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Calculus of Variations

Large conformal metrics with prescribed sign-changing Gauss curvature

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Abstract Let (M, g) be a two dimensional compact Riemannian manifold of genus $g(M) > 1$. Let f be a smooth function on M such that

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

Let $p_1, \ldots, p_n$ be any set of points at which $f(p_i) = 0$ and $D^2 f(p_i)$ is non-singular. We prove that for all sufficiently small $\lambda > 0$ there exists a family of "bubbling" conformal metrics $g_\lambda = e^{u_\lambda} g$ such that their Gauss curvature is given by the sign-changing function $K_{g_\lambda} = -f + \lambda^2$. Moreover, the family u_λ satisfies

$$u_\lambda(p_j) = -4 \log \lambda - 2 \log \left(\frac{1}{\sqrt{2}} \log \frac{1}{\lambda} \right) + O(1)$$

and

$$\lambda^2 e^{u_\lambda} \rightharpoonup 8\pi \sum_{i=1}^{n} \delta_{p_i}, \quad \text{as } \lambda \to 0,$$

where δ_p designates Dirac mass at the point p.

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1 Introduction

Let (M, g) be a two-dimensional compact Riemannian manifold. We consider in this paper the classical *prescribed Gaussian curvature problem*: Given a real-valued, sufficiently smooth funtion $\kappa(x)$ defined on M, we want to know if κ can be realized as the Gaussian curvature K_{g_1} of M for a metric g_1, which is in addition conformal to g, namely $g_1 = e^u g$ for some scalar function u on M.

It is well known, by the uniformization theorem, that without loss of generality we may assume that M has constant Gaussian curvature for g, $K_g =: -\alpha$. Besides, the relation $K_{g_1} = \kappa$ is equivalent to the following nonlinear partial differential equation

$$\Delta_g u + \kappa\, e^u + \alpha = 0, \quad \text{in } M, \tag{1.1}$$

where $\Delta_g = \mathrm{div}_g \nabla$ is the Laplace Beltrami operator on M. There is a considerable literature on necessary and sufficient conditions on the function κ for the solvability of the PDE (1.1). We refer the reader in particular to the classical references [3,7,12–14,17] and to [5] for a recent review of the state of the art for this problem.

Integrating Eq. (1.1), assuming that M has surface area equal to one, and using the Gauss-Bonet formula we obtain

$$\int_M \kappa e^u d\mu_g = \int_M K_g d\mu_g = -\alpha = 2\pi\chi(M), \tag{1.2}$$

where $\chi(M)$ is the Euler characteristic of the manifold M.

In what follows we shall assume that the surface M has **genus *g(M)* greater than one**, so that $\chi(M) = 2(1 - g(M)) < 0$ and hence

$$-K_g = \alpha > 0.$$

Then (1.2) tells us that a necessary condition for existence is that $\kappa(x)$ be negative somewhere on M. More than this, we must have that

$$\int_M \kappa d\mu_g < 0.$$

Indeed testing Eq. (1.1) against e^{-u} we get

$$\int_M \kappa d\mu_g = -\int_M (|\nabla_g u|^2 + \alpha) e^{-u} d\mu_g < 0. \tag{1.3}$$

Solutions u to Eq. (1.2) correspond to critical points in the Sobolev space $H^1(M, g)$ of the energy functional

$$E_\kappa(u) = \frac{1}{2}\int_M |\nabla_g u|^2 d\mu_g - \alpha \int_M u d\mu_g - \int_M \kappa e^u d\mu_g.$$

As observed in [3], since $\alpha > 0$, we have that if $\kappa \le 0$ and $\kappa \not\equiv 0$, then this functional is strictly convex and coercive in $H^1(M, g)$. It thus have a unique critical point which is a global minimizer of E_κ.

A natural question to ask is what happens when κ changes sign. A drastic change in fact occurs. If $\sup_M \kappa > 0$, then the functional E_κ is no longer bounded below, hence a global minimizer cannot exist. On the other hand, intuition would tell us that if κ is “not too positive” on a set “not too big”, then the global minimizer should persist in the form of a local minimizer. This is in fact true, and quantitative forms of this statement can be found in [1,4].

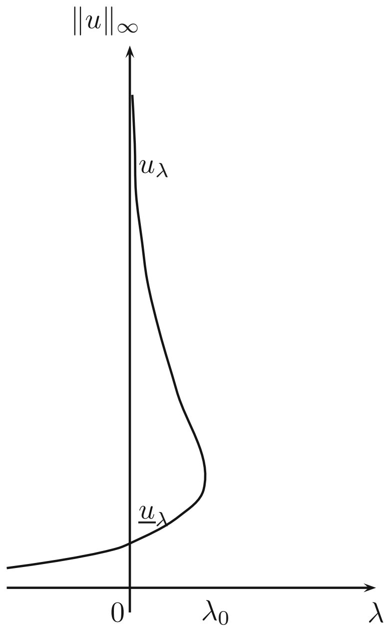

Fig. 1 Bifurcation diagram for solutions of problem (1.4)

We shall focus in what follows in a special class of functions $\kappa(x)$ which change sign being nearly everywhere negative. Let f be a function of class $C^3(M)$ such that

$$f \geq 0, \quad f \not\equiv 0, \quad \min_M f = 0.$$

For $\lambda > 0$ we let

$$\kappa_\lambda(x) = -f(x) + \lambda^2,$$

so that our problem now reads

$$\Delta_g u - f e^u + \lambda^2 e^u + \alpha = 0, \quad \text{in } M. \tag{1.4}$$

In [10], Ding and Liu proved that the global minimizer of E_{κ_0} persists as a local minimizer $\underline{u}_\lambda$ of E_{κ_λ} for any $0 < \lambda < \lambda_0$. From (1.3) we see that

$$\lambda_0 < \left(\int_M f \right)^{1/2}.$$

Moreover, they established the existence of a second, non-minimizing solution u_λ in this range. Uniqueness of the solution u_0 for $\lambda = 0$, and its minimizing character, tell us that we must have $\underline{u}_\lambda \to u_0$ as $\lambda \to 0$ while u_λ must become unbounded. The situation is depicted as a bifurcation diagram in Fig. 1.

The proof in [10] does not provide information on its asymptotic blowing-up behavior or about the number of such "large" solutions. Borer, Galimberti and Struwe [5] have recently provided a new construction of the mountain pass solution for small λ, which allowed them to identify further properties of it under the following generic assumption: points of global minima of f are non-degenerate. This means that if $f(p) = 0$ then $D^2 f(p)$ is positive definite. In [5] it is established that blowing-up of the family of large solutions u_λ occurs only near zeros of f, and the associated metric exhibits "bubbling behavior", namely Euclidean spheres emerge around some of the zero-points of f. In fact, the mountain-pass characterization let them estimate the number of bubbling points as no larger than four. More precisely, they find that along any sequence $\lambda = \lambda_k \to 0$, there exist points $p_1^k, \ldots, p_n^k$, $1 \le n \le 4$, converging to $p_1, \ldots, p_n$ points of global minima of f such that one of the following holds

(i) There exist $\varepsilon_\lambda^1, \ldots, \varepsilon_\lambda^k$, such that $\varepsilon_\lambda^i/\lambda \to 0$, $i = 1, \ldots, k$, and in local conformal coordinates around p_i there holds

$$u_\lambda(\varepsilon_\lambda^i x) - u_\lambda(0) + \log 8 \to w(x) := \log \frac{8}{(1+|x|^2)^2}, \tag{1.5}$$

smoothly locally in $\mathbb{R}^2$. We note that

$$\Delta w + e^w = 0, \quad \text{in } \mathbb{R}^2.$$

(ii) In local conformal coordinates around p_i, with a constant c_i there holds

$$u_\lambda(\lambda x) + 4\log(\lambda) + c_i \to w_\infty(x),$$

smoothly locally in $\mathbb{R}^2$, where w_∞ satisfies

$$\Delta w_\infty + [1 - (Ax, x)]e^{w_\infty} = 0, \quad \text{in } \mathbb{R}^2.$$

where $A = \frac{1}{2} D^2 f(p_i)$.

In this paper we will substantially clarify the structure of the set of large solutions of problem (1.4) with a method that yields both multiplicity and accurate estimates of their blowing-up behavior. Roughly speaking we establish that for any given collection of non-degenerate global minima of f, $p_1, \ldots, p_k$, there exist a solution u_λ blowing-up in the form (1.5) *exactly* at those points. Moreover

$$\varepsilon_\lambda^i \sim \frac{\lambda}{|\log \lambda|}, \quad u_\lambda(p_i) = -4\log\lambda - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + O(1).$$

In particular if f has exactly m non-degenerate global minimum points, then 2^m distinct large solutions exist for all sufficiently small λ.

In order to state our main result, we consider the singular problem

$$\Delta_g G - f e^G + 8\pi \sum_{i=1}^{n} \delta_{p_i} + \alpha = 0, \quad \text{in } M, \tag{1.6}$$

where δ_{p_i} designates the Dirac mass at the point p_i. We have the following result.

Lemma 1.1 *Problem* (1.6) *has a unique solution G which is smooth away from the singularities and in local conformal coordinates around p_i it has the form*

$$G(x) = -4\log|x| - 2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{|x|}\right) + \mathcal{H}(x), \tag{1.7}$$

where $\mathcal{H}(x) \in C(M)$.

Our main result is the following.

Theorem 1.1 *Let $p_1, \ldots, p_n$ be points such that $f(p_i) = 0$ and $D^2 f(p_i)$ is positive definite for each i. Then, there exists a family of solutions u_λ to* (1.4) *with*

$$\lambda^2 e^{u_\lambda} \rightharpoonup 8\pi \sum_{i=1}^{n} \delta_{p_i}, \quad as \quad \lambda \to 0,$$

and $u_\lambda \to G$ *uniformly in compacts subsets of* $M \setminus \{p_1, \dots, p_k\}$. *We define*

$$c_i = \frac{1}{2} e^{\mathcal{H}(p_i)/2}, \quad \delta_\lambda^i = \frac{c_i}{|\log \lambda|}, \quad \varepsilon_\lambda^i = \lambda \delta_\lambda^i$$

where $\mathcal{H}$ *is defined near* p_i *by relation* (2.2). *In local conformal coordinates around* p_i, *there holds*

$$u_\lambda(\varepsilon_\lambda^i x) + 4 \log \lambda + 2 \log \delta_\lambda^i \to \log \frac{8}{(1+|x|^2)^2},$$

uniformly on compact sets of $\mathbb{R}^2$ *as* $\lambda \to 0$.

Our proof consists of the construction of a suitable first approximation of a solution as required, and then solving by linearization and a suitable Lyapunov-type reduction There is a large literature in Liouville type equation in two-dimensional domains or compact manifold, in particular concerning construction and classification of blowing-up families of solutions. See for instance [6,9,11,15,16,18] and their references.

We shall present the detailed proof of our main result in the case of one bubbling point $n = 1$. In the last section we explain the necessary (minor, essentially notational) changes for general n. Thus, we consider the problem

$$\Delta_g u - f e^u + \lambda^2 e^u + \alpha = 0, \quad \text{in } M, \tag{1.8}$$

under the following hypothesis: there exists a point $p \in M$ such that $f(p) = 0$ and $D^2 f(p)$ is positive definite.

2 A nonlinear Green's function

We consider the singular problem

$$\Delta_g G - f e^G + 8\pi \delta_p + \alpha = 0, \quad \text{in } M, \tag{2.1}$$

where δ_p is the Dirac mass supported at p, which is assumed to be a point of global non-degenerate minimum of f. In this section we will establish the following result, which corresponds to the case $n = 1$ in Lemma 1.1.

Lemma 2.1 *Problem* (2.1) *has a unique solution* G *which is smooth away from the singularities and in local conformal coordinates around* p *it has the form*

$$G(x) = -4 \log |x| - 2 \log \left(\frac{1}{\sqrt{2}} \log \frac{1}{|x|} \right) + \mathcal{H}(x), \tag{2.2}$$

where $\mathcal{H}(x) \in C(M)$.

Proof In order to construct a solution to this problem, is important to consider the equation, in local conformal coordinates around p, for $\gamma \ll 1$

$$\Delta \mathcal{G} - f e^{\mathcal{G}} + 8\pi \delta_0 = 0, \quad \text{in } B(0, \gamma). \tag{2.3}$$

Since

$$-\Delta \log \frac{1}{|x|^4} = 8\pi \delta_0,$$

if we write $\mathcal{G} = -4\log|x| + h(x)$, then h satisfies

$$\Delta h - f(x)\frac{1}{|x|^4}e^h = 0, \quad \text{in } B(0,\gamma). \tag{2.4}$$

Since p is a non-degenerate point of minimum of f, we may assume that, in local conformal coordinates around p, there exist positive numbers β_1, β_2 such that

$$\beta_1|x|^2 \le f(x) \le \beta_2|x|^2, \tag{2.5}$$

for all $x \in B(0,\gamma)$, if γ is small enough. Letting $r = |x|$, it is thus important to consider the equation

$$\Delta V - \frac{1}{r^2}e^V = 0, \quad \text{in } B(0,\gamma). \tag{2.6}$$

For a radial function $V = V(r)$, this equation becomes

$$V''(r) + \frac{1}{r}V'(r) - \frac{1}{r^2}e^{V(r)} = 0, \quad 0 < r < \gamma. \tag{2.7}$$

We make the change of variables $r = e^t$, $v(t) = V(r)$, so that Eq. (2.7) transforms into

$$\frac{d^2}{dt^2}v(t) = e^{v(t)}, \quad -\infty < t < \log\gamma.$$

from where it follows that

$$\frac{d}{dt}\left(\frac{v'(t)^2}{2} - e^{v(t)}\right) = 0,$$

or $v'(t)^2 = 2(e^v + C)$, for some constant C. Choosing $C = 0$, we have

$$\frac{d}{dt}\left(e^{-v(t)/2}\right) = -\frac{1}{\sqrt{2}}.$$

Integrating and coming back to the original variable, we deduce that

$$V(r) = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{r}\right)$$

is a radial solution of Eq. (2.6). Note that, from condition (2.5) we readily find that $h_1(x) = V(|x|) - \log\beta_1$ is a supersolution of (2.4), while $h_2(x) = V(|x|) - \log\beta_2$ is a subsolution of (2.4).

Now we deal with existence of a solution of problem (2.1). The previous analysis suggests that the singular part of the Green's function, in local conformal coordinates around p, is

$$\Gamma(x) := -4\log|x| + V(|x|),$$

so we look for a solution of (2.1) of the form $u = \eta\Gamma + H$, where η is a smooth cut-off function such that $\eta \equiv 1$ in $B(p,\frac{\gamma}{2})$ and $\eta \equiv 0$ in $\mathbb{R}^2 \setminus B(p,\gamma)$. Therefore, H satisfies the equation

$$\Delta_g H - fe^{\eta\Gamma}e^H + \alpha = -\eta fe^\Gamma - 2\nabla_g\eta\nabla_g\Gamma - \Gamma\Delta_g\eta =: \Theta, \quad \text{in } M. \tag{2.8}$$

Observe that $fe^{\eta\Gamma} \in L^1(B(p,\gamma))$. Next we find ordered global sub and supersolutions for (2.8). Let us consider the problem

$$-\Delta_g h_0 + fh_0 = 1, \quad \text{in } M,$$

which has a unique non-negative solution of class $C^{2,\sigma}(M)$, $0<\sigma<1$. Observe that

$$\Delta_g \beta h_0 - f e^{\eta\Gamma} e^{\beta h_0} + \alpha - \Theta = -\beta + f\beta h_0 - f e^{\eta\Gamma} e^{\beta h_0} + \alpha - \Theta,$$

so if we choose $\beta = \beta_1 < 0$ small enough, then $\underline{H} := \beta_1 h_0$ is a subsolution of (2.8), while if we choose $\beta = \beta_2 > 0$ large enough, then $\overline{H} := \beta_2 h_0$ is a supersolution of (2.8).

We consider the space

$$X = \left\{ H \in H^1(M,g) \,\middle|\, \int_M f e^{\eta\Gamma} e^H < \infty \right\},$$

and the energy functional

$$E(H) = \frac{1}{2}\int_M |\nabla_g H|^2 + \int_M f e^{\eta\Gamma} F(H) + \int_M (-\alpha + \Theta) H, \tag{2.9}$$

where

$$F(H(x)) = \begin{cases} e^{\underline{H}(x)}(H - \underline{H}(x)) & H < \underline{H}(x), \\ e^H - e^{\underline{H}(x)} & H \in [\underline{H}(x), \overline{H}(x)], \\ e^{\overline{H}(x)}(H - \underline{H}(x)) & H > \overline{H}(x). \end{cases}$$

Observe that since $h_0 \in L^\infty(M,g)$ and $f e^{\eta\Gamma} \in L^1(B(p,\gamma))$, then $\overline{H}, \underline{H} \in X$, which means that the functional E is well defined in X. Since

$$\int_M -\Delta_g(\eta\Gamma) = -\lim_{a\to 0}\int_{\partial B(p,a)} \frac{\partial \Gamma}{\partial r} = 8\pi,$$

we conclude that

$$\int_M \Theta = \int_M (-\Delta_g(\eta\Gamma) - 8\pi\delta_p) = 0.$$

Besides $\alpha > 0$, so the functional E is coercive in X. We claim that E attains a minimum in X. In fact, taking $H_n \in X$ such that

$$\lim_{n\to\infty} E(H_n) = \inf_{H\in X} E(H) > -\infty,$$

and passing to a subsequence if necessary, we obtain

$$H_n \to \mathcal{H} \in X \text{ (in } L^2),\ \nabla_g H_n \rightharpoonup \nabla_g \mathcal{H} \text{ (weakly in } L^2),\ E(\mathcal{H}) = \inf_{H\in X} E(H).$$

Observe that if we take $\varphi \in C^\infty(M)$ then $\mathcal{H} + \varphi \in X$, we can differentiate and obtain

$$\frac{\partial}{\partial t} E(\mathcal{H} + t\varphi)\bigg|_{t=0} = 0, \quad \text{for all } \varphi \in C^\infty(M,g)$$

or

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} G(\mathcal{H})\varphi + \int_M (-\alpha + \Theta)\varphi = 0, \tag{2.10}$$

where

$$G(H) = \begin{cases} e^{\underline{H}(x)} & H < \underline{H}(x), \\ e^H & H \in [\underline{H}(x), \overline{H}(x)], \\ e^{\overline{H}(x)} & H > \overline{H}(x). \end{cases}$$

By suitably approximating $H_1 = (\underline{H} - \mathcal{H})_+$, we can use it as a test function in (2.10) and obtain

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g H_1 + \int_M f e^{\eta\Gamma} G(\mathcal{H}) H_1 + \int_M (-\alpha + \Theta) H_1 = 0.$$

Since $\underline{H}$ is a subsolution for Eq. (2.8), we have

$$\int_M \nabla_g \underline{H} \cdot \nabla_g H_1 + \int_M f e^{\eta\Gamma} e^{\underline{H}} H_1 + \int_M (-\alpha + \Theta) H_1 \leq 0.$$

Observe that

$$\int_M f e^{\eta\Gamma} G(\mathcal{H}) H_1 = \int_M f e^{\eta\Gamma} e^{\underline{H}} H_1.$$

From the above calculations we deduce

$$\int_M |\nabla_g H_1|^2 \leq 0,$$

hence $H_1 \equiv C$ for some constant C. If $C > 0$, necessarily $C \equiv H_1 \equiv \underline{H} - \mathcal{H}$ almost everywhere. Thus, $\underline{H} = \mathcal{H} + C$, and (2.10) traduces into

$$\int_M \nabla_g \underline{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} e^{\underline{H}} \varphi + \int_M (-\alpha + \Theta)\varphi = 0,$$

for all $\varphi \in C^\infty(M)$, which contradicts the fact that $\underline{H}$ solves

$$-\Delta_g \underline{H} + f \underline{H} = \beta_1,$$

or in other words, the fact that $\underline{H}$ is not a solution of problem (2.8). Hence $H_1 \equiv 0$, which implies $\underline{H} \leq \mathcal{H}$. In a similar way, we find $\mathcal{H} \leq \overline{H}$ and hence

$$\underline{H}(x) \leq \mathcal{H}(x) \leq \overline{H}(x), \quad \text{a.e. } x \in M.$$

Note that

$$\int_M \nabla_g \mathcal{H} \cdot \nabla_g \varphi + \int_M f e^{\eta\Gamma} e^{\mathcal{H}} \varphi + \int_M (-\alpha + \Theta)\varphi = 0, \tag{2.11}$$

for all $\varphi \in C^\infty(M, g)$. Besides, since the functional E is strictly convex and coercive, we conclude that $\mathcal{H}$ is the unique minimizer in X.

So far we have proven that problem (2.1) has a unique solution G which is smooth away from the singularity point p and in local conformal coordinates around p it has the form

$$G(x) = \eta \left[-4 \log |x| - 2 \log \left(\frac{1}{\sqrt{2}} \log \frac{1}{|x|} \right) \right] + \mathcal{H}(x),$$

where $\mathcal{H} \in X \cap L^\infty(M, g)$, is the unique minimizer of the functional E defined in X by (2.9).

Next we will further study the form of $\mathcal{H}$ near p, which in particular yields its continuity at p. For this purpose we use local conformal coordinates around p.

Let us consider the problem

$$\begin{cases} -\Delta_g \mathcal{J} = \alpha & \text{in } B(0, \frac{\gamma}{2}), \\ \mathcal{J} = \mathcal{H} & \text{on } \partial B(0, \frac{\gamma}{2}). \end{cases}$$

This problem has a unique solution $\mathcal{J}$, which is smooth in $B(0, \frac{\gamma}{2})$. So we can expand $\mathcal{J}$ as

$$\mathcal{J} = \sum_{k=0}^{\infty} b_k r^k = b_0 + O(r).$$

We write $\mathcal{H} = \mathcal{J} + \mathcal{F}$, therefore $\mathcal{F}$ solves

$$\begin{cases} -\Delta_g \mathcal{F} + \dfrac{f}{r^4}\dfrac{2}{\log^2 r} e^{\mathcal{J}} e^{\mathcal{F}} - \dfrac{1}{r^2}\dfrac{2}{\log^2 r} = 0 & \text{in } B(0, \frac{\gamma}{2}), \\ \mathcal{F} = 0 & \text{on } \partial B(0, \frac{\gamma}{2}), \end{cases}$$

because $\eta\Gamma \equiv \Gamma$ in $B(0, \frac{\gamma}{2})$. Since $\mathcal{F} \in L^2(B(0, \frac{\gamma}{2}))$ we can expand it as

$$\mathcal{F}(r, \theta) = \sum_{k=0}^{\infty} a_k(r) e^{ik\theta}.$$

Observe that

$$\frac{f(x)}{r^2} = \frac{\kappa_1 r^2 \cos^2(\theta) + \kappa_2 r^2 \sin^2(\theta) + \kappa_3 r^2 \sin\theta\cos\theta}{r^2} + O(r) = a(\theta) + O(r),$$

for $r \neq 0$. Besides, $\beta_1 \le a(\theta) \le \beta_2$. Thus

$$\frac{f(x)}{r^4}\frac{2}{\log^2 r} e^{\mathcal{J}} e^{\mathcal{F}} - \frac{1}{r^2}\frac{2}{\log^2 r} = \frac{1}{r^2}\frac{2}{\log^2 r}\left[(a(\theta) + O(r))e^{\mathcal{J}+\mathcal{F}} - 1\right].$$

Moreover, since $\mathcal{H} \in L^\infty(B(0, \frac{\gamma}{2}))$ we have $e^{\mathcal{J}+\mathcal{F}} \in L^2(B(0, \frac{\gamma}{2}))$, so

$$\frac{1}{r^2}\frac{2}{\log^2 r}\left[(a(\theta) + O(r))e^{\mathcal{J}+\mathcal{F}} - 1\right] = \sum_{k=0}^{\infty} m_k(r) e^{ik\theta},$$

where

$$|m_k(r)| \le \frac{C}{r^2}\frac{1}{\log^2 r}, \quad \forall k \ge 0,$$

for a constant C independent of k. Now, we study the behavior of the coefficients $a_k(r)$. For this purpose let us remember that

$$\Delta u(r, \theta) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

For $k \ge 1$, we see that $a_k(r)$ satisfies the ordinary differential equation

$$-\frac{\partial^2 a_k}{\partial r^2}(r) - \frac{1}{r}\frac{\partial a_k}{\partial r}(r) + \frac{k^2}{r^2} a_k(r) = m_k(r), \quad 0 < r < \frac{\gamma}{2}, \tag{2.12}$$

under the conditions

$$a_k\left(\frac{\gamma}{2}\right) = 0, \quad a_k(r) \in L^\infty\left(\left[0, \frac{\gamma}{2}\right]\right). \tag{2.13}$$

We recall that the L^∞-condition comes from the fact that $\mathcal{F} \in L^\infty(B(0, \frac{\gamma}{2}))$. Let us make the change of variables $r = e^t$, $A_k(t) = a_k(e^t)$, $M_k(t) = m_k(e^t)$, so the previous problem transform into

$$-\frac{d^2 A_k}{dt^2}(t) + k^2 A_k(t) = M_k(t), \quad -\infty < t < \log\frac{\gamma}{2}, \tag{2.14}$$

under the conditions

$$A_k\left(\log\frac{\gamma}{2}\right)=0,\quad A_k\in L^\infty\left(\left(-\infty,\log\frac{\gamma}{2}\right]\right). \tag{2.15}$$

Besides, $|M_k(t)|\le Ct^{-2}$ for all $k\ge 1$. All the solutions of the homogeneous equation are given by linear combinations of e^{kt} and e^{-kt} and a particular solution A_k^{part} of the non-homogeneous Eq. (2.14) is given by the variation of parameter formula. We conclude that this problem has a solution of the form

$$C_1e^{kt}+C_2e^{-kt}+A_k^{part}.$$

By the L^∞-condition we conclude that $C_2=0$ and by the boundary condition in (2.15) we deduce $C_1=0$. This implies that the null function is the only solution of the homogeneous equation under condition (2.15). Hence, this problem has a unique solution $A_k(t)$. We claim that for a constant C independent of k we have

$$|A_k(t)|\le C\frac{1}{k^2t^2}. \tag{2.16}$$

The proof of this fact is based on maximum principle: observe that since $k^2>0$, the operator

$$-\frac{d^2}{dt^2}+k^2$$

satisfies the weak maximum principle on bounded subsets of $(-\infty,\log\frac{\gamma}{2}]$. Let us prove that $\phi=\frac{C_1}{k^2t^2}+\rho e^{-kt}$ is a non-negative supersolution for this problem. Observe first that since $A_k(t)$ is bounded, there exist τ_ρ such that

$$A_k(t)\le\phi(t),\quad \text{for all } t\in(-\infty,\tau_\rho].$$

Besides,

$$\left(-\frac{d^2}{dt^2}+k^2\right)\phi=-6C_1\frac{1}{k^2t^4}+C_1\frac{1}{t^2}\ge M_k(t),\quad \forall t\in\left(\tau_\rho,\log\frac{\gamma}{2}\right),$$

where the last inequality is valid if we choose C_1 large enough. Observe also that $\phi(t)\ge A_k(t)$ for $t=\tau_\rho,\log\frac{\gamma}{2}$. Hence, by weak maximum principle we conclude that for all $\rho>0$

$$A_k(t)\le\frac{C_1}{k^2t^2}+\rho e^{-kt},\quad \forall t\in\left(-\infty,\log\frac{\gamma}{2}\right].$$

Taking the limit $\rho\to0$ in the last expression, we conclude that $A_k(t)\le C\frac{1}{k^2t^2}$. Analogously, we now prove that $\varphi=-\frac{C_2}{k^2t^2}-\rho e^{-kt}$ is a non-positive subsolution for this problem. Since $A_k(t)$ is bounded, there exist τ_ρ such that

$$\varphi(t)\le A_k(t),\quad \forall t\in(-\infty,\tau_\rho].$$

Besides,

$$\left(-\frac{d^2}{dt^2}+k^2\right)\varphi=6C_2\frac{1}{k^2t^4}-C_2\frac{1}{k^2t^2}\le M_k(t),\quad \forall t\in\left(\tau_\rho,\log\frac{\gamma}{2}\right),$$

where the last inequality is valid if we choose C_2 large enough. Observe also that $\varphi(t)\le A_k(t)$ for $t=\tau_\rho,\log\frac{\gamma}{2}$. Hence, by weak maximum principle we conclude that for all $\rho>0$

$$-\frac{C_2}{k^2t^2}-\rho e^{-kt}\le A_k(t),\quad \forall t\in\left(-\infty,\log\frac{\gamma}{2}\right].$$

Taking the limit $\rho \to 0$ in the last expression, we conclude (2.16). Finally, coming back to the variable r we conclude that there exist a unique solution $a_k(r)$ of problem (2.12–2.13), and for a constant C independent of k we have

$$|a_k(r)| \le C\frac{1}{k^2\log^2 r}, \quad 0 < r < \frac{\gamma}{2}.$$

Now we deal with $a_0(r)$. Observe that

$$e^{\mathcal{F}} = e^{a_0(r)}\left(1 + O\left(\frac{1}{\log^2 r}\right)\right), \quad e^{\mathcal{J}} = e^{b_0}(1 + O(r)),$$

and

$$a(\theta) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k e^{ik\theta}, \quad \text{with } \alpha_0 > 0,$$

so we conclude that $a_0(r)$ satisfies the ordinary differential equation

$$-\frac{\partial^2 a_0(r)}{\partial r^2} - \frac{1}{r}\frac{\partial a_0(r)}{\partial r} + 2\frac{\alpha_0 e^{b_0} e^{a_0(r)} - 1}{r^2\log^2 r} = O\left(\frac{1}{r^2\log^4 r}\right),$$

under the following conditions

$$a_0\left(\frac{\gamma}{2}\right) = 0, \quad a_0 \in L^\infty\left(\left[0, \frac{\gamma}{2}\right]\right).$$

We make the change of variables $r = e^t$, $\tilde{a}_0(t) = a_0(e^t)$, so the previous problem transform into

$$-\frac{d^2\tilde{a}_0}{dt^2} + 2\frac{\alpha_0 e^{b_0} e^{\tilde{a}_0} - 1}{t^2} = O\left(\frac{1}{t^4}\right), \tag{2.17}$$

under the conditions

$$\tilde{a}_0\left(\log\frac{\gamma}{2}\right) = 0, \quad \tilde{a}_0 \in L^\infty\left(\left(-\infty, \log\frac{\gamma}{2}\right]\right). \tag{2.18}$$

The L^∞-condition implies that there exist a sequence $t_n \to -\infty$ such that

$$\tilde{a}_0(t_n) \to L, \quad \text{as } n \to \infty,$$

where $L = -\log(\alpha_0 e^{b_0})$. If not there exist $M < 0$ such that

$$|\alpha_0 e^{b_0} e^{\tilde{a}_0} - 1| \ge \epsilon > 0, \quad \forall t < M,$$

which means that

$$\left|\frac{d^2\tilde{a}_0}{dt^2}\right| \ge C\frac{\epsilon}{t^2}, \quad \forall t < M.$$

Thus

$$|\tilde{a}_0| \ge C\epsilon\log|t|, \quad \forall t < M,$$

so $\tilde{a}_0$ is unbounded, a contradiction.

We claim that the problem (2.17, 2.18) has at most one solution. In fact, let us suppose by contradiction that u_1 and u_2 are two diferent solutions. We define $u = u_1 - u_2$, which satisfies the problem

$$-\frac{d^2u}{dt^2} + 2\alpha_0 e^{b_0} c(t)u = 0,$$

under the conditions,

$$u\left(\log\frac{\gamma}{2}\right)=0,\quad u\in L^{\infty}\left(\left(-\infty,\log\frac{\gamma}{2}\right]\right),$$

and where

$$c(t)=\begin{cases}0 & \text{if } u_1(t)=u_2(t),\\ \frac{1}{t^2}\frac{e^{u_1(t)-u_2(t)}}{u_1(t)-u_2(t)} & \text{if } u_1(t)\neq u_2(t).\end{cases}$$

Observe that $c(t)\geq 0$, so we can apply the strong maximum principle in bounded domains for this problem. Moreover, from the L^{∞} condition we deduce that there exists a sequence t_n such that $u(t_n)\to 0$ as $n\to\infty$ (the proof of this fact is the same that we gave before). From this two facts, we deduce easily that $u_1\equiv u_2$.

Let us make the change of variables $-t=e^s$, $A_0(s)=\tilde{a}_0(-e^s)$, so the previous problem transform into

$$-\frac{d^2A_0}{ds^2}+\frac{dA_0}{ds}+2(\alpha_0e^{b_0}e^{A_0}-1)=O(e^{-2s}),\tag{2.19}$$

under the conditions

$$A_0\left(\log\left(-\log\frac{\gamma}{2}\right)\right)=0,\quad A_0\in L^{\infty}\left(\left[\log\left(-\log\frac{\gamma}{2}\right),\infty\right)\right).$$

We look for a solution of this problem of the form $A_0(s)=L+\phi(s)$, so ϕ solves the differential equation

$$-\frac{d^2\phi}{ds^2}+\frac{d\phi}{ds}+2\phi=N(\phi)+O(e^{-2s}),$$

where

$$N(\phi)=-2(e^{\phi}-\phi-1).$$

Observe that $\phi_+=e^{2s}$, $\phi_-=e^{-s}$ are two linear independent solutions of the homogeneous equation.

From the previous analysis, we deduce that there exists a sequence $s_n\to\infty$ such that $\phi(s_n)=\delta_n\to 0$, as $n\to\infty$. We make the change of variables $\tilde{\phi}_n(\tau_n)=\phi(s)-\delta_n\phi_-(\tau_n)$, where $\tau_n=s-s_n$, so $\tilde{\phi}_n\in L^{\infty}$ solves the problem

$$\begin{cases}-\tilde{\phi}_n''+\tilde{\phi}_n'+2\tilde{\phi}_n &= N(\tilde{\phi}_n+\delta e^{-\tau_n})+e^{-2s_n}O(e^{-2\tau_n})\ \tau_n\in(0,\infty),\\ \tilde{\phi}_n(0) &= 0.\end{cases}\tag{2.20}$$

Let us study the linear problem

$$\begin{cases}-\varphi''+\varphi'+2\varphi=\omega & \text{in } (0,\infty),\\ \varphi(0)=0, & \varphi\in L^{\infty}(0,\infty)\end{cases}$$

for $\omega\in C([0,\infty))$ given. This problem has an explicit and unique solution $\varphi=T[g]$, in fact

$$\varphi(t)=C_1e^{\lambda_+t}+C_2e^{\lambda_-t}+e^{\lambda_+t}\int_0^t\frac{e^{\lambda_-s}\omega(s)}{3e^{2s}}ds-e^{\lambda_-t}\int_0^t\frac{e^{\lambda_+s}\omega(s)}{3e^{2s}}ds$$

and we deduce that $C_1=0$ and $C_2=0$ due to the L^{∞} condition and the value at 0 of φ, respectively. problem (2.20) can be written as

$$\tilde{\phi}_n=T[N(\tilde{\phi}_n+\delta e^{-\tau_n})+e^{-2s_n}O(e^{-2\tau_n})]:=A[\tilde{\phi}_n].\tag{2.21}$$

We consider the set

$$B_\epsilon = \{\varphi \in C([0,\infty)) \ : \ \|\varphi\|_\infty \le \epsilon\}\,.$$

It is easy to see that if s_n is large enough and δ_n small enough we have

$$\|A[\tilde{\phi}_n^1] - A[\tilde{\phi}_n^2]\|_\infty \le C\epsilon\|\tilde{\phi}_n^1 - \tilde{\phi}_n^2\|,$$
$$\|A[\tilde{\phi}_n]\| \le C\epsilon,$$

and where C is independent of n. It follows that for all sufficiently small ϵ we get that A is a contraction mapping of B_ϵ (provided n large enough), and therefore a unique fixed point of A exists in this region. We deduce that there exists a unique solution A_0 of problem (2.19), and it has the form $A_0(s) = L + \phi(s)$, where L is a fixed constant, and $\phi(s) \to 0$ as $s \to \infty$. This concludes the proof of Lemma 2.1. □

3 Construction of a first approximation

In this section we will build a suitable approximation for a solution of problem (1.8) which is large exactly near the point p. The "basic cells" for the construction of an approximate solution of problem (1.8) are the radially symmetric solutions of the problem

$$\begin{cases} \Delta w + \lambda^2 e^w = 0 & \text{in } \mathbb{R}^2, \\ w(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.1}$$

which are given by the one-parameter family of functions

$$w_\delta(|x|) = \log \frac{8\delta^2}{(\lambda^2\delta^2 + |x|^2)^2},$$

where δ is any positive number. We define $\varepsilon = \lambda\delta$. In order to construct the approximate solution we consider the equation

$$\Delta F - \frac{\delta^2}{r^2}e^F = 0, \tag{3.2}$$

in the variable $r = |x|/\varepsilon$ and we look for a radial solution $F = F(r)$, away from $r = 0$. For this purpose we solve problem (3.2) under the following initial conditions

$$F(1/\delta) = 0, \quad F'(1/\delta) = 0.$$

We make the change of variables $r = e^t$, $V(t) = F(r)$, so that Eq. (3.2) transforms into

$$V'' - \delta^2 e^V = 0.$$

We consider the transformation $V(s) = \tilde{V}(\delta s)$, so $\tilde{V}$ solves problem

$$\tilde{V}'' - e^{\tilde{V}} = 0, \quad \tilde{V}(\delta|\log\delta|) = 0, \quad \tilde{V}'(\delta|\log\delta|) = 0.$$

This problem has a unique regular solution, which blows-up at some finite radius $\gamma > 0$. Coming back to the variable $r = |x|/\varepsilon$, we conclude that the solution $F(r)$ is defined for all $1/\delta \le r \le Ce^{1/\delta} = \tilde{C}/\lambda$, for some constants C, $\tilde{C}$. Here we have used the definition of δ, see (3.3). Besides, we extend by 0 the function F for $r \in [0, 1/\delta)$, which means $F(r) = 0$, for all $r \in [0, 1/\delta)$ and we denote by $\tilde{F}(|x|) = F(|x|/\varepsilon)$. A first local approximation of the solution, in local conformal coordinates around p, is given by the radial function $u_\varepsilon(x) = w_\delta(|x|) + \tilde{F}(|x|)$.

In order to build a global approximation, let us consider η a smooth radial cutoff function such that $\eta(r) = 1$ if $r \le C_1\delta$ and $\eta(r) = 0$ if $r \ge C_2\delta$, for constants $0 < C_1 < C_2$. We consider as initial approximation $U_\varepsilon = \eta u_\varepsilon + (1-\eta)G$, where G is the Green function that we built in the previous section. In order to have a good approximation around p we have to adjust the parameter δ. The good choice of this number is

$$\log 8\delta^2 = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + \mathcal{H}(p), \tag{3.3}$$

where $\mathcal{H}$ is defined in Sect. 2. With this choice of the parameter δ, the function u_ε is approaching the Green function G around p.

A useful observation is that u satisfies problem (1.8) if and only if

$$v(y) = u(\varepsilon y) + 4\log\lambda + 2\log\delta$$

satisfies

$$\Delta_g v - \lambda^{-2} f(\varepsilon y)e^v + e^v + \varepsilon^2\alpha = 0, \quad y \in M_\varepsilon, \tag{3.4}$$

where $M_\varepsilon = \varepsilon^{-1}M$.

We denote in what follows $p' = \varepsilon^{-1}p$ and

$$\tilde{U}_\varepsilon(y) = U_\varepsilon(\varepsilon y) + 4\log\lambda + 2\log\delta,$$

for $y \in M_\varepsilon$. This means precisely in local conformal coordinates around p that

$$\begin{aligned}\tilde{U}_\varepsilon(y) =& \eta(\varepsilon|y|)\left(\log\frac{1}{(1+|y|^2)^2} + \tilde{F}(\varepsilon|y|)\right)\\ &+ (1-\eta(\varepsilon|y|))\left(G(\varepsilon y) + 4\log\lambda + 2\log\delta\right).\end{aligned}$$

Let us consider a vector $k \in \mathbb{R}^2$. We recall that $w_\delta(|x-k|)$ is also a solution of problem (3.1). To solve problem (3.4), we need to modify the first approximation of the solution, in order to have a new parameter related to translations. More precisely, we consider for $|k| \ll 1$ the new first approximation of the solution (in the expanded variable)

$$\begin{aligned}V_\varepsilon(y) =& \eta(\varepsilon|y|)\left(\log\frac{1}{(1+|y-k|^2)^2} + \tilde{F}(\varepsilon|y|)\right)\\ &+ (1-\eta(\varepsilon|y|))\left(G(\varepsilon y) + 4\log\lambda + 2\log\delta\right).\end{aligned}$$

We will denote by v_ε the first approximation of the solution in the original variable, which means

$$v_\varepsilon(x) = \eta(|x|)\left(\log\frac{8\delta^2}{(\varepsilon^2+|x-\varepsilon k|^2)^2} + \tilde{F}(|x|)\right) + (1-\eta(|x|))G(x).$$

Hereafter we look for a solution of problem (3.4) of the form $v(y) = V_\varepsilon(y) + \phi(y)$, where ϕ represent a lower order correction. In terms of ϕ, problem (3.4) now reads

$$L(\phi) = N(\phi) + E, \quad \text{in } M_\varepsilon, \tag{3.5}$$

where

$$\begin{aligned}L(\phi) :=& \Delta_g\phi - \lambda^{-2} f(\varepsilon y)e^{V_\varepsilon}\phi + e^{V_\varepsilon}\phi,\\ N(\phi) :=& \lambda^{-2} f(\varepsilon y)e^{V_\varepsilon}(e^\phi - 1 - \phi) - e^{V_\varepsilon}(e^\phi - 1 - \phi),\\ E :=& -(\Delta_g V_\varepsilon - \lambda^{-2} f(\varepsilon y)e^{V_\varepsilon} + e^{V_\varepsilon} + \varepsilon^2\alpha).\end{aligned}$$

4 The linearized operator around the first approximation

In this section we will develop a solvability theory for the second-order linear operator L defined in (3.5) under suitable orthogonality conditions. Using local conformal coordinates around p', then formally the operator L approaches, as ε, $|k| \to 0$, the operator in $\mathbb{R}^2$

$$\mathcal{L}(\phi) = \Delta\phi + \frac{8}{(1+|z|^2)^2}\phi,$$

namely, equation $\Delta w + e^w = 0$ linearized around the radial solution $w(z) = \log \frac{8}{(1+|z|^2)^2}$. An important fact to develop a satisfactory solvability theory for the operator L is the non-degeneracy of w modulo the natural invariance of the equation under dilations and translations. Thus we set

$$z_0(z) = \frac{\partial}{\partial s}[w(sz) + 2\log s]|_{s=1}, \tag{4.1}$$

$$z_i(z) = \frac{\partial}{\partial \zeta_i} w(z+\zeta)|_{\zeta=0}, \quad i = 1, 2. \tag{4.2}$$

It turns out that the only bounded solutions of $\mathcal{L}(\phi) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of the z_i, $i = 0, 1, 2$, see [2] for a proof. We define for $i = 0, 1, 2$,

$$Z_i(y) = z_i(y-k).$$

Additionally, let us consider R_0 a large but fixed number $R_0 > 0$ and χ a radial and smooth cut-off function such that $\chi \equiv 1$ in $B(k, R_0)$ and $\chi \equiv 0$ in $B(k, R_0+1)^c$.

Given h of class $C^{0,\beta}(M_\varepsilon)$, we consider the linear problem of finding a function ϕ such that for certain scalars c_i, $i = 1, 2$, one has

$$\begin{cases} L(\phi) = h + \sum_{i=1}^2 c_i \chi Z_i & \text{in } M_\varepsilon, \\ \displaystyle\int_{M_\varepsilon} \chi Z_i \phi = 0 & \text{for } i = 1, 2. \end{cases} \tag{4.3}$$

We will establish a priori estimates for this problem. To this end we define, given a fixed number $0 < \sigma < 1$, the norm

$$\|h\|_* = \|h\|_{*,p} := \sup_{M_\varepsilon} (\max\{\varepsilon^2, |y|^{-2-\sigma}\})^{-1}|h|. \tag{4.4}$$

Here the expression $\max\{\varepsilon^2, |y|^{-2-\sigma}\}$ is regarded in local conformal coordinates around $p' = \varepsilon^{-1}p$. Since local coordinates are defined up to distance $\sim\frac{1}{\varepsilon}$ that expression makes sense globally in M_ε.

Our purpose in this section is to prove the following result.

Proposition 4.1 *There exist positive numbers ε_0, C such that for any $h \in C^{0,\beta}(M_\varepsilon)$, with $\|h\|_* < \infty$ and for all k such that $|k| \le C\lambda/\delta$, there is a unique solution $\phi = T(h) \in C^{2,\beta}(M_\varepsilon)$ of problem (4.3) for all $\varepsilon < \varepsilon_0$, which defines a linear operator of h. Besides,*

$$\|T(h)\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*. \tag{4.5}$$

Observe that the orthogonality conditions in problem (4.3) are only taken respect to the elements of the approximate kernel due to translations.

The next Lemma will be used for the proof of Proposition 4.1. We obtain an a priori estimate for the problem

$$\begin{cases} L(\phi) = h & \text{in } M_\varepsilon, \\ \displaystyle\int_{M_\varepsilon} \chi Z_i \phi = 0 & \text{for } i = 1, 2. \end{cases} \tag{4.6}$$

We have the following estimate.

Lemma 4.1 *There exist positive constants ε_0, C such that for any ϕ solution of problem (4.6) with $h \in C^{0,\beta}(M_\varepsilon)$, $\|h\|_* < \infty$ and any k, $|k| \le C\lambda/\delta$*

$$\|\phi\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*,$$

for all $\varepsilon < \varepsilon_0$.

Proof We carry out the proof by a contradiction argument. If the above fact were false, there would exist sequences $(\varepsilon_n)_{n\in\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$, $|k_n| \to 0$ and functions ϕ_n, h_n with $\|\phi_n\|_\infty = 1$,

$$\log(\varepsilon_n^{-1})\|h_n\|_* \to 0,$$

such that

$$\begin{cases} L(\phi_n) = h_n & \text{in } M_{\varepsilon_n}, \\ \int_{M_{\varepsilon_n}} \chi Z_i \phi_n = 0 & \text{for } i = 1, 2. \end{cases} \tag{4.7}$$

A key step in the proof is the fact that the operator L satisfies a weak maximum principle in regions, in local conformal coordinates around p, of the form $A_\varepsilon = B(p', \varepsilon^{-1}\gamma/2)\backslash B(p', R)$, with R a large but fixed number. Consider the function $z_0(r) = \frac{r^2-1}{r^2+1}$, radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

We define a comparison function

$$Z(y) = z_0(a|y - p'|), \quad y \in M_\varepsilon.$$

Let us observe that

$$-\Delta Z(y) = \frac{8a^2(a^2|y-p'|^2 - 1)}{(1 + a^2|y-p'|^2)^3}.$$

So, for $100\, a^{-2} < |y - p'| < \varepsilon^{-1}\gamma/2$, we have

$$-\Delta Z(y) \ge 2\frac{a^2}{(1 + a^2|y-p'|^2)^2} \ge \frac{a^{-2}}{|y-p'|^4}.$$

On the other hand, in the same region,

$$e^{V_\varepsilon(y)} Z(y) \le C \frac{1}{|y-p'|^4}.$$

Hence if a is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this a, then

$$\Delta Z + e^{V_\varepsilon} Z < 0, \quad \text{in } A_\varepsilon.$$

Since $Z > 0$ in A_ε, we have

$$L(Z) < 0, \quad \text{in } A_\varepsilon.$$

We conclude that L satisfies weak maximum principle in A_ε, namely if $L(\phi) \le 0$ in A_ε and $\phi \ge 0$ on ∂A_ε, then $\phi \ge 0$ in A_ε.

We now give the proof of the Lemma in several steps.

STEP 1. We claim that

$$\sup_{y \in M_{\varepsilon_n} \setminus B(p/\varepsilon_n, \rho/\varepsilon_n)} |\phi_n(y)| = o(1),$$

where ρ is a fixed number. In fact, coming back to the original variable by the transformation

$$\hat{\phi}_n(x) = \phi_n\left(\frac{x}{\varepsilon_n}\right), \quad x \in M.$$

We can see that $\hat{\phi}_n$ satisfies the equation

$$\Delta_g \hat{\phi}_n - f e^{v_{\varepsilon_n}} \hat{\phi}_n + \lambda_n^2 e^{v_{\varepsilon_n}} \hat{\phi}_n = \frac{1}{\varepsilon_n^2} h_n\left(\frac{x}{\varepsilon_n}\right), \tag{4.8}$$

where

$$v_{\varepsilon_n}(x) = V_{\varepsilon_n}\left(\frac{x}{\varepsilon_n}\right) - 4\log\lambda_n - 2\log\delta,$$

is the approximation of the solution in the original variable. Taking $n \to \infty$, we can see that $\hat{\phi}_n$ converges uniformly over compacts of $M \setminus \{p\}$ to a function $\hat{\phi} \in H^1(M) \cap L^\infty(M)$ solution of the problem

$$\Delta_g \hat{\phi} - f e^J \hat{\phi} = 0, \quad \text{in } M \setminus \{p\} \tag{4.9}$$

where J is the limit of v_{ε_n}. We claim that $\hat{\phi} \equiv 0$, in fact, we consider the unique solution Φ of the problem

$$\Delta_g \Phi - \min\{f e^J, 1\}\Phi = -\delta_p, \quad \text{in } M.$$

Using local conformal coordinates around p we expand

$$\Phi(x) = -\frac{1}{2\pi}\log(|x|) + H(x)$$

for H bounded. Since $\hat{\phi} \in L^\infty(M)$, we conclude that for all sufficiently small ϵ and τ we have

$$|\hat{\phi}(x)| \le \epsilon\Phi(x), \quad x \in \partial B(0, \tau).$$

Multiplying (4.9) by $\varphi = (\hat{\phi} - \epsilon\Phi)_+$, and integrating by parts over $M_\tau = M \setminus U_\tau$, where U_τ is the neighborhood around p under the local conformal coordinates that we used, we have

$$\int_{M_\tau} |\nabla_g \varphi|^2 + \int_{M_\tau} f e^J \varphi^2 + \epsilon \int_{M_\tau} e^J \varphi \Phi = 0.$$

Since $\Phi \ge 0$, we have

$$\int_{M_\tau} |\nabla_g \varphi|^2 + \int_{M_\tau} f e^J \varphi^2 \le 0.$$

Hence $\varphi = (\hat{\phi} - \epsilon\Phi)_+ = 0$ in M_τ, so $\hat{\phi} \le \epsilon\Phi$ in M_τ. Multiplying by $\varphi = (\hat{\phi} + \epsilon\Phi)_-$ and integrating by parts, we have $(\hat{\phi} + \epsilon\Phi)_- = 0$, thus

$$|\hat{\phi}(x)| \le \epsilon\Phi(x), \quad x \in M_\tau.$$

Taking $\epsilon \to 0$ and $\tau \to 0$, we conclude that $\hat{\phi} \equiv 0$.

STEP 2. Let us consider the transformation

$$\tilde{\phi}_n(y) = \phi_n(y + p_n').$$

Thus $\tilde{\phi}_n$ satisfies the equation

$$\Delta_g \tilde{\phi}_n - \lambda_n^{-2} f(\varepsilon_n y + p_n) e^{V_{\varepsilon_n}(y+p_n')} \tilde{\phi}_n + e^{V_{\varepsilon_n}(y'+p_n')} = h_n(y + p_n'),$$

in $M_{\varepsilon_n} - \{p_n'\}$. Taking the limit $n \to \infty$ in the last equation [and also in problem (4.7)], we see that $\tilde{\phi}_n$ converges uniformly over compacts of $M_{\varepsilon_n} - \{p_n'\}$ to a bounded solution $\tilde{\phi}$ of the problem

$$\mathcal{L}(\tilde{\phi}) = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \chi Z_i \tilde{\phi} = 0, \quad i = 1, 2.$$

Hence $\tilde{\phi}(x) = C_0 Z_0(x)$.

In what follows we assume without loss of generality that $C_0 \ge 0$. If $C_0 < 0$, we work with $-\phi_n$ instead of ϕ_n and the following analysis is also valid.

STEP 3. In this step we will construct a non-negative supersolution in the region, in local conformal coordinates around p_n', $B_n = B(k_n, \rho) \setminus B(k_n, \varepsilon_n^{-1}\gamma/2)$, $\rho > 0$, where the weak maximum principle is valid. We work first in the case $C_0 > 0$. Let us consider the problem

$$\begin{cases} -\Delta\psi_n - e^{V_\varepsilon}\psi_n = 1 \text{ in } B_n, \\ \psi_n(y) = C_0 \qquad \text{on } \partial B(k_n, \rho), \\ \psi_n(y) = o(1) \qquad \text{on } \partial B(k_n, \varepsilon_n^{-1}\gamma/2). \end{cases} \tag{4.10}$$

We define $r = |y - k_n|$. A direct computation shows that

$$\psi_n(y) = C_0 Z_0(r) + C_\varepsilon Y(r) + W(r),$$

where

$$Y(r) = Z_0 \int_\rho^r \frac{1}{s Z_0^2(s)} ds, \quad W(r) = -Z_0(r) \int_\rho^r sY(s)ds + Y(r) \int_\rho^r s Z_0(s) ds,$$

and

$$C_\varepsilon = \frac{o(1) - C_0 Z_0(\varepsilon_n^{-1}\gamma/2) - W(\varepsilon_n^{-1}\gamma/2)}{Y(\varepsilon_n^{-1}\gamma/2)}.$$

We choose $\rho > R$, where R is the fixed minimal radio for which the weak maximum principle is valid in the region B_n. Observe that

$$L(\psi_n) = -1 - \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon} \psi_n \le h_n = L(\phi_n).$$

Moreover, from steps 1 and 2, we deduce that

$$\psi_n \ge \phi_n, \quad \text{on } \partial B_n, \tag{4.11}$$

which means that ψ_n is a supersolution for the problem

$$L(\phi_n) = h_n, \quad \text{in } B_n.$$

Since $\rho > R$, we can apply the weak maximum principle and we deduce that $\Psi_n \geq \phi_n$ in B_n. Observe that

$$\left|\frac{d\psi_n(\rho)}{dr}\right| \geq \varepsilon_n^{-1}. \tag{4.12}$$

In the other hand

$$\frac{dZ_0}{dr} = -C\frac{r}{(r^2-1)^2}, \tag{4.13}$$

where $C > 0$ is a constant independent of n. Since ϕ_n converges over compacts of the expanded variable to the function $C_0 Z_0$, we deduce from (4.11), (4.12) and (4.13) that the partial derivative of ϕ_n respect to r is discontinuous at $|y - k_n| = \rho$, for large values of n, which is a contradiction.

In the case $C_0 = 0$, ϕ_n converges to 0 over compacts of the expanded variable. Let us consider the problem

$$\begin{cases} -\Delta\psi_n - e^{V_\varepsilon}\psi_n = 1 & \text{in } B_n, \\ \psi_n(y) = 1/2 & \text{on } \partial B(k_n, \rho), \\ \psi_n(y) = o(1) & \text{on } \partial B(k_n, \varepsilon_n^{-1}\gamma/2). \end{cases}$$

It is easy to see that $\psi_n \leq 1/2$ in $\overline{B}_n$. Using the previous maximum principle argument we deduce that $\phi_n \leq \psi_n \leq 1/2$ Applying the same argument for the problem that $-\phi_n$ satisfies, we conclude $-\phi_n \leq 1/2$. Thus,

$$\|\phi_n\|_\infty \leq 1/2,$$

which is a contradiction with the fact $\|\phi_n\|_\infty = 1$. This finishes the proof of the a priori estimate. □

We are now ready to prove the main result of this section.

Proof of Proposition 4.1 We begin by establishing the validity of the a priori estimate (4.5). The previous lemma yields

$$\|\phi\|_\infty \leq C\log\left(\frac{1}{\varepsilon}\right)\left[\|h\|_* + \sum_{i=1}^{2}|c_i|\right], \tag{4.14}$$

hence it suffices to estimate the values of the constants $|c_i|$, $i = 1, 2$. We use local conformal coordinates around p, and we define again $r = |y|$ and we consider a smooth cut-off function $\eta(r)$ such that $\eta(r) = 1$ for $r < \frac{1}{\sqrt{\varepsilon}}$, $\eta(r) = 0$ for $r > \frac{2}{\sqrt{\varepsilon}}$, $|\eta'(r)| \leq C\sqrt{\varepsilon}$, $|\eta''(r)| \leq C\varepsilon$. We test the first equation of problem (4.3) against ηZ_i, $i = 1, 2$ to find

$$\langle L(\phi), \eta Z_i\rangle = \langle h, \eta Z_i\rangle + c_i\int_{M_\varepsilon}\chi|Z_i|^2. \tag{4.15}$$

Observe that

$$\langle L(\phi), \eta Z_i\rangle = \langle \phi, L(\eta Z_i)\rangle,$$

and

$$L(\eta Z_i) = Z_i\Delta\eta + 2\nabla\eta\cdot\nabla Z_i + \eta(\Delta Z_i + e^{V_\varepsilon}Z_i) - \eta\lambda^{-2}f(\varepsilon y)e^{V_\varepsilon}Z_i.$$

We have

$$\eta(\Delta Z_i + e^{V_\varepsilon} Z_i) = \varepsilon O((1+r)^{-3}).$$

Observe that

$$\lambda^{-2} f(\varepsilon y) e^{V_\varepsilon(y)} = \lambda^2 \delta^2 f(x) e^{v_\varepsilon(x)}, \quad \text{where } y = \frac{x}{\varepsilon}, x \in M,$$

thus

$$\eta \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon} Z_i = O(\varepsilon^2).$$

Since $\Delta\eta = O(\varepsilon)$, $\nabla\eta = O(\sqrt{\varepsilon})$, and besides $Z_i = O(r^{-1})$, $\nabla Z_i = O(r^{-2})$, we find

$$Z_i \Delta\eta + 2\nabla\eta \cdot \nabla Z_i = O(\varepsilon\sqrt{\varepsilon}).$$

From the previous estimates we conclude that

$$|\langle \phi, L(\eta Z_i)\rangle| \le C\sqrt{\varepsilon}\|\phi\|_\infty.$$

Combining this estimate with (4.14) and (4.15) we obtain

$$|c_i| \le C\left[\|h\|_* + \sqrt{\varepsilon}\log\frac{1}{\varepsilon}\right],$$

which implies

$$|c_i| \le C\|h\|_* \quad i = 1, 2.$$

It follows from (4.14) that

$$\|\phi\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right)\|h\|_*,$$

and the a priori estimate (4.5) has been thus proven. It only remains to prove the solvability assertion. For this purpose let us consider the space

$$H = \left\{\phi \in H^1(M_\varepsilon) \,:\, \int_{M_\varepsilon} \chi Z_i \phi = 0,\ i = 1, 2.\right\}$$

endowed with the inner product,

$$\langle \phi, \psi\rangle = \int_{M_\varepsilon} \nabla_g\phi\nabla_g\psi + \int_{M_\varepsilon} \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon}\phi\psi.$$

Problem (4.3) expressed in weak form is equivalent to that of finding $\phi \in H$ such that

$$\langle \phi, \psi\rangle = \int_{M_\varepsilon}\left[e^{V_\varepsilon}\phi + h + \sum_{i=1}^{2} c_i \chi Z_i\right]\psi, \quad \text{for all } \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where K is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K(\phi)$ has only zero as solution in H. This last equation is equivalent to problem (4.3) with $h \equiv 0$. Thus, existence of a unique solution follows from the a priori estimate (4.5). The proof is complete. □

5 The nonlinear problem

We recall that our goal is to solve problem (3.5). Rather than doing so directly, we shall solve first the intermediate problem

$$\begin{cases} L(\phi) = N(\phi) + E + \sum_{i=1}^{2} c_i \chi Z_i & \text{in } M_\varepsilon, \\ \displaystyle\int_{M_\varepsilon} \chi Z_i \phi = 0 & \text{for } i = 1, 2, \end{cases} \tag{5.1}$$

using the theory developed in the previous section. We assume that the conditions in Proposition (4.1) hold. We have the following result

Lemma 5.1 *Under the assumptions of Proposition* (4.1) *there exist positive number* C, ε_0 *such that problem* (5.1) *has a unique solution* ϕ *which satisfies*

$$\|\phi\|_\infty \le C\varepsilon \log \frac{1}{\varepsilon},$$

for all $\varepsilon < \varepsilon_0$.

Proof In terms of the operator T defined in Proposition (4.1), problem (5.1) becomes

$$\phi = T(N(\phi) + E) =: A(\phi). \tag{5.2}$$

For a given number $\vartheta > 0$, let us consider the space

$$H_\vartheta = \left\{ \phi \in C(M_\varepsilon) \ : \ \|\phi\|_\infty \le \vartheta \varepsilon \log \frac{1}{\varepsilon} \right\}.$$

From Proposition (4.1), we get

$$\|A(\phi)\|_\infty \le C \log \left(\frac{1}{\varepsilon}\right) (\|N(\phi)\|_* + \|E\|_*).$$

Let us first measure how well V_ε solves problem (3.4). Observe that

$$e^{V_\varepsilon(y)} = \lambda^4 \delta^2 e^{v_\varepsilon(x)}, \quad y = \frac{x}{\varepsilon}, x \in M, \tag{5.3}$$

so

$$\|e^{V_\varepsilon(y)}\|_* \le C\varepsilon.$$

As a consequence of the construction of the first approximation, the choice of the parameter δ, the expansion of the Green function G around p, and (5.3), a direct computation yields

$$\|E\|_* \le C\varepsilon.$$

Now we estimate

$$N(\phi) = \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon} (e^\phi - 1 - \phi) - e^{V_\varepsilon} (e^\phi - 1 - \phi).$$

In one hand, from (5.3) we deduce

$$\|e^{V_\varepsilon} (e^\phi - 1 - \phi)\|_* \le C\varepsilon \|\phi\|_\infty^2.$$

In the other hand

$$\lambda^{-2} f(\varepsilon y) e^{V_\varepsilon(y)} = \lambda^2 \delta^2 e^{v_\varepsilon(x)}, \quad y = \frac{x}{\varepsilon}, x \in M,$$

so

$$\|\lambda^{-2} f(\varepsilon y) e^{V_\varepsilon}(e^{\phi} - 1 - \phi)\|_* \leq C\varepsilon^{-\sigma}\|\phi\|_\infty^2.$$

We conclude,

$$\|N(\phi)\|_* \leq C\varepsilon^{-\sigma}\|\phi\|_\infty^2.$$

Observe that for $\phi_1, \phi_2 \in H_\vartheta$,

$$\|N(\phi_1) - N(\phi_2)\|_* \leq C\vartheta\varepsilon^{1-\sigma}\log\left(\frac{1}{\varepsilon}\right)\|\phi_1 - \phi_2\|_\infty,$$

where C is independent of ϑ. Hence, we have

$$\|A(\phi)\|_\infty \leq C\varepsilon\log\left(\frac{1}{\varepsilon}\right)[\vartheta^2\varepsilon^{1-\sigma}\log\left(\frac{1}{\varepsilon}\right) + 1],$$

$$\|A(\phi_1) - A(\phi_2)\|_\infty \leq C\varepsilon^{1-\sigma}\log\left(\frac{1}{\varepsilon}\right)\|\phi_1 - \phi_2\|_\infty.$$

It follows that there exist ε_0, such that for all $\varepsilon < \varepsilon_0$ the operator A is a contraction mapping from H_ϑ into itself, and therefore A has a unique fixed point in H_ϑ. This concludes the proof. □

With these ingredients we are now ready for the proof of our main result.

6 Proof of Theorem 1.1 for $n = 1$

After problem (5.1) has been solved, we find a solution to problem (3.5), and hence to the original problem, if $k = k(\varepsilon)$ is such that

$$c_i(k) = 0, \quad i = 1, 2. \tag{6.1}$$

Let us consider local conformal coordinates around p and define $r = |y|$. We consider a smooth cut-off function $\eta(r)$ such that $\eta(r) = 1$ for $r < \frac{1}{\sqrt{\varepsilon}}$, $\eta(r) = 0$ for $r > \frac{2}{\sqrt{\varepsilon}}$, $|\eta'(r)| \leq C\sqrt{\varepsilon}$, $|\eta''(r)| \leq C\varepsilon$. Testing the equation

$$L(\phi) = N(\phi) + E + \sum_{i=1}^{2} c_i \chi Z_i,$$

against ηZ_i, $i = 1, 2$, we find

$$\langle L(\phi), \eta Z_i\rangle = \int_{M_\varepsilon} [N(\phi) + E]\eta Z_i + c_i \int_{M_\varepsilon} \chi Z_i^2, \quad i = 1, 2.$$

Therefore, we have the validity of (6.1) if and only if

$$\langle L(\phi), \eta Z_i\rangle - \int_{M_\varepsilon} [N(\phi) + E]\eta Z_i = 0, \quad i = 1, 2.$$

We recall that in the proof of Proposition (4.1) we obtained

$$|\langle \phi, L(\eta Z_i)\rangle| \leq C\sqrt{\varepsilon}\|\phi\|_\infty,$$

thus

$$|\langle \phi, L(\eta Z_i)\rangle| \le C\varepsilon^{3/2} \log \frac{1}{\varepsilon}.$$

Observe that

$$\|N(\phi)\|_\infty \le C\varepsilon^2 \|\phi\|_\infty^2,$$

so

$$\left|\int_{M_\varepsilon} N(\phi)\eta Z_i\right| \le C\varepsilon\|\phi\|_\infty^2 \le C\varepsilon^3 \log^2 \frac{1}{\varepsilon}$$

Let us remember that

$$E = -\Delta V_\varepsilon + \lambda^{-2} f(\varepsilon y) e^{V_\varepsilon} - e^{V_\varepsilon} - \varepsilon^2 \alpha.$$

Using (5.3), we have

$$\int_{M_\varepsilon} e^{V_\varepsilon} \eta Z_i = O(\varepsilon^4).$$

We also have,

$$\int_{M_\varepsilon} \varepsilon^2 \alpha \eta Z_i = O(\varepsilon).$$

Observe that

$$\Delta_g V_\varepsilon(y) = \varepsilon^2 \Delta_g v_\varepsilon(x), \quad y = \frac{x}{\varepsilon}, x \in M,$$

thus

$$\int_{M_\varepsilon} \Delta V_\varepsilon \eta Z_i = O(\varepsilon^2).$$

Also, by change of variables we have

$$\int_{M_\varepsilon} f(\varepsilon y) e^{V_\varepsilon} \eta Z_i = \int_{\tilde{M}_\varepsilon} f(p + \varepsilon(y+k)) e^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_i(y+k+p'),$$

where $\tilde{M}_\varepsilon = M_\varepsilon - k + p'$. Using the fact that p is a local maximum of f of value 0, we have

$$f(p + \varepsilon(y+k)) = \varepsilon^2 \langle (y+k), D^2 f(p)(y+k)\rangle + O(\varepsilon^3),$$

where we used the fact that $f \in C^3(M)$. Thus

$$\lambda^{-2} \int_{M_\varepsilon} f(\varepsilon y) e^{V_\varepsilon} \eta Z_i = I_i + II_i,$$

where

$$I_i = \delta^2 \int_{\tilde{M}_\varepsilon} \langle (y+k), H_f(p)(y+k)\rangle e^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_i(y+k+p')$$

$$II_i = \int_{\tilde{M}_\varepsilon} O(\varepsilon) e^{V_\varepsilon(y+k+p')} \eta(|y+k|) Z_i(y+k+p').$$

Observe that $e^{V_\varepsilon(y+k+p')}\eta(|y+k|)Z_i(y+k+p') = O((1+|y|)^{-4})$, so

$$II_i = O(\varepsilon).$$

Finally, let us compute I_i. First, observe that $0 \in \tilde{M}_\varepsilon$. Let us consider a fixed number A_0, such that $\mathcal{B}_1 = B(0, A_0/\sqrt{\varepsilon}) \subset \tilde{M}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k)) =: \mathcal{B}$ and $\eta(\cdot + k) = 1$ in $\mathcal{B}_1$. We have the decomposition $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_2 = \tilde{\Omega}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k)) \setminus \mathcal{B}_1$. Also, observe that

$$Z_i(y+k+p') = C_0 \frac{y_i}{1+|y|}, \quad i = 1, 2,$$

where C_0 is a fixed constant independent of ε. We have the following computation

$$\begin{aligned}\langle (y+k), D^2 f(p)(y+k)\rangle &= f_{11}(p)(y_1+k_1)^2 + 2f_{12}(p)(y_1+k_1)(y_2+k_2)\\ &\quad + f_{22}(p)(y_2+k_2)^2,\end{aligned}$$

where $f_{11}(p) = \frac{\partial^2 f}{\partial y_1^2}(p)$, $f_{22}(p) = \frac{\partial^2 f}{\partial y_2^2}(p)$ and $f_{12}(p) = f_{21}(p) = \frac{\partial^2 f}{\partial y_1 \partial y_2}(p)$. We recall that

$$e^{V_\varepsilon(y+k+p')} = \frac{H_0}{(1+|y|^2)^2}(1 + C\sqrt{\varepsilon} + O(\varepsilon)), \tag{6.2}$$

in the region $\tilde{\Omega}_\varepsilon \cap \operatorname{supp}(\eta(\cdot + k))$.

Let us define $t_i(y) = e^{V_\varepsilon(y+k+p')}\eta(|y+k|)Z_i(y+k+p')$ and compute I_1. We have the following calculations for $i = 1$

$$\begin{aligned}\int_{\mathcal{B}} f_{11}(p)(y_1+k_1)^2 t_1(y) &= \int_{\mathcal{B}_1} f_{11}(p)(y_1+k_1)^2 t(y) + \int_{\mathcal{B}_2} f_{11}(p)(y_1+k_1)^2 t_1(y)\\ &= 2k_1 f_{11}(p)\int_{\mathcal{B}_1} C_0 \frac{y_1^2}{1+|y|}\frac{H_0}{(1+|y|^2)^2} + O(\varepsilon).\end{aligned}$$

In order to get the previous result, we used the fact that

$$\int_{\mathcal{B}_1} \frac{y_1}{1+|y|}\frac{dy}{(1+|y|^2)^2} = \int_{\mathcal{B}_1} \frac{y_1^3}{1+|y|}\frac{dy}{(1+|y|^2)^2} = 0,$$

and the expansion (6.2). We also have

$$\int_{\mathcal{B}} 2f_{12}(p)(y_1+k_1)(y_2+k_2)t_1(y) = 2k_2 f_{12}(p)\int_{\mathcal{B}_1} C_0\frac{y_1^2}{1+|y|}\frac{H_0}{(1+|y|^2)^2} + O(\varepsilon),$$

where we used the fact that

$$\int_{\mathcal{B}_1} \frac{y_1 y_2}{1+|y|}\frac{1}{(1+|y|^2)^2} = \int_{\mathcal{B}_1} \frac{y_1^2 y_2}{1+|y|}\frac{1}{(1+|y|^2)^2} = 0,$$

and also the expansion (6.2). Finally, we have

$$\int_{\mathcal{B}} f_{22}(p)(y_2+k_2)^2 t_1(y) = O(\varepsilon),$$

where we used

$$\int_{\mathcal{B}_1} \frac{y_1 y_2^2}{1+|y|}\frac{1}{(1+|y|^2)^2} = 0,$$

and also the expansion (6.2). From the above computations we conclude that

$$I_1 = 2\delta^2 I k_1 f_{11}(p) + 2\delta^2 I k_2 f_{12}(p) + O(\varepsilon),$$

where

$$I = \int_{\mathcal{B}_1} C_0 \frac{y_1^2}{1+|y|} \frac{H_0}{(1+|y|^2)^2} > 0.$$

Similar computations yield

$$I_2 = 2\delta^2 I k_1 f_{12}(p) + 2\delta^2 I k_2 f_{22}(p) + O(\varepsilon).$$

Summarizing, we have the system

$$\delta^2 D^2 f(p) k = \varepsilon b(k), \tag{6.3}$$

where b is a continuous function of k of size $O(1)$. Since p is a non-degenerate critical point of f, we know that $D^2 f(p)$ is invertible. A simple degree theoretical argument, yields that system (6.3) has a solution $k = O(\lambda\delta^{-1})$. We thus obtain $c_1(k) = c_2(k) = 0$, and we have found a solution of the original problem. The proof for the case $k = 1$ is thus concluded. □

7 Proof of Theorem 1.1 for general *n*

In this section we will detail the main changes in the proof of our main result, in the case of multiple bubbling.

Let $p_1, \ldots, p_n$ be points such that $f(p_j) = 0$ and $D^2 f(p_j)$ is positive definite for each j. We consider the singular problem

$$\Delta_g G - f e^G + 8\pi \sum_{j=1}^{k} \delta_{p_j} + \alpha = 0, \quad \text{in } M, \tag{7.1}$$

where δ_p designates the Dirac mass at the point p. A first remark we make is that the proof of Lemma 2.1 applies with no changes (except some additional notation) to find the result of Lemma 1.1. Indeed, the core of the proof is the local asymptotic analysis around each point p_j.

We define the first approximation in the original variable as

$$U_\varepsilon = \sum_{j=1}^{n} \eta_j u_\varepsilon^j + \left(1 - \sum_{j=1}^{n} \eta_j\right) G,$$

where η_j is defined around p_j as in Sect. 3 and, in local conformal coordinates around p_j, $u_\varepsilon^j(x) = w_{\delta_j}(|x - k_j|) + \tilde{F}_j(|x|)$, for parameters $k_j \in \mathbb{R}^2$. We make the following choice of the parameters δ_j

$$\log 8\delta_i^2 = -2\log\left(\frac{1}{\sqrt{2}}\log\frac{1}{\lambda}\right) + \mathcal{H}(p_i).$$

We also define the first approximation in the expanded variable around each p_j by

$$V_{\varepsilon_j}(y) = U_\varepsilon(\varepsilon_j y) + 4\log\lambda + 2\log\delta_j, \quad y \in M_{\varepsilon_j}$$

where $\varepsilon_j = \lambda\delta_j$ and $M_{\varepsilon_j} = \varepsilon_j^{-1} M$.

We look for a solution of problem (1.8) of the form $u(y) = U_\varepsilon(x) + \phi(x)$, where ϕ represent a lower order correction. By simplicity, we denote also by ϕ the small correction in the expanded variable around each p_j. In terms of ϕ, the expanded problem around p_j

$$\Delta_g v - \lambda^{-2} f(\varepsilon_j y) e^v + e^v + \varepsilon_j^2 \alpha = 0, \quad y \in M_{\varepsilon_j}$$

reads

$$L_j(\phi) = N_j(\phi) + E_j, \quad \text{in } M_{\varepsilon_j},$$

where

$$\begin{aligned} L_j(\phi) :=& \Delta_g \phi - \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} \phi + e^{V_{\varepsilon_j}} \phi, \\ N_j(\phi) :=& \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} (e^\phi - 1 - \phi) - e^{V_{\varepsilon_j}} (e^\phi - 1 - \phi), \\ E_j :=& -(\Delta_g V_{\varepsilon_j} - \lambda^{-2} f(\varepsilon_j y) e^{V_{\varepsilon_j}} + e^{V_{\varepsilon_j}} + {\varepsilon_j}^2 \alpha). \end{aligned}$$

Next we consider the linearized problem around our first approximation U_ε. Given h of class $C^{0,\beta}(M)$, which by simplicity we still denote by h in the expanded variable around each p_j, we consider the linear problem of finding a function ϕ such that for certain scalars c_i^j, $i = 1, 2;\ j = 1, \ldots, n$, one has

$$\begin{cases} L_j(\phi) = h + \sum_{i=1}^2 \sum_{j=1}^n c_i^j \chi_j Z_{ij} & \text{in } M_{\varepsilon_j}, \\ \displaystyle\int_{M_{\varepsilon_j}} \chi_j Z_{ij} \phi = 0 & \text{for all } i, j. \end{cases} \tag{7.2}$$

Here the definitions of Z_{ij} and χ_j are the same as before for Z_i and χ, with the dependence of the point p_j emphasized.

To solve this problem we consider now the norm

$$\|h\|_* = \sum_{j=1}^n \|h\|_{*,p_j}. \tag{7.3}$$

where $\|h\|_{*,p_j}$ is defined accordingly with (4.4). With exactly the same proof as in the case $n = 1$, we find the unique bounded solvability of problem (7.2) for all small $\varepsilon = \max \varepsilon_i$ by $\phi = T(h)$, so that

$$\|T(h)\|_\infty \le C \log\left(\frac{1}{\varepsilon}\right) \|h\|_*. \tag{7.4}$$

Then we argue as in the proof of Lemma 5.1 to obtain existence and uniqueness of a small solution ϕ of the projected nonlinear problem

$$\begin{cases} L_j(\phi) = N_j(\phi) + E_j + \sum_{i=1}^2 \sum_{j=1}^n c_i^j \chi_j Z_{ij} & \text{in } M_{\varepsilon_j}, \\ \displaystyle\int_{M_{\varepsilon_j}} \chi_j Z_{ij} \phi = 0 & \text{for all } i, j. \end{cases}$$

with

$$\|\phi\|_\infty \le C \varepsilon \log \frac{1}{\varepsilon}.$$

After this, we proceed as in Sect. 6 to choose the parameters k_j in such a way that $c_i^j = 0$ for all i, j. Summarizing, we have the system

$$D^2 f(p_j) k_j = \varepsilon_i \delta_i^{-2} b_j(k_1, \ldots, k_n), \tag{7.5}$$

which can be solved by the same degree-theoretical argument employed before. The proof is concluded. □

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