \epsilon$ for all $g \in G$. Let $\pi\colon Y \to X$ be a factor map between the topological dynamical systems (Y, G) and (X, G). We say that Y is an *equicontinuous extension* of X if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ and $\pi(x) = \pi(y)$ then $d(gx, gy) < \epsilon$ for all $g \in G$.

The following proposition provides the connection between equicontinuity and the property of being a factor of a product system:

Proposition 2.2.24. *Let (X, S, T) be a minimal system with commuting transformations S and T. If either S or T is equicontinuous, then (X, S, T) has a product extension.*

Proof. Suppose that T is equicontinuous. For any $\epsilon > 0$, let $0 < \delta < \epsilon$ be such that if two points are δ-close to each other, then they stay ϵ-close under the orbit of T. Suppose $(x, y) \in \mathcal{R}_S(X)$. Pick $x', a \in X$ and $n, m \in \mathbb{Z}$ such that $d(x, x') < \delta$, $d(S^n x', y) < \delta$, $d(T^m x', a) < \delta$, $d(S^n T^m x', a) < \delta$. By equicontinuity of T, we have that $d(T^{-m} S^n T^m x', T^{-m} a) < \epsilon$, $d(T^{-m} T^m x', T^{-m} a) < \epsilon$. Therefore $d(x, y) < 4\epsilon$. Hence, $\mathcal{R}_S(X)$ coincides with the diagonal and (X, S, T) has a product extension. $\square$

Specially, when $S = T$ we have:

Corollary 2.2.25. *Let (X, T) be a minimal system. Then (X, T) is equicontinous if and only if (X, T, T) has a product extension.*

Under the assumption that $\mathbf{Q}_T(X)$ is an equivalence relation, we have a better criterion:

Proposition 2.2.26. *Let (X, S, T) be a minimal system with commuting transformations S and T. Suppose that $\mathbf{Q}_T(X)$ is an equivalence relation. Then the system (X, S) is an equicontinuous extension of $(X/\mathbf{Q}_T(X), S)$ if and only if (X, S, T) has a product extension.*

Proof. Suppose that (X, S, T) has no product extensions. By Theorem 2.1.1, we can pick $x, y \in X, x \neq y$ such that $(x, y) \in \mathcal{R}_T(X)$. Denote $\epsilon = d(x, y)/2$. For any $0 < \delta < \epsilon/4$, there exist $z \in X, n, m \in \mathbb{Z}$ such that $d(z, x)$, $d(T^m z, y)$, $d(S^n z, S^n T^m z) < \delta$. Let $x' = S^n z, y' = S^n T^m z$. Then $(x', y') \in \mathbf{Q}_T(X)$, $d(x', y') < \delta$ and $d(S^{-n} x', S^{-n} y') = d(z, T^m z) > \epsilon - 2\delta > \epsilon/2$. So (X, S) is not an equicontinuous extension of $(X/\mathbf{Q}_T(X), S)$.

On the other hand, if (X, S) is not an equicontinuous extension of $(X/\mathbf{Q}_T(X), S)$, then there exists $\epsilon > 0$ and there exist sequences $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}$ in X and a sequence $(n_i)_{i \in \mathbb{N}}$ in $\mathbb{Z}$ with $d(x_i, y_i) < 1/i$, $(x_i, y_i) \in \mathbf{Q}_T(X)$, and $d(S^{n_i} x_i, S^{n_i} y_i) \geq \epsilon$. By passing to a subsequence, we may assume $(S^{n_i} x_i)_{i \in \mathbb{N}}$, $(S^{n_i} y_i)_{i \in \mathbb{N}}$, $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ converges to x_0, y_0, w and w respectively. Then $x_0 \neq y_0$. For any $\delta > 0$, pick $i \in \mathbb{N}$ such that $d(S^{n_i} x_i, x_0)$, $d(S^{n_i} y_i, y_0)$, $d(x_i, w)$, $d(y_i, w) < \delta$. Since $(x_i, y_i) \in \mathbf{Q}_T(X)$, we can pick $z \in X, m \in \mathbb{Z}$ such that $d(z, x_i)$, $d(T^m z, y_i)$, $d(S^{n_i} z, S^{n_i} x_i)$, $d(S^{n_i} T^m z, S^{n_i} y_i) < \delta$. So the distance between the corresponding coordinates of $(S^{n_i} z, z, S^{n_i} T^m z, T^m z)$ and (x_0, w, y_0, w) are all less than $C\delta$ for some uniform constant C. So $(x_0, y_0) \in \mathcal{R}_T(X)$, and (X, S, T) has not a product extension. $\square$

In the following we relativize the notion of being a product system to factor maps.

Definition 2.2.27. Let $\pi\colon Y \to X$ be a factor map between the systems of commuting transformations (Y,S,T) and (X,S,T). We say that π is *S-equicontinuous with respect to T* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $y, y' \in Y$ satisfy $(y,y') \in \mathbf{Q}_T(Y)$, $d(y,y') < \delta$ and $\pi(y) = \pi(y')$, then $d(S^n y, S^n y') < \epsilon$ for all $n \in \mathbb{Z}$.

Lemma 2.2.28. *Let (X,S,T) be a minimal system with commuting transformations S and T, and let π be the projection to the trivial system. Then π is S-equicontinous with respect to T if and only if (X,S,T) has a product extension.*

Proof. If π is not S-equicontinuous with respect to T, there exists $\epsilon > 0$ such that for any $\delta = \frac{1}{i} > 0$ one can find $(x_i, x_i') \in \mathbf{Q}_T(X)$ with $d(x_i,x_i') < \delta$ and $n_i \in \mathbb{Z}$ with $d(S^{n_i}x_i, S^{n_i}x_i') \geq \epsilon$. For a subsequence, $(x_i, S^{n_i}x_i, x_i', S^{n_i}x_i') \in \mathbf{Q}_{S,T}(X)$ converges to a point of the form $(a,x,a,x') \in \mathbf{Q}_{S,T}(X)$ with $x \neq x'$. We remark that this is equivalent to $(x,a,x',a) \in \mathbf{Q}_{S,T}(X)$ and hence $(x,x') \in \mathcal{R}_S(X)$. By Theorem 2.1.1 (X,S,T) has no product extension.

Conversely, if (X,S,T) has no product extension, by Theorem 2.1.1 we can find $x \neq x'$ with $(x,x') \in \mathcal{R}_S(X)$. Let $0 < \epsilon < d(x,x')$ and let $0 < \delta < \epsilon/4$. We can find $x'' \in X$ and $n,m \in \mathbb{Z}$ such that $d(x'',x) < \delta$, $d(S^n x'', x') < \delta$ and $d(T^m x'', S^n T^m x'') < \delta$. Writing $w = T^m x''$, $w' = S^n T^m x''$, we have that $(w,w') \in \mathbf{Q}_S(X)$, $d(w,w') < \delta$ and $d(T^{-m}w, T^{-m}w') > \epsilon/2$. Hence π is not S-equicontinuous with respect to T. $\square$

A connection between a magic system and a system which is S-equicontinuous with respect to T is:

Proposition 2.2.29. *For every minimal system with commuting transformations (X,S,T), the magic extension constructed in Theorem 2.2.11 is S-equicontinuous with respect to T.*

Proof. Let (X,S,T) be a minimal system with commuting transformations S and T. Recall that the magic extension Y of X is the orbit closure of a minimal point (z_1,z_2,x_0) under $\widehat{S}$ and $\widehat{T}$, and the factor map $\pi\colon Y \to X$ is the projection onto the last coordinate. Let $\vec{x} = (x_1,x_2,x_3), \vec{y} = (y_1,y_2,y_3) \in Y$ be such that $\pi(\vec{x}) = \pi(\vec{y})$ and $(\vec{x},\vec{y}) \in \mathbf{Q}_{\widehat{T}}(Y)$. Then we have that $x_1 = y_1$ and $x_3 = y_3$. Since $\widehat{S}^n\vec{x} = (S^n x_1, x_2, S^n x_3)$ and $\widehat{S}^n\vec{y} = (S^n x_1, y_2, S^n x_3)$, we conclude that $\widehat{S}$ preserves the distance between $\vec{x}$ and $\vec{y}$. $\square$

A direct corollary of this proposition is:

Corollary 2.2.30. *Let (X,S,T) be a minimal system with commuting transformations S and T. If (X,S,T) has a product extension, then it has a product extension which is S-equicontinuous with respect to T.*

Proof. If (X,S,T) has a product extension, by Theorem 2.1.1, we can build a magic extension which is actually a product system. This magic extension is S-equicontinuous with respect to T. $\square$

2.2.6. Changing the generators

Let (X, S, T) be a system with commuting transformations S and T. We remark that $\mathbf{Q}_{S,T}(X)$ depends strongly on the choice of the generators S and T. For instance, let (X, S) be a minimal system and consider the minimal systems (X, S, S) and (X, S, id) with commuting transformations. We have that (X, S, id) has a product extension, but (X, S, S) does not (unless (X, S) is equicontinous). However, there are cases where we can deduce some properties by changing the generators. Let (X, S, T) be a minimal system with commuting transformations S and T. Denote $S' = T^{-1}S$, $T' = T$. We have that (X, S', T') is a minimal system with commuting transformations S' and T'. Suppose now that (X, S', T') has a product extension. By Proposition 2.2.21, for any $x \in X$ we have that $(\mathbf{K}^x_{S',T'}, \widehat{S'}, \widehat{T'})$ is an extension of (X, S', T') and it is isomorphic to a product system. We remark that $(\mathbf{K}^x_{S',T'}, \widehat{T'}\widehat{S'}, \widehat{T'})$ is an extension of (X, S, T) and it is isomorphic to $(Y \times W, S \times T, T \times T)$, where $Y = \overline{\mathcal{O}_{S'}(x)}$ and $W = \overline{\mathcal{O}_{T'}(x)}$. It follows that (X, S, T) has an extension which is the Cartesian product of two systems with commuting transformations with different natures: one of the form (Y, S, id) where one of the transformations is the identity, and the other of the form (W, T, T) where the two transformations are the same.

2.3. Examples

In this section, we compute the $\mathcal{R}_{S,T}(X)$ relation in some minimal symbolic systems (X, S, T). We start by recalling some general definitions.

Let $\mathcal{A}$ be a finite alphabet. The *shift transformation* $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is the map $(x_i)_{i\in\mathbb{Z}} \mapsto (x_{i+1})_{i\in\mathbb{Z}}$. A *one dimensional subshift* is a closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$ invariant under the shift transformation. When there is more than one space involved, we let σ_X denote the shift transformation on the space X.

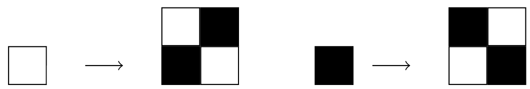
In the two dimensional setting, we define the *shift transformation* $\sigma_{(1,0)}\colon \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$, $(x_{i,j})_{i,j\in\mathbb{Z}} \mapsto (x_{i+1,j})_{i,j\in\mathbb{Z}}$ and $\sigma_{(0,1)}\colon \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$, $(x_{i,j})_{i,j\in\mathbb{Z}} \mapsto (x_{i,j+1})_{i,j\in\mathbb{Z}}$. Hence $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$ are the translations in the canonical directions. A *two dimensional subshift* is a closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ invariant under the shift transformations. We remark that $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$ are a pair of commuting transformations and therefore if $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is a subshift, $(X, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a system with commuting transformations $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$.

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^2}$ be a subshift and let $x \in X$. If B is a subset of $\mathbb{Z}^2$, we let $x|_B \in \mathcal{A}^B$ denote the restriction of x to B and for $\vec{n} \in \mathbb{Z}^2$, we let $B + \vec{n}$ denote the set $\{\vec{b} + \vec{n}\colon \vec{b} \in B\}$. When X is a subshift (one or two dimensional), we let $\mathcal{A}_X$ denote its alphabet.

In the following we compute the relation $\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X)$ in the Morse Tiling and then we state a general criteria for a $\mathbb{Z}^2$ shift space to have a product extension. See [103] for more background about tiling and substitutions.

2.3.1. The Morse tiling

Consider the Morse tiling system given by the substitution rule:



One can iterate this substitution in a natural way:

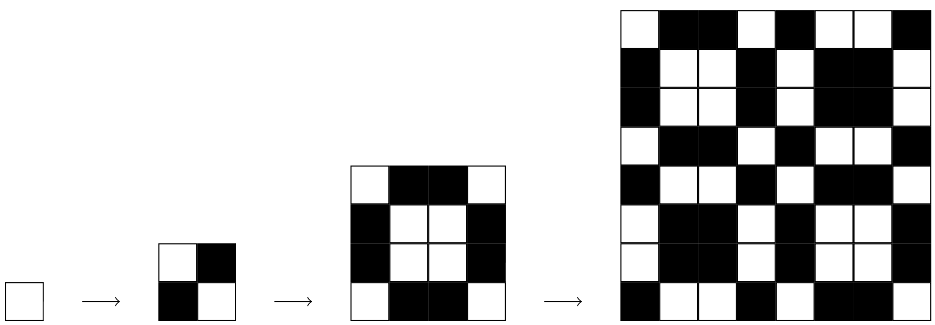


Figure 2.1: first, second and third iteration of the substitution

We identify 0 with the white square and 1 with the black one. Let $B_n = ([-2^{n-1}, 2^{n-1} - 1] \cap \mathbb{Z}) \times ([-2^{n-1}, 2^{n-1} - 1] \cap \mathbb{Z})$ be the square of size 2^n centered at the origin. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\{0,1\}^{\mathbb{Z}^2}$ such that the restriction of x_n to B_n coincides with the nth-iteration of the substitution. Taking a subsequence we have that $(x_n)_{n\in\mathbb{N}}$ converges to a point $x^* \in \{0,1\}^{\mathbb{Z}^2}$. Let $X_M \subseteq \{0,1\}^{\mathbb{Z}^2}$ be the orbit closure of x^* under the shift actions. We point out that X_M does not depend on the particular choice of x^* (we refer to Chapter 1 of [103] for a general reference about substitution tiling systems). Moreover, the *Morse system* $(X_M, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a minimal system with commuting transformations $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$.

Proposition 2.3.1. *For the Morse system,* $\mathcal{R}_{\sigma_{(1,0)}}(X_M) = \mathcal{R}_{\sigma_{(0,1)}}(X_M) = \Delta_{X_M}$. *Consequently, the Morse system has a product extension.*

Proof. Note that for $x = (x_{i,j})_{i,j\in\mathbb{Z}} \in X_M$, we have that $x_{i,j} + x_{i+1,j} = x_{i,j'} + x_{i+1,j'} \mod 2$ and $x_{i,j} + x_{i,j+1} = x_{i',j} + x_{i,j+1} \mod 2$ for every $i, j, i', j' \in \mathbb{Z}$. From this, we deduce that if $x_{0,0} = 0$ then $x_{i,j} = x_{i,0} + x_{0,j}$ for every $i, j \in \mathbb{Z}$. From now on, we assume that $x^*_{0,0} = 0$.

For $N \in \mathbb{N}$, let B_N denote the square $([-N, N] \cap \mathbb{Z}) \times ([-N, N] \cap \mathbb{Z})$. Suppose $(y, z) \in \mathcal{R}_{\sigma_{(1,0)}}(X_M)$ and let $w \in X_M$ be such that $(y, z, w, w) \in \mathbf{Q}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_M)$. We deduce that there exist $n, m, p, q \in \mathbb{Z}$ such that

$$\begin{aligned}
\sigma^p_{(1,0)}\sigma^q_{(0,1)}x^*|_{B_N} &= y|_{B_N};\\
\sigma^{p+n}_{(1,0)}\sigma^q_{(0,1)}x^*|_{B_N} &= z|_{B_N};\\
\sigma^p_{(1,0)}\sigma^{q+m}_{(0,1)}x^*|_{B_N} &= \sigma^{p+n}_{(1,0)}\sigma^{q+m}_{(0,1)}x^*|_{B_N} = w|_{B_N}.
\end{aligned}$$

Since $\sigma^p_{(1,0)}\sigma^{q+m}_{(0,1)}x^*|_{B_N} = \sigma^{p+n}_{(1,0)}\sigma^{q+m}_{(0,1)}x^*|_{B_N}$, we deduce that $x^*_{p+c,0} = x^*_{p+n+c,0}$ for all $c \leq N$. This in turn implies that $y|_{B_N} = \sigma^p_{(1,0)}\sigma^q_{(0,1)}x^*|_{B_N} = \sigma^{p+n}_{(1,0)}\sigma^q_{(0,1)}x^*|_{B_N} = z|_{B_N}$. Since N is arbitrary we deduce that $y = z$. Therefore $\mathcal{R}_{\sigma_{(1,0)}}(X_M) = \Delta_{X_M}$ and thus $(X_M, \sigma_{(1,0)}, \sigma_{(0,1)})$ has a product extension. □

Remark 2.3.2. In fact, let (Y, σ) be the one dimensional Thue-Morse system. This is the subshift generated by the one dimensional substitution $0 \mapsto 01$, $1 \mapsto 10$ (see [102]). Then we can define $\pi\colon Y \times Y \to X_M$ by $\pi(x, x')_{n,m} = x_n + x'_m$ and it turns out that this is a product extension of the two dimensional Morse system. Moreover, we have that $(\mathbf{K}^{x^*}_{S,T}, \widehat{S}, \widehat{T})$ is isomorphic to $(Y \times Y, T \times \mathrm{id}, \mathrm{id} \times T)$, where the isomorphism $\phi\colon \mathbf{K}^{x^*}_{S,T} \to Y \times Y$ is given by $\phi(a, b, c) = (a|_A, b|_B)$, where $A = \{(n, 0)\colon n \in \mathbb{Z}\}$ and $B = \{(0, n)\colon n \in \mathbb{Z}\}$. We show in the next subsection that this is a general procedure to build symbolic systems with a product extension.

2.3.2. Building factors of product systems

Let (X, σ_X) and (Y, σ_Y) be two minimal one dimensional shifts and let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be the respective alphabets.

Let $x \in X$ and $y \in Y$. Consider the point $\mathbf{z} \in (\mathcal{A}_X \times \mathcal{A}_Y)^{\mathbb{Z}^2}$ defined as $\mathbf{z}_{i,j} = (x_i, y_j)$ for $i, j \in \mathbb{Z}$ and let Z denote the orbit closure of $\mathbf{z}$ under the shift transformations. Then we can verify that $(Z, \sigma_{(1,0)}, \sigma_{(0,1)})$ is isomorphic to the product of (X, σ_X) and (Y, σ_Y) (and particularly $(Z, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a minimal system).

Let $\mathcal{A}$ be an alphabet and let $\varphi\colon \mathcal{A}_X \times \mathcal{A}_Y \to \mathcal{A}$ be a function. We can define $\phi\colon Z \to W \coloneqq \phi(Z) \subseteq \mathcal{A}^{\mathbb{Z}^2}$ such that $\phi(z)_{i,j} = \varphi(z_{i,j})$ for $i, j \in \mathbb{Z}$. Then $(W, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a minimal symbolic system with a product extension and we write $W = W(X, Y, \varphi)$ to denote this system. We show that this is the unique way to produce minimal symbolic systems with product extensions.

Proposition 2.3.3. *Let $(W, \sigma_{(1,0)}, \sigma_{(0,1)})$ be a minimal symbolic system with a product extension. Then, there exist one dimensional minimal subshifts (X, σ_X) and (Y, σ_Y) and a map $\varphi\colon \mathcal{A}_X \times \mathcal{A}_Y \to \mathcal{A}_W$ such that $W = W(X, Y, \varphi)$.*

Proof. We recall that $\mathcal{A}_W$ denotes the alphabet of W. For $n \in \mathbb{N}$ we let B_n denote $([-n, n] \cap \mathbb{Z}) \times ([-n, n] \cap \mathbb{Z})$. Let $\mathbf{w} = (w_{i,j})_{i,j \in \mathbb{Z}} \in W$. By Proposition 2.2.21, the last coordinate in

$\mathbf{K}^{\mathbf{w}}_{\sigma_{(1,0)},\sigma_{(0,1)}}(W)$ is a function of the two first coordinates. Since $\mathbf{K}^{\mathbf{w}}_{\sigma_{(1,0)},\sigma_{(0,1)}}(W)$ is a closed subset of X^3 we have that this function is continuous. Hence, there exists $n \in \mathbb{N}$ such that for every $i, j \in \mathbb{Z}$, $w_{i,j}$ is determined by $\mathbf{w}|_{B_n}$, $\mathbf{w}|_{B_n+(i,0)}$ and $\mathbf{w}|_{B_n+(0,j)}$. Let $\mathcal{A}_X = \{\mathbf{w}|_{B_n+(i,0)} : i \in \mathbb{Z}\}$ and $\mathcal{A}_Y = \{\mathbf{w}|_{B_n+(0,j)} : j \in \mathbb{Z}\}$. Then $\mathcal{A}_X$ and $\mathcal{A}_Y$ are finite alphabets and we can define $\varphi\colon \mathcal{A}_X \times \mathcal{A}_Y \to \mathcal{A}_W$ such that $\varphi(\mathbf{w}|_{B_n+(i,0)}, \mathbf{w}|_{B_n+(0,j)}) = w_{i,j}$.

We recall that since $(W, \sigma_{(1,0)}, \sigma_{(0,1)})$ has a product extension, $(\mathbf{K}^{\mathbf{w}}_{\sigma_{(1,0)},\sigma_{(0,1)}}(W), \widehat{\sigma_{(1,0)}}, \widehat{\sigma_{(0,1)}})$ is a minimal system. Let $\phi_1\colon \mathbf{K}^{\mathbf{w}}_{\sigma_{(1,0)},\sigma_{(0,1)}}(W) \to \mathcal{A}_X^{\mathbb{Z}}$ and $\phi_2\colon \mathbf{K}^{\mathbf{w}}_{\sigma_{(1,0)},\sigma_{(0,1)}}(W) \to \mathcal{A}_Y^{\mathbb{Z}}$ defined as $\phi_1(w_1, w_2, w_3) = (w_1|_{B_n+(i,0)})_{i\in\mathbb{Z}}$ and $\phi_2(w_1, w_2, w_3) = (w_2|_{B_n+(0,j)})_{j\in\mathbb{Z}}$. Let $X = \phi_1(W)$ and $Y = \phi_2(W)$. Then (X, σ_X) and (Y, σ_Y) are two minimal symbolic systems and $W = W(X, Y, \varphi)$. $\square$

The previous proposition says that for a minimal symbolic system $(W, \sigma_{(1,0)}, \sigma_{(0,1)})$, having a product extension means that the dynamics can be deduced by looking at the shifts generated by finite blocks in the canonical directions.

Remark 2.3.4. It was proved in [91] that two dimensional rectangular substitutions are sofic. It was also proved that the product of two one dimensional substitution is a two dimensional substitution and therefore is sofic. Moreover, this product is measurably isomorphic to a shift of finite type. Given Proposition 2.3.3, the natural question that one can formulate is what properties can be deduced for the subshifts (X, σ_X) and (Y, σ_Y)? For example, what happens with these subshifts when $(W, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a two dimensional substitution with a product extension? We do not know the answer to this question.

2.4. $\mathcal{R}_{S,T}(X)$ relation in the distal case

2.4.1. Basic properties

This section is devoted to the study of the $\mathcal{R}_{S,T}(X)$ relation in the distal case. We do not know if $\mathcal{R}_{S,T}(X)$, $\mathcal{R}_S(X)$ and $\mathcal{R}_T(X)$ are equivalence relations in the general setting. However, we have a complete description of these relations in the distal case.

Recall that a topological dynamical system (X, G) is *distal* if $x \neq y$ implies that

$$\inf_{g\in G} d(gx, gy) > 0.$$

Distal systems have many interesting properties (see [7], chapters 5 and 7). We recall some of them:

Theorem 2.4.1.

1. *The Cartesian product of distal systems is distal;*

2. *Distality is preserved by taking factors and subsystems;*

3. *A distal system is minimal if and only if it is transitive;*

4. *If (X, G) is distal and G' is a subgroup of G, then (X, G') is distal.*

The main property about distality is that it implies that cubes have the following transitivity property:

Lemma 2.4.2. *Let (X, S, T) be a distal minimal system with commuting transformations S and T. Suppose that R is either S or is T. Then*

1. *If $(x, y), (y, z) \in \mathbf{Q}_R(X)$, then $(x, z) \in \mathbf{Q}_R(X)$;*

2. *If $(a_1, b_1, a_2, b_2), (a_2, b_2, a_3, b_3) \in \mathbf{Q}_{S,T}(X)$, then $(a_1, b_1, a_3, b_3) \in \mathbf{Q}_{S,T}(X)$.*

Proof. We only prove (1) since the proof of (2) is similar. Let $(x, y), (y, z) \in \mathbf{Q}_R(X)$. Pick any $a \in X$. Then $(a, a) \in \mathbf{Q}_R(X)$. By Proposition 2.2.5, there exists a sequence $(g_n)_{n\in\mathbb{N}} = ((g'_n, g''_n))_{n\in\mathbb{N}}$ in $\mathcal{G}_R$ such that $g_n(x, y) = (g'_n x, g''_n y) \to (a, a)$, where $\mathcal{G}_R$ is the group generated by $\mathrm{id} \times R$ and $g \times g, g \in G$. We can assume (by taking a subsequence) that $g''_n z \to u$ and thus $(g''_n y, g''_n z) \to (a, u) \in \mathbf{Q}_R(X)$. Since $(g'_n, g''_n)(x, z) \to (a, u)$, by distality we have that (x, z) is in the closed orbit of (a, u) and thus $(x, z) \in \mathbf{Q}_R(X)$. □

Remark 2.4.3. It is worth noting that this transitivity lemma fails in the non-distal case, even if $S = T$ (see [114] for an example).

The following proposition gives equivalent definitions of $\mathcal{R}_{S,T}(X)$ in the distal case:

Proposition 2.4.4. *Let (X, S, T) be a distal system with commuting transformations S and T. Suppose $x, y \in X$. The following are equivalent:*

1. *$(x, y, y, y) \in \mathbf{Q}_{S,T}(X)$;*

2. *There exists $a, b, c \in X$ such that $(x, a, b, c), (y, a, b, c) \in \mathbf{Q}_{S,T}(X)$;*

3. *For every $a, b, c \in X$, if $(x, a, b, c) \in \mathbf{Q}_{S,T}(X)$, then $(y, a, b, c) \in \mathbf{Q}_{S,T}(X)$;*

4. *$(x, y) \in \mathcal{R}_{S,T}(X)$;*

5. *$(x, y) \in \mathcal{R}_S(X)$;*

6. *$(x, y) \in \mathcal{R}_T(X)$.*

Particularly, $\mathcal{R}_S(X) = \mathcal{R}_T(X) = \mathcal{R}_{S,T}(X)$.

Proof. (1)⇒(3). Suppose that $(x,a,b,c) \in \mathbf{Q}_{S,T}(X)$ for some $a,b,c \in X$. By (3),(4) and (5) of Proposition 2.2.3, $(x,a,b,c) \in \mathbf{Q}_{S,T}(X)$ implies that $(a,x,a,x) \in \mathbf{Q}_{S,T}(X)$, and $(x,y,y,y) \in \mathbf{Q}_{S,T}(X)$ implies that $(x,x,y,x) \in \mathbf{Q}_{T,S}(X)$. By Lemma 2.4.2, (a,x,a,x), $(x,x,y,x) \in \mathbf{Q}_{S,T}(X)$ implies that $(x,a,y,a) \in \mathbf{Q}_{S,T}(X)$. Again by Lemma 2.4.2, $(x,a,b,c),(x,a,y,a) \in \mathbf{Q}_{S,T}(X)$ implies that $(b,c,y,a) \in \mathbf{Q}_{S,T}(X)$ and thus $(y,a,b,c) \in \mathbf{Q}_{S,T}(X)$.

(3)⇒(2). Obvious.

(2)⇒(1). Suppose that $(x,a,b,c),(y,a,b,c) \in \mathbf{Q}_{S,T}(X)$ for some $a,b,c \in X$. Then $(b,c,y,a) \in \mathbf{Q}_{S,T}(X)$. By Lemma 2.4.2, $(x,a,y,a) \in \mathbf{Q}_{S,T}(X)$. By (4) and (5) of Proposition 2.2.3, $(y,a,y,a) \in \mathbf{Q}_{S,T}(X)$. Hence $(x,y,a,a),(y,y,a,a) \in \mathbf{Q}_{T,S}(X)$ and $(a,a,y,y) \in \mathbf{Q}_{T,S}(X)$. By Lemma 2.4.2, $(x,y,y,y) \in \mathbf{Q}_{T,S}(X)$ which is equivalent to $(x,y,y,y) \in \mathbf{Q}_{S,T}(X)$.

(1)⇒(4). Take $a = y$ and $b = y$.

(4)⇒(5) and (4)⇒(6) are obvious from the definition.

(5)⇒(1). Suppose $(x,y,a,a) \in \mathbf{Q}_{S,T}(X)$ for some $a \in X$. By (4) and (5) of Proposition 2.2.3, $(y,y,a,a) \in \mathbf{Q}_{S,T}(X)$. By Lemma 2.4.2, $(x,y,y,y) \in \mathbf{Q}_{T,S}(X)$ and thus $(x,y,y,y) \in \mathbf{Q}_{S,T}(X)$.

(6)⇒(1). Similar to (4)⇒(2). □

We can now prove that $\mathcal{R}_{S,T}(X)$ is an equivalence relation in the distal setting:

Theorem 2.4.5. *Let (X,S,T) be a distal system with commuting transformations S and T. Then $\mathbf{Q}_S(X)$, $\mathbf{Q}_T(X)$ and $\mathcal{R}_{S,T}(X)$ are closed equivalence relations on X.*

Proof. It suffices to prove the transitivity of $\mathcal{R}_{S,T}(X)$. Let $(x,y),(y,z) \in \mathcal{R}_{S,T}(X)$. Since (y,z,z,z) and $(x,y) \in \mathcal{R}_{S,T}(X)$, by (4) of Proposition 2.4.4, we have that $(x,z,z,z) \in \mathbf{Q}_{S,T}(X)$ and thus $(x,z) \in \mathcal{R}_{S,T}(X)$. □

We also have the following property in the distal setting, which allows us to lift an (S,T)-regionally proximal pair in a system to a pair in an extension system:

Proposition 2.4.6. *Let $\pi\colon Y \to X$ be a factor map between systems (Y,S,T) and (X,S,T) with commuting transformations S and T. If (X,S,T) is distal, then $\pi \times \pi(\mathcal{R}_{S,T}(Y)) = \mathcal{R}_{S,T}(X)$.*

Proof. The proof is similar to Theorem 6.4 of [110]. Let $(x_1,x_2) \in \mathcal{R}_{S,T}(X)$. Then there exist a sequence $(x_i)_{i\in\mathbb{N}} \in X$ and two sequences $(n_i)_{i\in\mathbb{N}},(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$(x_i, S^{n_i}x_i, T^{m_i}x_i, S^{n_i}T^{m_i}x_i) \to (x_1,x_1,x_1,x_2).$$

Let $(y_i)_{i\in\mathbb{N}}$ in Y be such that $\pi(y_i) = x_i$. By compactness we can assume that $y_i \to y_1$, $S^{n_i}y_i \to a$, $T^{m_i}y_i \to b$ and $S^{n_i}T^{m_i}y_i \to c$. Then $(y_1,a,b,c) \in \mathbf{Q}_{S,T}(Y)$ and $\pi^4(y_1,a,b,c) =$

(x_1,x_1,x_1,x_2). Particularly, $(y_1,a)\in \mathbf{Q}_S(Y)$. By minimality we can find $g_i\in G$ and p_i such that $(g_iy_1,g_iS^{p_i}a)\to(y_1,y_1)$. We can assume that $g_ib\to b'$ and $g_iS^{p_i}c\to c'$, so that $(y_1,y_1,b',c')\in \mathbf{Q}_{S,T}(Y)$ and $\pi^4(y_1,y_1,b',c')=(x_1,x_1,x_1,x_2')$, where $x_2'=\lim g_iS^{p_i}x_2$. Recall that $(x_1,x_2')\in \overline{\mathcal{O}_{G^\Delta}(x_1,x_2)}$, where $G^\Delta=\{g\times g\colon g\in G\}$. Since $(y_1,b')\in \mathbf{Q}_T(Y)$, we can find $(g_i')_{i\in\mathbb{N}}$ in G and $(q_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(g_i'y_1,g_i'T^{q_i}b')\to(y_1,y_1)$. We can assume without loss of generality that $g_i'T^{q_i}c'\to c''$ so that $(y_1,y_1,y_1,c'')\in\mathbf{Q}_{S,T}(Y)$ and $\pi^4(y_1,y_1,y_1,c'')=(x_1,x_1,x_1,x_2'')$, where $x_2''=\lim g_i'T^{q_i}x_2'$. Recall that $(x_1,x_2'')\in\overline{\mathcal{O}_{G^\Delta}(x_1,x_2')}$. So $(x_1,x_2'')\in\overline{\mathcal{O}_{G^\Delta}(x_1,x_2)}$. By distality, this orbit is minimal and thus we can find $(g_i'')_{i\in\mathbb{N}}$ in G such that $(g_i''x_1,g_i''x_2'')\to(x_1,x_2)$. We assume without loss of generality that $g_i''y_1\to y_1'$ and $g_i''c''\to y_2'$. Then $(y_1',y_1',y_1',y_2')\in\mathbf{Q}_{S,T}(Y)$ and $\pi^4(y_1',y_1',y_1',y_2')=(x_1,x_1,x_1,x_2)$. Particularly $(y_1',y_2')\in\mathcal{R}_{S,T}(Y)$ and $\pi\times\pi(y_1',y_2')=(x_1,x_2)$. □

These results allow us to conclude that cubes structures characterize factors with product extensions:

Theorem 2.4.7. *Let (X,S,T) be a minimal distal system with commuting transformations S and T. Then*

1. *$(X/\mathcal{R}_{S,T}(X),S,T)$ has a product extension, where $X/\mathcal{R}_{S,T}(X)$ is the quotient of X under the equivalence relation $\mathcal{R}_{S,T}(X)$. Moreover, it is the maximal factor with this property, meaning that any other factor of X with a product extension factorizes through it;*

2. *For any magic extension $(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T})$, $(\mathbf{K}_{S,T}^{x_0}/\mathcal{R}_{\widehat{S},\widehat{T}}(\mathbf{K}_{S,T}^{x_0}),\widehat{S},\widehat{T})$ is a product system. Moreover, both $(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T})$ and $(\mathbf{K}_{S,T}^{x_0}/\mathcal{R}_{\widehat{S},\widehat{T}}(\mathbf{K}_{S,T}^{x_0}))$ are distal systems.*

We have the following commutative diagram:

$$\begin{tikzcd}
(\mathbf{K}_{S,T}^{x_0},\widehat{S},\widehat{T}) \arrow[r] \arrow[d] & (X,S,T) \arrow[d] \\
(\mathbf{K}_{S,T}^{x_0}/\mathcal{R}_{\widehat{S},\widehat{T}}(\mathbf{K}_{S,T}^{x_0}),\widehat{S},\widehat{T}) \arrow[r] & (X/\mathcal{R}_{S,T}(X),S,T)
\end{tikzcd}$$

Proof. We remark that if (Z,S,T) is a factor of (X,S,T) with a product extension, then $\pi\times\pi(\mathcal{R}_{S,T}(X))=\mathcal{R}_{S,T}(Z)=\Delta_X$, meaning that there exists a factor map from $(X/\mathcal{R}_{S,T}(X),S,T)$ to (Y,S,T). It remains to prove that $X/\mathcal{R}_{S,T}(X)$ has a product extension. To see this, let π be the quotient map $X\to X/\mathcal{R}_{S,T}(X)$ and let $(y_1,y_2)\in\mathcal{R}_{S,T}(X/\mathcal{R}_{S,T}(X))$. By Proposition 2.4.6, there exists $(x_1,x_2)\in\mathcal{R}_{S,T}(X)$ with $\pi(x_1)=y_1$ and $\pi(x_2)=y_2$. Since $(x_1,x_2)\in\mathcal{R}_{S,T}(X)$, $y_1=\pi(x_1)=\pi(x_2)=y_2$. So $\mathcal{R}_{S,T}(X/\mathcal{R}_{S,T}(X))$ coincides with the diagonal. By Theorem 2.1.1, $(X/\mathcal{R}_{S,T}(X),S,T)$ has a product extension. This proves (1).

We now prove that the factor of the magic extension is actually a product system. By Theorem 2.4.5, we have that $\mathbf{Q}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0}), \mathbf{Q}_{\widehat{T}}(\mathbf{K}_{S,T}^{x_0})$ are equivalence relations and by Theorem 2.2.11 and Proposition 2.4.4, we have that $\mathbf{Q}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0}) \cap \mathbf{Q}_{\widehat{T}}(\mathbf{K}_{S,T}^{x_0}) = \mathcal{R}_{S,T}(\mathbf{K}_{S,T}^{x_0})$. Consequently $(\mathbf{K}_{S,T}^{x_0}/\mathcal{R}_{\widehat{S},\widehat{T}}(\mathbf{K}_{S,T}^{x_0}), \widehat{S}, \widehat{T})$ is isomorphic to $(\mathbf{K}_{S,T}^{x_0}/\mathbf{Q}_{\widehat{T}}(\mathbf{K}_{S,T}^{x_0}) \times \mathbf{K}_{S,T}^{x_0}/\mathbf{Q}_{\widehat{S}}(\mathbf{K}_{S,T}^{x_0}), \widehat{S} \times \mathrm{id}, \mathrm{id} \times \widehat{T})$, which is a product system.

Since (X, S, T) is distal, the distality of $(\mathbf{K}_{S,T}^{x_0}, \widehat{S}, \widehat{T})$ and $(\mathbf{K}_{S,T}^{x_0}/\mathcal{R}_{S,T}(\mathbf{K}_{S,T}^{x_0}), \widehat{S}, \widehat{T})$ follows easily from Theorem 2.4.1. □

2.4.2. Further remarks: The $\mathcal{R}_{S,T}(X)$ strong relation

Let (X, S, T) be a system with commuting transformations S and T. We say that x and y are *strongly $\mathcal{R}_{S,T}(X)$-related* if there exist $a \in X$ and two sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(x, y, a, a) = \lim_{i\to\infty}(x, S^{n_i}x, T^{m_i}x, S^{n_i}T^{m_i}x)$, and there exist $b \in X$ and two sequences $(n_i')_{i\in\mathbb{N}}$ and $(m_i')_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(x, b, y, b) = \lim_{i\to\infty}(x, S^{n_i'}x, T^{m_i'}x, S^{n_i'}T^{m_i'}x)$.

It is a classical result that when $S = T$, the $\mathcal{R}_{T,T}(X)$ relation coincides with the strong one (see [7], Chap 9). We show that this is not true in the commuting case even in the distal case, and give a counter example of commuting rotations in the Heisenberg group. We refer to [8] and [81] for general references about nilrotations.

Let $H = \mathbb{R}^3$ be the group with the multiplication given by $(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + ab')$ for all $(a, b, c), (a', b', c') \in H$. Let H_2 be the subgroup spanned by $\{ghg^{-1}h^{-1} \colon g, h \in H\}$. By a direct computation we have that $H_2 = \{(0, 0, c) \colon c \in \mathbb{R}\}$ and thus H_2 is central in H. Therefore H is a 2-step nilpotent Lie group and $\Gamma = \mathbb{Z}^3$ is a cocompact subgroup, meaning that $X_H \coloneqq H/\Gamma$ is a compact space. X_H is called the *Heisenberg manifold.* Note that $\mathbb{T}^3$ is a fundamental domain of X_H.

Lemma 2.4.8. *The map $\Phi \colon X_H \to \mathbb{T}^3$ given by*

$$\Phi((a, b, c)\Gamma) = (\{a\}, \{b\}, \{c - a\lfloor b\rfloor\})$$

is a well-defined homomorphism between X_H and $\mathbb{T}^3$. Here $\lfloor x\rfloor$ is the largest integer which does not exceed x, $\{x\} = x - \lfloor x\rfloor$, and $\mathbb{T}^3$ is viewed as $[0, 1)^3$ in this map. Moreover, $(a, b, c)\Gamma = (\{a\}, \{b\}, \{c - a\lfloor b\rfloor\})\Gamma$ for all $a, b, c \in \mathbb{R}$.

Proof. It suffices to show that $(a, b, c)\Gamma = (a', b', c')\Gamma$ if and only if $(\{a\}, \{b\}, \{c - a\lfloor b\rfloor\}) = (\{a'\}, \{b'\}, \{c' - a'\lfloor b'\rfloor\})$. If $(a, b, c)\Gamma = (a', b', c')\Gamma$, there exists $(x, y, z) \in \Gamma$ such that $(a', b', c') = (a, b, c) \cdot (x, y, z) = (x + a, y + b, z + c + ay)$. therefore,

$$x = a' - a, y = b' - b, z = c' - c - a(b' - b).$$

Since $x, y \in \mathbb{Z}$, we have that $\{a\} = \{a'\}, \{b\} = \{b'\}$. So $b - b' = \lfloor b \rfloor - \lfloor b' \rfloor$. Then

$$(c' - a'\lfloor b' \rfloor) - (c - a\lfloor b \rfloor) = (c' - c - a(b' - b)) - (a' - a)\lfloor b' \rfloor = z - x\lfloor b' \rfloor \in \mathbb{Z}.$$

So $(\{a\}, \{b\}, \{c - a\lfloor b \rfloor\}) = (\{a'\}, \{b'\}, \{c' - a'\lfloor b' \rfloor\})$.

Conversely, if $(\{a\}, \{b\}, \{c - a\lfloor b \rfloor\}) = (\{a'\}, \{b'\}, \{c' - a'\lfloor b' \rfloor\})$, suppose that

$$x = a' - a, y = b' - b, z = c' - c - a(b' - b).$$

Then $(a', b', c') = (a, b, c) \cdot (x, y, z)$. It remains to show that $(x, y, z) \in \Gamma$. Since $\{a\} = \{a'\}, \{b\} = \{b'\}$, we have that $x, y \in \mathbb{Z}$ and $b - b' = \lfloor b \rfloor - \lfloor b' \rfloor$. Then

$$(c' - a'\lfloor b' \rfloor) - (c - a\lfloor b \rfloor) = (c' - c - a(b' - b)) - (a' - a)\lfloor b' \rfloor = z - x\lfloor b' \rfloor \in \mathbb{Z}$$

implies that $z \in \mathbb{Z}$.

The claim that $(a, b, c)\Gamma = (\{a\}, \{b\}, \{c - a\lfloor b \rfloor\})\Gamma$ for all $a, b, c \in \mathbb{R}$ is straightforward. $\square$

Let $\alpha \in \mathbb{R}$ be such that $1, \alpha, \alpha^{-1}$ are linearly independent over $\mathbb{Q}$. Let $s = (\alpha, 0, 0)$ and $t = (0, \alpha^{-1}, \alpha)$. These two elements induce two transformations $S, T\colon X_H \to X_H$ given by

$$S(h\Gamma) = sh\Gamma, T(h\Gamma) = th\Gamma, \forall h \in H.$$

Lemma 2.4.9. *Let X_H, S, T be defined as above. Then (X_H, S, T) is a minimal distal system with commuting transformations S and T.*

Proof. We have that $st = (\alpha, \alpha^{-1}, \alpha + 1)$ and $ts = (\alpha, \alpha^{-1}, \alpha)$ and by a direct computation we have that they induce the same action on X_H. Therefore $ST = TS$.

It is classical that a rotation on a nilmanifold is distal [8] and it is minimal if and only if the rotation induced on its maximal equicontinuous factor is minimal. Moreover, the maximal equicontinuous factor is given by the projection on $H/H_2\Gamma$ which in our case is nothing but the projection in $\mathbb{T}^2$ (the first two coordinates). See [81] for a general reference on nilrotations.

Since $ST(h\Gamma) = (\alpha, \alpha^{-1}, \alpha) \cdot h\Gamma$ for all $h \in H$, we have that the induced rotation on $\mathbb{T}^2$ is given by the element (α, α^{-1}). Since 1, α and α^{-1} are linearly independent over $\mathbb{Q}$, by the Kronecker Theorem we have that this is a minimal rotation. We conclude that (X_H, ST) is minimal which clearly implies that (X_H, S, T) is minimal.

$\square$

In this example, we show that the relation $\mathcal{R}_{S,T}(X)$ is different from the strong one:

Proposition 2.4.10. *On the Heisenberg system (X_H, S, T), we have that*

$$\mathcal{R}_{S,T}(X_H) = \Big\{ \Big((a,b,c)\Gamma, (a,b,c')\Gamma \Big) \in X_H \times X_H \colon a,b,c,c' \in \mathbb{R} \Big\}.$$

However, for any $c \in \mathbb{R}\backslash\mathbb{Z}$, Γ and $(0,0,c)\Gamma$ are not strongly $\mathcal{R}_{S,T}(X_H)$-related.

Proof. Suppose that $((a,b,c)\Gamma, (a',b',c')\Gamma) \in \mathcal{R}_{S,T}(X_H)$. Then $((a,b,c)\Gamma, (a',b',c')\Gamma) \in \mathcal{R}_T(X_H)$. Projecting to the first coordinate, we have that $(\{a\}, v, \{a'\}, v) \in \mathbf{Q}_{\overline{S},id}(\mathbb{T})$ for some $v \in \mathbb{T}$, where in the system $(\mathbb{T}, \overline{S}, \text{id})$, $\overline{S}x = x + \alpha$ for all $x \in \mathbb{T}$ (we regard $\mathbb{T}$ as [0,1)). Since the second transformation is identity, we have that $\{a\} = \{a'\}$. Similarly, $\{b\} = \{b'\}$. So in order to prove the first statement, it suffices to show that $((a,b,c)\Gamma, (a,b,c')\Gamma) \in \mathcal{R}_{S,T}(X_H)$ for all $a,b,c,c' \in \mathbb{R}$. Since (X_H, S, T) is minimal, there exist a sequence $(g_i)_{i\in\mathbb{N}}$ in G and a sequence $(c_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that

$$\lim_{i\to\infty} g_i((0,0,0)\Gamma) = (a,b,c), \quad \lim_{i\to\infty} g_i((0,0,c_i)\Gamma) = (a,b,c').$$

Since $\mathcal{R}_{S,T}(X_H)$ is closed and invariant under $g \times g, g \in G$, it then suffices to show that Γ and $(0,0,c)\Gamma$ are $\mathcal{R}_{S,T}(X_H)$-related for all $c \in \mathbb{R}$. Fix $\epsilon > 0$. Let $n_i \to +\infty$ be such that $|\{n_i\alpha\}| < \epsilon$ and $\frac{c}{n_i\alpha} < \epsilon$. Let $x_i = (0, \frac{c}{n_i\alpha}, 0)\Gamma$. Then $d(x_i, \Gamma) < \epsilon$ and by Lemma 2.4.8, we have that

$$S^{n_i}x_i = (n_i\alpha, \frac{c}{n_i\alpha}, c)\Gamma = (\{n_i\alpha\}, \frac{c}{n_i\alpha}, \{c - n_i\alpha\lfloor\frac{c}{n_i\alpha}\rfloor\})\Gamma = (\{n_i\alpha\}, \frac{c}{n_i\alpha}, c)\Gamma.$$

So $d(S^{n_i}x_i, (0,0,c)\Gamma) < 2\epsilon$. We also have that $d(S^{n_i}(0,0,c)\Gamma, (0,0,c)\Gamma) < \epsilon$. Let $\delta > 0$ be such that if $d(h\Gamma, h'\Gamma) < \delta$, then $d(S^{n_i}h\Gamma, S^{n_i}h'\Gamma) < \epsilon$. Since the rotation on (α, α^{-1}) is minimal in $\mathbb{T}^2$, we can find m_i large enough such that $0 < \{m_i\alpha\} + \frac{c}{n_i} < \delta$ and $|\{m_i\alpha^{-1}\} - c| < \delta$. Hence, $d(T^{m_i}x_i, (0,0,c)\Gamma) < \delta$ and thus $d(S^{n_i}T^{m_i}x_i, (0,0,c)\Gamma) < 2\epsilon$. It follows that for large enough i, the distance between $(\Gamma, (0,0,c)\Gamma, (0,0,c)\Gamma, (0,0,c)\Gamma)$ and $(x_i, S^{n_i}x_i, T^{m_i}x_i, S^{n_i}T^{m_i}x_i)$ is less than 6ϵ. Since ϵ is arbitrary, we get that

$$(\Gamma, (0,0,c)\Gamma, (0,0,c)\Gamma, (0,0,c)\Gamma) \in \mathbf{Q}_{S,T}(X_H)$$

and thus Γ and $(0,0,c)\Gamma$ are $\mathcal{R}_{S,T}(X_H)$-related. This finishes the proof of the first statement.

For the second statement, let $h = (h_1, h_2, h_3) \in H$ with $h_i \in [0,1)$ for $i = 1,2,3$. We remark that $S^n\Gamma = (n\alpha, 0, 0)\Gamma = (\{n\alpha\}, 0, 0)\Gamma$. So if $(\Gamma, h\Gamma)$ are $\mathcal{R}_{S,T}(X_H)$-strongly related, then $h_2 = h_3 = 0$. Hence for $c \in (0,1)$, Γ and $(0,0,c)\Gamma$ are not $\mathcal{R}_{S,T}(X_H)$-strongly related. □

2.4.3. A Strong form of the $\mathcal{R}_{S,T}(X)$ relation

We say that $(x_1, x_2) \in X \times X$ are $\mathcal{R}^*_{S,T}(X)$-*related* if there exist $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ sequences in $\mathbb{Z}$ such that

$$(x_1, S^{n_i}x_1, T^{m_i}x_1, S^{n_i}T^{m_i}x_1) \to (x_1, x_1, x_1, x_2).$$

Obviously, $\mathcal{R}^*_{S,T}(X) \subseteq \mathcal{R}_{S,T}(X)$.

In this subsection, we prove that the relation generated by $\mathcal{R}^*_{S,T}(X)$ coincides with the $\mathcal{R}_{S,T}(X)$ relation. We start with some lemmas:

Remark 2.4.11. It is shown in [114] that, even in the case $S = T$, the relation generated by $\mathcal{R}^*_{S,T}(X)$ may not coincide with the $\mathcal{R}_{S,T}(X)$ relation in the non-distal setting. In fact, there exists a system with $\mathcal{R}^*_{T,T} = \Delta_X \neq \mathcal{R}_{T,T}$.

Lemma 2.4.12. *Let (X, S, T) be a minimal distal system with commuting transformations S and T. Then $\mathcal{R}_{S,T}(X) = \Delta_X$ if and only if $\mathcal{R}^*_{S,T}(X) = \Delta_X$.*

Proof. We only prove the non-trivial direction. Suppose that $\mathcal{R}^*_{S,T}(X)$ coincides with the diagonal. Fix $x_0 \in X$ and consider the system $(\mathbf{K}^{x_0}_{S,T}, \widehat{S}, \widehat{T})$. Let $\mathcal{R}_{\widehat{S},\widehat{T}}[(x_0, x_0, x_0)]$ be the set of points that are $\mathcal{R}_{\widehat{S},\widehat{T}}$ related with (x_0, x_0, x_0). Pick $(x_1, x_2, x_3) \in \mathcal{R}_{\widehat{S},\widehat{T}}[(x_0, x_0, x_0)]$. By definition, we have that $x_1 = x_2 = x_0$. Hence $(x_0, x_0, x_3) \in \mathbf{K}^{x_0}_{S,T}$ and thus (x_0, x_3) belongs to $\mathcal{R}^*_{S,T}(X)$. We conclude that $\#\mathcal{R}_{\widehat{S},\widehat{T}}[(x_0, x_0, x_0)] = 1$. By distality and minimality, the same property holds for every point in $\mathbf{K}^{x_0}_{S,T}$ and thus $\mathcal{R}_{\widehat{S},\widehat{T}}(\mathbf{K}^{x_0}_{S,T})$ coincides with the diagonal relation. Particularly, $(\mathbf{K}^{x_0}_{S,T}, \widehat{S}, \widehat{T})$ has a product extension and consequently so has (X, S, T). This is equivalent to saying that $\mathcal{R}_{S,T}(X) = \Delta_X$. $\square$

Let $\mathcal{R}(X)$ be the relation generated by $\mathcal{R}^*_{S,T}(X)$. We have:

Lemma 2.4.13. *Let $\pi \colon Y \to X$ be the factor map between two minimal distal systems (Y, S, T) and (X, S, T) with commuting transformations S and T. Then $\pi \times \pi(\mathcal{R}(Y)) \supseteq \mathcal{R}^*_{S,T}(X)$.*

Proof. Similar to the proof of Proposition 2.4.6. $\square$

We can now prove the main property of this subsection:

Proposition 2.4.14. *Let (X, S, T) be a distal minimal system with commuting transformations S and T. Then $\mathcal{R}(X) = \mathcal{R}_{S,T}(X)$.*

Proof. We only need to prove that $\mathcal{R}_{S,T}(X) \subseteq \mathcal{R}(X)$. Let $\pi \colon X \to X/\mathcal{R}(X)$ be the projection map. By Lemma 2.4.13, $\Delta_X = \pi \times \pi(\mathcal{R}(X)) \supseteq \mathcal{R}^*_{S,T}(X/\mathcal{R}(X))$. By Lemma 2.4.12, $\mathcal{R}_{S,T}(X/\mathcal{R}(X)) = \Delta_X$ and then $(X/\mathcal{R}(X), S, T)$ has a product extension. By Theorem 2.4.7 $(X/\mathcal{R}_{S,T}(X), S, T)$ is the maximal factor with this property and therefore $\mathcal{R}_{S,T}(X) \subseteq \mathcal{R}(X)$. $\square$

2.5. Properties of systems with product extensions

In this section, we study the properties of systems which have a product extension. We characterize them in terms of their enveloping semigroup and we study the class of systems which are disjoint from them. Also, in the distal case we study properties of recurrence and topological complexity.

2.5.1. The enveloping semigroup of systems with a product extension

Let (X, S, T) be a system with commuting transformations S and T, and let $E(X, S)$ and $E(X, T)$ be the enveloping semigroups associated to the systems (X, S) and (X, T) respectively. Hence $E(X, S)$ and $E(X, T)$ are subsemigroups of $E(X, G)$. We say that (X, S, T) is *automorphic* (or S and T are *automorphic*) if for any nets $u_{S,i} \in E(X, S)$ and $u_{T,i} \in E(X, T)$ with $\lim u_{S,i} = u_S$ and $\lim u_{T,i} = u_T$, we have that $\lim u_{S,i}u_{T,i} = u_S u_T$. Equivalently, S and T are automorphic if the map $E(X, S) \times E(X, T) \to E(X, G)$, $(u_S, u_T) \mapsto u_S u_T$ is continuous.

The following theorem characterizes the enveloping semigroup for systems with production extensions:

Theorem 2.5.1. *Let (X, S, T) be a system with commuting transformations S and T. Then (X, S, T) has a product extension if and only if S and T are automorphic. Particularly, $E(X, G) = E(X, S)E(X, T) := \{u_S u_T : u_S \in E(X, S), u_T \in E(X, T)\}$, and $E(X, S)$ commutes with $E(X, T)$.*

Proof. First, we prove that the property of being automorphic is preserved under factor maps. Let $\pi : Y \to X$ be a factor map between the systems (Y, S, T) and (X, S, T) and suppose that (Y, S, T) is automorphic. Suppose that (X, S, T) is not automorphic. Then there exist nets $u_{S,i} \in E(X, S)$ and $u_{T,i} \in E(X, T)$ such that $u_{S,i}u_{T,i}$ does not converge to $u_S u_T$. Taking a subnet, we can assume that $u_{S,i}u_{T,i}$ converges to $u \in E(X, G)$. Let $\pi^* : E(Y, G) \to E(X, G)$ be the map induced by π and let $v_{S,i} \in E(Y, S)$ and $v_{T,i} \in E(Y, T)$ be nets with $\pi^*(v_{S,i}) = u_{S,i}$ and $\pi^*(v_{T,i}) = u_{T,i}$. Assume without loss of generality that $v_{S,i} \to v_S$ and $v_{T,i} \to v_T$. Then $v_{S,i}v_{T,i} \to v_S v_T$. So $u_{S,i}u_{T,i} \to u_S u_T = u$, a contradiction. On the other hand, since a product system is clearly automorphic, we get the first implication.

Now suppose that S and T are automorphic.

Claim 1: $E(X, S)$ commutes with $E(X, T)$.

Indeed, let $u_S \in E(X, S)$ and $u_T \in E(X, T)$. Let (n_i) be a net such that $S^{n_i} \to u_S$. Then $S^{n_i}u_T \to u_S u_T$. On the other hand, since S commutes with $E(X, T)$ we have that $S^{n_i}u_T = u_T S^{n_i}$ for every i and this converges to $u_T u_S$ by the hypothesis of automorphy.

Claim 2 : For any $x \in X$, $\mathbf{K}^x_{S,T} = \{(u_S x, u_T x, u_S u_T x) : u_S \in E(X, S),\ u_T \in E(X, T)\}$.

We recall that $\mathbf{K}^x_{S,T}$ in invariant under $S \times \mathrm{id} \times S$ and $\mathrm{id} \times T \times T$. Since $\mathbf{K}^x_{S,T}$ is closed we have that is invariant under $u_S \times \mathrm{id} \times u_S$ and $\mathrm{id} \times u_T \times u_T$ for any $u_S \in E(X,S)$ and $u_T \in E(X,T)$. Hence $(u_S \times \mathrm{id} \times u_S)(\mathrm{id} \times u_T \times u_T)(x,x,x) = (u_S x, u_T x, u_S u_T x) \in \mathbf{K}^x_{S,T}$.

Conversely, let $(a,b,c) \in \mathbf{K}^x_{S,T}$. Let $(m_i)_{i\in\mathbb{N}}$ and $(n_i)_{\in\mathbb{N}}$ be sequences in $\mathbb{Z}$ such that $S^{m_i}x \to a$, $T^{n_i}x \to b$ and $S^{m_i}T^{n_i}x \to c$. Replacing these sequences with finer filters, we can assume that $S^{m_i} \to u_S \in E(X,S)$ and $T^{n_i} \to u_T \in E(X,T)$. By the hypothesis of automorphy, $S^{m_i}T^{n_i} \to u_S u_T$ and thus $u_S u_T x = c$ and $(a,b,c) = (u_S x, u_T x, u_S u_T x)$. The claim is proved.

Let (a,b,c) and $(a,b,d) \in \mathbf{K}^x_{S,T}$. We can take $u_S, u'_S \in E(X,S)$ and $u_T, u'_T \in E(X,T)$ such that $(a,b,c) = (u_S x, u_T x, u_S u_T x)$ and $(a,b,d) = (u'_S x, u'_T x, u'_S u'_T x)$. Since $E(X,S)$ and $E(X,T)$ commute we deduce that $c = u_S u_T x = u_S b = u_S u'_T x = u'_T u_S x = u'_T a = u'_T u'_S x = d$.

Consequently, the last coordinate of $\mathbf{K}^x_{S,T}$ is a function of the first two ones. By Proposition 2.2.21, (X,S,T) has a product extension.

□

2.5.2. Disjointness of systems with a product extension

We recall the definition of disjointness:

Definition 2.5.2. Let (X,G) and (Y,G) be two dynamical systems. A *joining* between (X,G) and (Y,G) is a closed subset Z of $X \times Y$ which is invariant under the action $g \times g$ for all $g \in G$ and projects onto both factors. We say that (X,G) and (Y,G) are *disjoint* if the only joining between them is their Cartesian product.

Definition 2.5.3. Let (X,S,T) be a minimal system with commuting transformations S and T. We say that a point $x \in X$ is S-T *almost periodic* if x is an almost periodic point of the systems (X,S) and (X,T). Equivalently, x is S-T almost periodic if $(\overline{\mathcal{O}_S(x)}, S)$ and $(\overline{\mathcal{O}_T(x)}, T)$ are minimal systems. The system (X,S,T) is S-T *almost periodic* if every point $x \in X$ is S-T almost periodic.

Remark 2.5.4. We remark that if $(\mathbf{K}^x_{S,T}, \widehat{S}, \widehat{T})$ is minimal, then x is S-T is almost periodic. Consequently, if (X,S,T) has a product extension we have that $(\mathbf{K}^x_{S,T}, \widehat{S}, \widehat{T})$ is minimal for every $x \in X$ and then (X,S,T) is S-T almost periodic.

The main theorem of this subsection is:

Theorem 2.5.5. *Let (X,S,T) be an S-T almost periodic system. Then (X,S) and (X,T) are minimal and weak mixing if and only if (X,S,T) is disjoint from all systems with product extension.*

We begin with a general lemma characterizing the relation of transitivity with the cube structure:

Lemma 2.5.6. *Let (X,T) be a topological dynamical system. Then (X,T) is transitive if and only if $\mathbf{Q}_T(X)=X\times X$.*

Proof. Let $x\in X$ be a transitive point. We have that $X\times X$ is the orbit closure of (x,x) under $T\times T$ and $\mathrm{id}\times T$. Since $\mathbf{Q}_T(X)$ is invariant under these transformations we conclude that $\mathbf{Q}_T(X)=X\times X$.

Conversely let U and V be two non-empty open subsets and let $x\in U$ and $y\in V$. Since $(x,y)\in\mathbf{Q}_T(X)$, there exist $x'\in X$ and $n\in\mathbb{Z}$ such that $(x',T^nx')\in U\times V$. This implies that $U\cap T^{-n}V\neq\emptyset$. □

We recall the following lemma ([98], page 1):

Lemma 2.5.7. *Let (X,T) be a topological dynamical system. Then (X,T) weakly mixing if and only if for every two non-empty open sets U and V there exists $n\in\mathbb{Z}$ with $U\cap T^{-n}U\neq\emptyset$ and $U\cap T^{-n}V\neq\emptyset$.*

The following lemma characterizes the weakly mixing property in terms of the cube structure:

Lemma 2.5.8. *Let (X,T) be a topological dynamical system. The following are equivalent:*

1. *(X,T) is weakly mixing;*
2. *$\mathbf{Q}_{T,T}(X)=X\times X\times X\times X$;*
3. *$(x,x,x,y)\in\mathbf{Q}_{T,T}(X)$ for every $x,y\in X$.*

Proof. (1) ⇒ (2). Let suppose that (X,T) is weakly mixing and let $x_0,x_1,x_2,x_3\in X$. Let $\epsilon>0$ and for $i=0,1,2,3$ let U_i be the open balls of radius ϵ centered at x_i. Since (X,T) is weak mixing there exists $n\in\mathbb{Z}$ such that $U_0\cap T^{-n}U_1\neq\emptyset$ and $U_2\cap T^{-n}U_3\neq\emptyset$. Since (X,T) is transitive we can find a transitive point in $x'\in U_0\cap T^{-n}U_1$. Let $m\in\mathbb{Z}$ such that $T^mx'\in U_2\cap T^{-n}U_3$. Then $(x',T^nx',T^mx',T^{n+m}x')\in U_0\times U_1\times U_2\times U_3$ and this point belongs to $\mathbf{Q}_{T,T}(X)$. Since ϵ is arbitrary we conclude that $(x_0,x_1,x_2,x_3)\in\mathbf{Q}_{T,T}(X)$.

(2) ⇒ (3). Clear.

(3) ⇒ (1). Let U and V be non-empty open sets and let $x\in U$ and $y\in V$. Since $(x,x,x,y)\in\mathbf{Q}_{T,T}(X)$, there exist $x'\in X$ and $n,m\in\mathbb{Z}$ such that $(x',T^nx',T^mx',T^{n+m}x')\in U\times U\times U\times V$. Then $x'\in U\cap T^{-n}U$ and $T^mx'\in U\cap T^{-n}V$ and therefore $U\cap T^{-n}U\neq\emptyset$ and $U\cap T^{-n}V\neq\emptyset$. By Lemma 2.5.7 we have that (X,T) is weak mixing. □

Remark 2.5.9. When (X,T) is minimal, a stronger results hold [110], Subsection 3.5.

The following is a well known result rephrased in our language:

Proposition 2.5.10. *Let (X,T) be a minimal system. Then $\mathcal{R}_{T,T}(X)=X\times X$ if and only if (X,T) is weakly mixing.*

Proof. If (X,T) is minimal we have that $(x,y) \in \mathcal{R}_{T,T}(X)$ if and only if $(x,x,x,y) \in \mathbf{Q}_{T,T}(X)$ [70], [110]. $\square$

Remark 2.5.11. If (X,T) is not minimal, it is not true that $\mathcal{R}_{T,T}(X) = X \times X$ implies that (X,T) is weakly mixing. For instance, let consider the set $X \coloneqq \{1/n \colon n > 1\} \cup \{1-1/n \colon n > 2\} \cup \{0\}$ and let T be the transformation defined by $T(0) = 0$ and for $x \neq 0$, $T(x)$ is the number that follows x to the right. If x and y are different from 0, then $(x,x,x,y) \in \mathbf{Q}_{T,T}$ implies $x = y$ and thus (X,T) is not weakly mixing. On the other hand, if x and y are different from 0, then there exists $n \in \mathbb{Z}$ with $y = T^n x$. Then $\lim\limits_{i\to\infty}(x, T^n x, T^i x, T^{n+i} x) = (x,y,0,0)$ meaning that $(x,y) \in \mathcal{R}_{T,T}(X)$. Since $\mathcal{R}_{T,T}(X)$ is closed we have that $\mathcal{R}_{T,T}(X) = X \times X$.

Lemma 2.5.12. *Let (X,S,T) be a minimal system with commuting transformations S and T. If S is transitive, then $\mathcal{R}_{T,T}(X) \subseteq \mathcal{R}_{S,T}(X) \subseteq \mathcal{R}_{S,S}(X)$.*

Proof. Suppose $(x,y) \in \mathcal{R}_{S,T}(X)$. For $\epsilon > 0$, there exist $z \in X$, $n,m \in \mathbb{Z}$ such that $d(x,z) < \epsilon$, $d(y, S^n z) < \epsilon$ and $d(T^m z, S^n T^m z) < \epsilon$. Pick $0 < \delta < \epsilon$ such that $d(x',y') < \delta$ implies $d(S^n x', S^n y') < \epsilon$ for all $x', y' \in X$. Since S is transitive, there exist $z' \in X$, $r \in \mathbb{Z}$ such that $d(z,z') < \delta$ and $d(T^m z, S^r z') < \delta$. So $d(S^n z, S^n z') < \epsilon$ and $d(S^n T^m z, S^{n+r} z') < \epsilon$. Thus $d(x,z') < 2\epsilon$, $d(y, S^n z') < 2\epsilon$ and $d(S^r z', S^{r+n} z') < 3\epsilon$. Since ϵ is arbitrary, $(x,y) \in \mathcal{R}_{S,S}(X)$.

Suppose $(x,y) \in \mathcal{R}_{T,T}(X)$. Then there exists $a \in X$ such that for any $\epsilon > 0$, there exists $z \in X, m,n \in \mathbb{Z}$ such that $d(x,z)$, $d(y, T^m z)$, $d(a, T^n z)$ and $d(a, T^{n+m} z) < \epsilon$. Pick $0 < \delta < \epsilon$ such that $d(x',y') < \delta$ implies $d(T^n x', T^n y') < \epsilon$ for all $x', y' \in X$. Since S is transitive, there exists $z' \in X$, $r \in \mathbb{Z}$ such that $d(z,z') < \delta$ and $d(T^m z, S^r z') < \delta$. So $d(T^n z, T^n z') < \epsilon$ and $d(T^{n+m} z, T^n S^r z') < \epsilon$. Thus $d(x,z') < \epsilon$, $d(y, S^r z') < \epsilon$, $d(a, T^n z') < \epsilon$, $d(a, T^n S^r z') < 2\epsilon$. Since ϵ is arbitrary, $(x,y,a,a) \in \mathbf{Q}_{S,T}(X)$. Similarly, $(x,b,y,b) \in \mathbf{Q}_{S,T}(X)$ for some $b \in X$. So $(x,y) \in \mathcal{R}_{S,T}(X)$. $\square$

Lemma 2.5.13. *Let (X,S,T) be a system with commuting transformations S and T such that both S and T are minimal. Then $\mathcal{R}_{S,T}(X) = X \times X$ if and only if both (X,S) and (X,T) are weakly mixing.*

Proof. If both (X,S) and (X,T) are weakly mixing, then $\mathcal{R}_{S,S}(X) = X \times X$ and T is transitive. By Lemma 2.5.12, $\mathcal{R}_{S,T}(X) = X \times X$.

Now suppose that $\mathcal{R}_{S,T}(X) = X \times X$. For any $x,y \in X$, since $(x,y) \in \mathcal{R}_{S,T}(X)$, we may assume that $(x,a,y,a) \in \mathbf{Q}_{S,T}(X)$ for some $a \in X$. For any $\epsilon > 0$, there exists $z \in X$, $n,m \in \mathbb{Z}$ such that $d(x,z) < \epsilon$, $d(a, S^n z) < \epsilon$, $d(y, T^m z) < \epsilon$, $d(a, S^n T^m z) < \epsilon$. Pick $0 < \delta < \epsilon$ such that $d(x',y') < \delta$ implies $d(S^n x', S^n y') < \epsilon$ for all $x', y' \in X$. Since $(z, T^m z) \in \mathcal{R}_{S,T}(X)$, there exist $z' \in X$, $r \in \mathbb{Z}$ such that $d(z,z') < \delta$, $d(T^m z, S^r z') < \delta$. So $d(S^n z, S^n z') < \epsilon$, $d(S^n T^m z, S^{n+r} z') < \epsilon$. Thus $d(x,z') < 2\epsilon$, $d(a, S^n z') < 2\epsilon$, $d(y, S^r z') < 2\epsilon$ and $d(a, S^{n+r} z') < 2\epsilon$. Since ϵ is arbitrary, $(x,y) \in \mathcal{R}_{S,S}(X)$. So $\mathcal{R}_{S,S}(X) = X \times X$ and since S is minimal we have that (X,S) is weakly mixing. Similarly, (X,T) is weakly mixing. $\square$

Shao and Ye proved [110] the following lemma in the case when $S = T$, but the same method works for the general case. So we omit the proof:

Lemma 2.5.14. *Let (X, S, T) be a system with commuting transformations S and T such that both S and T are minimal. Then the following are equivalent:*

1. $(x, y) \in \mathcal{R}_{S,T}(X)$*;*
2. $(x, y, y, y) \in \mathbf{K}^x_{S,T}$*;*
3. $(x, x, y, x) \in \mathbf{K}^x_{S,T}$*.*

Remark 2.5.15. We remark that a transformation is minimal if and only if it is both almost periodic and transitive.

Lemma 2.5.16. *Let (X, S, T) be a system with commuting transformations S and T such that (X, S) and (X, T) are minimal and weak mixing. Let (Y, S, T) be a minimal system with commuting transformations S and T such that (Y, S, T) has a product extension. Let $Z \subset X \times Y$ be a closed subset of $X \times Y$ which is invariant under $\overline{S} = S \times S$ and $\overline{T} = T \times T$. Let $\pi \colon Z \to X$ be the natural factor map. For $x_1, x_2 \in X$, if there exists $y_1 \in Y$ such that $z_1 = (x_1, y_1) \in Z$ is a S-T almost periodic point, then there exists $y \in Y$ such that $(x_1, y), (x_2, y) \in Z$.*

Proof. By Lemma 2.5.14, $(x_1, x_2, x_2, x_2) \in \mathbf{K}^{x_1}_{S,T}$. So there exists a sequence $(F_i)_{i\in\mathbb{N}} \in \mathcal{F}_{S,T}$ such that

$$\lim_{i\to\infty} F_i(x_1, x_1, x_1, x_1) = (x_1, x_2, x_2, x_2).$$

Recall that $z_1 = (x_1, y_1) \in \pi^{-1}(x_1)$. Without loss of generality, we assume that

$$\begin{aligned} &\lim_{i\to\infty} F_i(y_1, y_1, y_1, y_1) = (y_1, y_2, y_3, y_4); \\ &\lim_{i\to\infty} \overline{F}_i(z_1, z_1, z_1, z_1) = (z_1, z_2, z_3, z_4), \end{aligned}$$

where $\overline{F}_i = F_i \times F_i$ and $z_2 = (x_2, y_2), z_3 = (x_2, y_3), z_4 = (x_2, y_4)$ are points in Z. Since (x_1, y_1) is S-T almost periodic, there exists a sequence of integers $(n_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \overline{S}^{n_i} z_2 = z_1$. We can assume that $\lim_{i\to\infty} \overline{S}^{n_i} z_4 = z_4' = (x_1, y') \in Z$. Then

$$\lim_{i\to\infty} (\mathrm{id} \times \overline{S} \times \mathrm{id} \times \overline{S})^{n_i}(z_1, z_2, z_3, z_4) = (z_1, z_1, z_3, z_4').$$

This implies that $(y_1, y_1, y_3, y') \in \mathbf{Q}_{S,T}(Y)$ by Theorem 2.1.1 since $\mathcal{R}_S(Y) = \Delta_X$ we have that $y' = y_3$. Therefore $z_4' = (x_1, y_3)$ and $z_3 = (x_2, y_3)$ belong to Z. $\square$

We are now finally able to prove the main theorem of this subsection:

Proof of Theorem 2.5.5. Let (X,S,T) be a system such that (X,S) and (X,T) are minimal weak mixing and let (Y,S,T) be a system with a product extension. Suppose $Z \subseteq X \times Y$ is closed and invariant under $\overline{S} = S \times S, \overline{T} = T \times T$. We have to show that $Z = X \times Y$. Let $\mathcal{W} = \{Z \subseteq X \times Y \colon Z \text{ is closed invariant under } \overline{S} = S \times S, \overline{T} = T \times T\}$ with order $Z \leq Z'$ if and only if $Z' \subset Z$. Let $\{Z_i\}_{i\in I}$ be a totally ordered subset of $\mathcal{W}$ and denote $Z_0 = \cap_{i\in I} Z_i$. It is easy to see that $Z_0 \in \mathcal{W}$. By Zorn's Lemma, we can assume Z contains no proper closed invariant subset.

For any $x \in X$, denote $F_x = \{y \in Y \colon (x,y) \in Z\}$. Then $F_x \subseteq Y$ is a closed set of Y.

For any $g \in G$, let $Z_g = \{(x,y) \in X \times Y \colon y \in (F_x \cap gF_x)\}$. Then $Z_g \subseteq Z$ is closed invariant. Since Z contains no proper invariant subset, either $Z_g = \emptyset$ or $Z_g = Z$. Denote $U = \{x \in X \colon \exists y \in Y, (x,y) \text{ is an almost periodic point of } Z\}$. For any $x_0 \in U$, suppose $z_0 = (x_0, y_0) \in Z$ is an S-T almost periodic point. For any $g \in G, (x_0, gx_0) \in \mathcal{R}_{S,T}(X)$. By Proposition 2.5.16, there exists $y \in Y$ such that $(x_0,y),(gx_0,y) \in Z$. So $F_{x_0} \cap F_{gx_0} = F_{x_0} \cap gF_{x_0} \neq \emptyset$. Therefore $Z_g \neq \emptyset$. So $Z_g = Z$ for all $g \in G$. Thus $F_x = gF_x$ for every $x \in U$. Since g is arbitrary, F_x is closed invariant under G for every $x \in U$. Since (Y,G) is minimal, and $F_x \neq \emptyset$ we get that $F_x = Y$ for all $x \in U$.

It suffices to show that $U = X$. Fix $x \in X$. Since x is S-T-almost periodic, there exist minimal idempotents $u_S \in E(X,S)$ and $u_T \in E(X,T)$ such that $u_S x = x = u_T x$. These idempotents can be lifted to minimal idempotents in $E(Z,S)$ and $E(Z,T)$ which can be projected onto minimal idempotents in $E(Y,S)$ and $E(Y,T)$. We also denote these idempotents by u_S and u_T. By Theorem 2.5.1, these idempotents commute in $E(Y,G)$. So for $y \in Y$ such that $(x,y) \in Z$, we have that $u_S u_T(x,y) = (x, u_S u_T y) \in Z$, and $u_S(x, u_S u_T y) = (x, u_S u_T y)$, $u_T(x, u_S u_T y) = (x, u_S u_T y)$. This means that the point $(x, u_S u_T y) \in Z$ is S-T-almost periodic. Hence $U = X$ and therefore $Z = X \times Y$.

Conversely, let (X,S,T) be a system disjoint from systems with product extension. Let U and V be non-empty open subsets of X and let $x \in U$ and $y \in V$. Since X is S-T almost periodic, we have that $(\overline{\mathcal{O}_S(x)}, S)$ and $(\overline{\mathcal{O}_T(x)}, T)$ are minimal systems. By hypothesis, (X,S,T) is disjoint from $(\overline{\mathcal{O}_S(x)} \times \overline{\mathcal{O}_T(x)}, S \times \mathrm{id}, \mathrm{id} \times T)$. Since $(x,(x,x))$ and $(y,(x,x))$ belong to $X \times (\overline{\mathcal{O}_S(x)} \times \overline{\mathcal{O}_T(x)})$, we have that there exist sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(S^{n_i}T^{m_i}x, (S^{n_i}x, T^{m_i}x)) \to (y,(x,x))$. Particularly $(x, S^{m_i}x, T^{m_i}x, S^{n_i}T^{m_i}x) \in \mathbf{Q}_{S,T}(X)$ and this point converges to $(x,x,x,y) \in \mathbf{Q}_{S,T}(X)$. This implies that $(x,y) \in \mathbf{Q}_S(X)$, $(x,y) \in \mathbf{Q}_T(X)$ and $(x,y) \in \mathcal{R}_{S,T}(X)$ and since x and y are arbitrary we deduce that $\mathbf{Q}_S(X) = \mathbf{Q}_T(X) = \mathcal{R}_{S,T}(X) = X \times X$. By Lemma 2.5.6 we deduce that S and T are transitive and since (X,S,T) is S-T almost periodic we deduce that S and T are minimal. By Lemma 2.5.13 we deduce that (X,S) and (X,T) are minimal and weak mixing.

$\square$

2.5.3. Recurrence in systems with a product extension

We define the sets of return times in our setting:

Definition 2.5.17. Let (X, S, T) be a minimal distal system with commuting transformations S and T, and let $x \in X$. Let $x \in X$ and U be an open neighborhood of x. We define the *set of return times* $N_{S,T}(x, U) = \{(n, m) \in \mathbb{Z}^2 \colon S^n T^m x \in U\}$, $N_S(x, U) = \{n \in \mathbb{Z} \colon S^n x \in U\}$ and $N_T(x, U) = \{m \in \mathbb{Z} \colon T^m x \in U\}$.

A subset A of $\mathbb{Z}$ is a *set of return times for a distal system* if there exists a distal system (X, S), an open subset U of X and $x \in U$ such that $N_S(x, U) \subseteq A$.

A subset A of $\mathbb{Z}$ is a *Bohr*$_0$ set is here exists an equicontinuous system (X, S), an open subset U of X and $x \in U$ such that $N_S(x, U) \subseteq A$.

Remark 2.5.18. We remark that we can characterize $\mathbb{Z}^2$ sets of return times of distal systems with a product extension: they contain the Cartesian product of sets of return times for distal systems. Let (X, S, T) be a minimal distal system with a product extension $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$, and let U be an open subset of X and $x \in U$. By Theorem 2.4.7 we can assume that the product extension is also distal. Let π denote a factor map from $Y \times W \to X$. Let $(y, w) \in Y \times W$ such that $\pi(y, w) = x$ and let U_Y and U_W be neighborhoods of y and w such that $\pi(U_Y \times U_W) \subseteq U$. Then we have that that $N_\sigma(y, U_Y) \times N_\tau(w, U_W) \subseteq N_{S,T}(x, U)$.

Conversely, let (Y, σ) and (W, τ) be minimal distal systems. Let U_Y and U_W be non-empty open sets in Y and W and let $y \in U_Y$ and $w \in U_W$. Then $N_\sigma(y, U_Y) \times N_\tau(w, U_W)$ coincides with $N_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}((y, w), U_Y \times U_W)$.

Denote by $\mathcal{B}_{S,T}$ the family generated by Cartesian products of sets of return times for a distal system. Equivalently $\mathcal{B}_{S,T}$ is the family generated by sets of return times arising from minimal distal systems with a product extension.

Denote by $\mathcal{B}^*_{S,T}$ the family of sets which have non-empty intersection with every set in $\mathcal{B}_{S,T}$.

Lemma 2.5.19. *Let (X, S, T) be a minimal distal system with commuting transformations S and T, and suppose $(x, y) \in \mathcal{R}_{S,T}(X)$. Let (Z, S, T) be a minimal distal system with $\mathcal{R}_{S,T}(Z) = \Delta_Z$ and let J be a closed subset of $X \times Z$, invariant under $T \times T$ and $S \times S$. Then for $z_0 \in Z$ we have $(x, z_0) \in J$ if and only if $(y, z_0) \in J$.*

Proof. We adapt the proof of Theorem 3.5 [75] to our context. Let $W = Z^Z$ and $S^Z, T^Z \colon W \to W$ be such that for any $\omega \in W$, $(S^Z\omega)(z) = S(\omega(z))$, $(T^Z\omega)(z) = T(\omega(z))$, $z \in Z$. Let $\omega^* \in W$ be the point satisfying $\omega(z) = z$ for all $z \in Z$ and let $Z_\infty = \overline{\mathcal{O}_{G^Z}(\omega^*)}$, where G^Z is the group generated by S^Z and T^Z. It is easy to verify that Z_∞ is minimal distal. So for any $\omega \in Z_\infty$, there exists $p \in E(Z, G)$ such that $\omega(z) = p\omega^*(z) = p(z)$ for any $z \in Z$. Since (Z, S, T) is minimal and distal, $E(Z, G)$ is a group (see [7], Chapter 5). So $p \colon Z \to Z$ is surjective. Thus there exists $z_\omega \in Z$ such that $\omega(z_\omega) = z_0$.

Take a minimal subsystem $(A, S \times S^Z, T \times T^Z)$ of the product system $(X \times Z_\infty, S \times S^Z, T \times T^Z)$. Let $\pi_X\colon (A, S \times S^Z, T \times T^Z) \to (X, S, T)$ be the natural coordinate projection. Then π_X is a factor map between two distal minimal systems. By Proposition 2.4.6, there exists $\omega^1, \omega^2 \in W$ such that $((x, \omega^1), (y, \omega^2)) \in \mathcal{R}_{S',T'}(A)$, where $S' = S \times S^Z, T' = T \times T^Z$.

Let $z_1 \in Z$ be such that $\omega^1(z_1) = z_0$. Denote $\pi\colon A \to X \times Z$, $\pi(u, \omega) = (u, \omega(z_1))$ for $(u, \omega) \in A$, $u \in X$ and $\omega \in W$. Consider the projection $B = \pi(A)$. Then $(B, S \times S, T \times T)$ is a minimal distal subsystem of $(X \times Z, S \times S, T \times T)$ and since $\pi(x_0, \omega^1) = (x, z_0) \in B$ we have that J contains B. Suppose that $\pi(x, \omega^2) = (x, z_2)$. Then $((x, z_0), (y, z_2)) \in \mathcal{R}_{S\times S, T\times T}(B)$ and thus $(z_0, z_2) \in \mathcal{R}_{S,T}(Z)$. Since $\mathcal{R}_{S,T}(Z) = \Delta_{Z\times Z}$ we have that $z_0 = z_2$ and thus $(y, z_0) \in B \subseteq J$. □

Theorem 2.5.20. *Let (X, S, T) be a minimal distal system with commuting transformations S and T. Then for $x, y \in X$, $(x, y) \in \mathcal{R}_{S,T}(X)$ if and only if $N_{S,T}(x, U) \in \mathcal{B}^*_{S,T}$ for any open neighborhood U of y.*

Proof. Suppose $N(x, U) \in \mathcal{B}^*_{S,T}$ for any open neighborhood U of y. Since X is distal, $\mathcal{R}_{S,T}(X)$ is an equivalence relation. Let π be the projection map $\pi\colon X \to Y \coloneqq X/\mathcal{R}_{S,T}(X)$. By Theorem 2.4.7 we have that $\mathcal{R}_{S,T}(Y) = \Delta_Y$. Since (X, S, T) is distal, the factor map π is open and $\pi(U)$ is an open neighborhood of $\pi(x)$. Particularly $N_{S,T}(x, U) \subseteq N_{S,T}(\pi(x), \pi(U))$. Let V be an open neighborhood of $\pi(x)$. By hypothesis we have that $N_{S,T}(x, U) \cap N_{S,T}(\pi(x), \pi(U)) \neq \emptyset$ which implies that $N_{S,T}(\pi(x), \pi(U)) \cap N_{S,T}(\pi(x), V) \neq \emptyset$. Particularly $\pi(U) \cap V \neq \emptyset$. Since this holds for every V we have that $\pi(x) \in \overline{\pi(U)} = \pi(\overline{U})$. Since this holds for every U we conclude that $\pi(x) = \pi(y)$. This shows that $(x, y) \in \mathcal{R}_{S,T}(X)$.

Conversely, suppose that $(x, y) \in \mathcal{R}_{S,T}(X)$, let U be an open neighborhood of y and let A be a $\mathcal{B}^*_{S,T}$ set. Then, there exists a minimal distal system (Z, S, T) with $\mathcal{R}_{S,T}(Z) = \Delta_Z$, an open set $V \subseteq Z$ and $z_0 \in V$ such that $N_{S,T}(z_0, V) \subseteq A$. Let J be orbit closure of (x, z_0) under $S \times S$ and $T \times T$. By distality we have that $(J, S \times S, T \times T)$ is a minimal system and $(x, z_0) \in J$. By Lemma 2.5.19 we have that $(y, z_0) \in J$ and particularly, there exist sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(S^{n_i}T^{m_i}x, S^{n_i}T^{m_i}z_0) \to (y, z_0)$. This implies that $N_{S,T}(x, U) \cap N_{S,T}(z_0, V) \neq \emptyset$ and the proof is finished. □

Corollary 2.5.21. *Let (X, S, T) be a minimal distal system with commuting transformations S and T. Then (X, S, T) has a product extension if and only if for every $x \in X$ and every open neighborhood U of x, $N_{S,T}(x, U)$ contains the product of two set of return times for a distal system.*

Proof. We prove the non-trivial implication. Let suppose that there exists $(x, y) \in \mathcal{R}_{S,T}(X) \setminus \Delta_X$ and let U, V be open neighborhoods of x and y respectively such that $U \cap V = \emptyset$. By assumption $N_{S,T}(x, U)$ is a $\mathcal{B}_{S,T}$ set, and by Theorem 2.5.20 $N_{S,T}(x, V)$ has nonempty intersection with $N_{S,T}(x, U)$. This implies that $U \cap V \neq \emptyset$, a contradiction. We conclude that $\mathcal{R}_{S,T}(X) = \Delta_X$ and therefore (X, S, T) has a product extension. □

Specially, when $S = T$ we get

Corollary 2.5.22. *Let (X,T) be a minimal distal system. Then (X,T) is equicontinuous if and only if for every $x \in X$ and every open neighborhood U of x, $N_T(x,U)$ contains the sum of two sets of return times for distal systems.*

Proof. Suppose (X,T) is equicontinuous, then the system (X,T,T) with commuting transformations T and T has a product extension. So for every $x \in X$ and every open neighborhood U of x, we have that $N_{T,T}(x,U)$ contains a product of two sets A and B. In terms of the one dimensional dynamics, this means that $N_T(x,U)$ contains $A+B$.

Conversely, if $N_T(x,U)$ contains the sum of two sets of return times for distal systems A and B, we have that $N_{T,T}(x,U)$ contains the set $A \times B$. By Corollary 2.5.21, (X,T,T) has a product extension and by Corollary 2.2.25 (X,T) is an equicontinuous system. $\square$

Question 2.5.23. *A natural question arising from Corollary 2.5.22 is the following: is the sum of two set of return times for a distal system a Bohr$_0$ set?*

2.5.4. Complexity for systems with a product extension

In this subsection, we study the complexity of a distal system with a product extension. We start recalling some classical definitions.

Let (X,G) be a topological dynamical system. A finite cover $\mathcal{C} = (C_1,\dots,C_d)$ is a finite collection of subsets of X whose union is all X. We say that $\mathcal{C}$ is an *open cover* if every $C_i \in \mathcal{C}$ is an open set. Given two open covers $\mathcal{C} = (C_1,\dots,C_d)$ and $\mathcal{D} = (D_1,\dots,D_k)$ their *refinement* is the cover $\mathcal{C} \vee \mathcal{D} = (C_i \cap D_j \colon i = 1,\dots,d \quad j = 1,\dots,k)$. A cover $\mathcal{C}$ is *finer* than $\mathcal{D}$ if every element of $\mathcal{C}$ is contained in an element of $\mathcal{D}$. We let $\mathcal{D} \preceq \mathcal{C}$ denote this property.

We recall that if (X,S,T) is a minimal distal system with commuting transformations S and T then $\mathbf{Q}_S(X)$, $\mathbf{Q}_T(X)$ and $\mathcal{R}_{S,T}(X)$ are equivalence relations.

Let (X,S,T) be a minimal distal system with commuting transformations S and T, and let π_S be the factor map $\pi_S \colon X \to X/\mathbf{Q}_S(X)$. Denote $I_S = \{\pi_S^{-1} y \colon y \in X/\mathbf{Q}_S(X)\}$ the *set of fibers of* π_S.

Given a system (X,S,T) with commuting transformations S and T, and given a finite cover $\mathcal{C}$, denote $\mathcal{C}_0^{T,n} = \bigvee_{i=0}^{n} T^{-i}\mathcal{C}$. For any cover $\mathcal{C}$ and any closed $Y \subset X$, let $r(\mathcal{C},Y)$ be the minimal number of elements in $\mathcal{C}$ needed to cover the set Y. We remark that $\mathcal{D} \preceq \mathcal{C}$ implies that $r(\mathcal{D},Y) \leq r(\mathcal{C},Y)$.

Definition 2.5.24. Let $\mathcal{C}$ be a finite cover of X. We define the S-T complexity of $\mathcal{C}$ to be the non-decreasing function

$$c_{S,T}(\mathcal{C},n) = \max_{Y \in I_S} r(\mathcal{C}_0^{T,n}, Y).$$

Proposition 2.5.25. *Let (X, S, T) be a distal system with commuting transformations S and T. Then (X, S, T) has a product extension if and only if $c_{S,T}(\mathcal{C}, n)$ is bounded for any open cover $\mathcal{C}$.*

Proof. Suppose first that $\mathcal{R}_{S,T}(X) = \Delta_X$. Since $\mathbf{Q}_S(X)$ is an equivalence relation, by Proposition 2.2.26, we have that $\pi_S\colon (X, T) \to (X/\mathbf{Q}_S(X), T)$ is an equicontinuous extension. Let $\epsilon > 0$ be the Lebesgue number of the finite open cover $\mathcal{C}$, i.e. any open ball B with radius ϵ is contained in at least one element of $\mathcal{C}$. Then there exists $0 < \delta < \epsilon$ such that $d(x, y) < \delta$, $\pi_S(x) = \pi_S(y)$ implies that $d(T^n x, T^n y) < \epsilon$ for all $n \in \mathbb{Z}$. For any $Y \in I_S$, by compactness, let $x_1, \ldots, x_k \in Y$ be such that $Y \subset \bigcup_{i=1}^{k} B(x_i, \delta)$. Then $T^j(B(x_i, \delta) \cap Y) \subset B(T^j x_i, \epsilon) \cap Y \subset B(T^j x_i, \epsilon)$ for any $j \in \mathbb{N}$ (since $\mathbf{Q}_S(X)$ is invariant under $T \times T$). Let $U_{i,j}$ be an element of $\mathcal{C}$ containing $B(T^j x_i, \epsilon)$. Then $T^j(B(x_i, \delta) \cap Y) \subset U_{i,j}$. So $B(x_i, \delta) \cap Y \subset \bigcap_{j=0}^{n} T^{-j} U_{i,j}$. Thus $\{\bigcap_{j=0}^{n} T^{-j} U_{i,j} \colon 1 \leq i \leq k\}$ is a subset of $\mathcal{C}_0^{T,n}$ covering Y with cardinality k. Therefore $r(\mathcal{C}_0^{T,n}, Y)$ is bounded by the quantity of balls of radius δ needed to cover Y.

Suppose that $c_{S,T}(\mathcal{C}, n)$ is not bounded. For $Y, Y' \in I_S$, let $d_H(Y, Y)$ be the Hausdorff distance between Y and Y'. Since the factor map $X \to X/\mathbf{Q}_S$ is open, for any $\epsilon' > 0$, there exists $\delta' > 0$ such that if $y, y' \in X/\mathbf{Q}_S$ and $d(y, y') < \delta'$, then $d_H(\pi^{-1} y, \pi^{-1} y') < \epsilon'$.

Let $y \in Y$ and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcover of $Y = \pi^{-1}(y)$. Let $\epsilon' > 0$ be such that if $d(x, Y) < \epsilon'$, then x is covered by $\mathcal{C}'$. We can find $\delta' > 0$ such that if $d(y, y') < \delta'$, then $d_H(\pi^{-1} y, \pi^{-1} y') < \epsilon'$. Thus $\mathcal{C}'$ is also an open covering of $Y' = \pi^{-1}(y')$.

If $\pi^{-1} y \subset \bigcup_{i=1}^{k} B(x_i, \delta)$, then there exists $\delta' > 0$ such that $d(y, y') < \delta'$ implies that $\pi^{-1} y' \subset \bigcup_{i=1}^{k} B(x_i, \delta)$. If $c_{S,T}(\mathcal{C}, n)$ is not bounded, there exists $y_i \in Y$ such that $\pi^{-1}(y_i)$ can not be covered by i balls of radius $\delta > 0$. We assume with out loss of generality that $y_i \to y$ (by taking a subsequence). Since $\pi^{-1} y$ can be covered by a finite number K of balls of radius δ, we get that for large enough i, $\pi^{-1} y_i$ can also be covered by K balls of radius δ, a contradiction. Therefore $c_{S,T}(\mathcal{C}, n)$ is bounded.

Conversely, let suppose that $c_{S,T}(\mathcal{C}, n)$ is bounded for every open cover $\mathcal{C}$ and suppose that $\mathcal{R}_{S,T}(X) \neq \Delta_X$. We remark that if $\mathcal{C}$ is an open cover and $Y \in I_S$ then

$$r(\mathcal{C}_{-n}^{T,n}, Y) := r(\bigvee_{i=-n}^{n} T^{-i} \mathcal{C}, Y) = r(T^n \bigvee_{i=0}^{2n} T^{-i} \mathcal{C}, T^n T^{-n} Y) = r(\bigvee_{i=0}^{2n} T^{-i} \mathcal{C}, T^{-n} Y).$$

Since T commutes with S we have that $T^{-n} Y \in I_S$ and thus the condition that $c_{S,T}(\mathcal{C}, n)$ is bounded implies that $r(\bigvee_{i=-n}^{n} T^{-i} \mathcal{C}, Y)$ is bounded for any $Y \in I_S$.

Since $\mathcal{R}_{S,T}(X) \neq \Delta_X$ by Proposition 2.2.26, there exist $\epsilon > 0$ and $x \in X$ such that for any $\delta > 0$, one can find $y \in X$ and $k \in \mathbb{Z}$ such that $d(x, y) < \delta$, $\pi_S(x) = \pi_S(y)$ and $d(T^k x, T^k y) > \epsilon$. Pick any $Y \in I_S$ and let $\mathcal{C}'$ be a finite cover of open balls with radius $\epsilon/4$. Let $\mathcal{C}$ be the finite covering made up of the closures of the elements of $\mathcal{C}'$. Since $\mathcal{C} \prec \mathcal{C}'$ we have that $r(\mathcal{C}_{-n}^{T,n}, Y)$ is also bounded.

By a similar argument of Lemma 2.1 of [15], there exist closed sets $X_1, \ldots, X_c \subset X$ such that $Y \subset \bigcup_{i=1}^{c} X_i$, where each X_i can be written as $X_i = \bigcap_{j=-\infty}^{\infty} T^{-j} U_{i,j}$, with $U_{i,j} \in \mathcal{C}$. Then $y, z \in X_i$ implies that $d(T^j y, T^j z) < \epsilon/2$ for any $j \in \mathbb{Z}$.

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \delta_n = 0$. For any $n \in \mathbb{N}$ we can find $y_n \in X$ and $k_n \in \mathbb{Z}$ with $d(x, y_n) < \delta_n$, $\pi_S(x) = \pi_S(y_n)$ and $d(T^{k_n} x, T^{k_n} y_n) > \epsilon$. By taking a subsequence, we may assume that all y_n belong to the same set X_i. Since X_i is closed, $x \in X_i$. Thus $d(T^j x, T^j y_n) < \epsilon/2$ for any $j, n \in \mathbb{N}$, a contradiction.

$\square$

A pointwise cubic average for two commuting transformations

This chapter is based on the joint work with Wenbo Sun A pointwise cubic average for two commuting transformations *[35]. Huang, Shao and Ye recently studied pointwise multiple averages by using suitable topological models. Using the notion of dynamical cubes introduced in Chapter 2, the Huang-Shao-Ye strategy and the Host machinery of magic systems, we prove that for an ergodic system (X,μ,S,T) with commuting transformations S and T, the average*

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx)$$

converges a.e. as N goes to infinity for any $f_1,f_2,f_3\in L^\infty(\mu)$.

3.1. Introduction

3.1.1. Pointwise convergence for cube averages

A system $(X,\mathcal{X},\mu,S,T)$ with two commuting transformations S and T is a probability space $(X,\mathcal{X},\mu)$ endowed with two commuting measure preserving transformations $S,T\colon X\to X$. In this chapter, we study the pointwise convergence of a cubic average in such a system.

The existence of the limit in L^2 of the averages

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(T^ix)f_2(T^jx)f_3(T^{i+j}x) \tag{3.1.1}$$

was proved by Bergelson [12] and was generalized in [66] and [67] to higher orders averages.

There are two possible generalizations of these averages to systems with commuting transformations: one is to study averages of the form

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(R^{i+j}x) \tag{3.1.2}$$

for commuting transformations S, T and R. Another is to study averages of the form

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx) \tag{3.1.3}$$

for commuting transformations S and T.

The existence of the pointwise limit of (3.1.2) was proved by Assani [2] for three transformations and it was generalized to an arbitrary number of transformations by Chu and Frantzikinakis [22]. It is worth noting that in fact no assumption of commutativity of the transformations is required. In contrast, the average (3.1.3) has a very different nature. Leibman [81] showed that convergence of (3.1.3) fails (even in L^2) without commutativity assumptions. When the transformations commute, the L^2 convergence of (3.1.3) (and its higher order versions) was proved by Chu [20] based on the work of Host [64] which follows the works of Tao [112] and Austin [7]. In order to prove this result, Host introduced the notion of magic extensions, which allows one to study such averages in an extension system with convenient properties. It is natural to ask if the averages in (3.1.3) converges in the pointwise sense. In this chapter, we prove:

Theorem 3.1.1. *Let (X, μ, S, T) be an ergodic measure preserving system with commuting transformations S and T. Then the average*

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx)$$

converges a.e. as N goes to infinity for any $f_1, f_2, f_3 \in L^\infty(\mu)$.

Recently Huang, Shao and Ye [77] proved the pointwise convergence of multiple averages for a single transformation on a distal system. So a natural question arises from Theorem 3.1.1: If (X, μ, S, T) is an ergodic measure preserving system with commuting transformations S and T, does the average

$$\frac{1}{N}\sum_{i=0}^{N-1} f_1(S^ix)f_2(T^ix)$$

converge in the pointwise sense as N goes to infinity? The case when S and T are powers of some ergodic transformation was solved by Bourgain [16] but no further results were given until Huang, Shao and Ye result.

3.1.2. Strict ergodicity for dynamical cubes

The main ingredient in proving Theorem 3.1.1 is to find a suitable topological model for the original system. This means finding a measurable conjugacy to a space with a convenient topological structure. Jewett-Krieger's Theorem states that every ergodic system has a strictly ergodic model (see Section 3.2 for definitions) and it is known that one can add some additional properties to the topological model.

In this chapter, we are interested in the strict ergodicity property of the cube structure introduce in Chapter 2 of a topological model. Let X be a compact metric space and $S, T\colon X \to X$ be two commuting homeomorphisms. We recall that $\mathbf{Q}_{S,T}(X)$ is defined to be

$$\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^i x, T^j x, S^i T^j x)\colon x \in X, i, j \in \mathbb{Z}\}}.$$

This object was introduced in [34] motivated by Host's work [64] and results in a useful tool to study products of minimal systems and their factors. A classical argument using Birkhoff Ergodic Theorem (see, for example, the proof of Theorem 3.5.1) shows that the strict ergodicity property of $\mathbf{Q}_{S,T}(X)$ is connected to pointwise multiple convergence problems such as Theorem 3.1.1 and Theorem 3.5.1. We ask the following question:

Question 3.1.2. *For any ergodic system (X, μ, S, T) with two commuting transformations S and T, is there a topological model $(\widehat{X}, \widehat{S}, \widehat{T})$ of X such that $(\mathbf{Q}_{S,T}(\widehat{X}), \mathcal{G}_{\widehat{S},\widehat{T}})$ is strictly ergodic? Here $\mathcal{G}_{\widehat{S},\widehat{T}}$ is the group of action generated by* $\mathrm{id} \times \widehat{S} \times \mathrm{id} \times \widehat{S}$, $\mathrm{id} \times \mathrm{id} \times \widehat{T} \times \widehat{T}$ *and* $\widehat{R} \times \widehat{R} \times \widehat{R} \times \widehat{R}$, *where* $\widehat{R} = \widehat{S}$ *or* $\widehat{T}$.

Huang, Shao and Ye [75] gave an affirmative answer to this question for the case $S = T$. Although this question remains open in the general case, such a model always exists in an extension system of the original one. We prove the following theorem, which is the main tool to study Theorem 3.1.1:

Theorem 3.1.3. *For any ergodic system (X, μ, S, T) with two commuting transformations S and T, there exists an extension system (Y, ν, S, T) of X and a topological model $(\widehat{Y}, \widehat{S}, \widehat{T})$ of Y such that $(\mathbf{Q}_{S,T}(\widehat{Y}), \mathcal{G}_{S,T})$ is strictly ergodic.*

It is worth noting that since every measurable function on the original system can be naturally lifted to a function on the extension system, this result is already sufficient for our purposes.

3.1.3. Proof Strategy and organization

Conventions and background material are in Section 3.2. To prove Theorem 3.1.3, we refine the technique of Host in [64] to find a suitable magic extension of the original system in Section 3.3. Then we use the method of Huang, Shao and Ye [75] to find a desired model for this extension system in Section 3.4. The announced pointwise convergence result (Theorem 3.1.1) follows from Theorem 3.1.3, and we explain how this is achieved in Section 3.5.

3.2. Background Material

We start recalling some classical concepts. Let (X, G) be a topological dynamical system. The Krylov-Bogolyubov Theorem states that this systems always admits an invariant measure. When this measure is unique, we say that (X, G) is *uniquely ergodic*. In addition, we say that (X, G) is *strictly ergodic* if it is minimal and uniquely ergodic.

We state here a well known theorem for the case when G is spanned by d commuting transformations $T_1, \ldots, T_d$.

Theorem 3.2.1. *Let (X, G) be a topological dynamical system. The following are equivalent*

1. *(X, G) is uniquely ergodic.*
2. *For any continuous function f, the average*

$$\frac{1}{N^d} \sum_{i_1,\ldots,i_d \in [0,N-1]} f(T_1^{i_1} \cdots T_d^{i_d} x)$$

converges uniformly to $\int f d\mu$ as N goes to infinity.

A deep connection between measure preserving systems and topological dynamical systems is the Jewett-Krieger Theorem [78, 80] which asserts that every ergodic system (X, μ, T) is isomorphic to a strictly ergodic topological dynamical system $(\widehat{X}, \widehat{\mu}, \widehat{T})$, where $\widehat{\mu}$ is the unique ergodic measure of $(\widehat{X}, \widehat{T})$. We say that $(\widehat{X}, \widehat{T})$ is a *topological model* for (X, μ, T).

Further refinements have been given to the Jewett-Krieger Theorem. We state the one which is useful for our purposes.

Definition 3.2.2. Let (X, μ, G) be a measure preserving system. We say that G acts *freely* on X (or the system (X, μ, G) is *free*) if for any non-trivial $g \in G$ the set $\{x \in X : gx = x\}$ has measure 0. If (X, μ, G) is ergodic and G is abelian and this is equivalent to say that any non-trivial $g \in G$ defines a transformation different from the identity transformation on X.

Particularly, we say that an ergodic system (X, μ, S, T) with commuting transformations is *free* if $S^i T^j$ is not the identity transformation on X for any $(i, j) \neq (0, 0)$.

Theorem 3.2.3 (Weiss-Rosenthal [118])**.** *Let G be an amenable group and let $\pi \colon Y \to X$ be a factor map between two measure preserving systems (Y, ν, G) and (X, μ, G). Suppose that (X, μ, G) is free and $(\widehat{X}, \widehat{G})$ is a strictly ergodic model for (X, μ, G). Then there exits a strictly ergodic model $(\widehat{Y}, G)$ for (Y, ν, G) and a topological factor map $\widehat{\pi} : \widehat{Y} \to \widehat{X}$ such that the following diagram commutes:*

Here we mean that Φ and ϕ are measure preserving isomorphisms and $\pi \circ \Phi = \phi \circ \widehat{\pi}$.

In this case, we say that $\widehat{\pi} \colon \widehat{Y} \to \widehat{X}$ is a *topological model* for $\pi \colon Y \to X$.

$$\begin{tikzcd}
Y \arrow[r, leftrightarrow, "\Phi"] \arrow[d, "\pi"'] & \widehat{Y} \arrow[d, "\widehat{\pi}"] \\
X \arrow[r, leftrightarrow, "\phi"'] & \widehat{X}
\end{tikzcd}$$

3.2.1. Host magic extensions

The Host magic extension was first introduced in [64] to prove the L^2 convergence of multiple ergodic averages for systems with commuting transformations. Then Chu [21] used this tool to study the recurrence problems in the same setting of systems. We recall that this construction is valid for an arbitrary number of transformations, but for convenience we state it only for two transformations S and T.

Convention 3.2.4. *We recall that we implicitly assume that all functions are measurable and real valued but we remark that similar results hold for complex valued functions.*

The Host measure

Definition 3.2.5. For any measure preserving transformation R of the system $(X, \mathcal{X}, \mu)$, we let $\mathcal{I}_R$ denote the σ-algebra of R-invariant sets.

Let X^* denote the space X^4. Let μ_S be the relative independent square of μ over $\mathcal{I}_S$, meaning that for all $f_0, f_1 \in L^\infty(\mu)$ we have

$$\int_{X^2} f_0(x_0) f_1(x_1) d\mu_S = \int_X \mathbb{E}(f_1|\mathcal{I}_S)\mathbb{E}(f_1|\mathcal{I}_S) d\mu,$$

where $\mathbb{E}(f|\mathcal{I}_S)$ is the *conditional expectation* of f on $\mathcal{I}_S$. It is obvious that μ_S is invariant under $\mathrm{id} \times S$ and $g \times g$ for $g \in G$.

Let $\mu_{S,T}$ denote the relative independent square of μ_S over $\mathcal{I}_{T\times T}$. Hence for all $f_0, f_1, f_2, f_3 \in L^\infty(\mu)$ we have that

$$\int_{X^4} f_0(x_0) f_1(x_1) f_2(x_2) f_3(x_3) d\mu_{S,T} = \int_{X^2} \mathbb{E}(f_0 \otimes f_1 | \mathcal{I}_{T\times T}) \mathbb{E}(f_2 \otimes f_3 | \mathcal{I}_{T\times T}) d\mu_S.$$

The measure $\mu_{S,T}$ is invariant under $\mathrm{id} \times S \times \mathrm{id} \times S$, $\mathrm{id} \times \mathrm{id} \times T \times T$ and under $g \times g \times g \times g$ for all $g \in G$.

Let S^* and T^* denote the transformations $\mathrm{id} \times S \times \mathrm{id} \times S$ and $\mathrm{id} \times \mathrm{id} \times T \times T$ respectively. Then $(X^*, \mu_{S,T}, S^*, T^*)$ is a system with commuting transformations S^* and T^*. Let π denote the projection $(x_0, x_1, x_2, x_3) \to x_3$ from X^* to X. Then π defines a factor map between

$(X^*, \mu_{S,T}, S^*, T^*)$ and (X, μ, S, T). We remark that the system $(X^*, \mu_{S,T}, S^*, T^*)$ may not be ergodic even if (X, μ, S, T) is ergodic.

The Host seminorm

Let $f \in L^\infty(\mu)$. The Host seminorm [64] is defined to be the quantity

$$|||f|||_{\mu,S,T} = \left(\int_{X^4} f(x_0)f(x_1)f(x_2)f(x_3)d\mu_{S,T} \right)^{1/4}.$$

We have

Proposition 3.2.6 ([64], Proposition 2)**.**

1. *For $f_0, f_1, f_2, f_3 \in L^\infty(\mu)$, we have*

$$\int_{X^4} f_0 \otimes f_1 \otimes f_2 \otimes f_3 d\mu_{S,T} \leq |||f_0|||_{\mu,S,T} \, |||f_1|||_{\mu,S,T} \, |||f_2|||_{\mu,S,T} \, |||f_3|||_{\mu,S,T}$$

2. *$|||\cdot|||_{\mu,S,T}$ is a seminorm on $L^\infty(\mu)$.*

We recall some standard notation. For any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ of X, let $\mathcal{A} \vee \mathcal{B}$ denote the σ-algebra generated by $\{A \cap B \colon A \in \mathcal{A}, B \in \mathcal{B}\}$. If f is a measurable function on $(X, \mathcal{X}, \mu)$ and $\mathcal{A}$ is a sub-algebra of $\mathcal{X}$, let $\mathbb{E}(f|\mathcal{A})$ denote the conditional expectation of f over $\mathcal{A}$.

Definition 3.2.7. Let (X, μ, S, T) be a measure preserving system with commuting transformations S and T. We say that (X, μ, S, T) is *magic* if

$$\mathbb{E}(f|\mathcal{I}_S \vee \mathcal{I}_T) = 0 \text{ if and only if } |||f|||_{\mu,S,T} = 0.$$

The connection between the Host measure $\mu_{S,T}$ and magic systems is:

Theorem 3.2.8 ([64], Theorem 2)**.** *The system $(X^*, \mu_{S,T}, S^*, T^*)$ defined in Section 3.2.1 is a magic extension system of (X, μ, S, T).*

3.2.2. Dynamical cubes

The following notion of dynamical cubes for a system with commuting transformations was introduced and studied in [34] and presented in Chapter 2. We recall the definitions here.

Definition 3.2.9. Let (X, S, T) be a topological dynamical system with commuting transformations S and T. We let $\mathcal{G}_{S,T}$ denote the subgroup of G^4 generated by $\mathrm{id} \times S \times \mathrm{id} \times S$, $\mathrm{id} \times \mathrm{id} \times T \times T$ and $g \times g \times g \times g, g \in G$. For any $R \in G$, let $\mathcal{G}_R$ denote the subgroup of G^2 generated by $\mathrm{id} \times R$ and $g \times g, g \in G$.

Definition 3.2.10. Let (X, S, T) be a topological dynamical system with commuting transformations S and T and let $R \in G$. We define

$$\mathbf{Q}_{S,T}(X) = \overline{\{(x, S^i x, T^j x, S^i T^j x) \colon x \in X, i, j \in \mathbb{Z}\}};$$
$$\mathbf{Q}_R(X) = \overline{\{(x, R^i x) \in X \colon x \in X, i \in \mathbb{Z}\}}.$$

3.3. The existence of free magic extensions

In this section, we strengthen Theorem 3.2.8 for our purposes by requiring the magic extension to be also ergodic and free. We remark that there are a lot of interesting systems with commuting transformations where the action is not free. For example, the system (X, μ, S, S^i), where S is an ergodic measure preserving transformation of X and $i \in \mathbb{Z}, i \neq 1$. However, we have

Theorem 3.3.1. *Let (X, μ, S, T) be an ergodic system with commuting transformations S and T. Suppose that S^i and T^j are not the identity for any $i, j \in \mathbb{Z} \setminus \{0\}$. Then there exists a magic extension $(\widehat{X}, \nu, S^*, T^*)$ where the action of $\mathbb{Z}^2$ is free and ergodic.*

Remark 3.3.2. By Theorem 3.2.8, $(X^*, \mu_{S,T}, S^*, T^*)$ is a magic extension of X, but since $(X^*, \mu_{S,T}, S^*, T^*)$ may not be ergodic, we need to decompose the measure $\mu_{S,T}$ in order to get an ergodic magic extension of X.

Proof. Consider the measure $\mu_{S,T}$ on $X^* = X^4$. We claim that $\mu_{S,T}(\{\vec{x} : S^{*i} T^{*j} \vec{x} \neq \vec{x}\}) = 1$ for every $i, j \in \mathbb{Z}$. Let ${A^*}_{i,j}$ denote the set $\{\vec{x} : S^{*i} T^{*j} \vec{x} \neq \vec{x}\}$. Then the complement of ${A^*}_{i,j}$ is included in the union of the sets $X \times A \times X \times X$ and $X \times X \times B \times X$, where $A = \{x : S^i x = x\}$ and $B = \{x : T^j x = x\}$. Since the projection of $\mu_{S,T}$ onto any coordinate equals μ, we have that $\mu_{S,T}({A^{*c}}_{i,j}) \leq \mu(A) + \mu(B) = 0$. Therefore, writing $A^* = \bigcap_{i,j \in \mathbb{Z}} {A^*}_{i,j}$, we have that $\mu_{S,T}(A^*) = 1$.

Let

$$\mu_{S,T} = \int \mu_{S,T,\vec{x}} d\mu_{S,T}(\vec{x})$$

be the ergodic decompositions of $\mu_{S,T}$ under S^* and T^*. Then we have that $\mu_{S,T,\vec{x}}(A^*) = 1$ for $\mu_{S,T}$-a.e. $\vec{x} \in X^*$. By Proposition 3.13 of [21], for $\mu_{S,T}$-almost every $\vec{x} \in X^*$, the system $(X^*, \mu_{S,T,\vec{x}}, S^*, T^*)$ is a magic extension of (X, μ, S, T). Hence, we can pick $\vec{x}_0 \in A^*$ such that $(X^*, \mu_{S,T,\vec{x}_0}, S^*, T^*)$ is a magic extension. This is a magic ergodic free extension of (X, μ, S, T). □

We prove some properties for later use. In the rest of this section, we assume that (X, μ, S, T) is a free magic ergodic measure preserving system. Let $\mathcal{W}$ denote the σ-algebra $\mathcal{I}_S \vee \mathcal{I}_T$ and let $\mathcal{Z}_{S,T}$ be the factor associated to this σ-algebra.

Lemma 3.3.3. *$\mathcal{Z}_{S,T}$ is isomorphic to the product of two ergodic systems.*

Proof. Let $A \in \mathcal{I}_T$ and $B \in \mathcal{I}_S$. We have that $\lim_{N\to\infty} \frac{1}{N^2}\sum_{i,j=0}^{N-1} 1_A \circ S^i \circ T^j$ converges in $L^2(\mu)$ to $\mu(A)$. Since A is invariant under T, we have that $\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} 1_A \circ S^j$ converges to $\mu(A)$. Similarly $\lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{N-1} 1_B \circ S^j$ converges to $\mu(B)$. It follows that

$$\lim_{N\to\infty} \frac{1}{N^2}\sum_{i,j=0}^{N-1} 1_{A\cap B} \circ S^i \circ T^j = \lim_{N\to\infty} \frac{1}{N^2}\sum_{i,j=0}^{N-1} 1_A 1_B \circ S^i \circ T^j = \mu(A)\mu(B).$$

Since (X, μ, S, T) is ergodic, this limit equals $\mu(A \cap B)$ and therefore $\mu(A\cap B) = \mu(A)\mu(B)$.

We conclude that the map $A \cap B \to A \times B$ defines a measure preserving isomorphism between $(X, \mathcal{I}_T \vee \mathcal{I}_S, \mu, S, T)$ and $(X \times X, \mathcal{I}_T \otimes \mathcal{I}_S, \mu \otimes \mu, S \times \mathrm{id}, \mathrm{id} \times T)$.

□

For convenience, we write $(\mathcal{Z}_{S,T}, S, T) = (Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$.

Lemma 3.3.4. *The σ-algebra of $(T\times T)$-invariant sets on (X^2, μ_S) is measurable with respect to $\mathcal{W}^2$.*

Proof. We follow the proof of Proposition 4.7 of [67]. It suffices to show that

$$\mathbb{E}(f_0 \otimes f_1 | \mathcal{I}_{T\times T}) = \mathbb{E}(\mathbb{E}(f_0|\mathcal{W}) \otimes \mathbb{E}(f_1|\mathcal{W})|\mathcal{I}_{T\times T}).$$

It suffices to prove this equality when $\mathbb{E}(f_i|\mathcal{W}) = 0$ for $i = 0$ or 1. By Proposition 3.2.6, we have that

$$\int f_0 \otimes f_1 \otimes f_0 \otimes f_1 d\mu_{S,T} = \int |\mathbb{E}(f_0 \otimes f_1 | \mathcal{I}_{T\times T})|^2 d\mu_S \leq |||f_0|||^2_{\mu,S,T} \, |||f_1|||^2_{\mu,S,T} \,,$$

which implies that $\mathbb{E}(f_0 \otimes f_1|\mathcal{I}_{T\times T}) = 0$ whenever $|||f_i|||_{\mu,S,T} = 0$ for $i = 0$ or 1. Since the system is magic, this is equivalent to $\mathbb{E}(f_i|\mathcal{W}) = 0$ for $i = 0$ or 1, and we are done.

□

3.4. Strict ergodicity for dynamical cubes

This section is devoted to the proof of Theorem 3.1.3. By Theorem 3.3.1, it suffices to prove the following theorem:

Theorem 3.4.1. *For any free ergodic magic system (X, μ, S, T) with two commuting transformations S and T, there exists a topological model $(\widehat{X}, \widehat{S}, \widehat{T})$ of X such that $(\mathbf{Q}_{S,T}(\widehat{X}), \mathcal{G}_{S,T})$ is strictly ergodic.*

3.4.1. A special case: product systems

We start by proving a special case of Theorem 3.4.1.

Lemma 3.4.2. *Let (Y,σ) and (W,τ) be two strictly ergodic systems with unique measures ρ_Y and ρ_W. Then $(Y \times W, \sigma \times \mathrm{id}, \mathrm{id} \times \tau)$ is strictly ergodic with measure $\rho_Y \otimes \rho_W$.*

Proof. Let λ be an invariant measure on $Y \times W$. Since Y is uniquely ergodic, the projection onto the first coordinate of λ is ρ_Y. Using the disintegration with respect to Y, we have that

$$\lambda = \int_Y \delta_y \times \lambda_y d\rho_Y.$$

Since λ is invariant under $\mathrm{id} \times \tau$, we have that

$$(\mathrm{id} \times \tau)\lambda = \lambda = \int_Y \delta_y \times \tau\lambda_y d\rho_Y.$$

By the uniqueness of the disintegration, we get that $\tau\lambda_y = \lambda_y$ ρ_Y-a.e. Since (W,τ) is uniquely ergodic, a.e. we have that $\lambda_y = \rho_W$ and therefore

$$\lambda = \int_Y \delta_y \times \rho_W d\rho_Y = \rho_Y \otimes \rho_W.$$

$\square$

The next corollary follows similarly.

Corollary 3.4.3. *Let $((X_i, T_i))_{i=1}^n$ be strictly ergodic systems with measures $(\rho_i)_{i=1}^n$. For $j = 1, \ldots, n$ let $\widetilde{T}_j$ be the transformation on $\prod X_i$ defined as $(\widetilde{T}_j)_i = \mathrm{id}_{X_i}$ if $i \neq j$ and $(\widetilde{T}_j)_j = T_j$. Then the system $(\prod X_i, \widetilde{T}_1, \ldots, \widetilde{T}_n)$ is strictly ergodic with measure $\otimes \rho_i$.*

We are now ready to prove Theorem 3.4.1 for the case when the system is a product:

Proposition 3.4.4. *Let (Y,σ) and (W,τ) be two strictly ergodic systems with unique measures ρ_Y and ρ_W. Then $\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W)$ is uniquely ergodic with measure $\nu_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}$, where $\nu = \rho_Y \otimes \rho_W$. Particularly, $(\mathbf{Q}_{\sigma\times\mathrm{id}}(Y \times W), \mathcal{G}_{\sigma\times\mathrm{id}})$ is strictly ergodic with measure $\nu_{\sigma\times\mathrm{id}}$.*

Proof. By definition, we deduce that

$$\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W) = \{((y,w),(y',w),(y,w'),(y',w')) : y, y' \in Y, w, w' \in W\}$$

and $\mathcal{G}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}$ is the group spanned by

$$\begin{aligned}
(\sigma \times \mathrm{id}) \times (\sigma \times \mathrm{id}) \times (\sigma \times \mathrm{id}) \times (\sigma \times \mathrm{id})\\
(\mathrm{id} \times \tau) \times (\mathrm{id} \times \tau) \times (\mathrm{id} \times \tau) \times (\mathrm{id} \times \tau)\\
(\mathrm{id} \times \mathrm{id}) \times (\sigma \times \mathrm{id}) \times (\mathrm{id} \times \mathrm{id}) \times (\sigma \times \mathrm{id})\\
(\mathrm{id} \times \mathrm{id}) \times (\mathrm{id} \times \mathrm{id}) \times (\mathrm{id} \times \tau) \times (\mathrm{id} \times \tau).
\end{aligned}$$

We may identify $\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y\times W)$ with $Y\times Y\times W\times W$ under the map ϕ

$$((y,w),(y',w),(y,w'),(y',w'))\mapsto(y,y',w,w').$$

We remark that $\mathcal{G}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}$ is mapped to the group spanned by

$$\sigma\times\sigma\times\mathrm{id}\times\mathrm{id},\quad \mathrm{id}\times\mathrm{id}\times\tau\times\tau,\quad \mathrm{id}\times\sigma\times\mathrm{id}\times\mathrm{id}\quad\text{and}\quad \mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\tau.$$

This is the same as the group spanned by

$$\sigma\times\mathrm{id}\times\mathrm{id}\times\mathrm{id},\quad \mathrm{id}\times\mathrm{id}\times\tau\times\mathrm{id},\quad \mathrm{id}\times\sigma\times\mathrm{id}\times\mathrm{id}\quad\text{and}\quad \mathrm{id}\times\mathrm{id}\times\mathrm{id}\times\tau.$$

By Corollary 3.4.3, this system is uniquely ergodic with measure $\rho_Y\otimes\rho_Y\otimes\rho_W\otimes\rho_W$. Since $\nu_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}$ is an invariant measure on $\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y\times W)$, we have that it is the unique invariant measure and it coincides with $\phi^{-1}(\rho_Y\otimes\rho_Y\otimes\rho_W\otimes\rho_W)$.

□

3.4.2. Proof of the general case

Throughout this section, we consider (X,μ,S,T) as a fixed system which is magic, ergodic and free, and we follow the notations in the previous section. By Lemma 3.3.3, the factor associated to the σ-algebra $\mathcal{W}=\mathcal{I}_S\vee\mathcal{I}_T$ has the form $(Y\times W,\sigma\times\mathrm{id},\mathrm{id}\times\tau)$, where (Y,σ) and (W,τ) are ergodic systems.

Lemma 3.4.5. *There exists a strictly ergodic topological model for the factor map $\pi\colon X\to Y\times W$.*

Proof. By the Jewett-Krieger Theorem, we can find strictly ergodic models $(\widehat{Y},\widehat{\sigma})$ and $(\widehat{W},\widehat{\tau})$ for (Y,σ) and (W,τ), respectively. Let ρ_Y and ρ_W denote the unique ergodic measures on these systems. By Lemma 3.4.2, $(\widehat{Y}\times\widehat{W},\widehat{\sigma}\times\mathrm{id},\mathrm{id}\times\widehat{\tau})$ is a strictly ergodic model for $(Y\times W,\sigma\times\mathrm{id},\mathrm{id}\times\tau)$ with unique invariant measure $\rho_Y\otimes\rho_W$.

By Theorem 3.2.3, there exists a strictly ergodic model $\widehat{\pi}:\widehat{X}\to\widehat{Y}\times\widehat{W}$ for $\pi:X\to Y\times W$. □

We are now ready to prove Theorem 3.4.1:

Proof of Theorem 3.4.1. For any free ergodic magic system (X,S,T), let $\pi:X\to(Y\times W,\sigma\times\mathrm{id},\mathrm{id}\times\tau)$ be the factor map associated to the σ-algebra $\mathcal{W}=\mathcal{I}_S\vee\mathcal{I}_T$. Let $\widehat{\pi}\colon\widehat{X}\to\widehat{Y}\times\widehat{W}$ be the topological model given by Lemma 3.4.5. We claim that $(\mathbf{Q}_{S,T}(\widehat{X}),\mathcal{G}_{S,T})$ is strictly ergodic.

To simplify the notation, we replace $\widehat{X}$, $\widehat{W}$, $\widehat{Y}$, etc by X, W, Y etc. It was proved in Proposition 3.14 of [34] that $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is a minimal system. So it suffices to show unique ergodicity.

Claim 1: $(\mathbf{Q}_S(X), \mathcal{G}_S)$ is uniquely ergodic with measure μ_S.

We recall that the factor of X corresponding to $\mathcal{I}_S$ is (W, id, τ).

Suppose that the ergodic decomposition of μ under S is

$$\mu = \int_W \mu_\omega d\rho_W(\omega).$$

Then

$$\mu_S = \int_W \mu_\omega \times \mu_\omega d\rho_W(\omega).$$

Let $\pi_W \colon X \to W$ be the factor map and let λ be a $\mathcal{G}_S$-invariant measure on $\mathbf{Q}_S(X)$. For $i = 0, 1$, let $p_i \colon (\mathbf{Q}_S(X), \mathcal{G}_S) \to (X, G)$ be the projection onto the i-th coordinate. Then $p_i\lambda$ is a G-invariant measure of X. Therefore, $p_i\lambda = \mu$. Hence we may assume that

$$\lambda = \int_X \delta_x \times \lambda_x d\mu(x)$$

is the disintegration of λ over μ. Since λ is $(\mathrm{id} \times S)$-invariant, we have that

$$\lambda = (\mathrm{id} \times S)\lambda = \int_X \delta_x \times \lambda_{Sx} d\mu(x).$$

The uniqueness of disintegration implies that $\lambda_{Sx} = \lambda_x$ for μ-a.e. $x \in X$. So the map

$$F \colon X \to M(X) \colon x \mapsto \lambda_x$$

is an S-invariant function. Hence we can write $\lambda_x = \lambda_{\pi_W(x)}$ for μ-a.e. $x \in X$.

Then we have

$$\begin{aligned}
\lambda = \int_X \delta_x \times \lambda_x d\mu(x) &= \int_X \delta_x \times \lambda_{\pi_W(x)} d\mu(x) \\
&= \int_W \int_X \delta_x \times \lambda_\omega d\mu_\omega(x) d\rho_W(\omega) \\
&= \int_W \Big(\int_X \delta_x d\mu_\omega(x)\Big) \times \lambda_\omega d\rho_W(\omega) \\
&= \int_W \mu_\omega \times \lambda_\omega d\rho_W(\omega).
\end{aligned}$$

Recall that $\mathbf{Q}_{\mathrm{id}}(W) = \Delta_W$ and $\mathcal{G}_{\mathrm{id}}$ is spanned by (τ, τ). Therefore $(\mathbf{Q}_{\mathrm{id}}(W), \mathcal{G}_{\mathrm{id}})$ is isomorphic to (W, τ). Particularly, it is uniquely ergodic and for convenience we let P_W denote its invariant measure.

Let $\pi_Y^2\colon (\mathbf{Q}_S(X),\mathcal{G}_S)\to(\mathbf{Q}_{\mathrm{id}}(W),\mathcal{G}_{\mathrm{id}})$ be the natural factor map.

We have that

$$\pi_W^2(\lambda)=P_W.$$

Thus

$$\pi(\mu_\omega)=\pi(\lambda_\omega)=\delta_\omega. \tag{3.4.1}$$

On the other hand, $p_1(\lambda)=p_2(\lambda)=\mu$ implies that

$$\mu=\int_W \mu_\omega d\rho_W(\omega)=\int_W \lambda_\omega d\rho_W(\omega). \tag{3.4.2}$$

By (3.4.1), (3.4.2) and the uniqueness of disintegration, we have that $\lambda_\omega=\mu_\omega, \rho_W$-a.e. $\omega\in W$. So

$$\lambda=\int_W \mu_\omega\times\mu_\omega d\rho_W(\omega)=\mu_S.$$

This finishes the proof of Claim 1.

Claim 2: $(\mathbf{Q}_{S,T}(X),\mathcal{G}_{S,T})$ is uniquely ergodic with unique measure $\mu_{S,T}$.

Let λ be a $\mathcal{G}_{S,T}$-invariant measure on $\mathbf{Q}_{S,T}(X)$. Let $\overline{p}_1,\overline{p}_2\colon (\mathbf{Q}_{S,T}(X),\mathcal{G}_{S,T})\to(\mathbf{Q}_S(X),\mathcal{G}_S)$ be the projection onto the first two and last two coordinates, respectively. Then $\overline{p}_i\lambda$ is a $\mathcal{G}_S$-invariant measure of $\mathbf{Q}_S(X)$ and therefore, $\overline{p}_i\lambda=\mu_S$. Hence we may assume that

$$\lambda=\int_{\mathbf{Q}_S(X)}\delta_{\mathbf{x}}\times\lambda_{\mathbf{x}}d\mu_S(\mathbf{x})$$

is the disintegration of λ over μ_S. Since λ is $(\mathrm{id}\times\mathrm{id}\times T\times T)$-invariant, we have that

$$\lambda=(\mathrm{id}\times\mathrm{id}\times T\times T)\lambda=\int_{\mathbf{Q}_S(X)}\delta_{\mathbf{x}}\times\lambda_{(T\times T)\mathbf{x}}d\mu_S(\mathbf{x}).$$

The uniqueness of disintegration implies that $\lambda_{(T\times T)\mathbf{x}}=\lambda_{\mathbf{x}}$ for μ_S-a.e. $\mathbf{x}\in\mathbf{Q}_S(X)$. So the map

$$F\colon \mathbf{Q}_S(X)\to M(X^4)\colon \mathbf{x}\mapsto\lambda_{\mathbf{x}}$$

is a $(T\times T)$-invariant function and therefore F is $\mathcal{I}_{T\times T}$-measurable.

Let $(\Omega_{S,T},\overline{P})$ be the factor of $(X\times X,\mu_S)$ corresponding to the subalgebra $\mathcal{I}_{T\times T}$ and let ϕ denote the corresponding factor map. Suppose that the ergodic decomposition of μ_S under $T\times T$ is

$$\mu_S=\int_{\Omega_{S,T}}\mu_{S,\omega}d\overline{P}(\omega).$$

Then

$$\mu_{S,T} = \int_{\Omega_{S,T}} \mu_{S,\omega} \times \mu_{S,\omega} d\overline{P}(\omega).$$

Hence we can write $\lambda_{\mathbf{x}} = \lambda_{\phi(\mathbf{x})}$ for μ_S-a.e. $\mathbf{x} \in \mathbf{Q}_S(X)$. Then we have

$$\begin{aligned}\lambda &= \int_{\mathbf{Q}_S(X)} \delta_{\mathbf{x}} \times \lambda_{\mathbf{x}} d\mu_S(\mathbf{x}) = \int_{\mathbf{Q}_S(X)} \delta_{\mathbf{x}} \times \lambda_{\phi(\mathbf{x})} d\mu_S(\mathbf{x}) \\ &= \int_{\Omega_{S,T}} \int_{\mathbf{Q}_S(X)} \delta_{\mathbf{x}} \times \lambda_\omega d\mu_{S,\omega}(\mathbf{x}) d\overline{P}(\omega) \\ &= \int_{\Omega_{S,T}} (\int_{\mathbf{Q}_S(X)} \delta_{\mathbf{x}} d\mu_{S,\omega}(\mathbf{x})) \times \lambda_\omega d\overline{P}(\omega) \\ &= \int_{\Omega_{S,T}} \mu_{S,\omega} \times \lambda_\omega d\overline{P}(\omega).\end{aligned}$$

Recall that $\pi\colon X \to Y \times W$ is the factor map. Let

$$\pi^4\colon (\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T}) \to (\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W), \mathcal{G}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau})$$

be the natural factor map. By Lemma 3.3.4, there exists a factor map $\alpha\colon (Y \times W)^2 \to \Omega_{S,T}$ such that $\alpha \circ \pi^2 = \phi^2$.

Let $\nu = \rho_Y \otimes \rho_W$ denote the unique invariant measure on $Y \times W$. By Proposition 3.4.4, we have that $(\mathbf{Q}_{S,T}(Y \times W), G_{\sigma\times\mathrm{id},\mathrm{id}\times\tau})$ is uniquely ergodic and $\nu_{S,T}$ is its unique invariant measure.

Suppose that the ergodic decomposition of ν_S under $T \times T$ is

$$\nu_S = \int_{\Omega_{S,T}} \nu_{S,\omega} d\overline{P}(\omega).$$

Then we have

$$\nu_{S,T} = \int_{\Omega_{S,T}} \nu_{S,\omega} \times \nu_{S,\omega} d\overline{P}(\omega).$$

Since $\pi^4\lambda$ is an invariant measure on $\mathbf{Q}_{\sigma\times\mathrm{id},\mathrm{id}\times\tau}(Y \times W)$, we have that

$$\pi^4(\lambda) = \nu_{S,T} = \int_{\Omega_{S,T}} \nu_{S,\omega} \times \nu_{S,\omega} d\overline{P}(\omega).$$

Since $\phi^2 = \alpha \circ \pi^2$, we have that

$$\phi^2(\mu_{S,\omega}) = \phi^2(\lambda_\omega) = \alpha(\nu_{S,\omega}) = \delta_\omega. \tag{3.4.3}$$

On the other hand, $\overline{p_1}(\lambda) = \overline{p}_2(\lambda) = \mu$ implies that

$$\mu_S = \int_{\Omega_{S,T}} \mu_{S,\omega} d\overline{P}(\omega) = \int_{\Omega_{S,T}} \lambda_\omega d\overline{P}(\omega). \tag{3.4.4}$$

By (3.4.3), (3.4.4) and the uniqueness of disintegration, we have that $\lambda_\omega = \mu_{S,\omega}$, $\overline{P}$-a.e. $\omega \in \Omega_{S,T}$. So

$$\lambda = \int_{\Omega_{S,T}} \mu_{S,\omega} \times \mu_{S,\omega} d\overline{P}(\omega) = \mu_{S,T}.$$

Thus $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is strictly ergodic with unique measure $\mu_{S,T}$. □

3.5. Applications to pointwise results

We apply results in previous sections to deduce some convergence results. We remark that if S^i is the identity for some $i \neq 0$, the averages we consider in this section reduce to the Birkhoff ergodic theorem. So the difficult case is when the systems (X, μ, S) and (X, μ, T) are free, and we make this assumption throughout this section. Since the averages we consider can be deduced by proving them in an extension of X, by Theorem 3.3.1 we may assume that (X, μ, S, T) is a magic free ergodic system. By Theorem 3.4.1, we may take a strictly topological model $(\widehat{X}, \widehat{S}, \widehat{T})$ for X such that $(\mathbf{Q}_{S,T}(\widehat{X}), \mathcal{G}_{\widehat{S},\widehat{T}})$ is strictly ergodic. So (omitting the symbol $\widehat{\ }$ to simplify notation), throughout all this section we assume that (X, μ, S, T) is a magic free ergodic system and that $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is strictly ergodic.

Theorem 3.5.1. *Let $f_0, f_1, f_2, f_3 \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{N^4} \sum_{i,j,k,p=0}^{N-1} f_0(S^iT^jx) f_1(S^{i+k}T^jx) f_2(S^iT^{j+p}x) f_3(S^{i+k}T^{j+p}x)$$

converges almost everywhere to $\int f_0 \otimes f_1 \otimes f_2 \otimes f_3 d\mu_{S,T}$.

Proof. Recall that $\mathcal{G}_{S,T}$ is a $\mathbb{Z}^4$-action spanned by $S\times S\times S\times S$, $T\times T\times T\times T$, $\mathrm{id}\times S\times \mathrm{id}\times S$ and $\mathrm{id}\times \mathrm{id}\times T\times T$.

Let $f_0, f_1, f_2, f_3 \in L^\infty(\mu)$ and fix $\epsilon > 0$. Let $\widehat{f_0}, \widehat{f_1}, \widehat{f_2}, \widehat{f_3}$ be continuous functions on X such that $\|f_i - \widehat{f_i}\|_1 < \epsilon$ for $i = 0, 1, 2, 3$. We can assume that all functions are bounded by 1 in L^∞ norm. For simplicity, denote

$$I(h_0, h_1, h_2, h_3) = \int h_0 \otimes h_1 \otimes h_2 \otimes h_3 d\mu_{S,T}$$

and

$$\mathbb{E}_N(h_0 \otimes h_1 \otimes h_2 \otimes h_3)(x) = \frac{1}{N^4} \sum_{i,j,k,p=0}^{N-1} h_0(S^iT^jx) h_1(S^{i+k}T^jx) h_2(S^iT^{j+p}x) h_3(S^{i+k}T^{j+p}x)$$

for $x \in X, h_0, h_1, h_2, h_3 \in L^\infty(\mu)$.

By the telescoping inequality

$$\begin{aligned}
&|\mathbb{E}_N(f_0 \otimes f_1 \otimes f_2 \otimes f_3)(x) - I(f_0, f_1, f_2, f_3)| \\
\leq &\left|\mathbb{E}_N(f_0 \otimes f_1 \otimes f_2 \otimes f_3)(x) - \mathbb{E}_N(\widehat{f}_0 \otimes \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3)(x)\right| \\
+ &\left|\mathbb{E}_N(\widehat{f}_0 \otimes \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3)(x) - I(f_0, f_1, f_2, f_3)\right| \\
\leq &\frac{1}{N^2}\sum_{i,j} |f_0(S^iT^jx) - \widehat{f}_0(S^iT^jx)| + \frac{1}{N^3}\sum_{i,j,k} |f_1(S^{i+k}T^jx) - \widehat{f}_1(S^{i+k}T^jx)| \\
+ &\frac{1}{N^3}\sum_{i,j,p} |f_2(S^iT^{j+p}x) - \widehat{f}_2(S^iT^{j+p}x)| + \frac{1}{N^4}\sum_{i,j,k,p} |f_3(S^{i+k}T^{j+p}x) - \widehat{f}_3(S^{i+k}T^{j+p}x)| \\
+ &\left|\mathbb{E}_N(\widehat{f}_0 \otimes \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3)(x) - I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)\right| + \left|I(f_0, f_1, f_2, f_3) - I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)\right|.
\end{aligned}$$

Since $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is uniquely ergodic, we have that

$$\left|\mathbb{E}_N(\widehat{f}_0 \otimes \widehat{f}_1 \otimes \widehat{f}_2 \otimes \widehat{f}_3)(x) - I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)\right|$$

converges to 0 for every $x \in X$ as N goes to infinity.

On the other hand, by Birkhoff ergodic theorem, we have that the four first terms of the last inequality converge a.e. to $\|f_0 - \widehat{f}_0\|_1$, $\|f_1 - \widehat{f}_1\|_1$, $\|f_2 - \widehat{f}_2\|_1$ and $\|f_3 - \widehat{f}_3\|_1$, respectively.

Finally, using again the telescoping inequality and the fact that the marginals of $\mu_{S,T}$ are equal to μ we deduce that

$$\left|I(f_0, f_1, f_2, f_3) - I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)\right| \leq \|f_0 - \widehat{f}_0\|_1 + \|f_1 - \widehat{f}_1\|_1 + \|f_2 - \widehat{f}_2\|_1 + \|f_3 - \widehat{f}_3\|_1.$$

Therefore, we can find N large enough and a subset $X_N \subset X$ with measure larger than $1 - \epsilon$ such that for every $x \in X_N$,

$$|\mathbb{E}_N(f_0 \otimes f_1 \otimes f_2 \otimes f_3)(x) - I(f_0, f_1, f_2, f_3)| \leq 13\epsilon.$$

Since ϵ is arbitrary, we conclude that $\mathbb{E}_N(f_0 \otimes f_1 \otimes f_2 \otimes f_3)$ converges to $I(f_0, f_1, f_2, f_3)$ a.e. as N goes to infinity.

□

Since $(\mathbf{Q}_{S,T}(X), \mathcal{G}_{S,T})$ is uniquely ergodic, we also have:

Lemma 3.5.2. *Let $\widehat{f}_0$, $\widehat{f}_1$, $\widehat{f}_2$, $\widehat{f}_3$ be continuous functions on X. Then*

$$\frac{1}{N^4}\sum_{i,j=0}^{N-1}\sum_{k=-i}^{N-1-i}\sum_{p=-j}^{N-1-j} \widehat{f}_0(S^iT^jx)\widehat{f}_1(S^{i+k}T^jx)\widehat{f}_2(S^iT^{j+p}x)\widehat{f}_3(S^{i+k}T^{j+p}x)$$

converges to $I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)$.

Proof. Suppose that the averages does not converge to $I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)$. Then there exist $x \in X$, a sequence $N_m \to \infty$ and $\epsilon > 0$ such that the N_m-average at x and the integral differs at least ϵ. Take any weak*-limit of the sequence

$$\frac{1}{N^4} \sum_{i,j=0}^{N_m-1} \sum_{k=-i}^{N_m-1-i} \sum_{p=-j}^{N_m-1-j} (S^i T^j \times S^{i+k} T^j \times S^i T^{j+p} \times S^{i+k} T^{j+p}) \delta_{(x,x,x,x)}.$$

Such a limit is clearly invariant under $\mathcal{G}_{S,T}$ and therefore it equals to $\mu_{S,T}$ by unique ergodicity. Hence,

$$\frac{1}{N_m^4} \sum_{i,j=0}^{N_m-1} \sum_{k=-i}^{N_m-1-i} \sum_{p=-j}^{N_m-1-j} \widehat{f}_0(S^i T^j x) \widehat{f}_1(S^{i+k} T^j x) \widehat{f}_2(S^i T^{j+p} x) \widehat{f}_3(S^{i+k} T^{j+p} x)$$

converges to $I(\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3)$ as m goes to infinity, a contradiction. $\square$

For any $N \in \mathbb{N}$, denote

$$A_N := \{(i,j,k,p) \in \mathbb{Z}^4 \colon i,j \in [0, N-1], k \in [-i, N-1-i], p \in [-j, N-1-j]\}.$$

Let (X, μ, S, T) be a measure preserving system with commuting transformations S and T. For any $f \in L^\infty(X)$ and any $x \in X$, denote

$$S_N(f, x) := \Big| \frac{1}{N^4} \sum_{(i,j,k,p) \in A_N} f(S^i T^j x) f(S^{i+k} T^j x) f(S^i T^{j+p} x) f(S^{i+k} T^{j+p} x) \Big|.$$

Lemma 3.5.3. *Let (X, μ, S, T) be a measure preserving system with commuting transformations S and T and let $f_1, f_2, f_3 \in L^\infty(X)$ with $\|f_i\|_\infty \leq 1, i = 1, 2, 3$. Then there exists a universal constant C, such that for any $x \in X$ and any $N \in \mathbb{N}$, we have that*

$$\Big(\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f_1(S^i x) f_2(T^j x) f_3(S^i T^j x) \Big)^4 \leq C |S_N(f_3, x)|.$$

Proof. By Cauchy-Schwartz inequality and the boundedness of f_1, the expression inside the parenthesis on the left hand side is bounded by a multiple of the square of

$$\begin{aligned} &\frac{1}{N} \sum_{i=0}^{N-1} \Big(\frac{1}{N} \sum_{j=0}^{N-1} f_2(T^j x) f_3(S^i T^j x) \Big)^2 \\ &= \frac{1}{N^3} \sum_{j=0}^{N-1} \sum_{p=-j}^{N-1-j} \sum_{i=0}^{N-1} f_2(T^j x) f_2(T^{j+p} x) f_3(S^i T^j x) f_3(S^i T^{j+p} x). \end{aligned} \tag{3.5.1}$$

By Cauchy-Schwartz inequality and the boundedness of f_2, the square of (3.5.1) is bounded by a multiple of

$$
\begin{aligned}
&\frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{N}\sum_{p=-j}^{N-1-j}\Big(\frac{1}{N}\sum_{i=0}^{N-1} f_3(S^iT^jx)f_3(S^iT^{j+p}x)\Big)^2\\
&=\frac{1}{N}\sum_{j=0}^{N-1}\frac{1}{N}\sum_{p=-j}^{N-1-j}\frac{1}{N}\sum_{i=0}^{N-1}\frac{1}{N}\sum_{k=-i}^{N-1-i} f_3(S^iT^jx)f_3(S^iT^{j+p}x)f_3(S^{i+k}T^jx)f_3(S^{i+k}T^{j+p}x)\\
&=S_N(f_3,x).
\end{aligned}
$$

$\square$

Now we are able to prove the main result:

Theorem. *Let $f_1, f_2, f_3 \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx)$$

converges a.e.

Proof. We may assume without loss of generality that all the functions are bounded by 1 in L^∞ norm. Suppose first that $f_3 = h_3h_3'$, where h_3 is measurable with respect to $\mathcal{I}_T$ and h_3' is measurable with respect to $\mathcal{I}_S$. In this case, we have that $f_3(S^iT^jx) = h_3(S^ix)h_3'(T^jx)$ and thus

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx) = \frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)h_3(S^ix)f_2(T^jx)h_3'(T^jx),$$

and so the average converges by Birkhoff Theorem. Therefore the average converges a.e. for any f_3 in the subspace L spanned by those kind of functions. Any function f_3 measurable with respect to $\mathcal{W}$ can be approximated in the L^1 norm by functions in L. So, for f_3 measurable with respect to $\mathcal{W}$ we can take a sequence $(g_k)_{k\in\mathbb{N}}$ in L that converge to f_3 in L^1 norm. By Birkhoff Theorem, there exists a set A of full measure such that

$$\limsup_{N\to\infty}\left|\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)(f_3(S^iT^jx)-g_k(S^iT^jx))\right| \le \|f_3-g_k\|_1$$

for every $x \in A$ and $k \in \mathbb{N}$. Again by Birkhoff Theorem, let B be a set of full measure such that the average

$$\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)g_k(S^iT^jx)$$

converges for all $x \in B$ and all $k \in \mathbb{N}$. It is easy to check that for $x \in A\cap B$, the sequence $A_N = \frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^ix)f_2(T^jx)f_3(S^iT^jx)$ forms a Cauchy sequence and therefore it converges.

We then suppose that $\mathbb{E}(f_3|\mathcal{W}) = 0$. Let $\epsilon > 0$ and let $\widehat{f}_3$ be a continuous function on X such that $\|f_3 - \widehat{f}_3\|_1 < \epsilon$. We have that

$$\left|\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^i x) f_2(T^j x)(f_3(S^i T^j x) - \widehat{f}_3(S^i T^j x))\right| \leq \frac{1}{N^2}\sum_{i,j=0}^{N-1} \left|f_3(S^i T^j x) - \widehat{f}_3(S^i T^j x)\right|. \tag{3.5.2}$$

By Birkhoff Theorem, the right hand side converges a.e. to $\|f_3 - \widehat{f}_3\|_1$ as N goes to infinity. On the other hand, by Lemma 3.5.3, we have

$$\left(\frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f_1(S^i x) f_2(T^j x)\widehat{f}_3(S^i T^j x)\right)^4 \leq |S_N(\widehat{f}_3, x)|.$$

By Lemma 3.5.2, the right hand side converges to

$$\left|\left|\left|\widehat{f}_3\right|\right|\right|^4_{\mu,S,T} \leq \left(\left|\left|\left|\widehat{f}_3 - f_3\right|\right|\right|_{\mu,S,T} + |||f_3|||_{\mu,S,T}\right)^4 \leq \|f_3 - \widehat{f}_3\|_1^4 \leq \epsilon$$

as N goes to infinity. We deduce that a.e.

$$\limsup_{N\to\infty}\left|\frac{1}{N^2}\sum_{i,j=0}^{N-1} f_1(S^i x) f_2(T^j x) f_3(S^i T^j x)\right| \leq 2\epsilon.$$

Since ϵ is arbitrary, we have that this average goes to 0 a.e. $\square$

Enveloping semigroups of systems of order d

This chapter is based on the work Enveloping semigroups of systems of order d *[33], published in the journal Discrete and Continuous Dynamical Systems. We study the Ellis semigroup of a d-step nilsystem and the inverse limit of such systems. By using the machinery of cubes developed by Host, Kra and Maass, we prove that such a system has a d-step topologically nilpotent enveloping semigroup. In the case d* $= 2$*, we prove that these notions are equivalent, extending a previous result by Glasner.*

4.1. Introduction

In this chapter we consider a topological dynamical system (X, T), meaning that $T\colon X \to X$ is a homeomorphism of the compact metric space X to itself.

Several aspects of the dynamics of (X, T) can be deduced from algebraic properties of its enveloping semigroup $E(X, T)$. In particular, a topological dynamical system is a rotation on a compact abelian group if and only if its enveloping semigroup is an abelian group. Other interesting applications can be found in [7], [42] and [56].

In recent years the study of the dynamics of rotations on nilmanifolds and inverse limits of this kind of dynamics has drawn much interest. In particular, we point to the applications in ergodic theory [67], number theory and additive combinatorics (see for example [57]).

We recall that a minimal topological dynamical system is a system of order d if it is either a d-step nilsystem or an inverse limit of d-step nilsystems. It is revealed in [70] that they are a natural generalization of rotations on compact abelian groups and they play an important role in the structural analysis of topological dynamical systems. Particularly, systems of order 2 are the correct framework to study Conze-Lesigne algebras [67].

In this chapter we are interested in algebraic properties of the enveloping semigroup of a system of order d. A first question one can ask is if an enveloping semigroup is a d-step nilpotent group. Secondly, a deeper one : Does the property of having an enveloping semigroup that is a d-step nilpotent group characterize systems of order d?

Even when $E(X, T)$ is a compact group, multiplication needs not to be a continuous operation. For this reason we introduce the notion of topologically nilpotent group, which is a stronger condition than algebraically nilpotent, and it is more convenient to establish a characterization of systems of order d.

Using the machinery of cubes developed by Host, Kra and Maass [70], we prove:

Theorem 4.1.1. *Let (X,T) be a system of order d. Then, its enveloping semigroup is a d-step topologically nilpotent group and thus it is a d-step nilpotent group.*

Let A be an integer unipotent matrix (this means that $(A-I)^k=0$ for some $k\in\mathbb{N}$) and let $\alpha\in\mathbb{T}^d$. Let $X=\mathbb{T}^d$ and consider the transformation $Tx=Ax+\alpha$. We recall that the topological dynamical system (X,T) is an affine d-step nilsystem. In [99] it was proved that affine d-step nilsystems have nilpotent enveloping semigroups, and an explicit description of those semigroups was given. Theorem 4.1.1 generalizes this for more general systems, though does not give the explicit form of the enveloping semigroup.

The second question is more involved and has been tackled before by Glasner in [53]. There, in the case $d=2$, he proved that when (X,T) is an extension of its maximal equicontinuous factor by a torus K, the following are equivalent:

1. $E(X,T)$ is a 2-step nilpotent group;

2. There exists a 2-step nilpotent Polish group G of continuous transformations of X, acting transitively on X and there exists a closed cocompact subgroup $\Gamma\subseteq G$ such that: (i) $T\in G$, (ii) K is central in G, (iii) $[G,G]\subseteq K$ and the homogeneous space $(G/\Gamma,T)$ is isomorphic to (X,T).

The assumption that K is a torus can be removed, but one only obtain an extension of the system (X,T) where condition (2) is satisfied.

We proved that systems satisfying condition (2) are actually systems of order 2 (but not every system of order 2 needs to satisfy condition (2)). More generally we prove:

Theorem 4.1.2. *Let (X,T) be a minimal topological dynamical system. Then the following are equivalent:*

1. *(X,T) is a system of order 2;*

2. *$E(X,T)$ is a 2-step topologically nilpotent group;*

3. *$E(X,T)$ is a 2-step nilpotent group and (X,T) is a group extension of an equicontinuous system;*

4. *$E(X,T)$ is a 2-step nilpotent group and (X,T) is an isometric extension of an equicontinuous system.*

We do not know if the condition of having a 2-step nilpotent enveloping semigroup by itself is enough to guarantee that (X,T) is a system is of order 2.

The natural question that arises from this result is the converse of Theorem 4.1.1 in general:

Question 4.1.3. *Let (X,T) be a system with a d-step topologically nilpotent enveloping semigroup with $d>2$. Is (X,T) a system of order d?*

We recall a classical definition concerning factor and group extensions. Let (X,T) be a topological dynamical system and suppose that we have a compact group U of homeomorphism of X commuting with T (where U is endowed with the topology of uniform convergence). The quotient space $Y=X\backslash U=\{Ux : x\in X\}$ is a metric compact space and if we endow it with the action induced by T we get a topological dynamical system. By definition, the quotient map from X to Y defines a factor map. We say that (X,T) is an extension of (Y,T) by the group U.

Let (X,T) and (Y,T) be minimal topological dynamical systems and let $\pi\colon X\to Y$ be a factor map. We say that (X,T) is an isometric extension of (Y,T) if for every $y\in Y$ there exists a metric d_y in $\pi^{-1}(y)\times\pi^{-1}(y)$ with the following properties:

(I) (Isometry) If $x,x'\in\pi^{-1}(y)$ then $d_y(x,x')=d_{Ty}(Tx,Tx')$.

(II) (Compatibility of the metrics) If $(x_n,x_n')\in\pi^{-1}(y_n)$ and $(x_n,x_n')\to(x,x')\in\pi^{-1}(y)$ then $d_{y_n}(x_n,x_n')\to d_y(x,x')$.

Since we work with groups which are also topological spaces (but not necessarily topological groups), we can also consider a topological definition of nilpotent which is more suitable for our purposes. Let G be a topological space with a group structure. For $A,B\subseteq G$, we define $[A,B]_{\text{top}}$ as the *closed subgroup* spanned by $\{[a,b] : a\in A, b\in B\}$. The topological commutators subgroups G_j^{top}, $j\geq 1$, are defined by setting $G_1^{\text{top}}=G$ and $G_{j+1}^{\text{top}}=[G_j^{\text{top}},G]_{\text{top}}$. Let $d\geq 1$ be an integer. We say that G is *d-step topologically nilpotent* if G_{d+1}^{top} is the trivial subgroup.

Since $G_j\subseteq G_j^{\text{top}}$ for every $j\geq 1$, we have that if G is d-step topologically nilpotent, then G is also d-step nilpotent. In this sense Theorem 4.1.1 has stronger conclusions than the previous known particular cases.

For a distal system, we let $(E_j^{\text{top}}(X,T))_{j\in\mathbb{N}}$ denote the sequence of topological commutators of $E(X,T)$.

Let (X,T) and (Y,T) be topological dynamical systems and $\pi : X\to Y$ a factor map. We recall that there is a unique continuous semigroup homomorphism $\pi^*\colon E(X,T)\to E(Y,T)$ such that $\pi(ux)=\pi^*(u)\pi(x)$ for all $x\in X$ and $u\in E(X,T)$.

Note that if $\pi\colon X \to Y$ is a factor map between distal systems, we have that

$$\pi^*(E_j^{\mathrm{top}}(X,T)) = E_j^{\mathrm{top}}(Y,T) \text{ for every } j \geq 1.$$

4.2. Enveloping semigroups of systems of order d

In this section we prove Theorem 4.1.1. We introduce some notation.

Let $d \geq 1$ be an integer. For $0 \leq j \leq d$, let $J \subset [d]$, with cardinality $d-j$ and let $\eta \in \{0,1\}^J$. The subset

$$\alpha = \{\epsilon \in \{0,1\}^d : \epsilon_i = \eta_i \text{ for every } i \in J\} \subseteq \{0,1\}^d$$

is called a *face of dimension* j or equivalently, *a face of codimension* $d-j$.

Given $u\colon X \to X$, $d \in \mathbb{N}$ and $\alpha \subseteq \{0,1\}^d$ a face of a given dimension, we define $u_\alpha^{[d]}\colon X^{[d]} \to X^{[d]}$ as

$$u_\alpha^{[d]}\mathbf{x} = \begin{cases} (u_\alpha^{[d]}\mathbf{x})_\epsilon = ux_\epsilon, & \epsilon \in \alpha; \\ (u_\alpha^{[d]}\mathbf{x})_\epsilon = x_\epsilon, & \epsilon \notin \alpha. \end{cases}$$

Our theorem follows from the following lemma.

Lemma 4.2.1. *Let (X,T) be a distal topological dynamical system and let $E(X,T)$ be its enveloping semigroup. Then, for every $d,j \in \mathbb{N}$ with $j \leq d$ and $u \in E_j^{top}(X,T)$, we have that $\mathbf{Q}^{[d]}(X)$ is invariant under $u_\alpha^{[d]}$ for every face α of codimension j.*

Proof. Let $d \in \mathbb{N}$. Let $u \in E(X,T)$ and let (n_i) be a net with $T^{n_i} \to u$ pointwise. Let α be a face of codimension 1. Since $\mathbf{Q}^{[d]}(X)$ is invariant under $T_\alpha^{[d]}$, it is also invariant under $(T_\alpha^{n_i})^{[d]}$ for every i. Since $\mathbf{Q}^{[d]}(X)$ is closed and $(T_\alpha^{n_i})^{[d]} \to u_\alpha^{[d]}$ we get that $\mathbf{Q}^{[d]}(X)$ is invariant under $u_\alpha^{[d]}$. Let $1 < j \leq d$ and suppose that the statement is true for every $i < j$. Let α be a face of codimension j. We can see α as the intersection of a face β of codimension $j-1$ and a face γ of codimension 1. Let $u_{j-1} \in E_{j-1}^{\mathrm{top}}(X,T)$ and $v \in E(X,T)$ and remark that $[u_{j-1},v]_\alpha^{[d]} = [(u_{j-1})_\beta^{[d]}, v_\gamma^{[d]}]$. Since $(u_{j-1})_\beta^{[d]}$ and $v_\gamma^{[d]}$ leave invariant $\mathbf{Q}^{[d]}(X)$, so does $[u_{j-1},v]_\alpha^{[d]}$.

As $\mathbf{Q}^{[d]}(X)$ is closed, $E_\alpha = \{u \in E(X,T) : u_\alpha^{[d]} \text{ leaves invariant } \mathbf{Q}^{[d]}(X)\}$ is a closed subgroup of $E(X,T)$ and contains the elements of the form $[u_{j-1},v]$ for $u_{j-1} \in E_{j-1}^{\mathrm{top}}(X,T)$, $v \in E(X,T)$. We conclude that $E_j^{\mathrm{top}}(X,T) \subseteq E_\alpha$, completing the proof.

□

We use this to prove Theorem 4.1.1:

Proof of Theorem 4.1.1. Let (X,T) be a system of order d. Recall that $E(X,T)$ is a group since (X,T) is a distal system. Let $u \in E_{d+1}^{\mathrm{top}}(X,T)$ and $x \in X$. By Lemma 4.2.1 we have that $(x, \ldots, x, ux) \in \mathbf{Q}^{[d+1]}(X)$ and by Theorem 1.3.4 we have that $ux = x$. Since x and u are arbitrary, we conclude that $E_{d+1}^{\text{top}}(X,T)$ is the trivial subgroup.

□

4.3. Proof of Theorem 4.1.2

We start with some lemmas derived from the fact that $E(X,T)$ is topologically nilpotent.

Lemma 4.3.1. *Let (X,T) be a distal minimal topological dynamical system. Then the center of $E(X,T)$ is the group of elements of $E(X,T)$ which are continuous.*

Proof. Since T commutes with every element of $E(X,T)$ it is clear that every continuous element of $E(X,T)$ belongs to the center of $E(X,T)$. Conversely, let u be in the center of $E(X,T)$ and $x_0 \in X$. We prove that u is continuous at x_0. Suppose this is not true, and let $x_n \to x_0$ with $u(x_n) \to x' \neq u(x_0)$. By minimality, we can find $u_n \in E(X,T)$ such that $u_n(x_0) = x_n$. For a subnet we have that $u_n \to v$ and $v(x_0) = x_0$. Since u is central, we have $u(x_n) = u(u_n(x_0)) = u_n(u(x_0)) \to v(u(x_0)) = u(v(x_0)) = u(x_0)$, a contradiction. □

Recall the following classical theorem:

Theorem 4.3.2. *(See [7], Chapter 4) Let G be a group of homeomorphisms of a compact Hausdorff space X and suppose that G is compact in the pointwise topology. Then, the action of G on X is equicontinuous.*

A direct consequence is:

Corollary 4.3.3. *Let (X,T) be distal topological dynamical system. If $E(X,T)$ is d-step topologically nilpotent, then $E_d^{top}(X,T)$ is a compact group of automorphisms of (X,T) in the uniform topology.*

Proof. If $E(X,T)$ is d-step topologically nilpotent, then $E_d^{\text{top}}(X,T)$ is a compact group (in the pointwise topology) and by definition is included in the center of $E(X,T)$, meaning that every element is an automorphism of (X,T). By Theorem 4.3.2, $E_d^{\text{top}}(X,T)$ is a compact group of automorphisms in the uniform topology.

□

If a system has a 2-step topologically nilpotent enveloping semigroup we can describe the extension of its maximal equicontinuous factor.

For this, first we give a short proof of [106] in our context.

Theorem 4.3.4. *Let $\pi\colon X \to Y$ be a distal finite-to-one factor map between the minimal systems (X,T) and (Y,T). Then (Y,T) is equicontinuous if and only if (X,T) is equicontinuous.*

Proof. We prove the non trivial direction by studying the regionally proximal relation on X. We denote by d_X and d_Y the metrics on X and Y. We can assume that T is an isometry on Y. Since π is open and finite-to-one, there exists $\epsilon_0 > 0$ such that for every $y \in Y$ every ball of radius $2\epsilon_0$ in X intersects $\pi^{-1}(y)$ in at most one point. Let $\epsilon_1 < \epsilon_0$ such that $T(B(x, \epsilon_1)) \subseteq B(Tx, \epsilon_0)$. Since π is open, there exists $\delta > 0$ with the property that if $y, y' \in Y$ are such that $d_Y(y, y') < \delta$ then there exists $x, x' \in X$ with $d_X(x, x') < \epsilon_1$ and $\pi(x) = y$, $\pi(x') = y'$. Let $0 < \epsilon < \epsilon_1$ such that $\pi(B(x, \epsilon)) \subseteq B(\pi(x), \delta)$. Let (x, x') be a regionally proximal pair, and let $x'' \in X$ and $n_0 \in \mathbb{N}$ satisfying $d_X(x, x'') < \epsilon$ and $d_X(T^{n_0}x', T^{n_0}x'') < \epsilon$. We have that $d_Y(T^n\pi(x), T^n\pi(x'')) = d_Y(\pi(T^n x), \pi(T^n x'')) < \delta$ for every $n \in \mathbb{N}$ and by openness, we can find $x_n \in X$ such that $\pi(x_n) = \pi(T^n x)$ and $d_X(x_n, T^n x'') < \epsilon_1$.

We claim that $x_n = T^n x$. We proceed by induction. For $n = 0$ we have $d_X(x_0, x) < 2\epsilon_0$, $\pi(x) = \pi(x_0)$ and thus $x = x_0$. Suppose now that $x_n = T^n x$. We have that $d_X(T^n x, T^n x'') < \epsilon_1$ and then $d_X(T^{n+1}x, T^{n+1}x'') < \epsilon_0$. We conclude that $d_X(x_{n+1}, T^{n+1}x) < 2\epsilon_0$, and since they have the same projection, they are equal. This proves the claim.

Particularly, for $n = n_0$, we have $d_X(T^{n_0}x, T^{n_0}x') < 2\epsilon_0$ and since they are regionally proximal, they have the same projection and thus $x = x'$. We conclude that the regionally proximal relation is trivial and (X, T) is equicontinuous. $\square$

Lemma 4.3.5. *Let (X, T) be a topological dynamical system with a 2-step topologically nilpotent enveloping semigroup. Then, it is an extension of $Z_1(X)$ by the compact abelian group $E_2^{top}(X, T)$. Moreover, $E_2^{top}(X, T)$ is connected.*

Proof. By Corollary 4.3.3 we have that $E_2^{\mathrm{top}}(X, T)$ is a compact group of automorphisms of (X, T) and by Lemma 4.2.1 it acts trivially in every equicontinuous factor, meaning that there exists a factor map from $Z = X \backslash E_2^{\mathrm{top}}(X, T)$ to $Z_1(X)$. Denote by π the factor map from X to Z and note that if $u \in E_2^{\mathrm{top}}(X, T)$, then $\pi(x) = \pi(ux) = \pi^*(u)\pi(x)$ for every $x \in X$ and therefore $\pi^*(u)$ is trivial. Since $e = \pi^*(E_2^{\mathrm{top}}(X, T)) = E_2^{\mathrm{top}}(Z, T)$, we conclude that Z has an abelian enveloping semigroup and thus it is an equicontinuous factor. By maximality $Z_1(X) = X \backslash E_2^{\mathrm{top}}(X, T)$.

If $E_2^{\mathrm{top}}(X, T)$ is not connected, there exists an open (hence closed) subgroup $U \subseteq E_2^{\mathrm{top}}(X, T)$ such that $E_2^{\mathrm{top}}(X, T)/U$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ for some $n > 1$. Note that $X \backslash U$ is a finite-to-one extension of $Z_1(X)$ and therefore by Theorem 4.3.4 it is an equicontinuous system. By maximality we get that $X \backslash U = Z_1(X)$, a contradiction. $\square$

This lemma establish the implication (2) $\Rightarrow$ (3) of Theorem 4.1.2. A direct corollary is:

Corollary 4.3.6. *Let (X, T) be a system of order 2. Then, it is an extension of its maximal equicontinuous factor by the compact connected abelian group $E_2^{top}(X, T)$.*

We now prove the main implication in Theorem 4.1.2, namely implication (2) $\Rightarrow$ (1).

Proof of implication (2) $\Rightarrow$ (1) . We divide the proof into four parts. The first two parts follow, with some simplifications, the scheme proposed in [53], but the second two parts are new.

Step 1: Building a suitable extension of (X, T)**.**

Let (X, T) be a topological dynamical system with a 2-step topologically nilpotent enveloping semigroup. By Lemma 4.3.5, (X, T) is an extension of $(Z_1(X), T)$ by the compact abelian group $E_2^{\text{top}}(X, T)$. We denote this factor map by π. In order to avoid confusions, we denote the element of $Z_1(X)$ defining the dynamics by τ (instead of T). Let $\widehat{Z_1}$ be the dual group of $Z_1(X)$. Since $\{\tau^n : n \in \mathbb{Z}\}$ is dense in $Z_1(X)$ every $\chi \in \widehat{Z_1}$ is completely determined by its value at τ and thus we can identify $\widehat{Z_1}$ with a discrete subgroup of $\mathbb{S}^1$. Consider $\widehat{Z^*} = \{\lambda \in \mathbb{S}^1 : \exists n \in \mathbb{N}, \lambda^n \in \widehat{Z_1}\}$, the divisible group generated by $\widehat{Z_1}$. It is a discrete subgroup of $\mathbb{S}^1$, and we can consider its compact dual group $Z^* = \widehat{\widehat{Z^*}}$. Since $\widehat{Z^*}$ is a subgroup of the circle, Z^* is a monothetic group with generator the identity character τ^*: $\widehat{Z^*} \to \mathbb{S}$. Since $\widehat{Z_1} \subseteq \widehat{Z^*}$, there exists a homomorphism $\phi \colon Z^* \to Z_1(X)$. Since τ^* is projected to τ, ϕ also defines a factor map from (Z^*, τ^*) to $(Z_1(X), \tau)$. Consider $(X^*, T \times \tau^*)$ a minimal subsystem of

$$(\{(x, z^*) \in X \times Z^* : \pi(x) = \phi(z^*)\}, T \times \tau^*).$$

It is an extension of (X, T) and (Z^*, τ^*) and we can see $E(X^*, T \times \tau^*)$ as a subset of $E(X, T) \times E(Z^*, \tau^*) = E(X, T) \times Z^*$. It follows that $E_2^{\text{top}}(X^*, T \times \tau^*) = E_2^{\text{top}}(X, T) \times \{e\}$ and $E(X^*, T \times \tau^*)$ is 2-step topologically nilpotent. By Lemma 4.3.5 we have that $Z_1(X^*) = X^* \backslash E_2^{\text{top}}(X^*, T \times \tau^*) = Z^*$.

Step 2: Finding a transitive group in X^***.**

For simplicity we denote the transformation on X^* by T^*. Let $(x_0, x_1) \in X^* \times X^*$. We construct a homeomorphism h of X^* such that $h(x_0) = x_1$. For this, define Y as the closed orbit of (x_0, x_1) under $T^* \times T^*$. Since (X^*, T^*) is distal, $(Y, T^* \times T^*)$ is a minimal distal system and $E(Y, T^* \times T^*) = E(X^*, T^*)^{\triangle} := \{(u, u) : u \in E(X^*, T^*)\}$ (and we can identify $E(X^*, T^*)$ and $E(Y, T^* \times T^*)$). It follows that $E(Y, T^* \times T^*)$ is 2-step topologically nilpotent and by Lemma 4.3.5 $Z_1(Y) = Y \backslash E_2^{\text{top}}(Y, T^* \times T^*) = Y \backslash E_2^{\text{top}}(X^*, T^*)^{\triangle}$.

We obtain the following commutative diagram:

$$\begin{tikzcd}
(Y, T^* \times T^*) \arrow[r, "\pi_1"'] \arrow[d, "p_Y"'] & (X^*, T^*) \arrow[d, "p_{X^*}"'] \\
(Z_1(Y), \tau_Y{}^*) \arrow[r, "\rho"'] & (Z^*, \tau^*)
\end{tikzcd}$$

Since Z^* has a divisible dual group, we can identify $Z_1(Y)$ as a product group $Z^* \times G_0$ and we can write $p_Y(x, x') = (p_{X^*}(x), \Theta(x, x'))$ with $\Theta(x, x') \in G_0$. Since $Z_1(Y)$ is a product group, there exists $g_0 \in G_0$ such that $\tau_Y^* = \tau^* \times g_0$. We remark that if (x, x') and $(x, x'') \in Y$

then $(x,x')=u(x,x'')$ for some $u\in E(X^*,T^*)$. Writing $x'=vx$ for $v\in E(X^*,T^*)$ we deduce that $x''=[u,v]x'$. From this, we deduce that $G_0=\{\mathrm{id}\times u : u\in E_2^{\mathrm{top}}(X^*,T^*), (\mathrm{id}\times u)Y=Y\}$.

For $x\in X^*$, define $h(x)$ as the unique element in X^* such that

$$(x,h(x))\in Y \text{ and } p_Y(x,h(x))=(p_{X^*}(x),e). \tag{4.3.1}$$

By multiplying the second coordinate by a constant, we can suppose that $p_Y(x_0,x_1)=(p_{X^*}(x),e)$ and thus $h(x_0)=x_1$.

Claim 1: h is a homeomorphism of X^*:

- If $x_n\to x\in X^*$, then $p_Y(x_n,h(x_n))=(p_{X^*}(x_n),e)\to(p_{X^*}(x),e)=p_Y(x,h(x))$ and h is continuous (if $p_Y(x,x')=p_Y(x,x'')$ then $x'=x''$).
- If $h(x)=h(x')$ then $(x,h(x))$ and $(x',h(x))$ belong to Y and then $x'=ux$ for $u\in E_2^{\mathrm{top}}(X^*,T^*)$. We have that $p_Y(x',h(x))=(p_{X^*}(x),e)=p_Y(x,h(x))$ and thus $x=x'$.
- If $x'\in X^*$, we can find $x\in X$ such that $(x,x')\in Y$ and $p_Y(x,x')=(p_{X^*}(x),\Theta(x,x'))$. It follows that

  $$p_Y(x,x')=(\mathrm{id}\times\Theta(x,x'))(p_{X^*}(x),e) \text{ and } p_Y(\Theta^{-1}(x,x')x,x')=(p_{X^*}(\Theta^{-1}(x,x')x),e).$$

  By definition $h(\Theta^{-1}(x,x')x)=x'$ and therefore h is onto. This proves the claim.

Claim 2: h commutes with $E_2^{top}(X^*,T^*)$:

For $u\in E_2^{\mathrm{top}}(X^*,T^*)$ we have that $p_Y(ux,uh(x))=p_Y(x,h(x))=(p_{X^*}(x),e)=(p_{X^*}(ux),e)=p_Y(ux,h(ux))$ and we deduce that h commutes with $E_2^{\mathrm{top}}(X^*,T^*)$.

Claim 3: $[h,T^*]=g_0\in E_2^{top}(X^*,T^*)$:

By a simple computation we have that

$$p_Y(T^*x,T^*h(x))=\tau_Y^*(p_Y(x,h(x)))=(p_{X^*}(Tx),g_0)=(p_{X^*}(T^*x),g_0h(T^*x))$$

and $T^*h=g_0hT^*$. This proves the claim.

Define G as the group of homeomorphisms h of X^* such that

$$[h,T^*]\in E_2^{\mathrm{top}}(X^*,T^*) \text{ and } h \text{ commutes with } E_2^{\mathrm{top}}(X^*,T^*). \tag{4.3.2}$$

Then, for every pair of points in $X^*\times X^*$ we can consider a homeomorphism h as in (4.3.1) and this transformation belongs to G. Thus G is a group acting transitively on X^*.

Let Γ be the stabilizer of a point $x_0\in X^*$. We can identity (as sets) X^* and G/Γ.

Step 3: The application $g \to gx_0$ is open

Claim 4: There exists a group homomorphism $p : G \to Z^*$ such that $p(g)p_{X^*}(x) = p_{X^*}(gx)$.

Since $E_2^{\mathrm{top}}(X^*, T^*)$ is central in G, we have $gp_{X^*}(x) = gE_2^{\mathrm{top}}(X^*, T^*)x = E_2^{\mathrm{top}}(X^*, T^*)gx$ and so the action of $g \in G$ can descend to an action $p(g)$ in $X^* \backslash E_2^{\mathrm{top}}(X^*, T^*) = Z^*$. By definition this action satisfies $p(g)p_{X^*}(x) = p_{X^*}(gx)$ and $p(T^*) = \tau^*$. From this, we can see that $p(g)$ commutes with τ^* and thus $p(g)$ belongs to Z^*. Particularly, if $h_1, h_2 \in G$, we have that $p([h_1, h_2])$ is trivial and then $[h_1, h_1]x_0 = ux_0$ for some $u \in E_2^{\mathrm{top}}(X^*, T^*)$. By (4.3.2), $[h_1, h_2]$ commutes with T^* and thus $[h_1, h_2]$ coincides with u in every point. Therefore $[G, G] \subseteq E_2^{\mathrm{top}}(X^*)$ and thus G is a 2-step nilpotent Polish group. Since G is transitive in X^* we can check that p is an onto continuous group homomorphism. This proves the claim.

Since G and Z^* are Polish groups and p is onto, we have that p is an open map and the topology of Z^* coincides with the quotient topology of $G/\mathrm{Ker}(p) = G/E_2^{\mathrm{top}}(X^*, T^*)\Gamma$ (see [11], Chapter 1, Theorem 1.2.6).

Now we prove that the map $g \to gx_0$ is open. Consider a sequence $g_n \in G$ such that $g_n x_0$ is convergent in X^*. Projecting to Z^* we have that $p(g_n)p_{X^*}(x_0)$ is convergent and taking a subsequence we can assume that $p(g_n)$ is convergent in Z^*. Since p is open, we can find a convergent sequence $h_n \in G$ such that $p(g_n) = p(h_n)$. This implies that $p_{X^*}(g_n x_0) = p_{X^*}(h_n x_0)$ and therefore there exists $u_n \in E_2^{\mathrm{top}}(X^*, T^*)$ such that $g_n x_0 = u_n h_n x_0$. By the compactness of $E_2^{\mathrm{top}}(X^*, T^*)$ we can assume that u_n is convergent and $u_n h_n$ is convergent too. This proves that the map is open.

Step 4: Cubes of order 3 in X^* are completed in a unique way.

Let consider a sequence $\vec{n}_i = (n_i, m_i, p_i) \in \mathbb{Z}^3$ such that $T^{*\vec{n}_i \cdot \epsilon}x_0 \to x_0$ for every $\epsilon \neq \vec{1}$. We prove that $T^{*\vec{n}_i \cdot \vec{1}}x_0 \to x_0$. We see every transformation $T^{*\vec{n}_i \cdot \epsilon}$ as an element of G. Since the application $g \to gx_0$ is open, taking a subsequence, we can find h_i, h_i', h_i'' in G, converging to $h, h', h'' \in G$ such that $T^{*n_i}x_0 = h_i x_0$, $T^{*m_i}x_0 = h_i' x_0$ and $T^{*p_i}x_0 = h_i'' x_0$.

We have that

$$
\begin{aligned}
T^{*n_i+m_i}x_0 &= T^{*n_i}h_i'x_0 = [T^{*n_i}, h_i']h_i'h_ix_0 \\
T^{*n_i+p_i}x_0 &= T^{*n_i}h_i''x_0 = [T^{*n_i}, h_i'']h_i''h_ix_0 \\
T^{*m_i+p_i}x_0 &= T^{*m_i}h_i''x_0 = [T^{*m_i}, h_i'']h_i''h_i'x_0 \\
T^{*n_i+m_i+p_i}x_0 &= [T^{*m_i}, h_i''][T^{*n_i}, h_i''][T^{*n_i}, h_i']h_i''h_i'h_ix_0.
\end{aligned}
$$

Since $[G, G]$ is included in $E_2^{\mathrm{top}}(X^*, T^*)$, by taking a subsequence we can assume that $[T^{*n_i}, h_i'] \to g_1$, $[T^{*n_i}, h_i''] \to g_2$ and $[T^{*m_i}, h_i''] \to g_3$ and these limits belong to $E_2^{\mathrm{top}}(X^*, T^*)$. Taking limits we conclude that $g_1x_0 = g_2x_0 = g_3x_0 = x_0$ and since these transformations commute with T^*, we have that they are trivial.

We conclude that $\lim_{i\to\infty} T^{*n_i+m_i+p_i}x_0 = x_0$ and thus $(x_0,x_0,x_0,x_0,x_0,x_0,x_0) \in (X^*)^7$ can be completed in a unique way to an element of $\mathbf{Q}^{[3]}(X^*)$. If π_2 is the factor map from X^* to $Z_2(X^*)$, we have that $\#\pi_2^{-1}(\pi_2(x_0)) = 1$ and since (X^*,T^*) is distal, the same property holds for every element in X^*. We conclude that $X^* = Z_2(X^*)$ and thus (X^*,T^*) is a system of order 2.

Since being a system of order 2 is a property preserved under factor maps, (X,T) is a system of order 2.

□

We have established (1) ⇔ (2) and (2) ⇒ (3). Since implication (3) ⇒ (4) is obvious, we only have to prove (4) ⇒ (2).

For this, we first prove the following lemma:

Lemma 4.3.7. *Let $\pi\colon X\to Y$ be an isometric extension between the minimal distal systems (X,T_X) and (Y,T_Y). Then, there exists a minimal distal system (W,T_W) which is a group extension of X and a group extension of Y. If $E(X,T_X)$ is d-step nilpotent then $E(W,T_W)$ is also d-step nilpotent.*

Proof. Fix $y_0\in Y$ and let $F_0=\pi^{-1}(y_0)$. Define

$$\widetilde{Z} = \{(y,h) : y\in Y,\ h\in \mathrm{Isom}(F_0,\pi^{-1}(y))\}$$

It is compact metrizable space and we can define $T_{\widetilde{Z}}:\widetilde{Z}\to\widetilde{Z}$ as $T_{\widetilde{Z}}(y,g)=(T_Y(y),T_X\circ h)$. We remark that $(\widetilde{Z},T_{\widetilde{Z}})$ is a distal system and we can see $E(\widetilde{Z},T_{\widetilde{Z}})$ as a subset of $E(Y,T_Y)\times E(X,T_X)$. It follows that $E(\widetilde{Z},T_{\widetilde{Z}})$ is d-step nilpotent.

Let H denote the compact group of isometries of F_0 which are restrictions of elements of $E(X,T_X)$. We define the action of H on $\widetilde{Z}$ as $g(y,h)=(y,h\circ g^{-1})$ and we define the maps $\pi_Y(y,h)=y$ and $\pi_X(y,h)=h(x_0)$ from $\widetilde{Z}$ to X and Y. Define W as the orbit of (y_0,id) under $T_{\widetilde{Z}}$ and let T_W denote the restriction of $T_{\widetilde{Z}}$ to W. Since $(\widetilde{Z},T_{\widetilde{Z}})$ is a distal system, (W,T_W) is a minimal system and therefore the restrictions of π_Y and π_X define factor maps from (W,T_W) to (Y,T_Y) and (X,T_X). Since (X,T_X) is a distal system we have that $(E(X,T_X),T_X)$ is a minimal system and we have that $\{y_0\}\times H\subseteq W$. We conclude that (W,T_W) is an extension of (Y,T_Y) by the group H and thus it is an extension of (X,T) by the group $H_0=\{h\in H : h(x_0)=x_0\}$. Since (W,T_W) is a subsystem of $(\widetilde{Z},T_{\widetilde{Z}})$, we also have that $E(W,T_W)$ is d-step nilpotent. The lemma is proved. □

Now we prove the implication (4) ⇒ (2).

Proof of implication (4) ⇒ (2). Let (X,T) be system with a 2-step nilpotent enveloping semigroup and let $\pi: X\to Y$ be an isometric extension of the equicontinuous system (Y,T). By Lemma 4.3.7 we can find (W,T) which is an extension of (Y,T) by a group H and such that $E(W,T)$ is a 2-step nilpotent group. By Lemma 4.2.1, for $u \in E_2(W,T)$ and $w \in W$ we have that uw and w have the same projection on Y and therefore there exists $h_w \in H$ such that $uw = h_w w$. Since u and h_w are automorphisms of the minimal system (W,T), they are equal. Thus we have that $E_2(W,T)$ is a subgroup of H and therefore $E_2^{\text{top}}(W,T)$ is just the closure (pointwise or uniform) of $E_2(W,T)$. We conclude that $E_2^{\text{top}}(W,T)$ is included in H and therefore is central in $E(W,T)$. Since (X,T) is a factor of (W,T), $E_2^{\text{top}}(X,T)$ is also central in $E(X,T)$. This finishes the proof. □

4.4. Some further comments

We finish with some remarks about the structure of systems having topologically nilpotent enveloping semigroups.

Let (X,T) be a topological dynamical system and let $d > 2$. Suppose that $E(X,T)$ is a d-step topologically nilpotent group. By Corollary 4.3.3 $E_d^{\text{top}}(X,T)$ is a compact group of automorphisms of (X,T) and thus we can build the quotient $X_{d-1} = X \backslash E_d^{\text{top}}(X,T)$.

Lemma 4.4.1. *X_{d-1} has a $(d-1)$-step topologically nilpotent enveloping semigroup. Moreover it is the maximal factor of X with this property and consequently (X_{d-1},T) is an extension of $(Z_{d-1}(X),T)$*

Proof. Denote by $\pi\colon X \to X_{d-1}$ the quotient map. If $u \in E_d^{\text{top}}(X,T)$, by definition we have that $\pi(x) = \pi(ux) = \pi^*(u)\pi(x)$ and thus $\pi^*(u)$ is trivial. Since $\pi^*(E_d^{\text{top}}(X,T)) = E_d^{\text{top}}(X_{d-1},T)$ we have that $E_d^{\text{top}}(X_{d-1},T)$ is trivial.

Let (Z,T) be a topological dynamical system with a $(d-1)$-step topologically nilpotent enveloping semigroup and let $\phi : X \to Z$ be a factor map. Since $\phi^*(E_d^{\text{top}}(X,T)) = e$, for $u \in E_d^{\text{top}}(X,T)$ we have that $\phi(ux) = \phi^*(u)\phi(x) = \phi(x)$ and therefore ϕ can be factorized through X_{d-1}.

As $Z_{d-1}(X)$ has a $(d-1)$-step enveloping semigroup, we have that (X_{d-1},T) is an extension of $(Z_{d-1}(X),T)$. □

Iteratively applying Lemma 4.4.1, we construct a sequence of factors X_j, for $j \leq d-1$ with the property that X_j is an extension of $Z_j(X)$ and is an extension of X_{j-1} by the compact abelian group $E_j^{\text{top}}(X_j,T)$.

By Theorem 4.1.2, the factors X_2 and $Z_2(X)$ coincide and we obtain the following commutative diagram:

$$\begin{tikzcd}
(X,T) \arrow[r] \arrow[d] & (X_{d-1},T) \arrow[r] \arrow[d] & \cdots \arrow[r] & (X_3,T) \arrow[d] \arrow[dr] & & \\
(Z_d(X),T) \arrow[r] & (Z_{d-1}(X),T) \arrow[r] & \cdots \arrow[r] & (Z_3(X),T) \arrow[r] & (Z_2(X),T) \arrow[r] & (Z_1(X,T)
\end{tikzcd}$$

We conjecture that the factor X_j and $Z_j(X)$ also coincide for $j > 2$.

Part II

Automorphism groups of symbolic systems

Automorphism groups of low complexity symbolic systems

This chapter is mostly based on the article On automorphism groups of low complexity minimal subshifts *[36], joint work with Fabien Durand, Alejandro Maass and Samuel Petite, accepted for publication in the journal Ergodic Theory and Dynamical Systems. We study the automorphism group* $\mathrm{Aut}(X,\sigma)$ *of a minimal subshift* (X,σ) *of low* word *complexity. In particular, we prove that* $Aut(X,\sigma)$ *is virtually* $\mathbb{Z}$ *for aperiodic minimal subshifts with affine complexity on a subsequence, more precisely, the quotient of this group by the one generated by the shift map is a finite group. In addition, we provide examples to show that any finite group can be obtained in this way. The class considered includes minimal substitutions, linearly recurrent subshifts and even some minimal subshifts with polynomial complexity. In the case of polynomial complexity, first we prove that for minimal subshifts with polynomial recurrence any finitely generated subgroup of* $Aut(X,\sigma)$ *is virtually nilpotent. Then, we describe a variety of examples where we illustrate how to apply the methods we propose in this work to study automorphism groups. Some of the examples have polynomial complexity and are obtained by coding some nilrotations. Another ones are subshifts with subaffine complexity on a subsequence, but with a superpolynomial complexity. In all these examples we get a virtually* $\mathbb{Z}$ *group of automorphisms. The main technique in this work relies on the study of classical relations among points used in topological dynamics, in particular asymptotic pairs.*

In the last section we present a section of the article presented in Chapter 2, where we use the $\mathbf{Q}_{S,T}$ *cubes to study the group of automorphisms of the minimal part of the Robinson tiling.*

5.1. Introduction

We recall that an automorphism of a topological dynamical system (X,T), where $T\colon X\to X$ is a homeomorphism of the compact metric space X, is a homeomorphism from X to itself which commutes with T. We call $\mathrm{Aut}(X,T)$ the group of automorphisms of (X,T). There is a similar definition for measurable automorphisms when we consider an invariant measure μ for the system (X,T) or a general measure preserving system. The group of measurable automorphisms is historically denoted by $C(T)$ that stands for the *centralizer group* of (X, μ, T).

The study of automorphism groups is a classical and widely considered subject in ergodic theory. The group $C(T)$ has been intensively studied for mixing systems of finite rank. We refer to [43] for an interesting survey. Let us mention some key theorems. Ornstein proved in [94] that a mixing rank one dynamical system (X, μ, T) has a trivial (measurable) automorphism group: it consists of powers of T. Later, del Junco [31] showed that the famous weakly mixing (but not mixing) rank one Chacon subshift also shares this property. Finally, for mixing systems of finite rank King and Thouvenot proved in [79] that $C(T)$ is virtually $\mathbb{Z}$. That is, its quotient by the subgroup $\langle T \rangle$ generated by T is a finite group.

In the non weakly mixing case, Host and Parreau [74] proved, for a family of constant length substitution subshifts, that $C(\sigma)$ is also virtually $\mathbb{Z}$ and equals to $\mathrm{Aut}(X, \sigma)$, where σ is the shift map. Concomitantly, Lemańczyk and Mentzen [84] realized any finite group as the quotient of $C(\sigma)$ by $\langle \sigma \rangle$ with constant length substitution subshifts.

Priorly to these results, in the topological setting Hedlund in [61] described the automorphism groups for a family of binary substitutions including the Thue-Morse subshift. Precisely, he proved that $\mathrm{Aut}(X, \sigma)$ is generated by the shift and a flip map (a map which interchanges the letters). In the positive entropy situation, Boyle, Lind and Rudolph [17] obtained that the group of automorphisms of mixing subshifts of finite type contains various subgroups, so this group is large in relation to previous examples.

In this work we focus on the group of automorphisms $\mathrm{Aut}(X, \sigma)$ of minimal subshifts of subaffine complexity and, more generally, on zero entropy subshifts without assuming any mixing condition. All evidence described before in the measurable and topological context shows that we must expect that *low complexity* systems have a simple automorphism group. This is one of the main questions we want to address in this paper. Here, by complexity we mean the increasing function $p_X \colon \mathbb{N} \to \mathbb{N}$ that for $n \in \mathbb{N}$ counts the number of *words* of length n appearing in points of the subshift.

Recently Salo and Törmä in [107] proved that for subshifts generated by constant length or primitive Pisot substitutions the group of automorphisms is virtually $\mathbb{Z}$. This generalizes a result of Coven for constant length substitutions on two letters [28]. In [107] is asked whether the same result holds for any primitive substitution or more generally for linearly recurrent subshifts. In this paper we answer positively this question, proving the following more general theorem whose proof is given in Section 5.3.

Theorem 5.1.1. *Let (X, σ) be an aperiodic minimal subshift. If*

$$\liminf_{n \in \mathbb{N}} \frac{p_X(n)}{n} < \infty$$

then $\mathrm{Aut}(X, \sigma)$ is virtually $\mathbb{Z}$.

The class of systems satisfying the condition of Theorem 5.1.1 includes primitive substitutions, linearly recurrent subshifts [39] and more generally subaffine complexity subshifts or even some families with polynomial complexity (see Section 5.4). In addition, we illustrate this by realizing any finite group as the quotient group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$, where (X,σ) is a substitutive subshift. We observe that this result can be obtained combining the main results of [74] and [84] but we prefer to present here a different and straightforward proof.

Extending Theorem 5.1.1 for subshifts of polynomial complexity seems to be more intriguing. Nevertheless, several classes of examples still show that $\mathrm{Aut}(X,\sigma)$ has small growth rate. Indeed, in Sections 5.3 and 5.4 we give classes of minimal subshifts with polynomial complexity where $\mathrm{Aut}(X,\sigma)$ is virtually nilpotent (Theorem 5.3.8) and in most cases the finite group is abelian. Also, very recently, Cyr and Kra [29] proved the fact that for transitive subshifts with subquadratic complexity $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is periodic, meaning that any element in this group has finite order.

Their proof translates the question into a coloring problem of $\mathbb{Z}^2$ and uses a deep combinatorial result of Quas and Zamboni [101]. Our results arise from obstructions related to some classical and some less classical equivalence relations associated to fibers of special topological factors. This idea was already used by Olli in [93] to prove that $\mathrm{Aut}(X,\sigma)$ of Sturmian subshifts consists only in powers of the shift by studying the irrational rotation defining the subshift. Here, we consider the maximal nilfactor ([70],[110]) of a minimal subshift to find a class of examples with arbitrarily big polynomial complexity whose group of automorphisms is virtually $\mathbb{Z}$.

5.2. Preliminaries, notation and background

5.2.1. Topological dynamical systems

Let (X,T) be a topological dynamical system. We say that $x,y\in X$ are *proximal* if there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$\lim_{i\to+\infty} d(T^{n_i}x,T^{n_i}y)=0.$$

A stronger condition than proximality is asymptoticity. Two points $x,y\in X$ are said to be *asymptotic* if

$$\lim_{n\to+\infty} d(T^{n}x,T^{n}y)=0.$$

Nontrivial asymptotic pairs may not exist in an arbitrary topological dynamical system but it is well known that a nonempty aperiodic subshift always admits one [7].

Let $\pi\colon (Y,T)\to(X,T)$ be a factor map. We say that (Y,T) is a *proximal extension* of (X,T) if for $y,y'\in Y$ the condition $\pi(y)=\pi(y')$ implies that y,y' are proximal. For minimal systems, (Y,T) is an *almost one-to-one* extension of (X,T) via the factor map $\pi:(Y,T)\to(X,T)$ if there exists $x\in X$ with a unique preimage for the map π. The relation between these two notions is given by the following folklore lemma.

Lemma 5.2.1. *Let (Y,T) be an almost one-to-one extension of (X,T) via the factor map $\pi:(Y,T)\to(X,T)$. Then, (Y,T) is a proximal extension of (X,T).*

Proof. Let $x_0\in X$ be a point with a unique preimage by π and consider points $y,y'\in Y$ such that $\pi(y)=\pi(y')$. By minimality of (X,T), there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $T^{n_i}(\pi(y))$ $(=T^{n_i}(\pi(y')))$ converges to x_0 as i goes to infinity. By continuity of π and since T commutes with π, the sequences $(T^{n_i}y)_{i\in\mathbb{N}}$ and $(T^{n_i}y')_{i\in\mathbb{N}}$ converge to the same unique point in the preimage of x_0 by π. This shows that points y and y' are proximal. □

We recall that an automorphism of the topological dynamical system (X,T) is a homeomorphism ϕ of the space X such that $\phi\circ T=T\circ\phi$. We let $\mathrm{Aut}(X,T)$ denote the group of automorphisms of (X,T). We have,

Lemma 5.2.2. *Let (X,T) be a minimal topological dynamical system. Then, the action of $\mathrm{Aut}(X,T)$ on X is free, meaning that every nontrivial element in $\mathrm{Aut}(X,T)$ has no fixed points.*

Proof. Take $\phi\in\mathrm{Aut}(X,T)$ and $x\in X$ such that $\phi(x)=x$. Since ϕ commutes with T and is continuous, by minimality we deduce that $\phi(y)=y$ for all $y\in X$. Thus ϕ is the identity map. □

Lemma 5.2.3. *Let (X,T) be a minimal topological dynamical system. Let $x\in X$ and $\phi\in\mathrm{Aut}(X,T)$. Then x and $\phi(x)$ are proximal if and only if ϕ is the identity map.*

Proof. We prove the nontrivial direction. Let $x\in X$ and $\phi\in\mathrm{Aut}(X,T)$ such that x and $\phi(x)$ are proximal points. By definition, there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $\lim_{i\to+\infty}d(T^{n_i}x,T^{n_i}\phi(x))=0$. We can assume that $T^{n_i}x$ converges to some $y\in X$. Therefore $d(y,\phi(y))=0$. By Lemma 5.2.2 ϕ is the identity map. □

Let $\pi\colon(Y,T)\to(X,T)$ be a factor map between the minimal systems (Y,T) and (X,T), and let ϕ be an automorphism of (Y,T). We say that π is *compatible* with ϕ if $\pi(y)=\pi(y')$ implies $\pi(\phi(y))=\pi(\phi(y'))$ for all $y,y'\in Y$. We say that π is *compatible* with $\mathrm{Aut}(Y,T)$ if π is compatible with all $\phi\in\mathrm{Aut}(Y,T)$.

If the factor map $\pi\colon(Y,T)\to(X,T)$ is compatible with $\mathrm{Aut}(Y,T)$ we can define the projection $\widehat{\pi}(\phi)\in\mathrm{Aut}(X,T)$ by the equation $\widehat{\pi}(\phi)(\pi(y))=\pi(\phi(y))$ for all $y\in Y$. We have that $\widehat{\pi}\colon\mathrm{Aut}(Y,T)\to\mathrm{Aut}(X,T)$ is a group morphism.

Notice that $\widehat{\pi}$ might not be onto or injective. Indeed, for an irrational rotation of the circle, the group of automorphisms is the whole circle but for its Sturmian extension the group of automorphisms is $\mathbb{Z}$ [93]. We will show in Lemma 5.2.10 that this factor map is compatible, hence $\widehat{\pi}$ is well defined but is not onto. On the other hand, the map $\widehat{\pi}$ associated to the projection on the trivial system cannot be injective.

In the case of proximal extension between minimal systems we have.

Lemma 5.2.4. *Let $\pi\colon (Y,T) \to (X,T)$ be a proximal extension between minimal systems and suppose that π is compatible with $\mathrm{Aut}(Y,T)$. Then $\widehat{\pi}\colon \mathrm{Aut}(Y,T) \to \mathrm{Aut}(X,T)$ is injective.*

Proof. It suffices to prove that $\widehat{\pi}(\phi) = \mathrm{id}_X$, where id_X is the identity map on X, implies that $\phi = \mathrm{id}_Y$. Let ϕ be an automorphism with $\widehat{\pi}(\phi) = \mathrm{id}_X$. For $y \in Y$ we have that $\pi(\phi(y)) = \widehat{\pi}(\phi)\pi(y) = \pi(y)$. Since π is proximal, then y and $\phi(y)$ are proximal points. From Lemma 5.2.3 we conclude that ϕ is the identity map. $\square$

5.2.2. Subshifts

Let $\mathcal{A}$ be a finite set or *alphabet.* Elements in $\mathcal{A}$ are called *letters* or *symbols.* The set of finite sequences or *words* of length $\ell \in \mathbb{N}$ in $\mathcal{A}$ is denoted by $\mathcal{A}^\ell$ and the set of twosided sequences $(x_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ is denoted by $\mathcal{A}^\mathbb{Z}$. Also, a word $w = w_1 \dots w_\ell \in \mathcal{A}^\ell$ can be seen as an element of the free monoid $\mathcal{A}^*$ endowed with the operation of concatenation. The *length* of w is denoted by $|w| = \ell$.

The *shift map* $\sigma\colon \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is defined by $\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. To simplify notations we denote the shift map by σ independently of the alphabet, the alphabet will be clear from the context.

A *subshift* is a topological dynamical system (X,σ) where X is a closed σ-invariant subset of $\mathcal{A}^\mathbb{Z}$ (we consider the product topology in $\mathcal{A}^\mathbb{Z}$). For convenience, when we state general results about topological dynamical systems we use the notation (X,T), and to state specific results about subshifts we use (X,σ).

Let (X,σ) be a subshift. The *language* of (X,σ) is the set $\mathcal{L}(X)$ containing all words w such that $w = x_m \dots x_{m+\ell-1}$ for some $(x_n)_{n\in\mathbb{Z}} \in X$, $m \in \mathbb{Z}$ and $\ell \in \mathbb{N}$. We say that w *appears* in the sequence $(x_n)_{n\in\mathbb{Z}} \in X$. We denote by $\mathcal{L}_\ell(X)$ the set of words of length ℓ in $\mathcal{L}(X)$. The map $p_X\colon \mathbb{N} \to \mathbb{N}$ defined by $p_X(\ell) = \sharp\mathcal{L}_\ell(X)$ is called the *complexity function* of (X,σ).

In the proof of Theorem 5.1.1 we will need the following well-known notion that is intimately related to the concept of asymptotic pairs. A word $w \in \mathcal{L}(X)$ is said to be *left special* if there exist at least two distinct letters a and b such that aw and bw belong to $\mathcal{L}(X)$. In the same way we define *right special* words.

Let $\phi\colon (X,\sigma) \to (Y,\sigma)$ be a factor map between subshifts. By the Curtis-Hedlund-Lyndon Theorem, ϕ is determined by a *local map* $\hat{\phi}\colon \mathcal{A}^{2\mathbf{r}+1} \to \mathcal{A}$ in such way that $\phi(x)_n = \hat{\phi}(x_{n-\mathbf{r}} \dots x_n \dots x_{n+\mathbf{r}})$ for all $n \in \mathbb{Z}$ and $x \in X$, where $\mathbf{r} \in \mathbb{N}$ is called a *radius* of ϕ. The local map $\hat{\phi}$ naturally extends to the set of words of length at least $2\mathbf{r}+1$, and we also denote this map by $\hat{\phi}$.

5.2.3. Equicontinuous systems

We recall that a topological dynamical system (X, T) is *equicontinuous* if the family of transformations $\{T^n; n \in \mathbb{Z}\}$ is equicontinuous. Let (X, T) be an equicontinuous minimal system. It is well-known that the closure of the group $\langle T \rangle$ in the set of homeomorphisms of X for the uniform topology is a compact abelian group acting transitively on X (see [7]).

When X is a Cantor set, the dynamical system (X, T) is called an *odometer*. In this case one shows that X is a profinite group. More precisely, there exists a nested sequence of finite index subgroups $\ldots \subset \Gamma_{n+1} \subset \Gamma_n \subset \cdots \subset \Gamma_0 \subset \mathbb{Z}$ with trivial intersection such that X is isomorphic to the inverse limit

$$\lim_{\leftarrow n}(\mathbb{Z}/\Gamma_n, \pi_n) = \{(x_n)_{n\in\mathbb{N}}; x_n \in \mathbb{Z}/\Gamma_n,\ x_n = \pi_n(x_{n+1})\ \forall n \geq 0\},$$

where $\pi_n \colon \mathbb{Z}/\Gamma_{n+1} \to \mathbb{Z}/\Gamma_n$ denotes the canonical projection. The addition in this group is given by

$$(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (x_n +_n y_n)_{n\in\mathbb{N}}$$

for $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \in \lim_{\leftarrow n}(\mathbb{Z}/\Gamma_n, \pi_n)$, where $+_n$ stands for the addition in $\mathbb{Z}/\Gamma_n$. The group $\mathbb{Z}$ is a dense subgroup through the injection $i \colon k \mapsto (k \mod \Gamma_n)_{n\in\mathbb{N}}$. The action T is then given by the addition by $i(1)$ in the group X. It is a minimal and uniquely ergodic action on X.

5.2.4. Nilsystems

The class of nilsystems will allow us to compute the automorphism group of some interesting subshifts of polynomial complexity of arbitrary degree.

5.2.5. Automorphism group of d-step nilsystems

In this section we prove that the automorphism group of a proximal extension of a system of order d (and thus of a d-step nilsystem) is d-step nilpotent. This will be used later in the chapter to construct subshifts of polynomial complexity whose automorphism groups behave like subaffine complexity subshifts. Before we need some preliminary lemmas.

Let $\pi \colon (Y, T) \to (X, T)$ be a factor map between minimal systems. For $d \geq 1$ recall that $\pi_d \colon Y \to Z_d(Y)$ and $\tilde{\pi}_d \colon X \to Z_d(X)$ are the quotient maps induced by the regionally proximal relations of order d in each system. Since $(Z_d(Y), T)$ is the maximal d-step nilfactor of (Y, T) and $(Z_d(X), T)$ is a system of order d and a factor of (Y, T), then by Theorem 1.3.5 there exists a unique factor map $\varphi_d \colon (Z_d(Y), T) \to (Z_d(X), T)$.

Lemma 5.2.5. *Let $\pi \colon (Y, T) \to (X, T)$ be an almost one-to-one extension between minimal systems. Then, for any integer $d \geq 1$ the canonical induced factor map $\varphi_d : (Z_d(Y), T) \to$*

$(Z_d(X), T)$ *is a topological conjugacy (or the maximal d-step nilfactors of* (Y, T) *and* (X, T) *coincide).*

Proof. Let $\pi_d \colon Y \to Z_d(Y)$ and $\widetilde{\pi}_d \colon X \to Z_d(X)$ denote the quotient maps as above. First we prove that $\varphi_d : (Z_d(Y), T) \to (Z_d(X), T)$ is an almost one-to-one extension.

Let $y \in Y$ be such that $\pi^{-1}\{\pi(y)\} = \{y\}$. We claim that $\varphi_d^{-1}\{\varphi_d(\pi_d(y))\} = \{\pi_d(y)\}$. Let $y' \in Y$ be such that $\varphi_d(\pi_d(y)) = \varphi_d(\pi_d(y'))$. Then $\widetilde{\pi}_d(\pi(y)) = \widetilde{\pi}_d(\pi(y'))$ and thus $(\pi(y), \pi(y')) \in \mathbf{RP}^{[d]}(X)$. By Theorem 1.3.5, there exists a sequence $(\vec{n}_i)_{i \in \mathbb{N}}$ in $\mathbb{Z}^{d+1}$ such that $T^{\vec{n}_i \cdot \epsilon}\pi(y')$ converges to $\pi(y)$ for every $\epsilon \in \{0,1\}^{d+1} \setminus \{(0, \ldots, 0)\}$. Taking a subsequence we can assume that $T^{\vec{n}_i \cdot \epsilon}y'$ converges to y, the unique point in $\pi^{-1}\{\pi(y)\}$, for every $\epsilon \in \{0,1\}^{d+1} \setminus \{(0, \ldots, 0)\}$. Then, by Theorem 1.3.5, we deduce that $(y, y') \in \mathbf{RP}^{[d]}(Y)$. This implies that $\pi_d(y) = \pi_d(y')$ and then φ_d is an almost one-to-one extension.

Finally, by Lemma 5.2.1, π_d is a proximal extension. But $(Z_d(Y), T)$ is a distal system, so there are no proximal pairs. We conclude that φ_d must be a topological conjugacy. □

We deduce that,

Corollary 5.2.6. *Let* $\pi \colon (Y, T) \to (X, T)$ *be an almost one-to-one extension between minimal systems. If* (X, T) *is a system of order d, then it is the maximal d-step nilfactor of* (Y, T)*.*

For instance, since any Sturmian subshift is an almost one-to-one extension of a rotation on the circle [39], this rotation is its maximal 1-step nilsystem or more classically its maximal equicontinuous factor. Similarly, Toeplitz subshifts are the symbolic almost one-to-one extensions of odometers [37]. These odometers are hence their maximal 1-step nilsystems.

The next result is a characterization of the group of automorphisms of an equicontinuous system. In particular, we get that it is abelian.

Lemma 5.2.7. *Let* (X, T) *be an equicontinuous minimal system. Then* $\mathrm{Aut}(X, T)$ *is the closure of the group* $\langle T \rangle$ *in the set of homeomorphisms of* X *for the topology of uniform convergence. Moreover,* $\mathrm{Aut}(X, T)$ *is homeomorphic to* X*.*

Proof. Let G denote the closure in the set of homeomorphisms of X of the group $\langle T \rangle$ for the topology of uniform convergence. Clearly $G \subseteq \mathrm{Aut}(X, T)$. Moreover, by Ascoli's Theorem it is a compact abelian group.

Now we prove that $\mathrm{Aut}(X, T) \subseteq G$. Consider a point $x \in X$ and an automorphism $\phi \in \mathrm{Aut}(X, T)$. By minimality, there exists a sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that $(T^{n_i}x)_{i \in \mathbb{N}}$ converges to $\phi(x)$. Taking a subsequence we can assume that the sequence of maps $(T^{n_i})_{i \in \mathbb{N}}$ converges uniformly to a homeomorphism g in G. Combining both previous facts we get that $\phi(x) = g(x)$ and thus $g^{-1} \circ \phi(x) = x$. Since $g^{-1} \circ \phi \in \mathrm{Aut}(X, T)$, by Lemma 5.2.2 we conclude that $\phi = g$ and consequently $\phi \in G$.

To finish remark that Lemma 5.2.2 ensures that the map from G to X sending $g \in G$ to $g(x) \in X$ is a homeomorphism onto its image $Y \subset X$. Since Y is T invariant and T is minimal we get that $Y = X$. This proves that $\mathrm{Aut}(X,T)$ is homeomorphic to X. $\square$

We generalise previous result for systems of order d for any $d \in \mathbb{N}$.

Theorem 5.2.8. *Let (X,T) be a system of order d. Then, its automorphism group* $\mathrm{Aut}(X,T)$ *is a d-step nilpotent group.*

To prove this theorem, we need to introduce some notation. Given a function $\phi\colon X \to X$, for $k = 1,\dots,d$ we define the *k-face transformation* associated to ϕ as

$$\begin{cases} \phi^{[d],k}(\mathbf{x}) = & (\phi^{[d],k}\mathbf{x})_\epsilon = \phi x_\epsilon, \quad \epsilon_k = 1; \\ & (\phi^{[d],k}\mathbf{x})_\epsilon = x_\epsilon, \quad\ \ \epsilon_k = 0. \end{cases}$$

For example, for $d = 2$ the face transformations associated to $\phi\colon X \to X$ are $\phi^{[2],1} = \mathrm{id} \times \phi \times \mathrm{id} \times \phi$ and $\phi^{[2],2} = \mathrm{id} \times \mathrm{id} \times \phi \times \phi$. When $\phi = T$, the transformations $T^{[d],1}, T^{[d],2}, \dots, T^{[d],d}$ are called the *face transformations*. We let $\mathcal{F}_d$ denote the group spanned by the face transformations. We remark that $\mathbf{Q}^{[d]}(X)$ is invariant under $\mathcal{F}_d$ and under the diagonal transformation $T \times T \cdots \times T$ (2^d times). We denote by $\mathcal{G}_d$ the group spanned by $\mathcal{F}_d$ and the diagonal transformation.

We relate cube structures and automorphisms with the following lemma.

Lemma 5.2.9. *Let (X,T) be a minimal topological dynamical system and let $\phi \in \mathrm{Aut(X,T)}$. Then for every $d \in \mathbb{N}$, any face transformation $\phi^{[d],k}$, $k = 1,\dots,d$, leaves invariant $\mathbf{Q}^{[d]}(X)$.*

Proof. Let $\mathbf{x} \in \mathbf{Q}^{[d]}(X)$ and $k \in \{1,\dots,d\}$. By definition of $\mathbf{Q}^{[d]}(X)$, we can find $x \in X$ and a sequence $(g_i)_{i\in\mathbb{N}}$ in $\mathcal{G}_d$ such that $g_i x^{[d]} \to \mathbf{x}$. We remark that by minimality of (X,T), there exists a sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $T^{n_i}x \to \phi(x)$. Therefore $(T^{[d],k})^{n_i}(x^{[d]}) \to \phi^{[d],k}(x^{[d]})$ and thus $\phi^{[d],k}(x^{[d]}) \in \mathbf{Q}^{[d]}(X)$. Since ϕ commutes with T we have that $\phi^{[d],k}$ commutes with $\mathcal{G}_d$ and thus $\phi^{[d],k} g_i(x^{[d]}) = g_i \phi^{[d],k}(x^{[d]}) \in \mathbf{Q}^{[d]}(X)$. Taking the limit we conclude that $\phi^{[d],k}\mathbf{x} \in \mathbf{Q}^{[d]}(X)$ and then $\phi^{[d],k}$ leaves invariant $\mathbf{Q}^{[d]}(X)$. $\square$

Proof of Theorem 5.2.8. Let $\phi_1,\dots,\phi_{d+1} \in \mathrm{Aut}(X,T)$. Using Lemma 5.2.9 we have that $\phi_i^{[d+1],i}$ leaves invariant $\mathbf{Q}^{[d+1]}(X)$ for every $i = 1,\dots,d+1$. Therefore, their iterated commutator $[\cdots[\phi_1^{[d+1],1}, \phi_2^{[d+1],2}],\cdots,\phi_d^{[d+1],d}],\phi_{d+1}^{[d+1],d+1}]$ also leaves invariant $\mathbf{Q}^{[d+1]}(X)$. Let $h = [\cdots[\phi_1,\phi_2],\cdots,\phi_d],\phi_{d+1}]$ be the iterated commutator of $\phi_1,\dots,\phi_{d+1}$. A simple computation shows that

$$[\cdots[\phi_1^{[d+1],1}, \phi_2^{[d+1],2}],\cdots,\phi_d^{[d+1],d}],\phi_{d+1}^{[d+1],d+1}] = \mathrm{id} \times \mathrm{id} \cdots \times \mathrm{id} \times h.$$

Therefore, we have that $\mathrm{id} \times \mathrm{id} \cdots \times \mathrm{id} \times h(x^{[d]}) = (x, x, \ldots, x, hx) \in \mathbf{Q}^{[d+1]}(X)$ for every $x \in X$. By Theorem 1.3.5 we get that $hx = x$ for every $x \in X$. We conclude that h is the identity automorphism. $\square$

On the other hand, by definition of the regionally proximal relation of order d and the continuity of an automorphism we have that,

Lemma 5.2.10. *Let (X, T) be a minimal topological dynamical system. Let $\phi \in \mathrm{Aut}(X, T)$. Then $(x, y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(\phi(x), \phi(y)) \in \mathbf{RP}^{[d]}(X)$. Consequently, the projection $\pi_d \colon X \to Z_d(X)$ from X to its maximal d-step nilfactor is compatible with* $\mathrm{Aut}(X, T)$.

Combining Theorem 5.2.8, Lemma 5.2.10 and Lemma 5.2.4 we get,

Corollary 5.2.11. *Let (X, T) be a proximal extension of a minimal system of order d. Then,* $\mathrm{Aut}(X, T)$ *is a d-step nilpotent group.*

Since Sturmian and Toeplitz subshifts are almost one-to-one extensions of their maximal equicontinuous factors, they are also proximal extensions (Lemma 5.2.1). We obtain as a corollary that their automorphism groups are abelian. More precisely, Lemma 5.2.10 and Lemma 5.2.4 imply that their automorphism groups are subgroups of the automorphism group of their maximal equicontinuous factors, characterized in Lemma 5.2.7. In addition, it is not difficult to construct minimal symbolic almost one-to-one extensions of d-step nilsystems by considering codings on well chosen partitions. An example will be developed in Section 5.4.

5.3. On the automorphisms of subshifts with polynomial complexity

In this section we prove the main results of this paper. We start by proving Theorem 5.1.1 and in a second part we give new proofs of byproduct results from [74, 84]. Namely, a characterization of the automorphisms of bijective constant length substitutions and the realization of any finite group as the quotient $\mathrm{Aut}(X, T)/\langle T \rangle$. We end this section by presenting a tentative generalization of Theorem 5.1.1 to polynomial complexity by using a result on the growth rate of groups.

For the sequel, we recall that a group G satisfies *virtually* a property `P` (*e.g.*, nilpotent, solvable, isomorphic to a given group, ...) if there is a finite index subgroup $H \subset G$ satisfying the property `P`.

5.3.1. Proof of Theorem 5.1.1

Let (X, T) be a topological dynamical system. It is clear from the definition that for any proximal (asymptotic) pair $(x, y) \in X \times X$ and for any $\phi \in \mathrm{Aut}(X, T)$ we have that $(\phi(x), \phi(y))$ is a proximal (asymptotic) pair. We say that the asymptotic pairs (x, y) and (x', y') belong to the same class if they are in the same orbit, meaning that there exists $n \in \mathbb{Z}$ such that $(x', y') = (T^n x, T^n y)$. A class of asymptotic pairs is a (non closed) $T \times T$-invariant subset of $X \times X$. We denote by $[(x, y)]$ the class of the asymptotic pair (x, y). We say that two classes $[(x, y)]$, $[(x', y')]$ are *equivalent* if there is an asymptotic pair $(x'_1, y'_1) \in [(x', y')]$ such that $x = x'_1$ or x and x'_1 are asymptotic. This defines an equivalence relation and any class is called an *asymptotic component*. We denote by $\mathcal{AS}_{[(x,y)]}$ the asymptotic component of the class $[(x, y)]$ and by $\mathcal{AS}$ the collection of asymptotic components.

It is also plain to check for $\phi \in \mathrm{Aut}(X, T)$ and two equivalent asymptotic classes $[(x, y)]$ and $[(x', y')]$, that classes $[(\phi(x), \phi(y))]$ and $[(\phi(x'), \phi(y'))]$ are also equivalent. So the automorphism ϕ induces a permutation $j(\phi)$ on the collection $\mathcal{AS}$ of asymptotic components of (X, T). By denoting Per$\mathcal{AS}$ the set of such permutations, formally we have the group morphism

$$\begin{aligned} j\colon \mathrm{Aut}(X, T) &\to \mathrm{Per}\mathcal{AS} \\ \phi &\mapsto \left(\mathcal{AS}_{[x,y]} \mapsto \mathcal{AS}_{[(\phi(x),\phi(y))]}\right). \end{aligned} \tag{5.3.1}$$

In the case of subshifts, the following lemma is a key observation which relates the complexity of the subshift with asymptotic classes. The proof relies in classical ideas from [102].

Lemma 5.3.1. *Let (X, σ) be a subshift. If (X, σ) has a sublinear complexity, then there is a finite number of asymptotic classes. More generally, if the complexity $p_X(n)$ satisfies*

$$\liminf_{n\to+\infty} \frac{p_X(n)}{n} < +\infty \ ,$$

then there is a finite number of asymptotic classes.

In particular, this lemma provides a sufficient condition to bound the number of asymptotic components.

Proof. For the first statement see [102] Lemma V. 22. For the second claim we proceed as follows. The hypothesis implies the existence of a constant κ and an increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$ such that $p_X(n_i + 1) - p_X(n_i) \leq \kappa$. Indeed, if not, for any $A > 0$ and for any integer n large enough we have $p_X(n + 1) - p_X(n) \geq A$. It follows that $p_X(n) - p_X(m) = \sum_{i=m}^{n-1} p_X(i + 1) - p_X(i) \geq (n - m)A$ for any $n \geq m$ enough large. From here we get that $\liminf_{n\to+\infty} \frac{p_X(n)}{n} \geq A$ which is a contradiction since A is arbitrary.

Hence, the number of left special words of length n_i (see Section 5.2.2 for the definition) is bounded by κ. Any asymptotic pair defines a sequence with arbitrarily long special words, so there are at most κ asymptotic classes. $\square$

A second main ingredient for proving Theorem 5.1.1 is the following direct corollary of Lemma 5.2.3. We recall that an asymptotic pair is proximal and that the map j used in the following corollary has been defined in (5.3.1).

Corollary 5.3.2. *Let (X, T) be a minimal topological dynamical system with at least one asymptotic pair. We have the following exact sequence*

$$1 \longrightarrow \langle T \rangle \xrightarrow{\text{Id}} \mathrm{Aut}(X, T) \xrightarrow{j} \mathrm{Per}\mathcal{AS},$$

where Per$\mathcal{AS}$ denotes the set of permutations on the collection of asymptotic components of (X, T). Moreover, for any automorphism ϕ, the permutation $j(\phi)$ has a fixed point if and only if ϕ is a power of T.

As a byproduct of this result and Lemma 5.3.1 we get Theorem 5.1.1 that we recall and extend here.

Theorem. *Let (X, σ) be a minimal aperiodic subshift with $\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty$. Then,*

1. $\mathrm{Aut}(X, \sigma)$ *is virtually isomorphic to* $\mathbb{Z}$.
2. *The quotient group* $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ *is isomorphic to a finite subgroup of permutations without fixed points. In particular,* $\sharp \mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ *divides the number of asymptotic components.*

Proof. Only the second part of statement (2) is not straightforward from Corollary 5.3.2. The group $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ acts freely on the finite set of asymptotic component $\mathcal{AS}$: the stabilizer of any point is trivial. Thus, $\mathcal{AS}$ is decomposed into disjoint $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$-orbits, and any such orbit has the same cardinality as $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$. $\square$

Statement (2) of the theorem enables us to perform explicit computations of the automorphism group for easy cases. A first example comes from Sturmian subshifts. It is well-known that this system admits just one asymptotic component, so any automorphism is a power of the shift map. A bit more general case is when the number of asymptotic components is a prime p (*e.g.*, 2 for the Thue-Morse subshift), then the group $\mathrm{Aut}(X, \sigma)/\langle \sigma \rangle$ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$: either the trivial one or $\mathbb{Z}/p\mathbb{Z}$. In particular, since the Thue-Morse subshift admits an automorphism which is not the power of the shift map, then the quotient automorphism group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

One could ask whether the automorphism group is computable algorithmically, at least for substitution subshifts, or explicitly by theoretical arguments for some families of subshifts. This will be achieved in [41] for substitutive and linearly recurrent subshifts.

Statement (2) is not a real restriction. Given any finite group G, it acts on itself by left multiplication $L_g(h) = g \cdot h$ for $g, h \in G$. The map L_g defines then a permutation on the finite set G without fixed points. So G is a subgroup of elements of the permutation group on $\sharp G$ elements which verifies statement (2) in the theorem. Thus, it is natural to ask whether we can realize any finite group as $\mathrm{Aut}(X,T)/\langle T\rangle$ or if we can characterize those finite groups. This is done in the next subsection.

Finally, notice that the complexity condition of Theorem 5.1.1 is compatible with $\limsup_{n\to+\infty} \frac{p_X(n)}{n} = +\infty$. In Section 5.4.4 we construct a minimal subshift with subexponential complexity satisfying

$$\liminf_{n\to+\infty} p_X(n)/n < +\infty \text{ and } \limsup_{n\to+\infty} p_X(n)/n^d = +\infty \text{ for every } d > 1.$$

Thus, in this case, the automorphism group is virtually $\mathbb{Z}$ by Theorem 5.1.1.

5.3.2. A characterization of Aut$(X,\sigma)/\langle\sigma\rangle$ for constant length substitutions

In this section, by using the results of Section 5.3.1, we provide a characterization of the automorphism group for subshifts given by a constant length substitution $\tau : \mathcal{A} \to \mathcal{A}^*$ on a finite alphabet $\mathcal{A}$. Our characterization follows from the one of asymptotic components. We deduce then new and direct proofs of two already known results. The first one is due to Host and Parreau [74] on the characterization of the automorphism group of bijective constant length substitutions. The second one is a combination of results in [84] and [74], giving an explicit example of a substitutive minimal subshift (X,σ) such that $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is isomorphic to an arbitrary finite group G. Notice that in [84] the authors have a similar statement but in the measurable setting.

We recall that a substitution $\tau\colon \mathcal{A} \to \mathcal{A}^*$ is of constant length $\ell > 0$ if any word $\tau(a)$ for the letter $a \in \mathcal{A}$ is of length ℓ. A substitution of constant length is *bijective* if the corresponding letters at position $i \in \{0,\dots,\ell-1\}$ of all $\tau(a)$'s are pairwise distinct. We denote by X_τ the subshift

$$X_\tau = \{x \in \mathcal{A}^{\mathbb{Z}};\ \text{any word of } x \text{ appears in } \tau^n(a) \text{ for some } n \geq 0 \text{ and } a \in \mathcal{A}\}.$$

For constant length substitution, it is well known (*e.g.* see [102]) that the subshift (X_τ,τ) is minimal if and only if the substitution τ is *primitive*, that is, for some power $p \geq 0$ and any letter $a \in \mathcal{A}$, the word $\tau^p(a)$ contains all the letters of the alphabet. Recall that the substitution τ is *aperiodic* if and only if X_τ is infinite.

Lemma 5.3.3. *Let τ be a primitive aperiodic bijective constant length substitution. Let $(x, y) = ((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ be an asymptotic pair with $x_n = y_n$ for any $n \geq 0$ and $x_{-1} \neq y_{-1}$. Then, there exists an asymptotic pair $((x'_n)_{n\in\mathbb{Z}}, (y'_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ with $x'_n = y'_n$ for any $n \geq 0$ and $x'_{-1} \neq y'_{-1}$, such that*

$$\tau((x'_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{Z}} \text{ and } \tau((y'_n)_{n\in\mathbb{Z}}) = (y_n)_{n\in\mathbb{Z}}.$$

Proof. Let ℓ be the length of the substitution τ. By the classical result of Mossé [89, 90] on recognizability, the substitution $\tau\colon X_\tau \to \tau(X_\tau)$ is one-to-one. Moreover, the collection $\{\sigma^k\tau(X_\tau)\colon k = 0, \dots, \ell-1\}$ is a clopen partition of X_τ. So, there are $x' = (x'_n)_{n\in\mathbb{Z}}, y' = (y'_n)_{n\in\mathbb{Z}} \in X_\tau$ and $0 \leq k_x, k_y < \ell$ such that $\sigma^{k_x}\tau(x') = x$ and $\sigma^{k_y}\tau(y') = y$.

We claim that we have $k_x = k_y = 0$. Since the sequences x and y are asymptotic, there are integers $n \geq 0$, $k' \in \{0, \dots, \ell-1\}$ such that $\sigma^n(x), \sigma^n(y) \in \sigma^{k'}(\tau(X_\tau))$. The substitution τ is of constant length ℓ, so we have $\sigma^\ell \circ \tau = \tau \circ \sigma$. Therefore, we get x and y are in the same clopen set $\sigma^k(\tau(X_\tau))$ for some $k \in \{0, \dots, \ell-1\}$. Let us assume that $k \geq 1$. The words $x_{-1}x_0 \dots x_{k-1}$, $y_{-1}y_0 \dots y_{k-1}$ are then prefixes of the words $\tau(x'_{-1})$ and $\tau(y'_{-1})$ respectively. Since the substitution τ is bijective and $x_0 = y_0$, we have $x'_{-1} = y'_{-1}$. In particular, we get $x_{-1} = y_{-1}$: a contradiction.

To finish the proof, notice that the substitution τ is injective on the letters, so we obtain $x'_n = y'_n$ for any $n \geq 0$ and $x'_{-1} \neq y'_{-1}$. $\square$

Lemma 5.3.4. *Let τ be a primitive aperiodic bijective constant length substitution. Then, there exists an integer $p \geq 0$ such that for any asymptotic pair $((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) \in X_\tau^2$ the one-sided infinite sequences*

$$(x_{n+n_0})_{n\geq 0}, (y_{n+n_0})_{n\geq 0} \text{ are equal for some } n_0 \in \mathbb{Z} \text{ and fixed by } \tau^p.$$

Proof. Shifting the indices if needed by some σ^{n_0}, we can assume that for the asymptotic pair $(x, y) = ((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}})$ we have $x_n = y_n$ for any integer $n \geq 0$ and $x_{-1} \neq y_{-1}$. Let $p \geq 0$ be an integer such that for any letter $a \in \mathcal{A}$, any word in $\{\tau^{pn}(a)\}_{n\geq 1}$ starts with the same letter. Hence, the sequence of sequences $(\tau^{pn}(aa\cdots))_{n\geq 0}$ converges to a one-sided infinite word fixed by τ^p.

Applying inductively Lemma 5.3.3 to the substitution τ^p, we get a sequence of asymptotic pairs $((x^{(i)}, y^{(i)}))_{i\geq 0}$ verifying the conclusions of the lemma and such that $\tau^p(x^{(i+1)}) = x^{(i)}$, $\tau^p(y^{(i+1)}) = y^{(i)}$, $x^{(0)} = x$ and $y^{(0)} = y$. By the definition of p, all sequences $x^{(i)}$ and also $y^{(i)}$, $i \geq 0$, share the same letter a at index 0. The conclusion of the lemma follows straightforwardly since we assume that $\tau^{pn}(a \cdots)$ converges to a one-sided infinite word fixed by τ^p. $\square$

Thanks to this lemma we can obtain another proof of the following result due to Host and Parreau.

Theorem 5.3.5. *[74] Let τ be a primitive bijective constant length substitution. Then, any automorphism of the subshift X_τ is the composition of some power of the shift with an automorphism ϕ of radius 0. Moreover, its local rule $\hat{\phi}\colon \mathcal{A} \to \mathcal{A}$ satisfies*

$$\tau \circ \hat{\phi} = \hat{\phi} \circ \tau. \tag{5.3.2}$$

Conversely, notice that a local map satisfying (5.3.2) defines an automorphism of the subshift. Hence we obtain an algorithm to determine in this case the group of automorphisms since there is just a finite number of local rules of radius 0.

Proof. Notice first that when X_τ is finite, it is reduced to a finite orbit. Hence any automorphism is a power of the shift map by Lemma 5.2.7.

Let us assume now that the substitution τ is aperiodic and let $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in X_\tau$ be two asymptotic sequences. Lemma 5.3.4 provides a power $p \geq 0$ such that, shifting the sequences if needed by some σ^{n_0}, we can assume that $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ coincide and are fixed by τ^p.

Let ϕ be an automorphism of the subshift (X_τ, σ). The pair $(\phi(x), \phi(y))$ is also an asymptotic pair. Again, Lemma 5.3.4 ensures that for some integer $n_1 \in \mathbb{Z}$, the sequences $(\phi(x)_{n+n_1})_{n\geq 0}$ and $(\phi(y)_{n+n_1})_{n\geq 0}$ coincide and are also fixed by τ^p (observe as stated in Lemma 5.3.4, we can use the same power p for any couple of asymptotic pairs). In the following, we will consider the automorphism $\phi' = \sigma^{n_1} \circ \phi$, thus by definition, the sequence $(\phi'(x)_n)_{n\geq 0}$ is also fixed by τ^p.

Let $\mathbf{r}$ and $\hat{\phi}'$ denote the radius and the local map of ϕ' respectively. Taking a power of τ^p if needed, we can assume that the length ℓ of τ^p is greater than $2\mathbf{r} + 1$. Suppose now that $x_n = x_m$ for some $n, m \geq 0$. We have $\phi'(x)_{m\ell+\mathbf{r}} = \hat{\phi}'(x_{m\ell} \ldots x_{m\ell+2\mathbf{r}}) = \hat{\phi}'(\tau^p(x_m)_{[0,2\mathbf{r}]}) = \hat{\phi}'(\tau^p(x_n)_{[0,2\mathbf{r}]}) = \phi'(x)_{n\ell+\mathbf{r}}$, where for a word $u = u_0 \ldots u_{\ell-1}$, $u_{[0,2\mathbf{r}]}$ stands for the prefix $u_0 \ldots u_{2\mathbf{r}}$. Since $\phi'(x)_{n\ell+\mathbf{r}}$ and $\phi'(x)_{m\ell+\mathbf{r}}$ are the $r + 1^{\text{th}}$ letters of the words $\tau^p(\phi'(x)_n)$ and $\tau^p(\phi'(x)_m)$ respectively, and the substitution τ is bijective, we obtain that $\phi'(x)_n = \phi'(x)_m$.

Hence, we can define the local map $\hat{\psi}\colon \mathcal{A} \to \mathcal{A}$ by $\hat{\psi}(x_n) = \phi'(x)_n$ for any $n \geq 0$. This provides a shift commuting map $\psi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ such that for any word w in the language $\mathcal{L}(X_\tau)$, we have that $\psi(\tau^p(w)) = \tau^p(\psi(w))$. Thus $\psi(X_\tau) \subset X_\tau$. Since the substitution τ is bijective we also get relation (5.3.2). In the same way, using ϕ'^{-1} instead of ϕ' we obtain that ψ is invertible. By construction, we have that $\psi^{-1}\phi'(x)$ is asymptotic to x, so by Lemma 5.2.3, $\psi = \phi' = \sigma^{n_1} \circ \phi$. $\square$

A second consequence of Lemma 5.3.4 is the realization of any finite group as the group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ for a substitutive subshift of constant length.

Proposition 5.3.6. *Given a finite group G, there is a substitutive minimal subshift (X, σ) such that $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is isomorphic to G.*

Proof. The Fibonacci subshift is both a substitutive and a Sturmian subshift, then by previous discussion the quotient group $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is trivial. Then, let us assume that the finite group G is not trivial. We choose an enumeration of its elements $G = \{g_0, g_1, \ldots, g_{p-1}\}$ with $p \geq 2$ where g_0 denotes the neutral element.

For an element $h \in G$, we denote by $L_h \colon G \to G$ the bijection $g \mapsto hg$. We consider the alphabet G, viewed as a finite set, and define the substitution τ from the set of letters G into the set of words G^*, by

$$\tau \colon g \mapsto L_g(g_0)L_g(g_1)\cdots L_g(g_{p-1}).$$

Since the map L_g is a bijection on G, the substitution τ of constant length is primitive and bijective. Thus the associated subshift (X_τ, σ) is minimal.

Moreover observe that for any letter $g \in G$, the word $\tau(g)$ starts by the letter g, so any sequence $(\tau^n(gg\cdots))_{n\geq 1}$ converges to a τ-invariant infinite word.

We claim that the subshift (X_τ, σ) is not periodic, *i.e.*, not reduced to a periodic orbit. To show this it suffices to give an example of an asymptotic pair. The word g_0g_1 belongs to the language $\mathcal{L}(X_\tau)$ of the subshift X_τ. Hence the words $\tau(g_0)\tau(g_1)$ and its sub-word $g_{p-1}g_1$ (which is different from the word g_0g_1) also belong to $\mathcal{L}(X_\tau)$. It follows for any integer $n \geq 0$ that the words $\tau^n(g_0).\tau^n(g_1)$ and $\tau^n(g_{p-1}).\tau^n(g_1)$ are also in the language. Taking a subsequence if needed, these words converge as n goes to infinity to two sequences x and $y \in X_\tau$ that are, by construction, asymptotic.

Given an element $g \in G$ we extend the definition of the map L_g to G^* by defining for a word $w = h_1 \ldots h_n$, $L_g(w) := L_g(h_1) \ldots L_g(h_n)$. By concatenation, it defines a left continuous G-action on $G^{\mathbb{Z}}$. It is important to note that we have the relation for any $g, h \in G$

$$L_g(\tau(h)) = \tau(L_g(h)). \tag{5.3.3}$$

Hence any map L_g preserves the subshift X_τ and we have a left action of G on X_τ. It is plain to check that $L \colon g \mapsto L_g$ defines an injection of G into $\mathrm{Aut}(X_\tau, \sigma)$. Actually, we claim that we have a converse which allows to finish the proof.

Lemma 5.3.7. *For the subshift X_τ defined above the map*

$$\begin{aligned} \varphi \colon \mathbb{Z} \times G &\to \mathrm{Aut}(X_\tau, \sigma) \\ (n, g) &\mapsto \sigma^n \circ L_g \end{aligned}$$

is a group isomorphism.

Proof of Lemma 5.3.7. To show the injectivity of the map φ let us assume there are $g \in G$ and an integer k such that $L_g(x) = \sigma^k(x)$ for any $x \in X$. Necessarily $k = 0$, otherwise the infinite sequence $L_g\Big(\lim_{n\to+\infty} \tau^n(g^{-1}g^{-1}\cdots)\Big)$, which is equal to $\lim_n \tau^n(g_0g_0\cdots)$ by formula (5.3.3), is ultimately periodic. This is impossible since the subshift X_τ is not periodic. The injectivity of the map L implies finally that the map φ is injective.

To show it is also onto, it is enough to prove that any automorphism $\phi \in \mathrm{Aut}(X_\tau, \sigma)$ may be written as a power of the shift composed with a map of the kind L_g. Let (x, y) be an asymptotic pair. By Lemma 5.3.4 up to shift x, y and compose ϕ with a power of the shift map, there exist $g_1, g_2 \in G$ such that the sequences x, y are positively asymptotic to $\lim_{n\to+\infty} \tau^n(g_1)$, and $\phi(x), \phi(y)$ are positively asymptotic to $\lim_{n\to+\infty} \tau^n(g_2g_2\cdots)$. It follows from (5.3.3) that the points x and $L_{g_1(g_2^{-1})} \circ \phi(x)$ are asymptotic. So, by Lemma 5.2.3 the maps ϕ and $(L_{g_1(g_2^{-1})})^{-1} = L_{g_2(g_1^{-1})}$ coincide. □

□

5.3.3. Recurrence and growth rate of groups

We try to extend Theorem 5.1.1 to subshifts with higher complexity. For this, we need to introduce a stronger condition. We define, for a topologically transitive subshift (X, σ) and an integer $n \geq 1$, a *local recurrence time*:

$$N_X(n) := \inf\{|w|;\ w \in \mathcal{L}(X) \text{ contains any word of } X \text{ of length } n\}.$$

Clearly, this value is well defined and satisfies $N_X(n) \geq p_X(n) + n$. For instance, it is well-known that any primitive substitutive subshift is linearly repetitive meaning that $\sup_{n\geq 1} \frac{N_X(n)}{n} < +\infty$. We obtain the following result.

Theorem 5.3.8. *Let (X, σ) be a transitive subshift such that $\sup_{n\geq 1} \frac{N_X(n)}{n^d} < +\infty$ for some $d \geq 1$. Then, there is a constant C depending only on d, such that any finitely generated subgroup of $\mathrm{Aut}(X, \sigma)$ is virtually nilpotent of step at most C.*

Proof. Let $\mathcal{S} = \langle \phi_1, \ldots, \phi_\ell \rangle \subset \mathrm{Aut}(X, \sigma)$ be a finitely generated group. Let $\mathbf{r}$ be an upper bound of the radii of the local maps associated to all generators ϕ_i of $\mathcal{S}$ and their inverses. For $n \in \mathbb{N}$ consider

$$B_n(\mathcal{S}) = \{\phi_{i_1}^{s_1} \cdots \phi_{i_m}^{s_m}; 1 \leq m \leq n,\ i_1, \ldots, i_m \in \{1, \ldots, \ell\},\ s_1, \ldots, s_m \in \{1, -1\}\}\ .$$

Let w be a word of length $N_X(2n\mathbf{r} + 1)$ containing any word of length $(2n\mathbf{r} + 1)$ of X. If $\phi, \phi' \in B_n(\mathcal{S})$ are different then $\phi(w) \neq \phi'(w)$. Then, $B_n(\mathcal{S})$ can be injected into the set of words of length $N_X(2n\mathbf{r}+1)-2\mathbf{r}$ (the injection is just the valuation of ϕ on w). This implies that $\sharp B_n(\mathcal{S}) \leq p_X(N_X(2n\mathbf{r}+1)-2\mathbf{r})$. We deduce from the hypothesis on N_X that $\sharp B_n(\mathcal{S}) \leq n^{d^2+1}$ for all large enough integers $n \in \mathbb{N}$. Therefore, by the quantitative result of Shalom and Tao in [109] generalizing Gromov's classical result on the growth rate of groups, we get the conclusion. □

Notice that the constant C may be given explicitly in the result of [109]. It is clear that a subshift of polynomial local recurrence complexity has a polynomial complexity. The converse is not clear, but an additional possible condition is that the subshift has bounded repetitions of words. The natural question here is whether the automorphism group of a minimal subshift of polynomial local recurrence complexity, or just polynomial complexity, is finitely generated.

5.4. Gallery of examples

We present here examples of subshifts with various complexities. The first two examples are substitutive subshifts with superlinear complexity. Even if we can not apply straightforwardly the main results of the paper (*e.g.*, the substitutions are not primitive), we study their asymptotic components to prove their automorphism groups are isomorphic to $\mathbb{Z}$. Next, we define a coding of a nil-translation with a polynomial complexity of arbitrary high degree but having an automorphism group which is virtually $\mathbb{Z}$. To enlarge the zoology of automorphism groups we provide a subshift whose automorphism group is isomorphic to $\mathbb{Z}^d$. We end with a subshift whose complexity is, for infinitely many integers, subaffine and superpolynomial. Theorem 5.1.1 applies in this case.

5.4.1. Substitutions with superlinear complexity

Recall that substitutive subshifts have a prescribed complexity: with growth bounded or equivalent to n, $n\log\log n$, $n\log n$, or to n^2 (see [95]). Below we give two examples having a unique asymptotic component. This is enough to conclude that their automorphism groups are isomorphic to $\mathbb{Z}$.

A $n\log\log n$ complexity substitutive subshift

Let $\mathcal{A}=\{a,b\}$ and consider the substitution $\tau_1\colon \mathcal{A}\to\mathcal{A}^*$ defined by

$$\tau_1(a)=aba \text{ and } \tau_1(b)=bb.$$

We set

$$X_{\tau_1} = \{x \in \{a,b\}^{\mathbb{Z}};\ \text{ any word of } x \text{ appears in some } \tau_1^n(c), n \geq 0, c \in \{a,b\}\}.$$

It can be checked that (X_{τ_1}, σ) is a non minimal transitive subshift. Moreover, it is proven in [18] that its complexity is equivalent to $n \log_2 \log_2 n$.

In the sequel we need some specific notations. For a sequence $x \in \{a,b\}^{\mathbb{Z}}$ we set $x^- = \cdots x_{-2}x_{-1}$, $x^+ = x_0x_1\cdots$ and $x = x^-.x^+$. Let $b^{+\infty} = bbbbb\ldots \in \mathcal{A}^{\mathbb{N}}$ and $b^{-\infty} = \ldots bbbbb \in \mathcal{A}^{\mathbb{Z}_{<0}}$, where $\mathbb{Z}_{<0}$ is the set of negative integers. Thus the sequence $x = \ldots bb.bb\ldots \in \{a,b\}^{\mathbb{Z}}$ can be written $b^{-\infty}.b^{+\infty}$. In the same spirit we put $\tau_1^{+\infty}(c)$ for $\lim_{n\to+\infty}\tau_1^n(cc\ldots)$, when it exists in $\{a,b\}^{\mathbb{N}}$, and, $\tau_1^{-\infty}(c)$ for $\lim_{n\to+\infty}\tau_1^n(\cdots cc)$, when it exists in $\{a,b\}^{\mathbb{Z}_{<0}}$.

Let us ckeck (X_{τ_1}, σ) has a unique asymptotic component. We show that asymptotic points should end with $b^{+\infty}$.

Let (x,y) be an asymptotic pair. We can suppose, shifting if needed, that

$$\begin{aligned} x &= x^-a.x^+ = \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.x^+_0x^+_1x^+_2\cdots \\ y &= y^-b.x^+ = \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.x^+_0x^+_1x^+_2\cdots . \end{aligned}$$

Observe that $x^+_0 = b$ because aa does not belong to $\mathcal{L}(X_{\tau_1})$:

$$\begin{aligned} x &= \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.bx^+_1x^+_2x^+_3\cdots \\ y &= \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.bx^+_1x^+_2x^+_3\cdots . \end{aligned}$$

Suppose $x^+_1 = a$. Then, we should have $x^+_2x^+_3 = bb$ because aba is necessarily followed by bb. Thus, $bbabb$ should appear in some element of x which is not the case. Therefore $x^+_1 = b$:

$$\begin{aligned} x &= \cdots x^-_{-4}x^-_{-3}x^-_{-2}a.bbx^+_2x^+_3x^+_4\cdots \\ y &= \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.bbx^+_2x^+_3x^+_4\cdots . \end{aligned}$$

Suppose x^+ begins with $b^{2n+1}a$ for some $n \geq 1$. Then, $abab^{2n+1}aba$ should belong to the language of X_{τ_1}. But it should appear in some $\tau_1(u)$ and then we must have $abab^{2n+1}aba = \tau_1(ava)$ for some word $v \in \mathcal{L}(X_{\tau_1})$, hence $b^{2n+1} = \tau_1(v)$, which is not possible. Thus, x^+ begins with $b^{2n}a$ for some $n \geq 1$ or it is equal to $b^{+\infty}$. Suppose we are in the first situation:

$$\begin{aligned} x &= \cdots x^-_{-4}aba.b^{2n}abax^+_{2n+3}\cdots \\ y &= \cdots y^-_{-4}y^-_{-3}y^-_{-2}b.b^{2n}abax^+_{2n+3}\cdots . \end{aligned}$$

It can be checked that τ_1 is one-to-one on X_{τ_1}. Consequently, there are two unique sequences

$$x^{(1)} = x^{(1-)}a.b^n abax^{(1+)} \text{ and } y^{(1)} = y^{(1-)}b.b^n abax^{(1+)} \tag{5.4.1}$$

belonging to X_{τ_1} such that

$$x = \tau_1(x^{(1-)}a).\tau(b^n abax^{(1+)}) \text{ and } y = \tau_1(y^{(1-)}b).\tau(b^n abax^{(1+)}).$$

Thus, $(x^{(1)}, y^{(1)})$ is also an asymptotic pair. From the observation made before, n should be even and we can obtain a new asymptotic pair $(x^{(2)}, y^{(2)})$ having the shape given by (5.4.1). Of course n is decreasing at each step and we can continue until $n = 1$: we get an asymptotic pair $(x^{(k)}, y^{(k)})$ such that

$$\begin{aligned} x^{(k)} &= \cdots a.baba \cdots \\ y^{(k)} &= \cdots b.baba \cdots . \end{aligned}$$

But $ababa$ does not belong to $\mathcal{L}(X_{\tau_1})$. Consequently $x^+ = b^{+\infty}$ and (X_{τ_1}, σ) has a unique asymptotic component.

Furthermore, it can be checked, using already used arguments, that $z^-.b^{+\infty}$ is in $X_{\tau_1} \setminus \{b^{-\infty}.b^{+\infty}\}$ if and only if $z^- = \tau_1^{-\infty}(a)b^n$ for some non-negative integer n. Hence, if (x, y) is an asymptotic pair then x and y belong to

$$\{b^{-\infty}.b^{+\infty}, \sigma^i(\tau^{-\infty}(a).b^{+\infty}); i \in \mathbb{Z}\}.$$

A n^2 complexity substitutive subshift

Below we use the notation of the previous section. Consider the substitution $\tau_2 : \mathcal{A} \to \mathcal{A}^*$ defined by

$$\tau_2(a) = aab \text{ and } \tau_2(b) = b.$$

It is easy to check that the subshift (X_{τ_2}, σ) is transitive but not minimal. Moreover, from [95] its complexity is of the order n^2. Before showing it has a unique asymptotic component, let us introduce some key concepts for the treatment of this example.

Let x be a sequence of $\mathcal{B}^{\mathbb{N}}$, where $\mathcal{B}$ is an alphabet. We denote by $\mathcal{L}(x)$ the set of words having an occurrence in x. A *return word* to $u \in \mathcal{L}(x)$ is a word $w \in \mathcal{L}(x)$ such that wu belongs to $\mathcal{L}(x)$, contains exactly two occurrences of u and has u as a prefix. We denote by $\mathcal{R}_x(u)$ the set of return words to u.

In [19] is defined the notion of *sparse sequence* on the alphabet $\mathcal{B}$. It is an element x of $\mathcal{B}^{\mathbb{N}}$ satisfying:

$$\exists b \in A,\ \forall n \in \mathbb{N}, b^n \in \mathcal{L}(x) \text{ and } \#\mathcal{R}_x(b^n) = 2.$$

It is proven that $p_x(n)$ (the number of words of length n appearing in x) is less than or equal to $(n^2+n+2)/2$ whenever x is sparse. In Example 4.7.67 of [19] it is claimed that $x=\tau_2^{+\infty}(a)$ is sparse. Using Lemma 4.5.15 in [19] one can deduce that for all $n\geq 1$,

$$\mathcal{R}_x(b^n)=\{b,b^nu\}, \text{ where } \tau_2^n(a)=ub^n. \tag{5.4.2}$$

We show (X_{τ_2},σ) has a unique asymptotic component. Let (x,y) be an asymptotic pair. It suffices to prove that x and y end with $b^{+\infty}$. We can suppose that $x=x^-.x_0^+ax^+$ and $y=y^-.y_0^+bx^+$. We set $x^+=x_2^+x_3^+\cdots$.

Suppose that $x_2^+=a$. Then the only possibility to have bax_3^+ in $\mathcal{L}(X_{\tau_2})$ is $x_3^+=a$. Consequently, aaa would belong to $\mathcal{L}(X_{\tau_2})$, which is not the case. Therefore, $x_2^+=b$ and necessarily $x_{-1}^-x_0^-=ba$:

$$\begin{aligned} x &= \cdots x_{-1}^-b.aabx_3^+x_4^+x_4^+\cdots \\ y &= \cdots y_{-2}^-y_{-1}^-.y_0^+bbx_3^+x_4^+x_5^+\cdots . \end{aligned}$$

Suppose we are in the following situation:

$$\begin{aligned} x &= \cdots x_{-1}^-b.aab^naax_{n+4}^+\cdots \\ y &= \cdots y_{-2}^-y_{-1}^-.y_0^+bb^naax_{n+4}^+\cdots . \end{aligned}$$

From (5.4.2) one gets that

$$\begin{aligned} x &= \cdots x_{-1}^-b.aab^n\tau_2^{n+1}(a)\cdots \\ y &= \cdots y_{-2}^-y_{-1}^-.y_0^+bb^n\tau_2^{n+1}(a)\cdots . \end{aligned}$$

Then, using (5.4.2) again, x would have an occurrence of $w=\tau_2^n(a)\tau_2^n(a)\tau_2^n(a)$, but w does not belong to $\mathcal{L}(X_{\tau_2})$. Indeed, if it was the case, by a finite recurrence we prove that aaa should belong to $\mathcal{L}(X_{\tau_2})$, which is not the case. Hence, $x_3^+x_4^+x_4^+\cdots=b^{+\infty}$ and (X_{τ_2},σ) has a unique asymptotic component.

Observe that $(\sigma^{-n}(\tau_2^n(a^{-\infty})).b^{+\infty})$ converges in X_{τ_2}. Let z denote its limit. We can check that if (x,y) is an asymptotic pair then x and y belong to

$$\{b^{-\infty}.b^{+\infty},\sigma^i(z);i\in\mathbb{Z}\}.$$

We finish this section by proving that in both examples (X_{τ_1},σ) and (X_{τ_2},σ) the group of automorphisms is isomorphic to $\mathbb{Z}$.

Lemma 5.4.1. *Let τ denote either the substitution τ_1 or τ_2. Then, the group* $\mathrm{Aut}(X_\tau, \sigma)$ *is generated by the shift map σ.*

Observe that the main result of [29] gives only that the group $\mathrm{Aut}(X_{\tau_1}, \sigma)$ is a periodic group.

Proof. Let us first recall that for any asymptotic pair (x, y) of (X_τ, σ), x and y belong to

$$\{b^{-\infty}.b^{+\infty}, \sigma^i(z); i \in \mathbb{Z}\},$$

for some $z \in X_\tau$.

Notice that (X_τ, σ) has a unique minimal subsystem, namely $(\{b^{-\infty}.b^{+\infty}\}, \sigma)$. Moreover, it is clear that an automorphism ϕ of the subshift (X_τ, σ) maps any minimal subsystem onto a minimal subsystem, so ϕ fixes the sequence $b^{-\infty}.b^{+\infty}$. The morphism ϕ mapping asymptotic pairs onto asymptotic pairs, $\sigma^i(z)$ should be mapped to some $\sigma^j(z)$. The orbit $\{\sigma^k(z); k \in \mathbb{Z}\}$ being dense in X_τ one deduces that $\phi \circ \sigma^i = \sigma^j$. Thus, ϕ is a power of the shift map. $\square$

5.4.2. Coding a nil-translation

We introduce a class of examples of symbolic systems with polynomial complexity of arbitrarily high degree and with a group of automorphisms which is virtually $\mathbb{Z}$. We build these systems as symbolic extensions of minimal nilsystems.

We start by stating some general results we need and then review some generalities about the coded systems.

Let (X, T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite collection of subsets of X. We say that $\mathcal{U}$ covers X if $\bigcup_{i=1}^m U_i = X$. For two covers $\mathcal{U} = \{U_1, \dots, U_m\}$ and $\mathcal{V} = \{V_1, \dots, V_p\}$ of X we let $U \vee V$ denote the cover given by $\{U_i \cap V_j; i = 1, \dots m,\ j = 1, \dots p\}$.

Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be a finite cover of X and let $\mathcal{A}$ denote the set $\{1, \dots, m\}$. We say that $\omega = (w_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ is a $\mathcal{U}$-name of x if $x \in \bigcap\limits_{i \in \mathbb{Z}} T^{-i} U_{w_i}$. Let $X_{\mathcal{U}}$ denote the set

$$\{\omega \in A^{\mathbb{Z}}; \bigcap_{i \in \mathbb{Z}} T^{-i} U_{w_i} \neq \emptyset\} \subseteq \mathcal{A}^{\mathbb{Z}}.$$

It is easy to check that $X_{\mathcal{U}}$ is closed when each U_i is closed and if we let $\overline{\mathcal{U}}$ denote the collection $\{\overline{U}_1, \dots, \overline{U}_m\}$ we have that $\overline{X_{\mathcal{U}}} \subset X_{\overline{\mathcal{U}}}$. For $N \in \mathbb{N}$, let

$$\mathcal{U}_N = \bigvee_{i=-N}^{N} T^{-i} \mathcal{U}.$$

We say that the cover $\mathcal{U}$ separates points if every $\omega \in X_{\overline{\mathcal{U}}}$ is a name of exactly one $x \in X$.

If $\mathcal{U}$ separates points in X, we can build a factor map π between $(\overline{X_{\mathcal{U}}}, \sigma)$ and (X, T) where $\pi(\omega)$ is defined as the unique point in $\bigcap_{i \in \mathbb{Z}} T^{-i}\overline{U_{w_i}}$.

Lemma 5.4.2. *Let (X,T) be a minimal topological dynamical system and let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a partition which covers and separates points in X. Suppose that for every $N \in \mathbb{N}$ every atom of $\mathcal{U}_N$ has non-empty interior, then $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal system.*

Proof. Let $\omega, \omega' \in \overline{X_{\mathcal{U}}}$ and let $N \in \mathbb{N}$. We denote $x = \pi(\omega)$ and $x' = \pi(\omega')$. By definition we have that $\bigcap_{-N}^{N} T^{-i}U_{w_i} \neq \emptyset$ and therefore it has non-empty interior. Since (X,T) is minimal, there exists $n \in \mathbb{Z}$ such that $T^n x' \in \mathrm{int}(\bigcap_{-N}^{N} T^{-i}U_{w_i})$. This implies that $w'_{[n-N,n+N]} = w_{[-N,N]}$ and the proof is finished. $\square$

Now we compute the automorphism groups of symbolic extensions of some nilsystems. First we recall the construction of the systems studied in [1]. Let us consider the infinite matrix $A = (a_{i,j})_{i,j \in \mathbb{N}}$ where $a_{i,j} = \binom{j}{i}$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ & 1 & 2 & 3 & 4 & \cdots \\ & & 1 & 3 & 6 & \cdots \\ & & & 1 & 4 & \cdots \\ & & & & 1 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}.$$

It is proven in Section 4 of [1] that for all $i \in \mathbb{N}$, A^i is well defined and

$$A^i = \begin{pmatrix} 1 & i & i^2 & i^3 & i^4 & \cdots \\ & 1 & 2i & 3i^2 & 4i^3 & \cdots \\ & & 1 & 3i & 6i^2 & \cdots \\ & & & 1 & 4i & \cdots \\ & & & & 1 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}.$$

Let $\alpha \in [0,1]$ be an irrational number. For any $d \in \mathbb{N}$, consider A_{d+1} the restriction of A to $(d+1) \times (d+1)$ coordinates. We let $T_d \colon \mathbb{T}^d \to \mathbb{T}^d$ denote the function that maps $(x_0, \ldots x_{d-1})$ to the d first coordinates of $A_{d+1}(x_0, \ldots, x_{d-1}, \alpha)^t$. For example, T_2 is the function $(x_0, x_1) \mapsto (x_0 + x_1 + \alpha, x_1 + 2\alpha)$ and T_3 is the function $(x_0, x_1, x_2) \mapsto (x_0 + x_1 + x_2 + \alpha, x_1 + 2x_2 + 3\alpha, x_2 + 3\alpha)$.

We can represent the transformation T_d as $T_d(x) = A_d x + \vec{\alpha}$ where $\vec{\alpha}$ is the restriction to the first d-coordinates of the last column of A_{d+1} multiplied by α. This is the classical presentation of an affine nilsystem.

Fix $d \in \mathbb{N}$ and for $i, n \in \mathbb{Z}$ let $H_{i,n}$ be the plane given by the equation $\sum_{k=0}^{d-1} i^k x_k + i^d \alpha = n$. It can be proven that $H_{i,n} = T_d^{-i} H_{0,n}$ and for a fixed value of i, the planes $H_{i,n}$ are projected in $\mathbb{T}^d$ to the same plane $\widehat{H}_i$. We remark that

$$\widehat{H}_0 = \{(0, x_1, \ldots, x_{d-1}); (x_1, \ldots, x_{d-1}) \in \mathbb{T}^{d-1}\}.$$

We refer to Section 4 of [1] for further details.

We consider the partition $\mathcal{U}$ given by the cells generated by the planes $\widehat{H}_0, \ldots, \widehat{H}_{d-1}$. The partition $\bigvee_{i=0}^{n+d-1} T_d^{-i}\mathcal{U}$ coincides with the cells generated by the planes $\widehat{H}_0, \ldots, \widehat{H}_{n+d-1}$ (see Section 6 of [1]). Let $(x_0, \ldots, x_{d-1})$ and $(y_0, \ldots, y_{d-1})$ be different points in $\mathbb{T}^d$ and let $\overline{k} = \max\{k; x_k \neq y_k\}$. Then the difference (in $\mathbb{R}$) between $\sum_{k=0}^{d-1} i^k x_k + i^d\alpha$ and $\sum_{k=0}^{d-1} i^k y_k + i^d\alpha$ grows to infinity as i goes to infinity since this difference behaves like $i^{\overline{k}}(x_{\overline{k}} - y_{\overline{k}})$. This implies that for big enough N, $(x_0, \ldots, x_{d-1})$ and $(y_0, \ldots, y_{d-1})$ lie on different cells of $\bigvee_{i=-N}^{N} T_d^{-i}\mathcal{U}$ since for big enough i these points are separated by the cells generated by $\widehat{H}_i$.

We recall that $(\overline{X_{\mathcal{U}}}, \sigma)$ is the subshift associated to $\mathcal{U}$. By Lemma 5.4.2, one can see that $(\overline{X_{\mathcal{U}}}, \sigma)$ is a minimal system and it is an extension of $(\mathbb{T}^d, T_d)$ since $\mathcal{U}$ separates points. Moreover, the complexity function of $(\overline{X_{\mathcal{U}}}, \sigma)$ is given by

$$p(n) = \frac{1}{V(0, 1, \ldots, d-1)} \sum_{0 \leq k_1 < k_2 < \cdots < k_d \leq n+d-1} V(k_1, k_2, \cdots, k_d)$$

where $V(k_1, k_2, \cdots, k_d) = \prod_{1 \leq i < j \leq d} (k_j - k_i)$ is a Vandermonde determinant. We remark that varying $d \in \mathbb{N}$ we get an arbitrarily large complexity with a polynomial growth.

By construction and Corollary 5.2.6 we also get:

Lemma 5.4.3. *The maximal d-step nilfactor of* $(\overline{X_{\mathcal{U}}}, \sigma)$ *is the nilsystem* $(\mathbb{T}^d, T_d)$.

Proposition 5.4.4. *The group* $\mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma)$ *is virtually* $\mathbb{Z}$.

Proof. Let ϕ be an automorphism of $(\overline{X_{\mathcal{U}}}, \sigma)$ and let $\pi \colon \overline{X_{\mathcal{U}}} \to \mathbb{T}^d$ be the natural factor map. Let $W = \{\omega \in X; \#\pi^{-1}\{\pi(\omega)\} \geq 2\}$ be the set of points where π is not one-to-one. Since ϕ preserves the regionally proximal pairs of order d, we have that W is invariant under ϕ. We remark that the projection of W under π are the points which fall in $F := \widehat{H}_0 \cup \widehat{H}_1 \cup \cdots \cup \widehat{H}_{d-1}$ under some power of T, which is nothing but $\bigcup_{j\in\mathbb{Z}} T^j F = \bigcup_{j\in\mathbb{Z}} T^j \widehat{H}_0$. We have that the projection $\widehat{\pi}(\phi)$ is an automorphism that commutes with the affine ergodic transformation T which has eigenvalues equal to 1. By Theorem 2 and Corollary 1 in [116] we have that $\widehat{\pi}(\phi)$ has the form $(x_0, \ldots, x_{d-1})^t \mapsto B(x_0, \ldots, x_{d-1})^t + \beta$ where B is an integer matrix and $\beta \in \mathbb{T}^d$. Since W is invariant under ϕ we get that the projection $\widehat{\pi}(\phi)$ leaves invariant $\bigcup_{j\in\mathbb{Z}} T^j \widehat{H}_0$. Particularly, since $\widehat{H}_0$ is the restriction of a plane to $\mathbb{T}^d$, so is its image under $\widehat{\pi}(\phi)$ and therefore there exists $j \in \mathbb{Z}$ such that $\widehat{\pi}(\phi)\widehat{H}_0 = T^j \widehat{H}_0$. Hence, the automorphism $T^{-j}\widehat{\pi}(\phi)$ leaves invariant $\widehat{H}_0$. So we are left to study the automorphisms of $(\mathbb{T}, T_d)$ which leave invariant $\widehat{H}_0$. Let φ be such an automorphism. By [116] we can assume that φ has the form

$$\varphi \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} = B \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{pmatrix}$$

where the matrix $B = (b_{j,k})_{j,k=1\ldots,d}$ has integer entries and $\vec{\beta} = (\beta_0, \ldots, \beta_{d-1})^t \in \mathbb{T}^d$. Since φ commutes with T we have that B commutes with A_d (as real matrices) and $(B - Id)\vec{\alpha} = (A_d - Id)\vec{\beta}$ in $\mathbb{T}^d$.

Since $\varphi(0, x_1, \ldots, x_{d-1}) \in \widehat{H}_0$, for any $(x_1, \ldots, x_{d-1}) \in \mathbb{T}^{d-1}$ we deduce that $b_{1,2} = \cdots = b_{1,d} = 0 = \beta_0$. Since $A_d^i B = B A_d^i$ for any $i \in \mathbb{N}$, by looking at the first row of these matrices we deduce that for any $j = 1, \ldots, d$ and any $i \in \mathbb{N}$

$$\sum_{k=1,k\neq j}^{d} (b_{j,k}) i^{k-1} + (b_{j,j} - b_{1,1}) i^{j-1} = 0.$$

Since the vectors $(1, i, i^2, \ldots, i^{d-1})$ are linearly independent for different values of i we deduce that $B = b_{1,1} I_d$. Therefore, $(A_d - Id)\beta = (B - Id)\vec{\alpha} = (b_{1,1} - 1)\vec{\alpha}$. Since A_d is upper triangular with ones in the diagonal, this condition implies that $(b_{1,1} - 1)\alpha \in \mathbb{Q}$ and thus $b_{1,1} = 1$. We conclude that B is the identity matrix and then φ is the rotation by $\vec{\beta} := (0, \beta_1, \ldots, \beta_{d-1})^t$ and $(A_d - Id)\vec{\beta} \in \mathbb{Z}^d$. We can write this system as

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ & 0 & 2 & 3 & 4 & \cdots \\ & & 0 & 3 & 6 & \cdots \\ & & & 0 & 4 & \cdots \\ & & & & 0 & d \\ & & & & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \beta_1 \\ \beta_1 \\ \vdots \\ \beta_{d-1} \end{pmatrix} \in \mathbb{Z}.$$

This implies that $d\beta_{d-1} \in \mathbb{Z}$ and this is possible for finitely many $\beta_{d-1} \in \mathbb{T}$. Inductively, we deduce that there are finitely many (and rational) solutions $\vec{\beta} = (0, \beta_1, \ldots, \beta_{d-1})^t$ in $\mathbb{T}^d$. This means that the group of automorphisms that leaves invariant $\widehat{H}_0$ is a finite group of rational rotations. Therefore, $\widehat{\pi}(\mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma))$ is spanned by T_d and a finite set. Since the map $\widehat{\pi}\colon \mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma) \to \mathrm{Aut}(\mathbb{T}, T_d)$ is an injection we have that $\mathrm{Aut}(\overline{X_{\mathcal{U}}}, \sigma)$ is spanned by σ and a finite set. The result follows. $\square$

5.4.3. Example of a larger automorphism group

We remark that the statement of Theorem 5.1.1 is no longer valid for an arbitrary polynomial complexity, as the following shows.

Proposition 5.4.5. *For any $d \in \mathbb{N}$, there exists a minimal subshift (X, σ) with complexity satisfying* $\lim_{n\to+\infty} p_X(n)/n^{d+1} = 0$ *and where* $\mathrm{Aut}(X, \sigma)$ *is isomorphic to* $\mathbb{Z}^d$*.*

Proof. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{R} \setminus \mathbb{Q}$ be rationally independent numbers. For every $i = 1, \ldots, d$, let (X_i, σ_i) be the Sturmian extension of the rotation R_{α_i} by the angle α_i on the circle $\mathbb{S}^1$, and let $X = X_1 \times X_2 \cdots \times X_d$ and $\sigma = \sigma_1 \times \sigma_2 \cdots \times \sigma_d$. We remark that for any $i = 1, \ldots, d$, on (X_i, σ_i) the proximal relation and the regionally proximal relation coincide and thus the proximal relation is an equivalence relation. Since the maximal equicontinuous factor of (X_i, σ_i) is $(\mathbb{S}^1, R_{\alpha_i})$ via the factor map π_i, by [7], Chapter 11, theorems 7 and 9, we have that (X, σ) is a minimal system and the product system $((\mathbb{S}^1)^d, R_{\alpha_1} \times \cdots \times R_{\alpha_d})$ is its maximal equicontinuous factor.

The complexity function of any (X_i, σ_i) is $n + 1$, so we get that the complexity function of (X, σ) is $(n + 1)^d$. On the other hand, we observe that $\phi_1 \times \cdots \times \phi_d$ belongs to $\mathrm{Aut}(X, \sigma)$ for any choice of $\phi_i \in \mathrm{Aut}(X_i, \sigma_i)$. Since for every i, $\mathrm{Aut}(X_i, \sigma_i)$ is $\mathbb{Z}$, we conclude that $\mathbb{Z}^d$ can be embedded as a subgroup of $\mathrm{Aut}(X, \sigma)$.

We claim this embedding is actually an isomorphism. To prove this, recall that the Sturmian subshift X_i is an almost one-to-one extension of a rotation on the circle via an onto map $\pi_i \colon X_i \to \mathbb{S}^1$ that it is injective except on the orbit of the unit $\mathcal{O}_{R_{\alpha_i}}(1)$, where any point has two pre-images (*e.g.*, see [39]). By Lemma 5.2.10, for any automorphism $\phi \in \mathrm{Aut}(X, \sigma)$, the automorphism $\hat{\pi}(\phi)$ preserves the set of points in $(\mathbb{S}^1)^d$ having a maximum number (namely 2^d) of pre-images for the factor map $\pi = \pi_1 \times \cdots \times \pi_d$. This set is the product set $\mathcal{O}_{R_{\alpha_1}}(1) \times \cdots \times \mathcal{O}_{R_{\alpha_d}}(1)$. Clearly, the group of automorphisms of the form $\hat{\pi}(\sigma_1^{n_1} \times \cdots \times \sigma_d^{n_d})$, $n_1, \ldots, n_d \in \mathbb{Z}$, acts transitively on this set. Since the group $\mathrm{Aut}((\mathbb{S}^1)^d, R_{\alpha_1} \times \cdots \times R_{\alpha_d})$ acts freely and the morphism $\hat{\pi}$ is injective (Lemma 5.2.4), we get that any automorphism $\phi \in \mathrm{Aut}(X, \sigma)$ may be written as a product of automorphisms in $\mathrm{Aut}(X_i, \sigma_i)$. $\square$

5.4.4. Subshift with subexponential complexity

In this section we give an example of a minimal subshift (X, σ) generated by a uniformly recurrent sequence $x \in \{0, 1\}^{\mathbb{Z}}$ such that:

- There exists C such that for infinitely many n's one has $p_X(n) \leq Cn$.
- For any subexponential function ϕ there are infinitely many n's such that $p_X(n) \geq \phi(n)$, where subexponential means that $\lim_{n\to+\infty} \phi(n)/\alpha^n = 0$ for all $\alpha \in \mathbb{R}$.

As for subshifts $p_z(n)$ will stand for the number of words of length n occurring in the sequence $z \in \{0, 1\}^{\mathbb{Z}}$ or $z \in \{0, 1\}^{\mathbb{N}}$.

The proofs of the two following lemmas are left to the reader.

Lemma 5.4.6. *Let ξ be a substitution on $\{0,1\}^*$ of constant length L and τ be an endomorphism of $\{0,1\}^*$ having all words of length 2 in its images. Let $x \in \{0,1\}^{\mathbb{N}}$. Then, for any $y \in \{0,1\}^{\mathbb{N}}$ having occurrences of all words of length 2 and $0 \leq l \leq L$ we have*

$$p_{\xi\circ\tau(x)}(l) = p_{\xi(y)}(l).$$

Below ρ stands for the Morse substitution: $\rho(0) = 01$ and $\rho(1) = 10$.

Lemma 5.4.7. *Let ξ be a substitution on $\{0,1\}^*$ of constant length L. Let $x \in \{0,1\}^{\mathbb{N}}$. We have*

$$p_{\xi\circ\rho^3(x)}(2L) \leq 6L.$$

Below, when a substitution τ is of constant length L we set $|\tau| = L$. Let us construct inductively the sequence x. In fact, we will construct two increasing sequences of integers $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$, and a sequence of morphisms $(\tau_i)_{i\geq 1}$ such that

1. $x = \lim_{i\to\infty} \rho^3\tau_1 \dots \rho^3\tau_i(0^\infty)$, where $0^\infty = 00\cdots$,
2. $a_1 < b_1 < a_2 < b_2 < \dots$,
3. $p_x(a_i) \leq 3a_i$, $i \in \mathbb{N}$ and
4. $p_x(b_i) \geq \phi(b_i)$, $i \in \mathbb{N}$.

We start fixing $a_1 = 2$. Let $x^{(1)} = \rho^3(0^\infty)$. Then, $p_{x^{(1)}}(a_1) = 4$, which is less than $3a_1$.

Let k_1 be such that $2^{k_1} \geq \phi(k_1|\rho^3|)$ (observe it is always possible because ϕ has a subexponential growth) and τ_1 be a substitution of $\{0,1\}^*$ of length $L_1 = 2^{m_1}$ such that $\tau_1(0)$ starts with 0 and the number of words of length k_1 in $\tau_1(0)$ and $\tau_1(1)$ is 2^{k_1}. We set

$$b_1 = k_1|\rho^3| \text{ and } y^{(1)} = \rho^3\tau_1(0^\infty).$$

One gets

$$p_{y^{(1)}}(b_1) \geq \phi(b_1).$$

Moreover, notice that from Lemma 5.4.6 one has that

$$p_{y^{(1)}}(l) = p_{x^{(1)}}(l)$$

for all $l \leq |\rho^3|$. Now consider $x^{(2)} = \rho^3\tau_1\rho^3(0^\infty)$. Then from Lemma 5.4.7

$$p(2|\rho^3\tau_1|) \leq 6|\rho^3\tau_1|.$$

Setting $a_2 = 2|\rho^3\tau_1|$, one gets $p_{x^{(2)}}(a_2) \leq 3a_2$.

Let $k_2 \geq k_1$ be such that $2^{k_2} \geq \phi(k_2|\rho^3\tau_1\rho^3|)$ and τ_2 be a substitution of $\{0,1\}^*$ of length $L_2 = 2^{m_2}$ such that the number of words of length k_2 in $\tau_2(0)$ and $\tau_2(1)$ is 2^{k_2}. We set

$$b_2 = k_2|\rho^3\tau_1\rho^3| \text{ and } y^{(2)} = \rho^3\tau_1\rho^3\tau_2(0^\infty).$$

One gets that $p_{y^{(2)}}(b_2)$ is greater than $\phi(b_2)$. Moreover, notice that from Lemma 5.4.6 one has that

$$\begin{aligned} p_{y^{(2)}}(l) &= p_{x^{(2)}}(l) \quad \forall l \leq |\rho^3\tau_1\rho^3|, \\ p_{x^{(2)}}(l) &= p_{y^{(1)}}(l) \quad \forall l \leq |\rho^3\tau_1|, \\ p_{y^{(1)}}(l) &= p_{x^{(1)}}(l) \quad \forall l \leq |\rho^3|. \end{aligned}$$

Thus, $p_{y^{(2)}}(a_1) \leq 3a_1$, $p_{y^{(2)}}(b_1) \geq \phi(b_1)$ and $p_{y^{(2)}}(a_2) \leq 3a_2$.

Now suppose we have constructed:

1. morphisms τ_i of constant length such that $\tau_i(0)$ starts with 0, $1 \leq i \leq n$,
2. $x^{(i)} = \rho^3\tau_1 \dots \tau_{i-1}\rho^3(0^\infty)$,
3. $y^{(i)} = \rho^3\tau_1 \dots \tau_{i-1}\rho^3\tau_i(0^\infty)$, $1 \leq i \leq n$,
4. $a_1 < b_1 < a_2 < \cdots < a_n < b_n$, such that
   a) $a_i = 2|\rho^3\tau_1 \dots \rho^3\tau_{i-1}|$,
   b) $b_i \geq |\rho^3\tau_1 \dots \rho^3\tau_{i-1}\rho^3|$,
   c) $x^{(i)}[0, |\rho^3\tau_1 \dots \tau_{i-1}\rho^3|]$ is a prefix of $y^{(i)}$,
   d) $y^{(i)}[0, |\rho^3\tau_1 \dots \tau_{i-1}\rho^3\tau_i|]$ is a prefix of $x^{(i+1)}$,
   e) $p_{y^{(i)}}(l) = p_{x^{(i)}}(l)$ for all $l \leq |\rho^3\tau_1 \dots \tau_{i-1}\rho^3|$,
   f) $p_{y^{(n)}}(a_i) \leq 3a_i$, $1 \leq i \leq n$, and,
   g) $p_{y^{(n)}}(b_i) \geq \phi(b_i)$, $1 \leq i \leq n-1$.

We have seen this construction is realizable for $n = 2$. Proceeding as we did for the first cases, it is not difficult to see that it can be achieved for every $n \geq 1$.

To conclude, it suffices to observe that $(y^{(n)})_{n\geq 1}$ converges to the sequence x we are looking for. Indeed, the convergence follows from $(4c)$ and $(4d)$. Also observe that $y^{(n)}$ is a prefix of x. It is a classical exercise to show that x is uniformly recurrent. From $(4g)$ we get that $p_x(b_i) \geq \phi(b_i)$ for all $i \in \mathbb{N}$. For the last point, $p_x(a_i) \leq 3a_i$ for all $i \in \mathbb{N}$, it comes from $(4f)$ because it is true for all $n \geq 1$.

5.5. Comments and open questions

A standard question related to automorphisms is to determine if the transformation T has a root. That is, does it exist a transformation U such that $U^p = T$ for some integer $p \geq 0$. A classical way to deal with this problem is to notice that a root is an automorphism.

The automorphism group is also related to the collection of conjugacy maps between two systems. If π_1 and π_2 are two conjugacy maps between the same systems, then $\pi_1 \circ \pi_2^{-1}$ is an automorphism. Hence, a characterization of when the automorphism group is trivial, *i.e.* $\mathrm{Aut}(X,T)$ is generated by T, implies rigidity results in both problems.

5.5.1. Automorphisms and nilfactors

We have shown that a large family of minimal subshifts, either with sublinear or other type of polynomial complexity, have automorphisms groups that are virtually $\mathbb{Z}$. Even in the case of minimal subshifts obtained as extensions of minimal systems whose automorphism group is much complex (the case of extensions of nilsystems). So a natural question is whether this behaviour is generally true just because the fibres over particular topological factors are constrained.

5.5.2. Eigenvalues, roots of T and automorphisms.

We obtain, in the good cases, that the group of automorphisms is a subgroup of the corresponding one of a maximal nilfactor. This proves that there are connections between automorphisms and continuous eigenvalues. To study these relations we can focus on rational eigenvalues. So it is natural to ask: does a Cantor minimal system (X,T) admit a non trivial automorphism with finite order or have some roots, are there constraints on the rational continuous eigenvalues of (X,T) ?

Classical examples of Toeplitz sequences with a unique asymptotic component (so $\mathrm{Aut}(X,\sigma)$ is generated by σ) show that the converse is false: a system may have rational eigenvalues and no automorphisms of finite order.

5.5.3. Complexity versus group of automorphisms

The results of [29] and of this paper show a relation between complexity and the growth rate of the groups. Is it possible to be more precise ? For instance, is it true that for a transitive subshift with a subquadratic complexity the group $\mathrm{Aut}(X,\sigma)/\langle\sigma\rangle$ is finite? Very recently, Salo [108] showed a Toeplitz subshift with subquadratic complexity and whose automorphism group is not finitely generated, answering negatively this question. So, in the polynomial complexity case one cannot expect to have always virtually $\mathbb{Z}$ groups of automorphisms. It is an interesting question to describe automorphism groups of subshifts with polynomial complexity.

5.5.4. Measurable versus continuous automorphisms

The main result in [74] shows a rigidity result, any measurable automorphism is almost everywhere continuous for bijective constant length substitutions. Is it possible to enlarge this class of subshifts with the same rigidity property ? A first answer is negative: This is not true for substitution of non-constant length and even for Pisot substitution on the alphabet $\{0, 1\}$. Consider the two substitutions τ and ξ defined by $\tau(0) = 010$, $\tau(1) = 01$, $\xi(0) = 001$ and $\xi(1) = 10$. Let (X_τ, σ) and (X_ξ, σ) be the subshift they generate. It can be shown that they are both measure theoretically isomorphic to $(\mathbb{S}^1, R_\alpha)$ (see [10]), where R_α is the rotation of angle $\alpha = (1 + \sqrt{5})/2$, and, thus (X_τ, σ) and (X_ξ, σ) are measure theoretically isomorphic. But they cannot be topologically isomorphic because their dimension groups are not isomorphic (see [38] for their computations).

5.5.5. Realization of automorphism groups

By the Curtis-Hedlund-Lyndon theorem, the collection of automorphisms of a subshift is countable. We leave open the realization of any countable group as an automorphism group. More precisely,

Question. Given a countable group G (not necessarily finitely generated). Does it exist a minimal subshift (X, σ) such that $\mathrm{Aut}(X, \sigma)/\langle\sigma\rangle$ is isomorphic to G ?

Notice that Toeplitz sequences can also be realized on residually finite groups [27]. A priori, they may provide interesting solutions in this class. But, as stated in the remark below Corollary 5.2.11, their automorphism group is abelian. This kills any non commutative group realization by this way.

If we restrict to some families of subshifts (*e.g.* Sturmian or Toeplitz sub shifts), we prove that their automorphism groups are subgroups of their maximal equicontinuous factors. Can we characterize these groups for the Sturmian and Toeplitz cases ?

5.6. Computing the group of automorphisms of tilings by using cubes

In this section we show an application of the cubes introduced in Chapter 2 to study automorphism groups of two dimensional tilings. Even if one wants to understand the automorphism group of a one dimensional subshift, the study of the two (or higher) dimensional setting could provide useful information for the one dimensional one. The work of Cyr and Kra [29] illustrates this fact.

In what follows, we compute a special factor (built using cubes) of the minimal part of the Robinson tiling. The Robinson tiling was introduced by Robinson [105] in the 70's to study undecidability problems and showed how to tile the plane in a nonperiodical way. This tile has been well studied in symbolic dynamics, specially in the context of theoretical computer science. We refer to [103] and [52] for further details.

We give a useful general result and then we briefly introduce the Robinson tiling.

Lemma 5.6.1. *Let (X, S, T) be a minimal system with commuting transformations S and T, and let ϕ be an automorphism of (X, S, T). Then $\phi\times\phi\times\phi\times\phi(\mathbf{Q}_{S,T}(X)) = \mathbf{Q}_{S,T}(X)$. Particularly, if $(x, y) \in \mathcal{R}_S(X)$ (or $\mathcal{R}_T(X)$ or $\mathcal{R}_{S,T}(X)$), then $(\phi(x), \phi(y)) \in \mathcal{R}_S(X)$ (or $\mathcal{R}_T(X)$ or $\mathcal{R}_{S,T}(X)$).*

Proof. We recall that G denotes the $\mathbb{Z}^2$ action spanned by S and T. Let $\mathbf{x} \in \mathbf{Q}_{S,T}(X)$ and let $x \in X$. There exist sequences $(g_i)_{i\in\mathbb{N}}$ in G and $(n_i)_{i\in\mathbb{N}}$, $(m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ such that $(g_i x, g_i S^{n_i}x, g_i T^{m_i}x, g_i S^{n_i}T^{m_i}x) \to \mathbf{x}$. Since $(\phi(x), \phi(x), \phi(x), \phi(x)) \in \mathbf{Q}_{S,T}(X)$ we have that

$$\begin{aligned}
&(g_i\phi(x), g_i S^{n_i}\phi(x), g_i T^{m_i}\phi(x), g_i S^{n_i}T^{m_i}\phi(x)) \in \mathbf{Q}_{S,T}(X)\\
=&(\phi(g_i x), \phi(g_i S^{n_i}x), \phi(g_i T^{m_i}x), \phi(g_i S^{n_i}T^{m_i}x)) \in \mathbf{Q}_{S,T}(X)\\
\to&(\phi\times\phi\times\phi\times\phi)(\mathbf{x}) \in \mathbf{Q}_{S,T}(X).
\end{aligned}$$

Hence $\phi\times\phi\times\phi\times\phi(\mathbf{Q}_{S,T}(X)) = \mathbf{Q}_{S,T}(X)$.

If $(x, y) \in \mathcal{R}_S(X)$, then there exists $a \in X$ with $(x, y, a, a) \in \mathbf{Q}_{S,T}(X)$ and thus $(\phi(x), \phi(y), \phi(a), \phi(a)) \in \mathbf{Q}_{S,T}(X)$. This means that $(\phi(x), \phi(y)) \in \mathcal{R}_S(X)$. The proof for the cases $\mathcal{R}_S(X)$ and $\mathcal{R}_{S,T}(X)$ are similar. □

Remark 5.6.2. In general we do not know if $\mathcal{R}_S(X)$ (or $\mathcal{R}_T(X)$ or $\mathcal{R}_{S,T}(X)$) is an equivalence relation. In any case, if $\sigma(\mathcal{R}_S(X))$ is the smallest closed and $T\times T$-invariant relation generated by $\mathcal{R}_S(X)$ one easily check that $(x, y) \in \sigma(\mathcal{R}_S(X))$ if and only if $(\phi(x), \phi(y)) \in \sigma(\mathcal{R}_S(X))$. Therefore the factor map $\pi\colon X \to X/\sigma(\mathcal{R}_S(X))$ is compatible with $\mathrm{Aut}(X, T)$.

5.6.1. The Robinson Tiling

Consider the following set of tiles and their rotations and reflections:

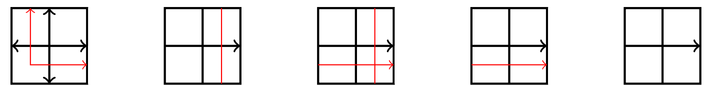


Figure 5.1: The Robinson Tiles (up to rotation and reflection). The first tile and its rotations are called crosses.

Let $\mathcal{A}$ be the set of the 28 Robinson tiles. Let $Y \subseteq \mathcal{A}^{\mathbb{Z}^2}$ be the subshift defined by the following rules:

1. The outgoing arrows match with the ingoing arrows;
2. There exists $\vec{n} \in \mathbb{Z}^2$ such that there is a cross in every position of the form $\{\vec{n}+(2i,2j)\}$ for $i,j \in \mathbb{Z}$ (this means that there is a 2-lattice of crosses).

This system is not minimal but it has a unique minimal subsystem [52]. We let X_R denote this unique minimal subsystem. Then $(X_R, \sigma_{(1,0)}, \sigma_{(0,1)})$ is a minimal system with commuting transformations $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$ and we call it the *minimal Robinson system*. For $n \in \mathbb{N}$ we define *supertiles of order* n inductively. Supertiles of order 1 correspond to crosses and if we have defined supertiles of order n, supertiles of order $n+1$ are constructed putting together 4 supertiles of order n in a consistent way and adding a cross in the middle of them (see Figure 5.6.1). We remark that supertiles of order n have size $2^n - 1$ and they are completely determined by the cross in the middle. Particularly, for every $n \in \mathbb{N}$ there are four supertiles of order n. It can be proved [52], [103] that for every $x \in X_R$, given $n \in \mathbb{N}$, supertiles of order n appear periodically (figure 5.3 illustrates this phenomenon).

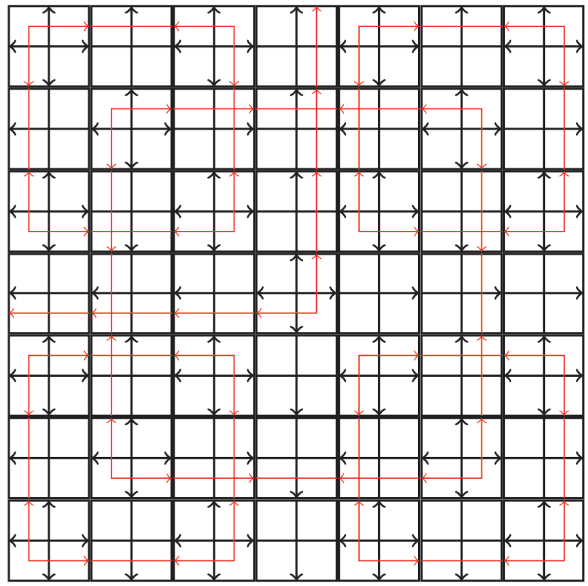


Figure 5.2: A supertile of order 3. The four 3x3 squares of the corners are supertiles of order 2.

Let $x \in X_R$. A horizontal line in x is the restriction of x to a set of the form $\{(i, j_0) \colon i \in \mathbb{Z}\}$ where $j_0 \in \mathbb{Z}$. Similarly, a vertical line in x is the restriction of x to a set of the form $\{(i_0, j) \colon j \in \mathbb{Z}\}$ where $i_0 \in \mathbb{Z}$. We remark that a line passing through the center of a supertile of order n has only one cross restricted to the supertile. The presence of supertiles of any order, forces the the existence of lines (vertical or horizontal) with at most one cross that are called *fault lines*. A point $x \in X_R$ can have 0,1 or 2 fault lines. When x is a point with two fault lines, then these lines divide the plane in four quarter planes (one line is horizontal and the other is vertical). On each one of these quarter planes the point is completely determined. The tile in the intersection of two fault lines determines completely the fault lines and therefore this tile determines x. See [103], Chapter 1, Section 4 for more details.

Given a point $x \in X_R$ and $n \in \mathbb{N}$, supertiles of order n appear periodically, leaving lines between them (which are not periodic). We remark that the center of one of the supertiles of order n determines the distribution of all the supertiles of order n. We say that we *decompose x into supertiles of order n* if we consider the distribution of its supertiles of order n, ignoring the lines between them.

Let $B_n \coloneqq ([-2^{n-1}, 2^{n-1}] \cap \mathbb{Z}) \times ([-2^{n-1}, 2^{n-1}] \cap \mathbb{Z})$ be the square of side of size $2^n + 1$. Recall that $x|_{B_n} \in \mathcal{A}^{B_n}$ is the restriction of x to B_n. Then, looking at $x|_{B_n}$, we can find the center of at least one supertile of order n, and therefore we can determine the distribution of supertiles of order n in x. We remark that if x and y are points in X such that $x|_{B_n} = y|_{B_n}$, then we can find the same supertile of order n in the same position in x and y, and therefore

x and y have the same decomposition into tiles of order n.

We study the $\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R)$ relation in the minimal Robinson system. We have:

Proposition 5.6.3. *Let $(X_R, \sigma_{(1,0)}, \sigma_{(0,1)})$ be the minimal Robinson system. Then $(x, y) \in \mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R)$ if and only if they coincide in the complement of its fault lines. Particularly, points which have no fault lines are not related to any point by $\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R)$.*

Proof. We start computing the $\mathcal{R}_{\sigma_{(1,0)}}(X_R)$ relation. Let $x, y \in \mathcal{R}_{\sigma_{(1,0)}}(X_R)$ with $x \neq y$ (the case $\mathcal{R}_{\sigma_{(0,1)}}(X_R)$ is similar). Let $p \in \mathbb{N}$ be such that $x|_{B_p} \neq y|_{B_p}$ and let $x' \in X$, $n, m \in \mathbb{Z}$ and $z \in X_R$ with $x'|_{B_p} = x|_{B_p}$, $\sigma^n_{(1,0)}x'|_{B_p} = y|_{B_p}$, $\sigma^m_{(0,1)}x'|_{B_p} = z|_{B_p}$ and $\sigma^n_{(1,0)}\sigma^m_{(0,1)}x'|_{B_p} = z|_{B_p}$. Then $\sigma^n_{(1,0)}\sigma^m_{(0,1)}x'|_{B_p} = \sigma^m_{(0,1)}x'|_{B_p}$ and thus $\sigma^n_{(1,0)}\sigma^m_{(0,1)}x'$ and $\sigma^m_{(0,1)}x'$ have the same decomposition into supertiles of order p, which implies that x and y have also the same decomposition. Particularly, the difference between x and y must occur in the lines which are not covered by the supertiles of order p (we remark that these lines have at most one cross). Let L_p be such a line on x. For q larger than p, we decompose into tiles of order q and we conclude that L_p lies inside L_q. Taking the limit in q, we deduce that x and y coincide everywhere except in one or two fault lines.

Now suppose that x and y coincide everywhere except in fault lines. For instance, suppose that x and y have two fault lines and let $n \in \mathbb{N}$. We can find $z \in X_R$ with no fault lines and $p \in \mathbb{Z}$ such that $z|_{B_n} = x|_{B_n}$ and $\sigma^p_{(1,0)}z|_{B_n} = y_{B_n}$. Then, we can find a supertile of large order containing $z|_{B_n}$ and $\sigma^p_{(0,1)}z|_{B_n}$. Hence, along the horizontal we can find $q \in \mathbb{Z}$ such that $\sigma^q_{(0,1)}z|_{B_n} = \sigma^q_{(0,1)}\sigma^p_{(1,0)}z|_{B_n}$. Since n is arbitrary, we have that $(x, y) \in \mathcal{R}_{\sigma_{(1,0)}}(X_R)$. $\square$

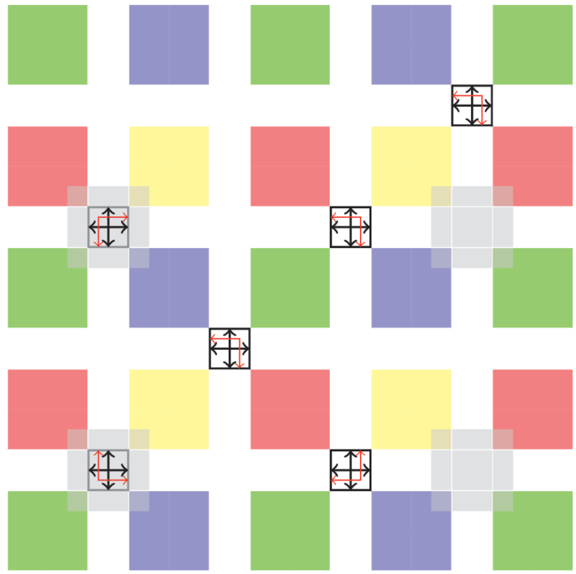


Figure 5.3: For an arbitrary $n \in \mathbb{N}$, the colored squares represent tiles of order n. In this picture we illustrate how points with two fault lines, with different crosses in the middle are related.

Let $\pi \colon X_R \to X_R/\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R)$ be the quotient map. Then in the minimal Robinson system we can distinguish three types of fibers for π: fibers with cardinality 1 (tilings with no fault lines), fibers with cardinality 6 (tilings with one fault line), and fibers with cardinality 28 (tilings with 2 fault lines).

Corollary 5.6.4. *The group of automorphisms of the minimal Robinson system is spanned by $\sigma_{(1,0)}$ and $\sigma_{(0,1)}$.*

Proof. Let $\pi \colon X_R \to X_R/\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R)$ be the quotient map. By Proposition 5.6.3 and Lemma 5.2.4 we have that that $\widehat{\pi} \colon \mathrm{Aut}(X_R, \sigma_{(1,0)}, \sigma_{(0,1)}) \to \mathrm{Aut}(X_R/\mathcal{R}_{\sigma_{(1,0)},\sigma_{(0,1)}}(X_R))$ is an injection. Let ϕ be a automorphism of the minimal Robinson system and let F be a fiber with maximum cardinality. Since π is a compatible factor map, we have that $\phi(F)$ is also a fiber with maximum cardinality, but there is only one (up to shift) fiber with maximum cardinality. This implies that $\phi(F) = \sigma^n_{(1,0)}\sigma^m_{(0,1)}(F)$ and therefore $\widehat{\pi}(\phi) = \widehat{\pi}(\sigma^n_{(1,0)}\sigma_{(0,1)})$. Since $\widehat{\pi}$ is an injection we get the result. $\square$

The Robinson tiling is a tiling space which has a "hierarchical structure", meaning that patterns that "look similar"appear with an arbitrary big size. This concept has not been mathematically formalized but many people use it when describing this kind of phenomenon. We believe our methods can be used to study automorphism groups of other tilings, or families of tilings having this property.

Perspectives

In this Chapter we present some open questions and comments that stem from the discussion in this thesis document. All these problems conform a future plan of research.

In Chapter 2 and 3 we have derived applications from introducing cube structures for a dynamical system given by two commuting transformations. The results discovered in this case let us think in the natural generalization of the new cubes when one introduce a larger number of transformations. There is a natural, and even obvious, way to do this: suppose that X is a compact metric space and that $T_1,\dots,T_d\colon X\to X$ are commuting transformations on X (*i.e.* $T_i\circ T_j=T_j\circ T_i$ for $i,j=1,\dots,d$). We should define the space of dynamical cubes $\mathbf{Q}_{T_1,\dots,T_d}(X)$ as the closure of the points

$$\{(T_1^{\epsilon_1 n_1}\cdots T_d^{\epsilon_d n_d}x)_{\epsilon\in\{0,1\}^d}:\ x\in X,\ n_1\dots,n_d\in\mathbb{Z}\}.$$

The space $\mathbf{Q}_{T_1,\dots,T_d}(X)$ is a topological dynamical system as well. It is invariant under the *diagonal transformations* $\widetilde{T}_i:=T_i\times\cdots\times T_i$ (2^d times), $i=1,\dots,d$ and under the *face transformations* $\widehat{T}_i$, $i=1,\dots,d$ defined as

$$\widehat{T}_i(\mathbf{x})=\begin{cases}(\widehat{T}_i\mathbf{x})_\epsilon=T_ix_\epsilon, & \epsilon_i=1;\\ (\widehat{T}_i\mathbf{x})_\epsilon=x_\epsilon, & \epsilon_i=0.\end{cases}$$

For example, for three commuting transformations T_1,T_2,T_3 on X, $\mathbf{Q}_{T_1,T_2,T_3}(X)$ is the closure of the set

$$\{(x,T_1^nx,T_2^mx,T_1^nT_2^mx,T_3^px,T_1^nT_3^px,T_2^mT_3^px,T_1^nT_2^mT_3^px):\ x\in X,\ n,m,p\in\mathbb{Z}\}.$$

Having proposed this space of dynamical cubes, beyond natural topological properties, the main question to be understood is “what means to deduce the last (or any) coordinate of a dynamical cube looking at the other ones ?”. The answer to this property must reflect the topological structure of the underlying dynamical system together with its structural factors. Then, the next step is to use the cube structures to build invariant closed relations and use them to build factors. Hopefully, those factors will have the property that a coordinate in a cube is determined by the other ones and thus it will have an understandable topological structure.

A second very involving question is whether the study of these cube structures could help to deduce other pointwise convergence results, as shown in Chapter 3. Given a probability space $(X, \mathcal{X}, \mu)$ and measure preserving commuting transformations $T_1, \ldots, T_d$, the structure $\mathbf{Q}_{T_1,\ldots,T_d}(X)$ (defined in a suitable topological representation of X) should help to study the average

$$\frac{1}{N^d} \sum_{0\leq n_1,\ldots,n_d<N} \prod_{\epsilon\in\{0,1\}^d\setminus\{\vec{0}\}} f_\epsilon(T_1^{\epsilon_1 n_1}\cdots T_d^{\epsilon_d n_d}x)$$

for bounded functions f_ϵ, $\epsilon \in \{0,1\}^d \setminus \{\vec{0}\}$.

For example, the cube structure $\mathbf{Q}_{T_1,T_2,T_3}$ should help to understand the average

$$\frac{1}{N^3} \sum_{\substack{0\leq n<N\\ 0\leq m<N\\ 0\leq p<N}} f_1(T_1^n x) f_2(T_2^m x) f_3(T_1^n T_2^m x) f_4(T_3^p x) f_5(T_1^n T_3^p x) f_6(T_2^m T_3^p x) f_7(T_1^n T_2^m T_3^p x)$$

for bounded functions $f_1, f_2, f_3, f_4, f_5, f_6, f_7$.

This of course requires to have a better understanding in the measure theoretical situation. For example, in order to study pointwise convergence of higher order cubic averages, one should study the sigma algebra $\bigvee \mathcal{I}_{T_i}$ in a convenient extension of the original system, like Host's magic extensions in [64]. Up to now, it is not clear what is the structure of this σ-algebra, and no topological representations are known. This is because at the time they were studied, they were used to get L^2 convergence of multiple averages and no representations were needed to achieve this result.

A very ambitious question that remained open during this thesis is the study of the pointwise convergence of the average

$$\frac{1}{N} \sum_{i=0}^{N-1} f_1(S^i x) f_2(T^i x), \tag{6.0.1}$$

where (X, μ, S, T) is an ergodic system with commuting transformation S and T. This question appeared naturally when studying the convergence of averages in Chapter 3. In fact, the hope was to use the new cubes or some modifications to solve the problem. Nevertheless, it was clear that we need to have a better understanding of another structures. In order to study the average 6.0.1 we propose to study the structure $N_{S,T}(X)$ defined as the closure in X^3 of

$$\{(x, S^i x, T^i x) : x \in X, i \in \mathbb{Z}\}.$$

Then translate the strategy proposed by Huang, Shao and Ye in [76] to this setting. That is, produce a topological representation of the system where our new structure is uniquely ergodic. Here we remark that the structure $N_{S,T}(X)$ is a topological dynamical: the transformations $S \times S \times S$, $T \times T \times T$ and $\mathrm{id} \times S \times T$ act on it. In order to get unique ergodicity of $N_{S,T}(X)$ we have to understand several σ-algebras like $\mathcal{I}_S \vee \mathcal{I}_T \vee \mathcal{I}_{S^{-1}T}$, $\mathcal{I}_S \vee \mathcal{I}_T$, $\mathcal{I}_T \vee \mathcal{I}_{S^{-1}T}$ and $\mathcal{I}_S \vee \mathcal{I}_{S^{-1}T}$. This is work in progress and we do not present further details here.

Another direction of research derived from Chapter 2 is to look for more applications of the $\mathbf{Q}_{S,T}$ cubes to the theory of tiling systems, as was done at the end of Chapter 5. A first step is to pass from $\mathbb{Z}^2$ actions to $\mathbb{R}^2$ actions (mainly because people interested in tiling theory consider $\mathbb{R}^2$ actions instead of $\mathbb{Z}^2$ actions). Then we will explore examples or classes of examples where the structure of fibers over the special factors produced by cubes can shed light of the structure of their automorphism groups.

In Chapter 4, we study the enveloping semigroup of a system of order d. We left open the converse of Theorem 4.1.1, namely: Does some property of the enveloping semigroup characterize systems or order d? We think we need some new tools coming from a pure topological analogue of the theory developed by Host and Kra [67] in measure preserving setting. Here we remark that the result by Host, Kra and Maass [70] uses results from the measure preserving context. To make a pure topological proof of the structure theorem in [70] does not seem to be an easy task. Very recently Gutman, Manners and Varjú have claimed to have a purely topological proof of the Host-Kra-Maass structure theorem, so we expect to apply some of their methods in the resolution of our problem.

Another problem we would like to tackle is to understand the automorphism group of one dimensional minimal subshifts with polynomial complexity. In this direction Cyr and Kra showed that a better understanding of the dynamics of multidimensional subshifts helps to deduce results about the automorphism group of one dimensional ones. Because of this, we are also interested in the study of asymptoticity and related notions in multidimensional subshifts. For example, we are interested in study relations given by special factors built through cube structures, as was done in Chapter 2. Many of these relations may result proximal (meaning that two points that are related need to be proximal) which is the case when the factor defined by the relation is an almost one-to-one extension. As was shown in Chapter 5 these kind of results allow to inject the group of automorphisms into the group of automorphisms of the factor and then one can have a better understanding by studying properties in the factor. In this topic we will also study what kind of countable groups can appear as the automorphism group of a minimal subshift. In particular, which groups are automorphisms groups of Toeplitz subshifts. In Chapter 5 we have shown that such groups are always abelian, but we do not know if any abelian group can be realized in this way.

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