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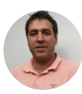

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Abstract

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz wedge- like domain (see figure 1). We construct positive weak solutions of the problem

$$\Delta u + u^p = 0 \quad \text{in } \Omega,$$

which vanish in a suitable trace sense on $\partial\Omega$, but which are singular at prescribed "edge" of Ω if p is equal or slightly above a certain exponent $p_0 > 1$ which depends on Ω. Moreover, in the case which Ω is unbounded, the solutions have fast decay at infinity.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary $\partial\Omega$. A model of nonlinear elliptic boundary value problem is the classical Lane-Emden-Fowler equation,

$$\begin{cases} -\Delta u & = |u|^p \quad \text{in} \quad \Omega, \\ u & > 0 \quad \text{in} \quad \Omega, \\ u & = 0 \quad \text{in} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$. Following Brezis and Turner [3] and Quittner and Souplet [13], we will say that a positive function u is a very weak solution of problem (1.1), if u and $\mathrm{dist}(x, \partial\Omega)u^p \in L^1(\Omega)$, and

$$\int_\Omega u\Delta v + |u|^p v \mathrm{d}x = 0, \qquad \forall v \in C^2(\overline{\Omega}), \text{ with } v = 0 \text{ on } \partial\Omega.$$

From the results in [3, 13], it follows that if p satisfies the constraint

$$1 < p < \frac{n+1}{n-1}, \tag{1.2}$$

then $u \in C^2(\overline{\Omega})$, i.e. u is a classical solution of problem (1.1).

It is well known that, if $1 < p < \frac{n+2}{n-2}$, one can use Sobolev's embedding and standard variational techniques to prove the existence of a positive very weak solution of problem (1.1). However, if $\frac{n+1}{n-1} < p < \frac{n+2}{n-2}$, this very weak solution may not be bounded. A result in the understanding of very weak solutions was achieved by Souplet [14]. He constructed an example of a positive function

$a \in L^\infty(\Omega)$ such that problem (1.1), with u^p replaced by $a(x)u^p$ for $p > \frac{n+1}{n-1}$, has a very weak solution which is unbounded, developing a point singularity on the boundary. This shows that the exponent $p = \frac{n+1}{n-1}$ is truly a critical exponent. Let us mention that the study of the behavior near an isolated boundary singularity of any positive solution of (1.1) when the exponent $p \geq \frac{n+1}{n-1}$ was achieved by Bidaut-Véron-Ponce-Véron in [2]. Finally, del Pino-Musso-Pacard [5] showed the existence of $\varepsilon > 0$ such that for any $p \in [\frac{n+1}{n-1}, \frac{n+1}{n-1} + \varepsilon)$ an unbounded, positive, very weak solution of (1.1) exists which blows up at a prescribed point of $\partial\Omega$. For the respective problem with interior singularity see for example [4, 6, 11, 12].

Let us give some definitions for convenience to the reader. Let $n \geq 2$ and $(r,\theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ be the spherical-coordinates of $x \in \mathbb{R}^n$ abbreviated by $x = (r,\theta)$. Given an open Lipschitz spherical cap $\omega \subsetneq \mathbb{S}^{n-1}$ let

$$C_\omega = \{x = (r,\theta) : \ r > 0, \ \theta \in \omega\},$$

be the corresponding infinite cone. The set

$$C_\omega^R = C_\omega \cap B_R(0) \subset \mathbb{R}^n$$

is called a conical piece with spherical cap ω and radius R.

A bounded Lipschitz domain $\Omega \subset C_\omega$ is called a domain with a conical boundary piece if there exists a conical piece C_ω^R such that $\Omega \cap B_R(0) = C_\omega^R$.

We denote by λ and $\phi_1(\theta)$ to be respectively the first eigenvalue and the corresponding eigenfunction of the problem

$$\begin{cases} -\Delta_{\mathbb{S}^{n-1}} u &= \lambda u \qquad \text{in } \omega \\ \qquad\quad u &= 0 \qquad \text{on } \partial\omega, \end{cases} \tag{1.3}$$

with $\int_\omega \phi_1^2 \mathrm{d}S_x = 1$.

Finally, we define the exponent

$$p^* = \frac{n+\gamma}{n+\gamma-2}; \qquad \text{with} \qquad \gamma = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda}, \tag{1.4}$$

and note that p^* depends on ω.

In the same spirit as above, McKennab-W. Reichel [9] generalized the results of Souplet [14] to domain with conical boundary piece, and they showed that the exponent p^* is a truly critical exponent, in the sense that, if $1 < p < p^*$, then every very weak solution of problem (1.1) is bounded (see also [1]). Finally, Horák-McKennab-Reichel [8] considered a bounded Lipschitz domain Ω with a conical boundary piece of spherical cap $\omega \subset \mathbb{S}^{n-1}$, at $0 \in \partial\Omega$, and they proved the existence of $\varepsilon > 0$ such that for any $p \in (p^*, p^* + \varepsilon)$ an unbounded, positive, very weak solution of (1.1) exists which blows up at $0 \in \partial\Omega$.

Let us consider the following problem

$$\begin{cases} \Delta_x u + u^p &= 0, \qquad \text{in } C_\omega \\ u &> 0, \qquad \text{in } C_\omega \\ u &= 0, \qquad \text{on } \partial C_\omega \setminus \{0\}. \end{cases} \tag{1.5}$$

The authors in [8] proved that problem (1.5) admits a positive solution of the form $w(\theta) = |x|^{-\frac{2}{p-1}}\phi_p(\theta)$, where ϕ_p solves the problem

$$\begin{aligned} \Delta_{\mathbb{S}^{n-1}}\phi - \frac{2}{p-1}\left(-\frac{2}{p-1} + n - 2\right)\phi + \phi^p = 0, &\qquad \text{in } \omega \\ \phi = 0, &\qquad \text{on } \partial\omega, \end{aligned} \tag{1.6}$$

for any $p \in (p^*, \infty)$ if $n = 2,\ 3$ and any $p \in (p^*, \frac{n+1}{n-3})$ if $n \geq 4$. But this solution does not have fast decay at infinity.

We note here that if $\omega = \mathbb{S}^{n-1}_+$, then $\gamma = 1$, thus the critical exponent $p^* = \frac{n+1}{n-1}$ and $C_\omega = \mathbb{R}^n_+$. In [5], del Pino-Musso-Pacard constructed a solution of problem (1.5) in $\mathbb{R}^n_+$ with fast decay. More precisely they showed that there exists $\varepsilon > 0$ such that for any $p \in (\frac{n+1}{n-1}, \frac{n+1}{n-1} + \varepsilon)$ problem (1.5) in $\mathbb{R}^n_+$ admits a solution $u \in C^2(\mathbb{R}^n_+)$ satisfying

$$u(x) \approx |x|^{-\frac{2}{p-1}} \phi_p(\theta), \qquad \text{as } |x| \to 0$$

and

$$u(x) \approx |x|^{-(n-1)} \phi_1(\theta), \qquad \text{as } |x| \to \infty.$$

The first result of this work is the construction of a singular solution at 0 with fast decay at infinity, for problem (1.5). In particular we prove

Theorem 1.1. *There exists a number $p(n, \lambda) > p^*$, such that for any*

$$p \in (p^*, p(n, \lambda)),$$

there exists a solution $u_1(x)$ to problem (1.5) such that

$$u_1(x) = |x|^{-\frac{2}{p-1}} \phi_p(\theta)(1 + o(1)) \qquad \text{as } |x| \to 0,$$

where ϕ_p solves (1.6), and

$$u_1(x) = |x|^{2-\gamma-n} \phi_1(\theta)(1 + o(1)) \qquad \text{as } |x| \to \infty,$$

where γ is defined in (1.4). In addition, we have the pointwise estimate

$$|u_1(x)| \leq C|x|^{-\frac{2}{p-1}} ||\phi_p||_{\mathcal{C}^2(\omega)},$$

for some constant $C > 0$ which does not depend on p.

To describe our main result let us introduce some new notations.

Let $x \in \mathbb{R}^n$ with $n \geq 2$. Given $\tau \in \mathbb{R}$, we let $\omega(\tau) \subsetneq \mathbb{S}^{n-1}$ to be the corresponding Lipschitz spherical cap. We set

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

where $\sigma : \mathbb{R} \to \mathbb{R}^n$ is a smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\} < C < \infty.$$

Now, given τ, we let $(r_{\sigma(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ to be the spherical-coordinates of $x \in \mathbb{R}^n$ centered at $\sigma(\tau)$ abbreviated by $x = (r_{\sigma(\tau)}, \theta)$. We define

$$\widetilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta) : \ r_{\sigma(\tau)} > 0, \ \theta \in \omega(\tau)\} \subset \mathbb{R}^n$$

and we set

$$\Omega_{\tau_1,\tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \widetilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1}, \qquad \text{(See figure 1)}$$

$$\Omega^R_{\tau_1,\tau_2} = \Omega_{\tau_1,\tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : \ x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1},$$

and

$$S_{\tau_1,\tau_2} = \{(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^n : r_{\sigma(\tau)} = 0\}.$$

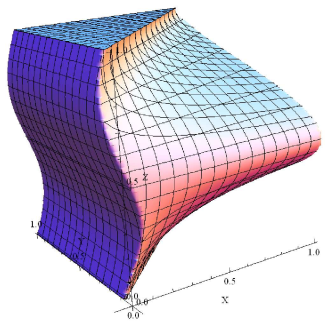

Figure 1: $\Omega_{0,1}$

Finally we define $\lambda^* = \inf_{\tau\in\mathbb{R}} \lambda(\tau)$ and $\gamma^* = \inf_{\tau\in\mathbb{R}} \gamma(\tau)$.

In this work we assume that $\omega(\tau)$ depends smoothly on τ, i.e. $\lambda(\tau)$ is a smooth bounded function with respect τ with bounded derivatives. We also assume that $\inf_{\tau\in\mathbb{R}} \lambda(\tau) > 0$. Finally, we suppose that there exists $\varepsilon > 0$, such that for any $p \in \left(\sup_{\tau\in\mathbb{R}} p^*(\tau), \sup_{\tau\in\mathbb{R}} p^*(\tau) + \varepsilon\right)$, there exists a solution $u_1(\tau, x)$ of theorem 1.1. That means, $\mathrm{osc}_{\tau\in\mathbb{R}}\lambda(\tau)$ is small enough.

Theorem 1.2. *Let $\varepsilon > 0$ be small enough. Then there exists a number $p_0 > \sup_{\tau\in\mathbb{R}} p^*$ such that, given $p \in (\sup_{\tau\in\mathbb{R}} p^*, p_0)$, and $\frac{2}{p-1} \le -\rho < n + \gamma^* - 2$, the following problem*

$$\begin{cases} -\Delta u & = u^p \quad \text{in} \quad \Omega_{-\infty,\infty}, \\ u & > 0 \quad \text{in} \quad \Omega_{-\infty,\infty} \\ u & = 0 \quad \text{on} \quad \partial\Omega_{-\infty,\infty} \setminus S_{-\infty,\infty} \end{cases}$$

possesses very weak solutions u. In addition we have that

$$u(\tau, x) \approx u_1\left(\tau, \frac{x - \sigma(\tau)}{\varepsilon}\right) \quad \text{as} \quad r_{\sigma(\tau)} \to 0,$$

where u_1 is in theorem 1.1. And

$$u(\tau, x) \le C\, r_{\sigma(\tau)}^{\rho} \quad \text{as} \quad r_{\sigma(\tau)} \to \infty.$$

Our third and final result of this paper is the following

Theorem 1.3. *Let $\alpha > 0$ be small enough and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded Lipschitz domain such that*

$$\Omega \cap \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} = \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \subset \mathbb{R}^{n+1}.$$

There exists a number $p_0 > \sup_{\tau\in\mathbb{R}} p^*$ such that, given $p \in (\sup_{\tau\in\mathbb{R}} p^*, p_0)$, there exist very weak solutions u to the problem

$$\begin{cases} -\Delta u & = u^p, \quad \text{in} \quad \Omega, \\ u & > 0\ , \quad \text{in} \quad \Omega \\ u & = 0\ , \quad \text{on} \quad \partial\Omega \setminus S_{\tau_1-\alpha,\tau_2+\alpha}. \end{cases}$$

Moreover, $\forall(\tau,x)\in\Omega^R_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}}$

$$u(\tau,x)\approx u_1\left(\tau,\frac{x-\sigma(\tau)}{\varepsilon}\right)\qquad \text{as}\ \ r_{\sigma(\tau)}\to 0.$$

The paper is organized as follows. In section 3 we prove theorem 1.1. In subsection 3.1, we prove some regularity results with respect τ, for the function $u_1(\tau,x)$ in theorem 1.1. Section 4 will be devoted to the proofs of theorems 1.2 and 1.3.

2 The eigenvalue problem on spherical caps.

Let $n\geq 2$, $\tau\in\mathbb{R}$, and $\omega(\tau)\subsetneq\mathbb{S}^{n-1}$ be the corresponding open Lipschitz spherical cap. We denote by $\lambda(\tau)$ and $\phi_1(\tau,\theta)$ to be respectively the first eigenvalue and eigenfunction of the eigenvalue problem

$$\begin{cases}-\Delta_{\mathbb{S}^{n-1}}u &= \lambda(\tau)u, \qquad \text{in } \omega(\tau)\\ \qquad\quad u &= 0\ , \qquad \text{on } \partial\omega,\end{cases} \tag{2.1}$$

with $\int_{\omega(\tau)}\phi_1^2 dS_x = 1$.

We assume that $\omega(\tau)$ depends smoothly on τ, i.e. $\lambda(\tau)$ is a smooth bounded function with respect τ with bounded derivatives. We also assume that $\inf\limits_{\tau\in\mathbb{R}}\lambda(\tau)>0$.

Now note that, without loss of generality, we can set $\theta_1=\cos t$, with $0<t<\beta(\tau)$, where $\beta(\tau)$ is a smooth function with bounded derivatives satisfying

$$\begin{cases} 0<\inf\limits_{\tau\in\mathbb{R}}\beta(\tau)<\sup\limits_{\tau\in\mathbb{R}}\beta(\tau)<2\pi \qquad \text{for} \quad n=2\\ \text{and}\\ 0<\inf\limits_{\tau\in\mathbb{R}}\beta(\tau)<\sup\limits_{\tau\in\mathbb{R}}\beta(\tau)<\pi \qquad \text{for} \quad n\geq 3.\end{cases}$$

Then problem (2.1) is equivalent to the following one

$$\begin{cases} -\sin^{2-n}t\frac{d}{dt}\left(\sin^{n-2}t\frac{d\phi_1}{dt}\right) &= \lambda\phi_1 \quad \text{in}\ \ (0,\beta(\tau)).\\ \phi_1(\beta(\tau)) &= 0\\ \frac{d\phi_1}{dt}(0) &= 0,\end{cases} \tag{2.2}$$

with

$$C(n)\int_0^{\beta(\tau)}\sin^{n-2}(t)|u|^2\mathrm{d}t=\int_\omega|\phi_1|^2\mathrm{d}S=1.$$

We note here that, for $n=2$ in problem (2.2), we may have $\phi_1(0)=0$ instead of $\frac{d\phi_1}{dt}(0)=0$.

We have the following lemma

Lemma 2.1. *Let $\phi_1(\tau,\theta)$ be the first eigenfunction of the following eigenvalue problem*

$$\begin{aligned}-\Delta_{\mathbb{S}^{n-1}}u = \lambda u, &\qquad \text{in } \omega(\tau)\\ u=0\ , &\qquad \text{on } \partial\omega(\tau),\end{aligned} \tag{2.3}$$

with $\int_{\omega(\tau)}\phi_1^2 dS=1$. Then there exists a positive constant C such that

$$\sup_{\tau\in\mathbb{R}}\left\|\,|\phi_1|+\left|\frac{\partial\phi_1}{\partial\tau}\right|+\left|\frac{\partial^2\phi_1}{\partial\tau^2}\right|\,\right\|_{L^\infty(\omega(\tau))}<C. \tag{2.4}$$

We postpone the proof of this lemma to the appendix.

3 Positive singular solution in the Cone

We keep the assumptions and notations of the previous section, and we consider the cone

$$C_{\omega(\tau)} = \{(r,\theta):\ r>0,\ \theta\in\omega(\tau)\},$$

where $r=|x|$ and $\theta=\frac{x}{|x|}$. We define the critical exponent

$$p^*(\tau)=\frac{n+\gamma(\tau)}{n+\gamma(\tau)-2} \quad \text{with} \quad \gamma(\tau)=\frac{2-n}{2}+\sqrt{\left(\frac{n-2}{2}\right)^2+\lambda(\tau)}.$$

We consider the problem

$$\begin{cases} \Delta_x u+u^p &=0, \qquad \text{in } C_{\omega(\tau)}\\ u &>0, \qquad \text{in } C_{\omega(\tau)}\\ u &=0, \qquad \text{on } \partial C_{\omega(\tau)}\setminus\{0\}. \end{cases} \tag{3.1}$$

If we set $w=|x|^{-\frac{2}{p-1}}\phi(\theta)$, we arrive at the problem

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}}\phi-\frac{2}{p-1}\left(-\frac{2}{p-1}+n-2\right)\phi+\phi^p &=0, \qquad \text{in } \omega(\tau)\\ \phi &=0, \qquad \text{on } \partial\omega(\tau). \end{cases} \tag{3.2}$$

By lemma 9 in [8], problem (3.2) has a positive solution $\phi_p\in H_1(\omega(\tau))\cap L^\infty(\omega(\tau))$ for any $p\in(p^*,\infty)$ if $n=2$ or 3 and for any $p\in(p^*(\tau),\frac{n+1}{n-3})$ if $n\geq 4$. Also as $p\downarrow p^*(\tau)$ then $-\frac{2}{p-1}\left(-\frac{2}{p-1}+n-2\right)\uparrow\lambda(\tau)$ and

$$\phi_p=\left(\frac{\lambda-\frac{2}{p-1}\left(-\frac{2}{p-1}+n-2\right)}{c_p}\right)^{\frac{1}{p-1}}(\phi_1+o(1)),$$

where $c_p=\int_{\omega(\tau)}\phi_1^{p+1}d\theta$.

In addition, for the same range on p, by theorem 10 in [8], the function

$$w_p(\tau,r,\theta)=r^{-\frac{2}{p-1}}\phi_p(\tau,\theta)$$

is a positive solution of (3.1).

In the rest of this section, for convenience, we omit dependence on the parameter τ writing $\lambda=\lambda(\tau)$, $\phi_1(\theta)=\phi_1(\tau,\theta)$ and so on.

Let $p\in(p^*,\frac{n+2}{n-2})$, we look for solutions of (3.1) of the form

$$u_1(x)=|x|^{-\frac{2}{p-1}}\phi(-\log|x|,\theta), \tag{3.3}$$

where $\theta=\frac{x}{|x|}$, so that the equation $\Delta u+u^p=0$ reads in terms of the function ϕ defined for $t\in\mathbb{R}$ and $\theta\in\omega$, as

$$\partial_t^2\phi+A\phi_t-\varepsilon\phi+(\Delta_{\mathbb{S}^{n-1}}\phi+\lambda\phi)+\phi^p=0, \tag{3.4}$$

where $t=-\log r$, $A=-\left(n-2\frac{p+1}{p-1}\right)$ and $\varepsilon=\lambda+\frac{2}{p-1}(n-\frac{2p}{p-1})$.

Let $\mu=\int_\omega\phi_1^{p+1}\mathrm{d}\theta$, we define a_∞ by

$$\mu a_\infty^{p-1}=\varepsilon.$$

We look for a positive function a which is a solution of

$$a''(t) + Aa'(t) - \varepsilon a(t) + \mu a^p(t) = 0, \tag{3.5}$$

which converges to 0 as t tends to $-\infty$ and converges to a_∞ as t tends to $+\infty$. Observe that, when $p \in (p^*, \frac{n+2}{n-2})$, the coefficients A and ε are positive and, therefore, in this range, classical ODE techniques yield the existence of a, a positive heteroclinic solution of (3.5) tending to 0 at $-\infty$ and tending to a_∞ at $+\infty$.

Observe that since the equation (3.5) is autonomous, the function a is not unique and a can be normalized so that $a(0) = \frac{1}{2}a_\infty$. For more informations about the function a, we refer the reader to lemmas 2.3, 2.4, 2.5 and appendix in [5].

Proposition 3.1. *Let $0 \leq p_0 < \infty$ and ε be small enough, then there exists a unique operator*

$$G_{p_0} : \ a^{p_0} L^\infty(\mathbb{R} \times \omega) \mapsto a^{p_0} L^\infty(\mathbb{R} \times \omega),$$

such that for any $a^{-p_0} g \in L^\infty(\mathbb{R} \times \omega)$, the function $u = G_{p_0}(g)$ is the unique solution of

$$L_p u = \left(\partial_t^2 + A\partial_t - \varepsilon + (\Delta_{\mathbb{S}^{n-1}} + \lambda) + p\phi_0^{p-1}\right) u = g; \quad \phi_0 = a(t)\phi_1(\theta),$$

with zero Dirichlet boundary data.

Furthermore,

$$\left|\left|d^{-1} a^{-p_0}(t)\psi\right|\right|_{L^\infty(\mathbb{R}\times\omega)} \leq \frac{C}{\varepsilon} \left|\left|a^{-p_0}(t) g\right|\right|_{L^\infty(\mathbb{R}\times\omega)}. \tag{3.6}$$

If in addition $g(t, \cdot)$ is L^2-orthogonal to ϕ_1 for a.e. t, then we have

$$\left|\left|d^{-1} a^{-p_0}(t)\psi\right|\right|_{L^\infty(\mathbb{R}\times\omega)} \leq C \left|\left|a^{-p_0}(t) g\right|\right|_{L^\infty(\mathbb{R}\times\omega)}$$

where $d : \omega \to (0, \infty)$ denotes the distance function to $\partial\omega$.

Proof. The proof follows the same lines as in lemma 2.6 in [5], so we will only focus on the differences. We first define ϕ_* to be the positive solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}}\phi_* + \lambda\phi_* + \delta(\delta - n - 2\gamma + 2)\phi_* = & -1 & \text{in } \omega \\ \phi_* = & 0 & \text{on } \partial\omega \end{cases} \tag{3.7}$$

see the proof of lemma 2.6 in [5] with obvious modifications. Using the function $(t, \theta) \to e^{-\delta t}\phi_*(\theta)$ as a barrier, as done in the paper [5], we can show that, given any function g such that $a^{-p_0} g \in L^\infty(\mathbb{R} \times \omega)$ and given $t_1 < -1 < 1 < t_2$, we can solve the equation

$$L_p u = g$$

in $(t_1, t_2) \times \omega$ with 0 boundary conditions.

To prove the estimate (3.6), we argue by contradiction, assuming that

$$||a^{-p_0}\psi_i||_{L^\infty} = 1$$

and

$$\lim_{i\to\infty} ||a^{-p_0} f_i|| = 0$$

we get a contradiction using similar argument as in lemma 2.6 in [5]. The rest of the proof is the same as in lemma 2.6 in [5] with obvious modifications so we omit it here. □

Proof of theorem 1.1. We look for a solution to problem (3.4) of the form

$$\phi = a(t)\phi_1(\theta) + \psi(t,\theta),$$

and we let G_p to be the operator defined in proposition 3.1. To conclude the proof, it is enough to find a function ψ solution of the fixed point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)),$$

where

$$\begin{aligned}\phi_0(t,\theta) &= a(t)\phi_1(\theta),\\ \mathcal{M}(\phi_0) &= a^p\left(\phi_1^p - \mu\phi_1\right)\\ \mathcal{Q}(\psi) &= |\phi_0+\psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.\end{aligned}$$

The rest of the proof is the same as in [5]. We recall here that $\psi << a\phi_1$. Also in [5], they have proven that if ε is small enough then there exists t_0 such that for any $t \le -\frac{t_0}{\varepsilon}$,

$$\frac{1}{2}e^{\delta^- t} \le a(t) \le e^{\delta^- t},$$

with $\delta^- = \frac{1}{2}\left(\sqrt{A^2+4\varepsilon} - A\right)$. And the result follows, since

$$\frac{1}{2}\left(\sqrt{A^2+4\varepsilon} - A\right) + \frac{2}{p-1} = n + \gamma - 2.$$

□

Remark 3.2.

If $1 < p_0 < p$ is close enough to p, we can apply a fix point argument like in the proof of theorem 1.1, for the operator G_{p_0}.

In view of the proof of lemma 2.1, $\phi_* = \phi_*(t, \cos(s\beta(\tau)))$.

Thus if the function g in proposition 3.1 is of the form $g = g(t, \cos(s\beta(\tau)))$, we have that the solution $u = G_{p_0}(g)$ is of the form $u = u(t, \cos(s\beta(\tau)))$. Hence we obtain, that the solution u_1 in theorem 1.1 is of the form

$$u_1 = r^{-\frac{2}{p-1}} u_1(r, \cos(s\beta(\tau))).$$

3.1 Regularity of the solution u_1 with respect τ

We first recall some definitions and known results, see the book of Gilbarg and Trudinger [7] for the proofs.

Let

$$Lu = a^{i,j}(x)D_{i,j}u + b^i(x)D_iu + c(x)u = g(x), \qquad a^{i,j} = a^{j,i},$$

where the coefficients $a^{i,j}$, b^i, c and the function g are defined in an open bounded domain $\Omega \subset \mathbb{R}^n$ and

$$a^{i,j}\xi_i\xi_j \le \mu|\xi|^2; \quad \mu > 0.$$

We assume that

$$||a^{i,j}||_{C^{2,a}},\ ||b^i||_{C^{2,a}},\ ||c||_{C^{2,a}} \le \Lambda.$$

Definition 3.3. *We say that a bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary $\partial\Omega$ are of class $C^{k,a}$, $0 \leq a \leq 1$, if at each point $x \in \partial\Omega$ there is a ball $B_r(x)$ and a one-to-one mapping ψ from $B_r(x)$ onto $D \subset \mathbb{R}^n$ such that:*

$$\psi(B_r(x) \cap \Omega) \subset \mathbb{R}^n_+,\ \psi(B_r(x) \cap \partial\Omega) \subset \partial\mathbb{R}^n_+,\ \psi \in C^{k,a}(B_r(x)) \text{ and } \psi^{-1} \in C^{k,a}(D).$$

A domain Ω will be said to have a boundary portion $T \subset \partial\Omega$ of class $C^{k,a}$, if at each point $x \in T$ there is a ball $B_r(x)$ in which the above conditions are satisfied and such that $B_r(x) \cap \partial\Omega \subset T$.

Proposition 3.4. ***(Lemma 6.18 in [7]).*** *Let $0 < a \leq 1$ and Ω be a domain with a $C^{2,a}$ boundary portion T, and let $\phi \in C^{2,a}(\overline{\Omega})$. Suppose that u is a $C^2(\Omega) \cap C_0(\overline{\Omega})$ function satisfying $Lu = g$ in Ω, $u = \phi$ on T, where g and the coefficients of the strictly elliptic operator L belong to $C^a(\overline{\Omega})$. Then $u \in C^{2,a}(\Omega \cup T)$.*

Proposition 3.5. ***(Corollary 6.7 in [7]).*** *Let $0 < a \leq 1$ and Ω be a domain with a $C^{2,a}$ boundary portion T, and let $\phi \in C^{2,a}(\overline{\Omega})$. Suppose that u is a $C^{2,a}(\Omega \cup T)$ function satisfying $Lu = g$ in Ω, $u = \phi$ on T. Then, if $x \in T$ and $B = B_\rho(x)$ is a ball with radius $\rho < \mathrm{dist}(x, \partial\Omega - T)$, we have*

$$||u||_{C^{2,a}(B\cap\Omega)} \leq C(n, \mu, \Lambda, \Omega \cap B_\rho(x)) \left(||u||_{C(\Omega)} + ||\phi||_{C^{2,a}(\overline{\Omega})} + ||g||_{C^a(\Omega)} \right).$$

We first prove the following result

Lemma 3.6. *Let $\tau \in \mathbb{R}$ be fixed, $x \in \mathbb{R}^n$, $n \geq 2$, $g \in C^a(\overline{C_\omega} \setminus \{0\})$ and $u = G_p(g)$ be the operator in proposition 3.1. Then*

$$\begin{aligned} |\nabla_x u(\tau, x)| &\leq C(n, p, \lambda, C_{\omega(\tau)}, g)\ |x|^{-1} \\ |D^2_x u(\tau, x)| &\leq C(n, p, \lambda, C_{\omega(\tau)}, g)\ |x|^{-2}. \end{aligned} \tag{3.8}$$

Proof. First we note that $||u(\tau, \cdot)||_{L^\infty(C_\omega(\tau))} \leq C||g(t, \cdot)||_{L^\infty(C_\omega(\tau))}$ and u is a solution of

$$\begin{cases} -\Delta_x u + \frac{4}{p-1}\frac{x\cdot\nabla_x u}{|x|^2} + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{u}{|x|^2} - p\frac{\phi_0^{p-1}u}{|x|^2} = -\frac{g}{|x|^2}, & \text{in } C_{\omega(\tau)} \\ u = 0 \quad \text{in } \partial C_{\omega(\tau)} \setminus \{0\}. \end{cases} \tag{3.9}$$

Set $R = |x|$, consider the domain

$$\Omega_R = \{y \in C_\omega : \ \frac{R}{4} < |y| < 4R\},$$

and let $y = \frac{x}{R}$ and define $v(y) = u(\tau, Ry)$. Then $y \in \Omega_1$ and v is a solution of

$$\begin{cases} -\Delta v + \frac{4}{p-1}\frac{y\cdot\nabla v}{|y|^2} + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{v}{|y|^2} - p\frac{\phi_0^{p-1}v}{|y|^2} = -\frac{g}{|y|^2}, & \text{in } \Omega_1 \\ v = 0 \quad \text{in } T, \end{cases} \tag{3.10}$$

where we have set

$$T = \partial\Omega_1 \setminus \{y \in C_\omega : \ |y| = \frac{1}{4} \text{ or } |y| = 4\}.$$

Let $0 < \varepsilon < \frac{\rho}{4}$ be small enough, where ρ is the defined in proposition 3.5 with $\Omega = \Omega_1$. Let $y_0 \in \partial\Omega_1 \setminus \{y \in C_\omega : \ |y| = \frac{1}{6} \text{ or } |y| = \frac{8}{3}\}$ then by propositions 3.4 and 3.5 we have

$$||v||_{C^2(B_\rho(\psi_0)\cap\Omega_{\frac{2}{3}})} \leq C(n, \mu, \Lambda, \Omega_1 \cap B_\rho(y_0))||g||_{C^a(\overline{\Omega_1})}$$

where in the last inequality we have used the estimate in proposition 3.1.

We note here that ρ depends only on Ω_1 and not on y_0. Thus if we apply a covering argument and standard interior Schauder estimates we have

$$||v||_{C^2(\Omega_{\frac{1}{2}})} \leq C\left(n, \mu, \Lambda, \Omega_1, \rho\right) ||g(x)||_{C^a(\overline{\Omega_1})}.$$

Using the facts that $x \in \Omega_{\frac{R}{2}}$, $\nabla v(y) = R\,\nabla u(x)$, $D_{i,j}v = R^2 D_{i,j}u$, $R = |x|$ and the above estimate, the result follows at once. □

In the rest of this paper we assume that the Lipschitz spherical cap $\omega(\tau)$ has the property: *there exists* $\widetilde{\varepsilon} > 0$, *such that for any* $p \in (\sup_{\tau\in\mathbb{R}} p^*(\tau), \sup_{\tau\in\mathbb{R}} p^*(\tau) + \widetilde{\varepsilon})$, *there exists a solution* u_1 *of theorem 1.1. Thus* $\varepsilon(\tau)$ *is a smooth bounded function with bounded derivatives and there exist* $\varepsilon_0, \varepsilon_1 > 0$ *such that* $\varepsilon_0 \leq \varepsilon(\tau) \leq \varepsilon_1$, $\forall \tau \in \mathbb{R}$.

Now, we recall some facts from the proof of theorem 1.1. Let $a(\tau, t)$ be the solution of the problem

$$\partial_t^2 a + A\partial_t a - \varepsilon(\tau)a + \mu(\tau)a^p = 0, \tag{3.11}$$

where $A = -\left(n - 2\frac{p+1}{p-1}\right)$, $\varepsilon(\tau) = \lambda(\tau) + \frac{2}{p-1}(n - \frac{2p}{p-1})$, $\mu(\tau) = \int_{\omega(\tau)} \phi_1^{p+1}(\tau,\theta)\mathrm{d}\theta$ and $\mu(\tau)a_\infty^{p-1}(\tau) = \varepsilon(\tau)$. Recall also that we have chosen $a(\tau, t)$ such that

$$a(\tau, 0) = \frac{1}{2}a_\infty(\tau), \qquad \lim_{t\to\infty} a(\tau, t) = a_\infty(\tau), \quad \text{and} \qquad \lim_{t\to-\infty} a(\tau, t) = 0.$$

We next prove the following lemma

Lemma 3.7. *Let* a *be the solution of (3.11),* $\varepsilon_0 = \inf_{\tau\in\mathbb{R}} \varepsilon(\tau)$,

$$\widetilde{\delta}^+(\tau) = \frac{-A + \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2} \qquad \text{and} \qquad \delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

Then there exists $\widetilde{t} > 0$ *such that*

$$\begin{aligned}
\left|\frac{\partial a}{\partial \tau}(\tau, t)\right| &\leq C(\varepsilon_0, p, n)|t|e^{\delta^-(\tau)t}, && \forall(\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\widetilde{t}}{\varepsilon_0}),\\
\left|\frac{\partial^2 a}{\partial \tau^2}(\tau, t)\right| &\leq C(\varepsilon_0, p, n)|t|^2 e^{\delta^-(\tau)t}, && \forall(\tau, t) \in \mathbb{R} \times (-\infty, -\frac{\widetilde{t}}{\varepsilon_0})\\
\left|\frac{\partial a}{\partial \tau}(\tau, t)\right| &\leq C(\varepsilon_0, p, n)|t|e^{\widetilde{\delta}^+(\tau)t}, && \forall(\tau, t) \in \mathbb{R} \times (\frac{\widetilde{t}}{\varepsilon_0}, \infty),\\
\left|\frac{\partial^2 a}{\partial \tau^2}(\tau, t)\right| &\leq C(\varepsilon_0, p, n)|t|^2 e^{\widetilde{\delta}^+(\tau)t}, && \forall(\tau, t) \in \mathbb{R} \times (\frac{\widetilde{t}}{\varepsilon_0}, \infty).
\end{aligned}$$

And

$$\begin{aligned}
\left|\frac{\partial a}{\partial \tau}(\tau, t)\right| &\leq C(\varepsilon_0, p, n), && \forall(\tau, t) \in \mathbb{R} \times [-\frac{\widetilde{t}}{\varepsilon_0}, \frac{\widetilde{t}}{\varepsilon_0}],\\
\left|\frac{\partial^2 a}{\partial \tau^2}(\tau, t)\right| &\leq C(\varepsilon_0, p, n), && \forall(\tau, t) \in \mathbb{R} \times [-\frac{\widetilde{t}}{\varepsilon_0}, \frac{\widetilde{t}}{\varepsilon_0}].
\end{aligned}$$

Proof. By our assumptions and lemma 2.5 in [5] there exists a constant $\bar{t} < 0$ (independent on p, μ and τ) such that

$$\frac{1}{2}e^{\delta^-(\tau)t} \le \frac{a(\tau,t)}{a_\infty(\tau)} \le e^{\delta^-(\tau)t}, \qquad \forall t \le \frac{\bar{t}}{\varepsilon_0},$$

where

$$\delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

Choose $\tau_0 \in \mathbb{R}$ and set $a(\tau,t) = a_\infty(\tau)(e^{\delta^-(\tau)t} + w(\tau,t))$. Then w is a solution of the fixed point problem

$$\begin{aligned} w &= -\varepsilon e^{\delta^-(\tau)t} \int_{-\infty}^{t} e^{-2\delta^-(\tau)\zeta - A\zeta} \left(\int_{-\infty}^{\zeta} e^{\delta^-(\tau)s + As} \left(e^{\delta^-(\tau)s} + w\right)^p \mathrm{d}s \right) \mathrm{d}\zeta \\ &:= T[w]. \end{aligned} \tag{3.12}$$

Indeed, let $1 < p_0 < p$ and ρ be sufficiently small such that for any $\tau \in O_{\tau_0} = \{\tau \in \mathbb{R} : \ |\tau - \tau_0| < \rho\}$ we have

$$p\delta^-(\tau) \ge p_0\delta^-(\tau_0) \qquad \text{and} \qquad p\delta^-(\tau_0) \ge p_0\delta^-(\tau).$$

Thus, it is easy to find a fixed point in the set of functions defined in $(-\infty, \frac{\bar{t}}{\varepsilon_0})$ and satisfying

$$|w| \le \frac{1}{2}e^{p_0\delta^-(\tau_0)t}$$

provided $|\bar{t}|$ is fixed large enough (independent of p and τ).

Now let

$$G = \{g : (-\infty, \frac{\bar{t}}{\varepsilon_0}) \mapsto \mathbb{R} : \ ||e^{-p_0\delta^-(\tau_0)t} g||_{L^\infty(-\infty, \frac{\bar{t}}{\varepsilon_0})} < C\}$$

and define $F(\tau, g) = g - T(g)$. By (3.12) we can apply the Implicit Function theorem in the domain $O_{\tau_0} \times G$ to obtain that there exists a unique function w such that

$$F(\tau, w(\tau,t)) = 0 \qquad \text{for any} \quad |\tau - \tau_0| < \rho_0 < \rho$$

for some ρ_0 small enough. On the other hand since $T(g)$ is smooth with respect τ we have that $w(\tau,t)$ is smooth with respect τ.

Notice that

$$0 = F_\tau(\tau, w(\tau,t)) + F_g(\tau, w(\tau,t))\frac{\partial w}{\partial \tau}$$

thus we have

$$\left|\frac{\partial w}{\partial \tau}(\tau,t)\right| \le C(\varepsilon_0, p, n)|t|e^{\delta^- t}, \tag{3.13}$$

provided $|\bar{t}|$ is fixed large enough. Similarly we have

$$\left|\frac{\partial^2 w}{\partial \tau^2}(\tau,t)\right| \le C(\varepsilon_0, p, n)|t|^2 e^{\delta^- t}. \tag{3.14}$$

By (3.12) and the above inequalities we have that the derivatives $\frac{\partial^2 w}{\partial\tau\partial t}$, $\frac{\partial^3 w}{\partial^2\tau\partial t}$ exist and are bounded.

Since the choice of τ_0 is abstract, we conclude that the functions a, $\partial_t a \in C^2$ with respect τ, for any $t \le \frac{\bar{t}}{\varepsilon_0}$. We also have

$$\begin{aligned} \left|\frac{\partial a}{\partial \tau}(\tau,t)\right| &\le C(\varepsilon_0, p, n)|t|e^{\delta^-(\tau)t}, \qquad \forall (\tau,t) \in \mathbb{R} \times (-\infty, -\frac{\widetilde{t}}{\varepsilon_0}), \\ \left|\frac{\partial^2 a}{\partial \tau^2}(\tau,t)\right| &\le C(\varepsilon_0, p, n)|t|^2 e^{\delta^-(\tau)t}, \qquad \forall (\tau,t) \in \mathbb{R} \times (-\infty, -\frac{\widetilde{t}}{\varepsilon_0}). \end{aligned} \tag{3.15}$$

Let $t_0 \in \left(-\infty, \frac{\bar{t}}{\varepsilon_0}\right)$ such that $a(\tau, t_0)$, $\frac{\partial a(\tau,t_0)}{\partial t} \in C^2$ with respect τ. Using standard ODE techniques we can prove that, if $|h|$ is sufficiently small then

$$|a(\tau,t) - a(\tau+h,t)| \leq C(t)h, \qquad \forall t \in \mathbb{R}, \tag{3.16}$$

where $C(t)$ is a positive smooth function such that $\lim\limits_{t\to\infty} C(t) = \infty$.

Choose $|h|$ sufficiently small and set $v_h = \frac{a(\tau+h,t)-a(\tau,t)}{h}$ and $a(\tau) = a(\tau,t)$. Then v_h satisfies

$$\begin{aligned}
&\frac{\partial^2 v_h}{\partial t^2} + A\frac{\partial v_h}{\partial t} - \varepsilon(\tau+h)v_h = -\mu(\tau+h)\frac{a^p(\tau+h) - a^p(\tau)}{h} \\
&\qquad -\frac{\mu(\tau+h)-\mu(\tau)}{h}a^p(\tau) + \frac{\varepsilon(\tau+h)-\varepsilon(\tau)}{h}a(\tau), \qquad \text{in } (t_0,\infty), \\
&v_h(\tau,t_0) = \frac{a(\tau+h,t_0) - a(\tau,t_0)}{h}, \\
&\frac{\partial v_h(\tau,t_0)}{\partial t} = \frac{\frac{\partial a(\tau+h,t_0)}{\partial t} - \frac{\partial a(\tau,t_0)}{\partial t}}{h}.
\end{aligned} \tag{3.17}$$

Using the following expansion

$$\begin{aligned}
a^p(\tau+h) &= a^p(\tau) + pa^{p-1}(\tau,t)\left(a(\tau+h) - a(\tau)\right) \\
&\quad + \frac{1}{2}\int_{a(\tau)}^{a(\tau+h)} p(p-1)t^{p-2}(a(\tau+h)-t)\mathrm{d}t,
\end{aligned}$$

thus by the properties of initial data in (3.17), our assumptions on μ, ε, (3.16) and above equality, we can obtain by using standard ODE techniques in (3.17) that

$$|v_h|, \quad |\frac{\partial v_h}{\partial t}| < C(t),$$

where $C(t)$ is a positive smooth function such that $\lim\limits_{t\to\infty} C(t) = \infty$. Thus by Arzela Ascoli theorem, there exist a subsequence $\{v_{h_n}\}$ such that $v_{h_n} \to v$ locally uniformly and v satisfies

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} + A\frac{\partial v}{\partial t} - \varepsilon(\tau)v = -\mu(\tau)pa^{p-1}(\tau,t)v - \mu'(\tau)a^p(\tau) + \varepsilon'(\tau)a(\tau) \quad \text{in } (t_0,\infty) \\
&v(\tau,t_0) = \frac{\partial a(\tau,t_0)}{\partial \tau} \\
&\frac{\partial v(\tau,t_0)}{\partial t} = \frac{\partial^2 a(\tau,t_0)}{\partial\tau\partial t}.
\end{aligned}$$

By uniqueness of the above problem, we have that $\lim\limits_{h\to 0} v_h = v$ for all $\tau \in \mathbb{R}$ and $t \geq t_0$. And thus $\frac{\partial}{\partial\tau}a(\tau,t)$ exists for any $(\tau,t) \in \mathbb{R}^2$. Applying the same argument we can obtain also that $\frac{\partial^2}{\partial\tau^2}a(\tau,t)$ exists for any $(\tau,t) \in \mathbb{R}^2$. The only difference is that we should use the fact that $a(\tau,t) > c > 0$ for any $(\tau,t) \in \mathbb{R} \times (t_0,\infty)$.

Set $a = a_\infty w$ then w satisfies

$$\partial_t^2 w + A\partial_t w - \varepsilon(\tau)w + \varepsilon(\tau)w^p = 0. \tag{3.18}$$

Let us now recall some facts from lemma 2.5 in [5]. Set

$$\widetilde{\delta}^+(\tau) = \frac{-A + \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2} \text{ and } \widetilde{\delta}^-(\tau) = \frac{-A - \sqrt{A^2 - 4(p-1)\varepsilon(\tau)}}{2}.$$

There exists a $\widehat{t} > 0$ (independent on p and τ) such that , $\forall\, t \geq \frac{\widehat{t}}{\varepsilon_0}$

$$\begin{aligned} &\frac{1}{2} e^{\widetilde{\delta}^-(\tau)t} \leq 1 - w(\tau,t) \leq 2e^{\widetilde{\delta}^-(\tau)t} \\ &\frac{1}{C(\varepsilon_0)} w(1-w) \leq \frac{\partial w}{\partial t} \leq C(\varepsilon_0) w(1-w). \end{aligned} \tag{3.19}$$

Notioce that the function $\frac{\partial w}{\partial \tau}$ is a solution of

$$\frac{\partial^2 v}{\partial t^2} + A\frac{\partial v}{\partial t} - \varepsilon(\tau)v + pw^{p-1}(\tau,t)v = \varepsilon'(\tau)w^p(\tau) + \varepsilon'(\tau)w(\tau), \tag{3.20}$$

but the function $\frac{\partial a}{\partial t}$ is one solution of the corresponding homogeneous problem. For the other solution of the homogeneous problem ψ we can easily prove by using (3.19) that

$$|\psi(t,\tau)| \leq C(\varepsilon_0) e^{\widetilde{\delta}^-(\tau)t}.$$

Thus by the representation formula and the properties of w, we can easily get

$$\left|\frac{\partial w}{\partial \tau}\right| \leq C(\varepsilon_0, p, n)|t| e^{\widetilde{\delta}^+(\tau)t}, \qquad \forall\, t \geq \frac{\widetilde{t}}{\varepsilon_0}.$$

Using the estimates (3.19) and the fact that w is a solution of (3.18), we can prove that

$$\left|\frac{\partial^2 w}{\partial t^2}\right| < C(\varepsilon_0, n, p) e^{\widetilde{\delta}^+(\tau)t}.$$

Setting $w = \left(1 - e^{\widetilde{\delta}^+(\tau)t} + v\right)$, then v can be written (see appendix in [5])

$$\begin{aligned} v = {} & \varepsilon e^{\widetilde{\delta}^-(\tau)t} \int_{t_p}^{t} e^{-2\widetilde{\delta}^-(\tau)\zeta - A\zeta} \left(\int_{\zeta}^{\infty} e^{\widetilde{\delta}^-(\tau)s + As} \mathcal{Q}\left(-e^{\widetilde{\delta}^+(\tau)s} + v\right) \mathrm{d}s \right) \mathrm{d}\zeta \\ & + \lambda_p e^{\widetilde{\delta}^-(\tau)t}. \end{aligned} \tag{3.21}$$

where $Q(x) = |1+x|^p - 1 - px$, t_p is large enough and $\lambda_p(\tau)$ is a smooth bounded function. Thus by (3.21) and the definition of v we can prove that there exists a constant $C > 0$ such that

$$\frac{1}{C} e^{\widetilde{\delta}^+(\tau)t} \leq -\partial_t^2 w(\tau,t) \leq C e^{\widetilde{\delta}^+(\tau)t}, \qquad \forall t \geq t_p.$$

By the same argument we can prove that

$$\left|\frac{\partial^2 w}{\partial \tau^2}(\tau,t)\right| \leq C(\varepsilon_0, p, n)|t|^2 e^{\widetilde{\delta}^+(\tau)t}, \qquad \forall\, t \geq \frac{\widetilde{t}}{\varepsilon_0}.$$

This ended the proof. □

Lemma 3.8. *Let u_1 be the solution given by theorem 1.1, then the following estimates hold*

$$|\partial_\tau u_1(\tau,x)| \leq C|x|^{-\frac{2}{p-1}} \qquad \text{and} \qquad |\partial_\tau^2 u_1(\tau,x)| \leq C|x|^{-\frac{2}{p-1}},$$

where the constant C does not depend on τ and x.

Proof. In view of the proof of theorem 1.1,

$$u_1 = |x|^{-\frac{2}{p-1}} f(\tau,\theta) = |x|^{-\frac{2}{p-1}} \left(a(\tau,t)\phi_1(\tau,\theta) + \psi(\tau,\theta)\right),$$

where ψ is a solution of the fixed point problem

$$\psi = -G_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi)), \tag{3.22}$$

where $\phi_0(\tau,\theta) = a(\tau,t)\phi_1(\tau,\theta)$, $\mathcal{M}(\phi_0) = a^p\left(\phi_1^p - \mu\phi_1\right)$ and

$$\mathcal{Q}(\psi) = |\phi_0 + \psi|^p - \phi_0^p - p\phi_0^{p-1}\psi.$$

We recall here that $|\psi(t,\theta)| << a(\tau,t)\phi_1(\tau,\theta)$.

Here we will only treat the case $n \geq 3$. For $n = 2$ the proof is the same.

By uniqueness, our assumptions on $\omega(\tau)$, and remark 3.2. $\psi = \psi(t,\widetilde{s})$, $\widetilde{s} \in (0,\beta(\tau))$, $\theta_1 = \cos\widetilde{s}$, where $\beta(\tau)$ is a positive smooth function such that

$$0 < \inf_{\tau\in\mathbb{R}} \beta(\tau) \leq \sup_{\tau\in\mathbb{R}} \beta(\tau) < \pi.$$

Then ψ satisfies

$$\begin{aligned}&\left(\partial_t^2 + A\partial_t - \varepsilon(\tau)\right)\psi + \sin^{2-n}(\widetilde{s})\partial_{\widetilde{s}}\left(\sin^{n-2}(\widetilde{s})\partial_{\widetilde{s}}\psi\right) + \lambda(\tau)\psi + p\phi_0^{p-1}\psi\\ &= -\mathcal{M}(\phi_0) - Q(\psi),\end{aligned}$$

for any $(t,\widetilde{s}) \in \mathbb{R}\times(0,\beta(\tau))$, and $\psi(t,\beta(\tau)) = 0$.

Setting now $s = \frac{\widetilde{s}}{\beta(\tau)}$, we have that $\psi(\tau,t,s)$ satisfies

$$\begin{aligned}\widetilde{L}_p\psi :&= \left(\partial_t^2 + A\partial_t - \varepsilon(\tau)\right)\psi + \frac{1}{\beta^2(\tau)}\partial_s^2\psi\\ &+ (n-2)\frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s\psi + \lambda\psi + p\phi_0^{p-1}\psi = -\mathcal{M}(\phi_0) - Q(\psi),\end{aligned} \tag{3.23}$$

for any $(t,s) \in \mathbb{R}\times(0,1)$, and $\psi(\tau,t,1) = 0$.

Let $1 < p_0 < p$ such that $p - p_0$ is small enough and let $g : \mathbb{R}\times(0,1) \to \mathbb{R}$ such that $g \in C^a(\mathbb{R}\times[0,1])$ for some $0 < a \leq 1$, and

$$\sup_{\tau\in\mathbb{R}} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |a^{-p}(\tau,t)g(t,s)| < \infty.$$

Let $u(\tau,t,s) = -\widetilde{G}_p(\mathcal{M}(\phi_0) + \mathcal{Q}(g))$ be the solution of (3.23). This solution exists since problem (3.23) is equivalent to (3.22). In addition, by proposition 3.1 we have the following estimate

$$\begin{aligned}\sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|d^{-1}a^{-p_0}(\tau,t)u(\tau,t)\right| &\leq C \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|a^{-p_0}(\tau,t)\mathcal{M}(\phi_0)(\tau,t,s)\right|\\ &+ \frac{C}{\varepsilon} \sup_{(t,s)\in\mathbb{R}\times(0,1)} \left|a^{-p_0}(\tau,t)\mathcal{Q}(g)(\tau,t)\right|,\end{aligned} \tag{3.24}$$

for some constant $C > 0$ which does not depend on τ.

We can easily prove that

$$\lim_{h\to 0} \sup_{(t,s)\in\mathbb{R}\times(0,1)} |u(\tau+h,t,s) - u(\tau,t,s)| = 0.$$

Recall the definitions

$$u_h(\tau,t,s) = \frac{u(\tau+h,t,s) - u(\tau,t,s)}{h}, \qquad u(\tau) = u(\tau,t,s), \cdots$$

Clearly u_h satisfies

$$\begin{aligned}
&\left(\partial_t^2 + A\partial_t - \varepsilon(\tau+h)\right) u_h(\tau) + \frac{1}{\beta^2(\tau+h)}\partial_s^2 u_h(\tau) \\
&+ \frac{(n-2)\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)}\partial_s u_h(\tau) + \lambda(\tau+h)u_h + p\phi_0^{p-1}(\tau+h)u_h(\tau) \\
&= -\frac{\frac{1}{\beta^2(\tau+h)} - \frac{1}{\beta^2(\tau)}}{h}\partial_s^2 u(\tau) + \frac{\varepsilon(\tau+h)-\varepsilon(\tau)}{h}u(\tau) - \frac{\lambda(\tau+h)-\lambda(\tau)}{h}u(\tau) \\
&- (n-2)\frac{\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}}{h}\partial_s u(\tau) - p\frac{\phi_0^{p-1}(\tau+h) - \phi_0^{p-1}(\tau)}{h}u(\tau) \\
&- \frac{\mathcal{M}(\phi_0)(\tau+h) - \mathcal{M}(\phi_0)(\tau)}{h} - \frac{\mathcal{Q}(g)(\tau+h) - \mathcal{Q}(g)(\tau)}{h}.
\end{aligned}$$

Now notice that $u(\tau,t,s) = w(t,\cos(s\beta(\tau))) = v(\tau,x)$, where $x_1 = |x|\cos(s\beta(\tau))$. In addition, $v(\tau,x)$ satisfies

$$\begin{cases}
-\Delta_x v + \frac{4}{p-1}\frac{x\cdot\nabla_x v}{|x|^2} + \frac{2}{p-1}\left(n - \frac{2}{p-1} - 2\right)\frac{v}{|x|^2} - p\frac{\phi_0^{p-1}v}{|x|^2} = -\frac{g}{|x|^2}, & \text{in } C_\omega(\tau) \\
v = 0 \quad \text{in } \partial C_\omega(\tau)\setminus\{0\}.
\end{cases}$$

Thus by lemma 3.6 we have

$$\left|\frac{1}{\sin s\beta(\tau)}\frac{\partial u}{\partial s}\right| \le \frac{1}{\inf\limits_{\tau\in\mathbb{R}}\beta(\tau)}|x||v_{x_1}| < C.$$

Similarly we can obtain $\left|\frac{\partial^2 u}{\partial s^2}\right| < C$ for some constant $C > 0$ which does not depend on τ.

Thus we have

$$\begin{aligned}
\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{1}{\sin s\beta(\tau)}\frac{\partial u}{\partial s}(\tau,t,s)\right| < C \\
\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{\partial^2 u}{\partial s}(\tau,t,s)\right| < C,
\end{aligned} \tag{3.25}$$

where the constant $C > 0$ does not depend on τ. Now we have

$$\begin{aligned}
&\lim_{h\to 0}\sup_{\tau\in\mathbb{R}}\left|\frac{\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}}{h}\partial_s u(\tau)\right| \\
&= \sup_{\tau\in\mathbb{R}}\left|\left(-\frac{\beta'(\tau)}{\beta^2(\tau)}\cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2\beta(\tau)s}\right)\partial_s u(\tau)\right| < C,
\end{aligned}$$

where in the last inequality we have used the fact that

$$0 < \inf_{\tau\in\mathbb{R}}\beta(\tau) \le \sup_{\tau\in\mathbb{R}}\beta(\tau) < \pi$$

and (3.25). Using the fact that

$$\begin{aligned}
&a^p(\tau+h)\phi_1^p(\tau+h) - a^p(\tau)\phi_1^p(\tau) \\
&= \left(a^p(\tau+h) - a^p(\tau)\right)\phi_1^p(\tau+h) + a^p(\tau)\left(\phi_1^p(\tau+h) - \phi_1^p(\tau)\right),
\end{aligned}$$

and

$$a^p(\tau+h) = a^p(\tau) + pa^{p-1}(\tau)(a^p(\tau+h) - a^p(\tau)) + \frac{p(p-1)}{2}\int_{a^p(\tau)}^{a^p(\tau+h)} t^{p-2}(a^p(\tau+h)-t)\mathrm{d}t,$$

(the same for ϕ_1), and lemmas 2.1, 3.7, we have that

$$\left|\lim_{h\to 0}\frac{\mathcal{M}(\phi_0)(\tau+h) - \mathcal{M}(\phi_0)(\tau)}{h}\right| = \left|\frac{\partial \mathcal{M}(\phi_0)}{\partial \tau}\right| < C.$$

Similarly we have that

$$\left|\lim_{h\to 0}\frac{\mathcal{Q}(g)(\tau+h) - \mathcal{Q}(g)(\tau)}{h}\right| = \left|\frac{\partial \mathcal{Q}(g)}{\partial \tau}\right| < C.$$

By proposition 3.1 we have

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}|u_h| < C$$

and thus by Arzela Ascoli theorem, there exist a subsequence $\{u_{h_n}\}$ such that $u_{h_n}\to v$ locally uniformly and $v(\tau,t,s)$ satisfies

$$\left(\partial_t^2 + A\partial_t - \varepsilon(\tau)\right)v + \frac{1}{\beta^2(\tau)}\partial_s^2 v + \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s v + \lambda(\tau)u + p\phi_0^{p-1}(\tau)v = H(\phi_1,a,g),$$

with $v(\tau,t,1)=0$. Notice that

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}|H(\tau,t,s)| < C,$$

thus by proposition 3.1 v is a unique solution. Furthermore,

$$\lim_{h\to 0} u_h = v = \frac{\partial u}{\partial \tau},$$

and

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{\partial u}{\partial \tau}(\tau,s,t)\right| < C, \tag{3.26}$$

for some constant C independent on g.

Similarly as (3.25) we can prove,

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{1}{\sin s\beta(\tau)}\frac{\partial^2 u}{\partial\tau\partial s}(\tau,t,s)\right| < C$$
$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{\partial^3 u}{\partial\tau\partial s\partial s}(\tau,t,s)\right| < C$$

and by the same argument as above

$$\sup_{\tau\in\mathbb{R}}\sup_{(t,s)\in\mathbb{R}\times(0,1)}\left|\frac{\partial^2 u}{\partial\tau\partial\tau}(\tau,t,s)\right| < C, \tag{3.27}$$

where C is a constant which depends on g.

Now we consider the fix point problem (3.23). Let $\tau_0 \in \mathbb{R}$ and ρ be small enough such that for any $\tau \in O_{\tau_0} = \{\tau \in \mathbb{R} : \ |\tau - \tau_0| < \rho\}$ we have $p\delta^-(\tau) \geq p_0\delta^-(\tau_0)$, where

$$\delta^-(\tau) = \frac{-A + \sqrt{A^2 + 4\varepsilon(\tau)}}{2}.$$

We can easily show that $a^p(\tau, t) \leq Ca^{p_0}(\tau_0, t), \ \forall \tau \in O_{\tau_0}$, for some positive constant C independent on τ and t.

Now since $0 < p - p_0$ is small enough, we can use a fix point argument like in [5] (see remark 3.2) in the Banach space

$$\mathbf{X} = \{g \in L^\infty(\mathbb{R} \times (0,1)) : \sup_{(t,s) \in \mathbb{R} \times (0,1)} |a^{-p_0}(\tau_0, t) g(t,s)| < \infty\}$$

to prove that there exists a unique solution

$$\psi(\tau, t, s) = -\widetilde{G}_p(\mathcal{M}(\phi_0) + \mathcal{Q}(\psi(\tau, t, s))), \qquad \forall \tau \in O_{\tau_0}.$$

Now, let $(\tau, g) \in O_{\tau_0} \times \mathbf{X}$, we set the bounded operator

$$T(\tau, g) = g + \widetilde{G}_p(g),$$

We can apply the Implicit Function theorem to $O_{\tau_0} \times \mathbf{X}$ to obtain that:

let $0 < \rho_0 \leq \rho$ be small enough, then for any $\tau \in \{\tau \in \mathbb{R} : \ |\tau - \tau_0| < \rho_0\} \subset O_{\tau_0}$ there exists a function $\psi(\tau, t, s)$ such that

$$T(\tau, \psi(\tau, t, s)) = 0.$$

Using (3.26), (3.27) and again the Implicit Function theorem, we can also prove that $\partial_\tau \psi, \ \partial_\tau^2 \psi$ exist. Furthermore using the fact that

$$0 = T_\tau(\tau, \psi(\tau)) + T_g(\tau, \psi(\tau))\partial_\tau \psi,$$

and the estimate (3.26) we have that

$$\sup_{\tau \in (\tau_0 - \rho_0, t_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial u}{\partial \tau}(\tau, t, s) \right| < C.$$

Similarly we have

$$\sup_{\tau \in (\tau_0 - \rho_0, t_0 + \rho_0)} \sup_{(t,s) \in \mathbb{R} \times (0,1)} \left| \frac{\partial^2 u}{\partial \tau \partial \tau}(\tau, t, s) \right| < C.$$

And the result follows since τ_0 is abstract. $\square$

4 The proof of theorems 1.2 and 1.3

Let $x \in \mathbb{R}^n, \ n \geq 2, \ R > 0, \ B_R(0) \subset \mathbb{R}^n$ and

$$r_{\sigma(\tau)} = |x - \sigma(\tau)|,$$

where $\sigma : \mathbb{R} \to \mathbb{R}^n$ is a smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \left\{ |\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)| \right\} < C < \infty.$$

Define

$$\widetilde{r}^2 = \sum_{i=1}^n |(x_i - \sup |\sigma(\tau)|)^2 :$$

Given τ, let $(r_{\sigma(\tau)}, \theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ be the spherical-coordinates of $x \in \mathbb{R}^n$ centered at $\sigma(\tau)$ abbreviated by $x = (r_{\sigma(\tau)}, \theta)$. We define the cone

$$\widetilde{C}_{\omega(\tau)} = \{x = (r_{\sigma(\tau)}, \theta) :\ r_{\sigma(\tau)} > 0,\ \theta \in \omega(\tau)\} \subset \mathbb{R}^n.$$

and we denote by

$$\Omega_{\tau_1,\tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \widetilde{C}_{\omega(\tau)}\} \subset \mathbb{R}^{n+1}.$$

$$\Omega^R_{\tau_1,\tau_2} = \Omega_{\tau_1,\tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n :\ x \in B_R(\sigma(\tau))\} \subset \mathbb{R}^{n+1},$$

and

$$S_{\tau_1,\tau_2} = \{(\tau, x) \in [\tau_1, \tau_2] \times \mathbb{R}^n : r_{\sigma(\tau)} = 0\}.$$

Let $C_{\delta,\rho}\left(\Omega^R_{\tau_1,\tau_2}\right)$ be the set of continuous function $f \in C\left(\Omega^R_{\tau_1,\tau_2}\right)$ with norm

$$||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} := \sup_{(\tau,x)\in\Omega^R_{\tau_1,\tau_2}} \left(\chi_{[0,1]}(r_{\sigma(\tau)}) r^{-\delta}_{\sigma(\tau)} |f| + \chi_{[1,\infty)}(r_{\sigma(\tau)}) \widetilde{r}^{\,-\rho} |f|\right).$$

Let $\delta \in (-n-\gamma+2, \gamma)$, we define $\phi_\delta(\tau, \theta)$ to be the unique positive solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi_\delta + \lambda \phi_\delta + (\delta(\delta+n-2) - \lambda)\, \phi_\delta & = -1, \qquad \text{in } \omega(\tau) \\ \phi_\delta & = \ \ 0, \qquad \text{on } \partial\omega(\tau). \end{cases}$$

Notice here that $\lambda = \gamma^2 + \gamma(n-2)$, thus $\delta(\delta+n-2) - \lambda < 0$ if and only if $\delta \in (-n-\gamma+2, \gamma)$. A direct computation shows that

$$-\Delta_x \left(|x|^\delta \phi_\delta\right) = |x|^{\delta-2}.$$

In view of lemma 2.1 we have that $\phi_\delta = \phi_\delta(t)$ where $t \in (0, \beta(\tau))$ and it satisfies

$$\begin{cases} \sin^{2-n} t \frac{d}{dt}\left(\sin^{n-2} t \frac{d\phi_\delta}{dt}\right) + \lambda \phi_\delta + (\delta(\delta+n-2) - \lambda)\, \phi_\delta & = -1 \quad \text{in } (0, \beta(\tau)) \\ \phi_\delta(\beta(\tau)) & = \ \ 0. \end{cases}$$

We next set $\beta^* = \sup_{\tau\in\mathbb{R}} \beta(\tau)$, and $\lambda^* = \inf_{\tau\in\mathbb{R}} \lambda(\tau)$, $\gamma^* = \inf_{\tau\in\mathbb{R}} \gamma(\tau)$ and we let ϕ^*_δ be the solution of

$$\begin{cases} \sin^{2-n} t \frac{d}{dt}\left(\sin^{n-2} t \frac{d\phi^*_\delta}{dt}\right) + \lambda^* \phi^*_\delta + (\delta(\delta+n-2) - \lambda^*)\, \phi^*_\delta & = -1 \quad \text{in } (0, \beta^*) \\ \phi_\delta(\beta^*) & = \ \ 0 \end{cases}$$

with $\gamma \in (-n-\gamma^*+2, \gamma^*)$.

Thus ϕ^*_δ is the unique solution of the problem

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}} \phi^*_\delta + \lambda^* \phi^*_\delta + (\delta(\delta+n-2) - \lambda^*)\, \phi^*_\delta & = -1, \qquad \text{in } \omega^* \\ \phi_\delta & = \ \ 0, \qquad \text{on } \partial\omega^* \end{cases}$$

where $\omega^* = \bigcup_\tau \omega(\tau)$ and by assumptions we have that $\omega^* \subsetneq \mathbb{S}^{n-1}$.

Proposition 4.1. *Assume that* $\delta, \rho \in (-n-\gamma^*+2, 0]$, *and*

$$\sup_{\tau\in\mathbb{R}} \left\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\right\} < \varepsilon, \tag{4.1}$$

where $\varepsilon > 0$ *is small enough. Then, for all* $\tau_1 < \tau_2 \in \mathbb{R}$, *and* $R > 0$, *there exists a unique operator*

$$G_{\delta,\rho,R,\tau_1,\tau_2} :\ C_{\delta,\rho}\left(\Omega^R_{\tau_1,\tau_2}\right) \to C_{\delta,\rho}\left(\Omega^R_{\tau_1,\tau_2}\right),$$

such that, for each $f \in C_{\delta,\rho}\left(\Omega^R_{\tau_1,\tau_2}\right)$, the function $G_{\delta,\rho,R,\tau_1,\tau_2}(f)$ is a solution of problem

$$\begin{cases} \Delta u &= \frac{1}{r^2_{\sigma(\tau)}} f, \qquad \text{in} \qquad \Omega^R_{\tau_1,\tau_2}, \\ u &= \quad 0, \qquad \text{on} \qquad \partial\Omega^R_{\tau_1,\tau_2} \setminus S_{\tau_1,\tau_2}. \end{cases} \tag{4.2}$$

Moreover the norm of $G_{\delta,\rho,R,\tau_1,\tau_2}$ is bounded by a constant $c > 0$ which does not depend on R, τ_1 and τ_2.

Proof. Without loss of generality we can assume that $R > 4$.

We first solve, for each $r \in (0, \frac{1}{4})$, the problem

$$\begin{cases} \Delta u &= \frac{1}{|x-\sigma(\tau)|^2} f, \qquad \text{in} \qquad \Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}, \\ u &= \quad 0, \qquad \text{on} \qquad \partial\left(\Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}\right). \end{cases} \tag{4.3}$$

and call u_r its unique solution.

A straightforward calculations show that

$$-\Delta(r^\delta_{\sigma(\tau)}\phi^*_\delta) \geq r^{\delta-2}_{\sigma(\tau)}(1 - |\delta|(|\delta|+1)|\sigma'|) - |\delta||\sigma''|r^{\delta-1}_{\sigma(\tau)}.$$

We choose ε small enough such that

$$-\Delta(r^\delta_{\sigma(\tau)}\phi^*_\delta) \geq \frac{1}{2}\left(r^{\delta-2}_{\sigma(\tau)} - r^{\delta-1}_{\sigma(\tau)}\right).$$

Let ψ be the solution of

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}}\psi = -C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} \qquad \text{in} \quad \omega^* \\ \psi = \quad 0, \qquad \text{on} \qquad \partial\omega* \end{cases}$$

for some constant $C > 0$ and we define the following cut-of function $\eta: \mathbb{R}^n \to [0,1]$ by $\eta = 1$ in $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and $\eta \in C^\infty_0(B_1(0))$.

We next set

$$\Phi(\tau, x) = C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\eta(x)r^\delta_{\sigma(\tau)}\phi^*_\delta + \psi.$$

If we choose the uniform constant $C > 0$, large enough, we have by the maximum principle

$$\begin{aligned} |u_r(\tau,x)| &\leq \Phi(\tau,x) \leq C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta|x|^\delta + \psi \\ &\leq C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta)(|x|^\delta + 1), \qquad \forall(\tau,x) \in \Omega^R_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2} \end{aligned} \tag{4.4}$$

where in the last inequality we have used the fact that

$$\psi(\theta) \leq C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta), \qquad \forall\theta \in \omega^*.$$

Using (4.4) and again the maximum principle we get

$$|u_r(\tau,x)| \leq C||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}\phi^*_\delta(\theta)|x|^\delta, \qquad \forall(\tau,x) \in \Omega^{\frac{1}{2}}_{\tau_1,\tau_2} \setminus \Omega^r_{\tau_1,\tau_2}. \tag{4.5}$$

Set now $\psi_0 = \widetilde{r}^\rho\phi^*_\rho$, then

$$\Delta_{\mathbb{S}^{n-1}}\psi_0 = -\widetilde{r}^{\rho-2}.$$

Thus using (4.5) and the maximum principle we obtain,

$$|u_r| \leq C(\sup_{\tau\in\mathbb{R}}|\sigma|)||f||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})}||\phi^*_\rho||_{L^\infty(\omega)}|x|^\rho, \quad \forall r_{\sigma(\tau)} > \frac{1}{2}. \tag{4.6}$$

By standard interior elliptic estimates and Arzela Ascoli theorem, there exists a subsequence $\{u_{r_j}\}$, such that $r_j \downarrow 0$ and $u_{r_j} \to u$ locally uniformly. By standard elliptic theory, (4.5) and (4.6), we have that $u \in C^2(\Omega^R_{\tau_1,\tau_2})$ and is unique. □

Proof of theorem 1.2. We choose $\delta = -\frac{2}{p-1}$ and we set

$$u_\varepsilon(x,\tau) = \eta(x)\varepsilon^{-\frac{2}{p-1}}u_1(\frac{x-\sigma}{\varepsilon}),$$

where u_1 is the function given in theorem 1.1 and $\eta:\ \mathbb{R}^n \to [0,1]$ is a cut-of function such that $\eta = 1$ in $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and $\eta \in C_0^\infty(B_1(0))$.

By construction of $u_1(x)$ and lemma 3.6 we have

$$\begin{aligned} |\nabla_x u_1(\tau,x)| &\leq C(n,p,\lambda,C_{\omega(\tau)})|x|^{-1} \\ |D_x^2 u(\tau,x)| &\leq C(n,p,\lambda,C_{\omega(\tau)})|x|^{-2}. \end{aligned} \tag{4.7}$$

First we assume that

$$\sup_{\tau\in\mathbb{R}} \left\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\right\} < \widetilde{\varepsilon}, \tag{4.8}$$

where $\widetilde{\varepsilon} > 0$ is small enough. Then by the above two estimates (4.7), (4.8) and lemma 3.8 we have

$$|\partial_\tau^2 u_\varepsilon(x,\tau)| \leq Cr^{-\frac{2}{p-1}}(\tau) + C(n,\gamma^*)\widetilde{\varepsilon}\left(r_{\sigma(\tau)}^{-\frac{2}{p-1}-2} + r_{\sigma(\tau)}^{-\frac{2}{p-2}-1}\right). \tag{4.9}$$

Now, let $R > 4$, $\tau_1 < \tau_2 \in \mathbb{R}$ and define the following problem

$$\begin{cases} -\Delta u &= u^p, \quad \text{in} \quad \Omega^R_{\tau_1,\tau_2}, \\ u &> 0\ , \quad \text{in} \quad \Omega^R_{\tau_1,\tau_2} \\ u &= 0\ , \quad \text{on} \quad \partial\Omega^R_{\tau_1,\tau_2} \setminus S_{\tau_1,\tau_2}. \end{cases} \tag{4.10}$$

We then look for a solution of the form $u = u_\varepsilon + v$. By virtue of proposition 4.1 we can rewrite this equation as the fixed point problem

$$v = -G_{\delta,\rho,R,\tau_1,\tau_2}\left(|x|^2\left(\Delta u_\varepsilon + |u_\varepsilon + v|^p\right)\right) \tag{4.11}$$

$$\Delta v = -|u_\varepsilon + v|^p - \Delta u_\varepsilon.$$

We assume that ε is small enough, then by (4.9) we have for some constant $C_0(n,\gamma) > 0$,

$$\begin{aligned} |||u_\varepsilon|^p + \Delta u_\varepsilon||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} &\leq C_0\left(\varepsilon^{n+\gamma-2-\frac{p-3}{p-1}} + \varepsilon^2 + \varepsilon + \widetilde{\varepsilon}\right) \\ &\leq C_0\left(\varepsilon + \widetilde{\varepsilon}\right), \end{aligned}$$

we recall here that $\delta = -\frac{2}{p-1}$.

Then, using theorem 1.1 one can easily see that

$$\begin{aligned} &\left|\left||x|^2|v_\varepsilon + v_1|^p - |v_\varepsilon + v_2|^p\right|\right|_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2})} \\ &\leq C_1(n,\gamma^*,p)\left(\sup_{\tau\in\mathbb{R}}||\phi_p||_{L^\infty(\omega)} + \widetilde{\varepsilon}\right)^{p-1}||v_1 - v_2||_{C_{\delta,\rho}(\Omega^1_{\tau_1,\tau_2})} \\ &+ C(n,\gamma^*,p)(\varepsilon + \widetilde{\varepsilon})^{p-1}||v_1 - v_2||_{C_{\delta,\rho}(\Omega^R_{\tau_1,\tau_2}\setminus\Omega^1_{\tau_1,\tau_2})}, \end{aligned} \tag{4.12}$$

for all $v_1, v_2 \in C_{\delta,\beta}\left(C^R_\omega \setminus \{0\} \times (\tau_1,\tau_2)\right)$ such that

$$||v_i||_{C_{\delta,\beta}(C^R_\omega\setminus\{0\}\times(\tau_1,\tau_2))} \leq 2C_0(\varepsilon + \widetilde{\varepsilon}).$$

We recall that all the constants above do not depend on R, t_1, t_2, ε and $\widetilde{\varepsilon}$. To obtain a contraction mapping is enough to take ε, $\widetilde{\varepsilon}$ small enough and p close enough to $\sup_{\tau\in\mathbb{R}} p^*$ to ensure that

$\sup_{\tau\in\mathbb{R}} ||\phi_p(\tau,\cdot)||_{L^\infty(\omega(\tau))}$ is as small as we need. The above estimates allow an application of contraction mapping principle in the ball of radius $2C_0(\varepsilon+\widetilde{\varepsilon})$ in $\Omega^R_{\tau_1,\tau_2}$ to obtain a solution to the problem (4.11), which we denote by

$$u_{R,\tau_1,\tau_2} = u_\varepsilon + v_{R,\tau_1,\tau_2}.$$

In view of the fix point argument, we have that $|v_{R,t_1,t_2}| \leq \frac{u_\varepsilon}{4}$ near S_{τ_1,τ_2}, thus the solution u_{R,t_1,t_2} is singular along S_{τ_1,τ_2} and positive near S_{τ_1,τ_2}. The maximum principle then implies that

$$u_{R,t_1,t_2} > 0 \qquad \text{in} \quad \Omega^R_{\tau_1,\tau_2}.$$

Moreover we have that

$$||v_{R,\tau_1,\tau_2}||_{C_{\delta,\beta}(\Omega^R_{\tau_1,\tau_2})} \leq 2C_0(\varepsilon+\widetilde{\varepsilon}).$$

That is , v_{R,τ_1,τ_2} is uniformly bounded by a constant which depend only on n, γ^*, p. By standard interior elliptic estimates and Arzela-Ascoli theorem, there exists a subsequence $\{u_{R_j,-\tau_j,\tau_j}\}$, such that $R_j \uparrow \infty$, $\tau_j \uparrow \infty$ and $u_{R_j,-\tau_j,\tau_j} \to u$ locally uniformly. Again standard elliptic theory yields $u \in C^2(\Omega_{-\infty,\infty})$.

For the general case

$$\sup_{\tau\in\mathbb{R}} \big\{|\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)|\big\} < C,$$

set $\widetilde{\sigma} = \frac{\sigma}{k}$, where $k > 0$ is large enough such that

$$\sup_{\tau\in\mathbb{R}} \big\{|\widetilde{\sigma}(\tau)| + |\widetilde{\sigma}'(\tau)| + |\widetilde{\sigma}''(\tau)|\big\} < \widetilde{\varepsilon}.$$

As before we can find a solution $u(x)$ of the problem with singularity along $\{(\tau,x) \in \mathbb{R}\times\mathbb{R}^n : |x-\widetilde{\sigma}(\tau)| = 0\}$. But the function $v(y) = k^{\frac{2}{p-1}} u(ky)$, where $y = kx$, is a singular solution of the problem and has singularity along $S_{-\infty,\infty}$, and the result follows. □

Let $\alpha > 0$, Ω be a bounded Lipschitz domain such that

$$\Omega \cap \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} = \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \subset \mathbb{R}^{n+1}.$$

Let $C_\delta\left(\Omega^R_{\tau_1,\tau_2}\right)$ be the set of continuous function $f \in C\left(\Omega^R_{\tau_1,\tau_2}\right)$ with norm

$$||f||_{C_\delta(\Omega^R_{\tau_1,\tau_2})} = \sup_{(\tau,x)\in\Omega^R_{\tau_1,\tau_2}} \left(r^{-\delta}(\tau)|f|\right).$$

We define $C_\delta(\Omega)$ to be the space of the continuous function in Ω with the norm

$$||f||_{C_\delta(\Omega)} = ||f||_{C_\delta\left(\Omega^R_{\tau_1-\alpha,\tau_2+\alpha}\right)} + ||f||_{L^\infty\left(\overline{\Omega}\setminus\Omega^{\frac{R}{2}}_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}}\right)}.$$

We consider a smooth, positive bounded function $\nu : \overline{\Omega} \to (0,\infty)$, which is equal to $r_{\sigma(\tau)}$ in $\Omega^{\frac{R}{2}}_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}}$ and satisfying

$$0 < \sup_{x\in\overline{\Omega}\setminus\Omega^R_{\tau_1-\frac{\alpha}{2},\tau_2+\frac{\alpha}{2}}} \nu < C.$$

We obtain the following proposition

Proposition 4.2. *Let $\tau_1 < \tau_2 \in \mathbb{R}$ and $\alpha > 0$ be small enough. Assume that Ω is a bounded Lipschitz domain such that*

$$\Omega \cap \Omega^R_{\tau_1-2\alpha,\tau_2+2\alpha} = \Omega^R_{\tau_1-2\alpha,\tau_2+2\alpha} \subset \mathbb{R}^{n+1},$$

$\delta \in (-n-\gamma^*+2, 0]$ and

$$\sup_{\tau \in \mathbb{R}} \left\{ |\sigma(\tau)| + |\sigma'(\tau)| + |\sigma''(\tau)| \right\} < \varepsilon, \tag{4.13}$$

for some $\varepsilon > 0$ small enough. Then, there exists a unique operator

$$G_{\delta,\tau_1,\tau_2} : \; C_\delta(\Omega) \to C_\delta(\Omega),$$

such that, for each $f \in C_\delta(\Omega)$, the function $G_{\delta,\tau_1,\tau_2}(f)$ is a solution of the problem

$$\begin{cases} \Delta u & = \frac{1}{\nu^2} f, \qquad \text{in} \qquad \Omega, \\ u & = \quad 0, \qquad \text{on} \qquad \partial\Omega \setminus S_{\tau_1-\alpha,\tau_2+\alpha}. \end{cases} \tag{4.14}$$

Moreover the norm of G_{δ,τ_1,τ_2} is bounded by a constant $c > 0$ which does not depend on R, τ_1 and τ_2.

Proof. Let $\widehat{\sigma}(t)$ be a bounded smooth curve such that

$$\sup_{\tau \in \mathbb{R}} \left\{ |\widehat{\sigma}(\tau)| + |\widehat{\sigma}'(\tau)| + |\widehat{\sigma}''(\tau)| \right\} < 2\varepsilon,$$

$$r_{\widehat{\sigma}(\tau)} = r_{\sigma(\tau)}, \qquad \forall (\tau, x) \in \Omega^R_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}},$$
$$r_{\widehat{\sigma}(\tau)} \geq r_{\sigma(\tau)}, \quad \forall (\tau, x) \in \Omega,$$

and

$$r_{\widehat{\sigma}(\tau)} > c > 0, \quad \forall (\tau, x) \in \Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \setminus \overline{\Omega^R_{\tau_1-\frac{\alpha}{2},\tau_2+\frac{\alpha}{2}}}.$$

Given τ, we let $\widehat{\omega}(\tau) \subsetneq \mathbb{S}^{n-1}$ be the corresponding Lipschitz spherical cap and $(r_{\widehat{\sigma}(\tau)}, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ be the spherical-coordinates of $x \in \mathbb{R}^n$ centered at $\widehat{\sigma}(\tau)$ abbreviated by $x = (r_{\widehat{\sigma}(\tau)}, \theta)$.

We set

$$\widehat{C}_{\widehat{\omega}(\tau)} = \{(r_{\widehat{\sigma}(\tau)}, \theta) : \; \widehat{r}(\tau) > 0, \; \theta \in \widehat{\omega}(\tau)\},$$
$$\widehat{\Omega}_{\tau_1,\tau_2} = \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : x \in \widehat{C}_{\widehat{\omega}(\tau)}\}$$

and $\widehat{\Omega}^R_{\tau_1,\tau_2} = \widehat{\Omega}_{\tau_1,\tau_2} \cap \{(\tau, x) \in (\tau_1, \tau_2) \times \mathbb{R}^n : \; x \in B_R(\widehat{\sigma}(\tau))\} \subset \mathbb{R}^{n+1}$. We construct $\widehat{\omega}(\tau)$ such that

$$\Omega^R_{\tau_1-\alpha,\tau_2+\alpha} \subsetneq \widehat{\Omega}^{2R}_{\tau_1-\alpha,\tau_2+\alpha},$$

$$\widehat{\Omega}^R_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}} = \Omega^R_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}}.$$

We next define η be a cut-off function satisfying $\eta = 1$ in $\Omega^{\frac{R}{2}}_{\tau_1-\frac{\alpha}{2},\tau_2+\frac{\alpha}{2}}$ and $\eta = 0$ in $\Omega \setminus \Omega^R_{\tau_1-\alpha,\tau_2+\alpha}$. We write $\widehat{f} = \eta f$ and we let $u_1 = G_{\delta,\rho,R,\tau_1,\tau_2}(\widehat{f})$ be the function given by proposition 4.1 in $\widehat{\Omega}^{2R}_{\tau_1-\alpha,\tau_2+\alpha}$.

Set

$$\widetilde{f} = f - \nu\Delta(\eta u_1),$$

then $\widetilde{f}$ has support in $\Omega \setminus \Omega^{\frac{R}{2}}_{\tau_1-\frac{\alpha}{4},\tau_2+\frac{\alpha}{4}}$, and $\widetilde{f} \in C(\Omega)$. Furthermore we have

$$||\widetilde{f}||_{C_\delta(\Omega)} \leq C||f||_{C_\delta(\Omega)},$$

for some positive constant $C > 0$.

Finally, let u_2 be a solution of

$$\begin{cases} \Delta u & = \frac{1}{\nu^2}\widetilde{f}, \qquad \text{in} \qquad \Omega, \\ u & = \quad 0, \qquad \text{on} \qquad \partial\Omega, \end{cases}$$

which clearly satisfy the bound

$$||u_2||_{L^\infty(\Omega)} \le C||\widetilde{f}||_{C_\delta(\Omega)} \le C||f||_{C_\delta(\Omega)}.$$

The desired result then follows by looking for a solution of (4.14) of the form $u = \eta u_1 + u_2$. □

Proof of theorem 1.3. We choose $\delta = -\frac{2}{p-1}$ and we set

$$u_\varepsilon(x,\tau) = \eta(x)\varepsilon^{-\frac{2}{p-1}}u_1(\frac{x-\sigma}{\varepsilon}),$$

where u_1 is the function given by theorem 1.1 and $\eta : \mathbb{R}^n \to [0,1]$ is a cut-of function such that $\eta = 1$ in $\Omega^{\frac{R}{2}}_{\tau_1-\frac{\alpha}{2},\tau_2+\frac{\alpha}{2}}$ and $\eta = 0$ in $\Omega \setminus \Omega^R_{\tau_1-\alpha,\tau_2+\alpha}$.

The rest of the proof is the same as in theorem 1.2, the only difference is that we use proposition 4.2 instead of proposition 4.1. □

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Proof of lemma 2.1 To prove lemma 2.1 we need the following inequality whose the proof can be found in [10] (theorem 2, page 43).

Lemma .3. *Let $A(r)$, $B(r)$ be nonnegative functions such that $1/A(r)$, $B(r)$ are integrable in (r,∞) and $(0,r)$, respectively, for all positive $r<\infty$. Then, for $q \ge 2$ the Sobolev inequality*

$$\left[\int_0^s B(t)|u(t)|^q \mathrm{d}t\right]^{1/q} \le C\left[\int_0^s A(t)|u'(t)|^2 \mathrm{d}t\right]^{1/2} , \tag{.15}$$

is valid for all $u \in C^1[0,s]$ such that $u(s) = 0$ (or vanish near infinity, if $s = \infty$), if and only if

$$K = \sup_{r\in(0,s)} \left[\int_0^r B(t)\mathrm{d}t\right]^{1/q} \left[\int_r^s (A(t))^{-1}\mathrm{d}t\right]^{1/2}$$

is finite. The best constant in (.15) satisfies the following inequality

$$K \le C \le K\left(\frac{q}{q-1}\right)^{1/2} q^{1/q}.$$

Proof of lemma 2.1. Let $n \ge 3$, (for $n = 2$ the proof is easy and we omit it). By our assumptions on $\omega(\tau)$ and without loss of generality, we can set $\theta_1 = \cos t$, with $0 < t < \beta(\tau)$, where $\beta(\tau)$ is a smooth function with bounded derivatives such that

$$0 < \inf_{\tau\in\mathbb{R}} \beta(\tau) < \sup_{\tau\in\mathbb{R}} \beta(\tau) < \pi.$$

Then problem (2.1) is clearly equivalent to

$$\begin{cases} -\sin^{2-n}t\frac{d}{dt}\left(\sin^{n-2}t\frac{d\phi_1}{dt}\right) & =\lambda\phi_1, \qquad \text{in } (0,\beta(\tau)). \\ \phi_1(\beta(\tau)) & =0 \\ \partial_t\phi_1(0) & =0. \end{cases} \tag{.16}$$

We denote by $\mathcal{H}((0,\beta(\tau)))$ the completion of $C^\infty([0,\beta(\tau)])$ under the norm

$$||v||^2_{\mathcal{H}((0,\beta(\tau)))}=\int_0^{\beta(\tau)}\sin^{n-2}(t)|\partial_t v|^2\mathrm{d}t<\infty,$$

and the property $v(\beta(\tau))=\partial_t v(0)=0$.

The space $\mathcal{H}(\omega(\tau))$ is a Hilbert space with inner product

$$(u,v)=\int_0^{\beta(\tau)}\sin^{n-2}(t)\partial_t u\partial_t v dt.$$

Indeed, by lemma .3 and our assumptions on $\beta(\tau)$, we can easily obtain that

$$\int_0^{\beta(\tau)}v^2\sin^{n-3}t\mathrm{d}t\le C(n)\int_0^{\beta(\tau)}\sin^{n-2}(t)|\partial_t v|^2\mathrm{d}t. \tag{.17}$$

By above inequality we can prove that the space $\mathcal{H}(\omega(\tau))$ is compactly embedded in

$$L^2_{\sin t}((0,\beta(\tau))):=\left\{u:\ (0,\beta(\tau))\to\mathbb{R}:\ \int_0^{\beta(\tau)}u^2\sin^{n-2}(t)\mathrm{d}t<\infty\right\}.$$

Thus using standard arguments we can prove that the eigenvalue problem

$$0<\lambda(\tau)=\inf_{u\in\mathcal{H}((0,\beta(\tau)))}\frac{\int_0^{\beta(\tau)}\sin^{n-2}(t)\left|\frac{du}{dt}\right|^2\mathrm{d}t}{\int_0^{\beta(\tau)}u^2\sin^{n-2}(t)\mathrm{d}t},$$

has a positive minimizer $\phi_1(\tau,t)\in\mathcal{H}(0,\beta(\tau))$.

But,

$$\begin{aligned} C(n)\int_0^{\beta(\tau)}\sin^{n-2}(t)|\partial_t\phi_1|^2\mathrm{d}t&=\int_\omega|\nabla\phi_1|^2\mathrm{d}S,\\ C(n)\int_0^{\beta(\tau)}\sin^{n-2}(t)|u|^2\mathrm{d}t&=\int_\omega|\phi_1|^2\mathrm{d}S=1, \end{aligned} \tag{.18}$$

thus $\phi_1\in H^1_0(\omega(\tau))$ and is a weak solution of the eigenvalue problem (2.1). Hence by standard elliptic arguments we can prove that $\phi_1\in L^\infty(\omega(\tau))$. In addition by our assumption we have that

$$\sup_{\tau\in\mathbb{R}}\sup_{t\in(0,\beta(\tau))}|\phi_1(\tau,t)|<C. \tag{.19}$$

By the ODE equation (.16) and the estimate (.19), we can write

$$\phi_1(\tau,t)=\lambda\int_t^{\beta(\tau)}\frac{1}{\sin^{n-2}s}\int_0^s\sin^{n-2}(r)\phi_1(\tau,r)\mathrm{d}r\mathrm{d}s. \tag{.20}$$

Thus we have the following estimates

$$\begin{aligned} \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} \left|\frac{1}{\sin t}\partial_t\phi_1(\tau,t)\right| &\le C \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} |\phi_1(\tau,t)| \\ \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} \left|\partial_t^2\phi_1(\tau,t)\right| &\le C \sup_{\tau\in\mathbb{R}} \sup_{t\in(0,\beta(\tau))} |\phi_1(\tau,t)|\,. \end{aligned} \tag{.21}$$

Setting now $s = \frac{t}{\beta(\tau)}$, we have that $\phi_1 = \phi_1(\tau, s)$ satisfies

$$\begin{cases} \frac{1}{\beta^2(\tau)}\partial_s^2\phi_1(\tau,s) + \frac{(n-2)\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}\partial_s\phi_1(\tau,s) + \lambda(\tau)\phi_1(\tau,s) &= 0 \qquad \text{in } (0,1) \\ \phi_1(1) &= 0 \\ \partial_t\phi_1(0) &= 0. \end{cases}$$

It is easy to see that $\lim_{h\to 0}\phi_1(\tau+h,s) = \phi_1(\tau,s)$ in $L^\infty(\mathbb{R}\times(0,1))$. We set

$$u_h(\tau) = \frac{\phi_1(\tau+h,s)-\phi_1(h,s)}{h}, \qquad \phi_1(\tau) = \phi_1(\tau,t),$$

then u_h satisfies

$$\begin{aligned} &\frac{1}{\beta^2(\tau+h)}\partial_s^2 u_h(\tau) + \frac{(n-2)\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)}\partial_s u_h(\tau) + \lambda(\tau+h)u_h(\tau) \\ = &\ -\frac{\frac{1}{\beta^2(\tau+h)}-\frac{1}{\beta^2(\tau)}}{h}\partial_s^2\phi_1(\tau) - \frac{\lambda(\tau+h)-\lambda(\tau)}{h}\phi_1(\tau) \\ - &\ (n-2)\frac{\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}}{h}\partial_s\phi_1(\tau) = F_h(\tau,s), \end{aligned} \tag{.22}$$

with $u_h(\tau,1) = \partial_s u_h(\tau,0) = 0$. On the other hand notice that

$$\begin{aligned} &\sup_{\tau\in\mathbb{R}}\left|(n-2)\frac{\frac{\cos(\beta(\tau+h)s)}{\beta(\tau+h)\sin(\beta(\tau+h)s)} - \frac{\cos(\beta(\tau)s)}{\beta(\tau)\sin(\beta(\tau)s)}}{h}\partial_s\phi_1(\tau,s)\right| \\ \le &\ \sup_{\tau\in\mathbb{R}}\left|(n-2)\left(-\frac{\beta'(\tau)}{\beta^2(\tau)}\cot(\beta(\tau)s) - \frac{s\beta'(\tau)}{\sin^2\beta(\tau)s}\right)\partial_s\phi_1(\tau,s)\right| \\ < &\ C(n, \inf_{\tau\in\mathbb{R}}\beta(\tau)), \end{aligned} \tag{.23}$$

where in the last inequality we have used (.21) and our assumptions on β. Also using our assumption on λ we have that

$$\sup_{h\in\mathbb{R}}\sup_{\tau\in\mathbb{R}} F_h(\tau,s) < C(n, \inf_{\tau\in\mathbb{R}}\beta(\tau)). \tag{.24}$$

Finally combining above estimates(.22)-(.24) we have

$$\lim_{h\to 0}\sup_{\tau\in\mathbb{R}}\int_0^1 u_h^2(\tau,s)\sin^{n-2}(\beta(\tau)s)\mathrm{d}s < C < \infty. \tag{.25}$$

By (.25) we can prove

$$\sup_{\tau\in\mathbb{R}}\sup_{\tau\in\omega(\tau)} |u_h| < C$$

and we have the following representation formula

$$\begin{aligned}\frac{u_h(\tau,s)}{\beta^2(\tau+h)} &= \lambda(\tau+h)\int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau+h)\xi)}\int_0^\xi \sin^{n-2}(\beta(\tau+h)r)u_h(\tau,r)drd\xi \\ &- \int_s^1 \frac{1}{\sin^{n-2}(\beta(\tau+h)\xi)}\int_0^\xi \sin^{n-2}(\beta(\tau+h)r)F_h(\tau,r)drd\xi.\end{aligned}$$

The rest of the proof is standard and we omit it. □

References

[1] H. Beresticky, I. Capuzzo-Dolcetta, L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems, Topol. Methods Nonlinear Anal. 4 (1994) 59-78.*

[2] M.F. Bidaut-Véron, A. Ponce, L. Véron, *Boundary singularities of positive solutions of some nonlinear elliptic equations, C. R. Acad. Sci. Paris Sr. I 344 (2) (2007) 83-88.*

[3] H. Brezis, R. Turner, *On a class of superlinear elliptic problems, Comm. Partial Differential Equations 2 (1977) 601-614.*

[4] C.C. Chen, C.S. Lin, *Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equations, J. Geom. Anal. 9 (2) (1999) 221-246.*

[5] M. del Pino, M. Musso, F. Pacard, *Boundary singularities for weak solutions of semilinear elliptic problems, Journal of Functional Analysis 253 (2007) 241-272.*

[6] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (4) (1981) 525-598.*

[7] G. Gilbarg, N. S. Trudinger,*Elliptic partial differential equations of second orders. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.*

[8] J. Horák, P.J. McKenna, W. Reichel, *Very weak solutions with boundary singularities for semilinear elliptic Dirichlet problems in domains with conical corners, J. Math. Anal. Appl. 352 (2009), no. 1, 496-514.*

[9] P.J. McKenna, W. Reichel, *A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains, J. Funct. Anal. 244 (1) (2007) 220-246.*

[10] V.G. Maz'ya, *Sobolev Spaces, Springer-Verlag, Berlin/New York, 1985.*

[11] R.Mazzeo, F. Pacard, *A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis, J. Differential Geom. 44 (2) (1996) 331-370.*

[12] F. Pacard, *Existence and convergence of positive weak solutions of* $-\Delta u = u^{\frac{n}{n-2}}$ *in bounded domains of* $\mathbb{R}^n$, $n \geq 3$, *Calc. Var. Partial Differential Equations 1 (1993) 243-265.*

[13] P. Quittner, P. Souplet, *A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces, Arch. Ration. Mech. Anal. 174 (1) (2004) 49-81.*

[14] P. Souplet, *Optimal regularity conditions for elliptic problems via* L^p_δ*−spaces, Duke Math. J. 127 (1) (2005) 175-192.*