The precedence constrained knapsack problem: Separating maximally violated inequalities.

Daniel Espinoza
Department of Industrial Engineering, Universidad de Chile

Marcos Goycoolea
School of Business, Universidad Adolfo Ibañez

Eduardo Moreno
Faculty of Engineering and Sciences, Universidad Adolfo Ibañez

Received: December 5, 2012/ Accepted: –

Abstract

We consider the problem of separating maximally violated inequalities for the precedence constrained knapsack problem. Though we consider maximally violated constraints in general, special emphasis is placed on induced cover inequalities and induced clique inequalities. Our contributions include a new partial characterization of maximally violated inequalities, a new safe shrinking technique, and new insights on strengthening and lifting. This work follows on the work of Boyd (1993), Park and Park (1997), van de Leesel et al (1999) and Boland et al (2011).

1 Introduction

Given a directed graph $G = (V, A)$, vectors $a \in \mathbb{Z}_+^V, c \in \mathbb{Z}^V$, and $b \in \mathbb{Z}_+$, the precedence-constrained knapsack problem (PCKP) consists in solving a problem of the form,

$$\begin{array}{ll} \max & cx \\ s.t., & x \in P(G, a, b) \\ & x \in \{0, 1\}^V \end{array}$$

where

$$P(G, a, b) = \left\{x \in [0, 1]^V : ax \le b,\ x_i \le x_j\ \forall (i, j) \in A\right\}.$$

Aside from being an interesting problem in itself, PCKP is an important relaxation of many common, more complex, integer programming problems. Important examples arise in the context of production scheduling problems, where a number of jobs must be scheduled for processing subject to limited resources, and where precedence relationships dictate that in order for some jobs to be processed other jobs must be processed as well. An example of such a problem that has received much attention in recent years is the open-pit mine production scheduling problem. In this problem, jobs represent discretized units of rock that must be extracted, and precedence relationships establish that units must be extracted from the surface on downwards. Integer programming formulations for open pit mine production scheduling problems having PCKP as a substructure appear as early as in the 1960s and 1970s in the works of Johnson [17], Dagdelen and Johnson [14] and others. Since then, a number of articles have addressed PCKP specifically. These articles can broadly be subdivided in two groups: Those articles which develop algorithms to solve PCKP as an optimization problem, and those articles that analyze the polyhedral structure of $P(G, a, b)$.

Practical open pit mining problems are very large, easily having tens or hundreds of millions of variables. As shown by Johnson and Neimi [16], however, PCKP is strongly $\mathcal{NP}$-complete. Thus, efforts to solve this problem to provable optimality gaps have either been based on approximation algorithms [12, 23] or on enumerative methods such as branch-and-bound. An important component of any branch-and-bound solver is the LP relaxation solver. Much progress has been made in recent years solving the LP relaxations of PCKP generalizations with customized algorithms. Important examples include the work of Boland et al. [8], Chicoisne et al [10] and Bienstock and Zuckerberg [5]. Heuristics for these problems that use PCKP as a subproblem, or that use the PCKP linear programming relaxation as a subproblem, are described in Amaya et al [1], Bley et al [6], Chicoisne et al [10] and Cullenbine et al [13].

Polyhedral analyses, on the other hand, have focused on developing useful cutting planes. Boyd [9], Park and Park [22] and van de Leesel et al [18] describe characterizations, separation algorithms, and strengthening techniques for an important class of cutting planes known as induced minimal cover inequalities. Boland et al [7] extends previous results using clique inequalities. Bley [6] tests many of these ideas on open pit mine production scheduling problems. Despite important work in this problem, cutting plane techniques to date are still limited in terms of the instance sizes that can effectively be tackled computationally. This is a problem, because if there is any hope of being able to solve large integer programming formulations of PCKP generalizations, such as those that appear in the context of open pit mining, cutting planes are likely to play an important role as they have in solving other large combinatorial optimization problems [2, 3].

In this article, given a fractional point $x^* \in P(G, a, b)$, we are interested in efficiently finding an inequality $\alpha x \leq \beta$ that is valid for $x \in P(G, a, b) \cap \mathbb{Z}^n$ and as violated as possible by x^*. Specifically, we are interested in tackling algorithmic aspects of separating maximally violated inequalities for the precedence constrained knapsack problem, and builds on the work of Boyd [9], Park and Park [22], van de Leesel et al [18] and Boland et al [7] so that it can be applied on very large instances of PCKP and on generalizations of PCKP. For this we introduce a new shrinking technique that can be used to reduce the separation problem in any given instance of PCKP to an equivalent separation problem in a smaller instance. This shrinking procedure is safe, in the sense that it guarantees that the most violated cuts in the original problem can be mapped to equally violated cuts in the shrunken problem, and vice-versa. Furthermore, we introduce a new way of strengthening general valid inequalities for PCKP, and remark how the lifting techniques of Park and Park [22], originally proposed for minimal induced cover inequalities, can be generalized to broader classes of inequalities.

This article is organized as follows. In Section 2 we review important results in the literature and introduce the notation we will use throughout the paper. In Section 3 we characterize maximally violated inequalities and introduce the concept of *break-points*, which will be used throughout the paper. In Section 4 we show how to shrink the original graph in order to find maximally violated inequalities in a smaller problem. Moreover, we show that if this shrinking operations is obtained using break-points, then we can map maximally violated inequalities obtained in the shrunken problem to maximally violated inequalities of the original one. In Section 5 we show that it is possible to obtain even further reductions. Finally, in Section 6 we show how to obtain strengthened inequalities by using lifting procedures. Our lifting results builds on the results of Boyd [9], Park and Park [22] and van de Leesel et al [18].

2 Definitions, assumptions, and background material

In this section we establish the assumptions and notations that we will use throughout the article. More importantly, we survey some important prior results concerning two classes of valid inequalities for PCKP, the minimal induced cover inequalities and the clique inequalities. These results, all of which can be found in previous literature, will be the starting point for our developments in later sections.

Definition 1. Consider a directed graph $G = (V, A)$ with no directed cycles. We say that $C \subseteq V$ is a *closure* in G if $i \in C$ implies $j \in C$ for all $(i, j) \in A$. That is, if set C is closed under the precedence relationships defined by graph G. Given any set $S \subseteq V$ we define the smallest closure containing S in graph $G = (V, A)$ as follows,

$$cl(G, S) = \{j \in V : \text{ there is a path in } G \text{ from some } i \in S \text{ to } j \ \}.$$

To simplify notation, when graph G is clear from context we will write $cl(S)$ instead of $cl(G, S)$. For $i \in V$, we will write $cl(i)$ instead of $cl(\{i\})$.

Definition 2. Consider a directed graph $G = (V, A)$ with no directed cycles. We say that $R \subseteq V$ is a *reverse closure* in G if $j \in R$ implies $i \in R$ for all $(i, j) \in A$. That is, if set R is closed under the reverse of precedence relationships defined by graph G. Given any set $S \subseteq V$ we define the smallest reverse closure containing S in graph $G = (V, A)$ as follows,

$$rcl(G, S) = \{i \in V : \text{ there is a path in } G \text{ from } i \text{ to some } j \in S \}.$$

To simplify notation, when graph G is clear from context we will write $rcl(S)$ instead of $rcl(G, S)$. For $i \in V$, we will write $rcl(i)$ instead of $rcl(\{i\})$.

Definition 3. We say that (G, a, b) defines an instance of PCKP if,

- $G = (V, A)$ is a directed graph,
- $a \in \mathbb{Z}^V$ represents non-negative node-weights,
- $b \in \mathbb{Z}$ is a non-negative scalar,

In order to simplify notation and proofs we will make the following working assumptions regarding instances of PCKP. Given a set $C \subseteq V$ and a vector $y \in \mathbb{R}^V$ we will convene that $y(C) = \sum_{i \in C} y_i$.

Definition 4. We say that an instance of PCKP (G, a, b) satisfies our working assumptions if,

- $G = (V, A)$ has no directed cycles,
- For every $i, j, k \in V$, $(i, j), (j, k) \in A$ implies $(i, k) \in A$,
- There are no arcs of the form (i, i) with $i \in V$,
- $a(cl(i)) \leq b$ for all $i \in V$,
- $a(V) > b$.

Note that there is no loss of generality from our working assumptions. In fact, if G were not acyclic, we could iteratively collapse all of the variables associated to nodes in any directed cycle into a single variable. In this way we would obtain an equivalent instance of PCKP. Adding (or removing) arcs to satisfy the second and third conditions does not change the solution of PCKP, nor does it change the set $P(G, a, b)$. If $a(cl(k)) > b$ for $k \in V$, then $x_k = 0$ for all $x \in P(G, a, b) \cap \{0, 1\}^n$. In this case variable x_k may as well be eliminated. If $a(V) \leq b$, then $P(G, a, b)$ is fully described by inequalities $x_i \leq x_j$ for $(i, j) \in A$ and $x_i \geq 0$ for $i \in V$, making all valid inequalities trivial. Finally note that under our working assumptions, sets $cl(S)$ and $rcl(S)$ can be defined as follows,

$$cl(S) = \{j \in V : (i, j) \in A \text{ for some } i \in S\}.$$
$$rcl(S) = \{i \in V : (i, j) \in A \text{ for some } j \in S\}.$$

Theorem 1 (Boyd, [9])**.** *If (G, a, b) defines an instance of PCKP satisfying our working assumptions, then $P(G, a, b)$ is full dimensional.*

Given an instance of PCKP (G, a, b) satisfying our working assumptions, we focus on finding strong inequalities $\alpha x \leq b$ that are valid for $P(G, a, b) \cap \mathbb{Z}^V$. In what follows, we formalize two different notions of strong valid inequalities.

Definition 5. Let (G, a, b) define an instance of PCKP satisfying our working assumptions. Consider two inequalities $\alpha x \leq \beta$, and $\alpha' x \leq \beta'$, both valid for $P(G, a, b) \cap \mathbb{Z}^V$. We say that inequality $\alpha' x \leq \beta'$ is *stronger* than inequality $\alpha x \leq \beta$ if (a) for every $\hat{x} \in P(G, a, b)$ we have $\alpha' \hat{x} \leq \beta'$ implies $\alpha \hat{x} \leq \beta$, and (b) there exists $\hat{x} \in P(G, a, b) \cap \mathbb{Z}^V$ such that $\alpha' \hat{x} = \beta'$ and $\alpha \hat{x} < \beta$. We say that inequality $\alpha' x \leq \beta'$ is *stronger in a weak-sense* than inequality $\alpha x \leq \beta$ if, instead of condition (b), we have (b') there exists $\hat{x} \in P(G, a, b)$ such that $(\alpha' \hat{x} - \beta')/||(\alpha', b')|| < (\alpha \hat{x} - \beta)/||(\alpha, \beta)||$.

From a polyhedral analysis point of view it is only natural to try and generalize known families of inequalities for the knapsack problem (KP) to PCKP. The first to do so was Boyd [9], who extending the cover inequalities of Balas and Jeroslow [4], and $(1, k)$-configurations of Padberg [20], first introduced the notion of induced cover inequalities.

Definition 6. Let (G, a, b) define an instance of PCKP satisfying our working assumptions. We say that $C \subseteq V$ is an *induced cover* of (G, a, b) if $a(cl(C)) > b$. If C is an induced cover, then valid inequality

$$x(C) \leq |C| - 1, \tag{1}$$

is known as the *induced cover inequality* associated to C. In this document we will define an induced cover C to be *minimal* if $a\,(cl(C \setminus \{i\})) \leq b$, $\forall i \in C$. If C is a minimal induced cover, we say that inequality (1) is a *minimally induced cover inequality*, or MIC inequality for short.

Note that the definition of MIC inequalities that we use coincides with that used by Park and Park [22]. However, Boyd [9] and van de Leensel et al [18], require that an induced cover $C \subseteq V$ satisfy $a\,(cl(C) \setminus \{i\}) \leq b$ $\forall i \in C$ in order to be minimal.

Remark 1. Let (G, a, b) represent an instance of PCKP satisfying our working assumption. If C is a minimal induced cover, then $i, j \in C$ implies $(i, j) \notin A$. In fact, if $i, j \in C$ and $(i, j) \in A$, then $cl(C) = cl(C - \{j\})$, thus contradicting the minimality of C.

As observed by Bley et al [6], given $x^* \in P(G, a, b)$ it is possible to find a maximally violated MIC inequality by solving a simple integer programming problem.

Given any induced cover C, there always exists a minimal induced cover $C' \subseteq C$. Further, if C' is an induced cover and $C' \subsetneq C$, then the induced cover inequality associated to C' is stronger than that associated to C. As observed by Boyd this does not imply that minimal induced cover inequalities are facet-defining. To see this, consider the example illustrated in Figure 1 and the MIC inequality $x_{15} + x_{16} + x_{17} \leq 2$. All feasible points satisfying this inequality at equality also satisfy $x_{10} = x_{11} = 1$. Since we know that the feasible region is full dimensional (Theorem 1), it follows that the inequality cannot be facet defining. Boyd [9] and later Park and Park [22] describe conditions under which MIC inequalities are facet-defining.

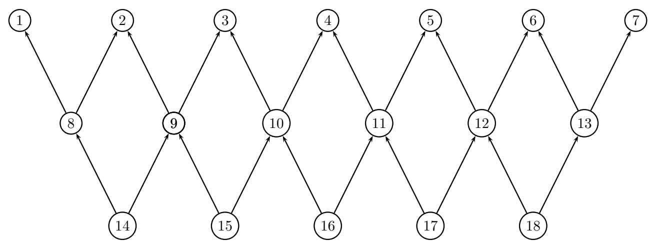

Figure 1: Example PCKP: $a_i = 1$, $i = 1, \ldots, 18$, $b = 10$, note that $C = \{15, 16, 17\}$ is a MIC, but not a facet for the problem. In particular, x_3, x_4, x_5 must be one whenever $x_{15} + x_{16} + x_{17} = 2$.

In the case that a MIC inequality is not facet-defining it can be strengthened through lifting techniques to obtain a constraint of form,

$$\sum_{i \in C} x_i + \sum_{i \in cl(C)} \gamma_i (1 - x_i) + \sum_{j \in V \setminus cl(C)} \eta_j x_j \leq |C| - 1, \tag{2}$$

where η, γ are non-negative vectors in $\mathbb{R}^C$. Going back to the example illustrated in Figure 1, we have that if we strengthen the MIC inequality $x_{15}+x_{16}+x_{17} \leq 2$ on x_{10} and then x_{11} we obtain the facet-defining inequality $x_{15} + x_{16} + x_{17} - x_{10} - x_{11} \leq 0$.

Park and Park [22] showed that given a minimally induced cover C, it is possible to lift the corresponding MIC inequality on variables x_i with $i \in cl(C)$ in polynomial time. Moreover, they showed that this was possible in such a way as to guarantee that the resulting inequality was as violated as possible. van de Leesel et al [18] showed that it was possible to speed-up the algorithm of Park and Park to run in $\mathcal{O}(|V| \log^* |V|)$. Park and Park [22] also introduced a heuristic for lifting x_i with $i \in rcl(C)$. van de Leesel et al [18] studied the optimal lifting problem associated to these variables as well as the conditions required to ensure that the resulting inequality would be facet-defining. van de Leesel et al [18] also propose a pseudo-polynomial time algorithm for the case in which G is a tree.

Definition 7. Let (G, a, b) define an instance of PCKP satisfying our working assumptions. We say that $C \subseteq V$ is an *induced clique* of (G, a, b) if $a(cl(\{i, j\})) > b$ for all $i, j \in C$ such that $i \neq j$. If C is an induced clique, then valid inequality

$$x(C) \leq 1, \tag{3}$$

is known as the *induced clique inequality* associated to C.

Proposition 1. *(Boland et al. [7]) Let (G, a, b) define an instance of PCKP satisfying our working assumptions. If C is an induced clique in (G, a, b) then $i, j \in C$ implies $(i, j) \notin A$. Further, $i, j \in C$ implies that there is no $k \in V$ such that $(k, i) \in A$ and $(k, j) \in A$.*

Proof. First, suppose $i, j \in C$ and $(i, j) \in A$. Since $i, j \in C$ we know $a(cl(\{i, j\})) > b$. However, $(i, j) \in A$ implies $cl(j) \subseteq cl(i)$, thus $cl(\{i, j\}) \subseteq cl(\{i\})$. This in turn implies $a(cl(\{i\})) > b$ which contradicts our working assumptions.
Second, suppose $i, j \in C$ and that there exists $k \in V$ such that $(k, i) \in A$ and $(k, j) \in A$. In this case $cl\{i, j\} \subseteq cl(k)$. However, in this case, $a(cl(\{i, j\})) > b$ implies $a(cl(\{k\})) > b$, which contradicts our working assumptions. ■

Boland et al. [7] observe that it is possible to obtain facet-defining clique inequalities by up-strengthening induced covers of size 2. Based on this observation they propose a polynomial-time running heuristic that can generate a special class of induced clique inequalities.

3 Maximally violated inequalities

In this section we characterize maximally violated inequalities for PCKP. We begin by describing a partial characterization of maximally violated inequalities in general, and continue by characterizing some specific properties of maximally violated induced cover and clique inequalities.

Definition 8. Let (G, a, b) define an instance of PCKP satisfying our working assumptions, and consider $x^* \in P(G, a, b)$. Define,

$$B[x^*] = \{i \in V : x^*_j < x^*_i, \ \forall (j, i) \in A\}.$$

We say that $B[x^*]$ is the set of precedence *break-points* associated to G and x^*. If $j \in V$ is such that there is no arc $(i, j) \in A$, then we assume $j \in B[x^*]$. For each $i \in V$ define,

$$B[x^*, i] = \{j \in B(x^*) : (j, i) \in A\} = B[x^*] \bigcap rcl(i).$$

Proposition 2. *Let (G, a, b) define an instance of PCKP satisfying our working assumptions, and consider $x^* \in P(G, a, b)$.*

- *For every $i \notin B[x^*]$, there exists $j \in B[x^*, i]$ such that $x^*_j = x^*_i$.*
- *If $i, j \in V$ are such that $B[x^*, i] = B[x^*, j]$ then $x^*_i = x^*_j$.*

Definition 9. Let (G, a, b) define an instance of PCKP satisfying our working assumptions. We say that an inequality $\alpha x \leq \beta$ is *weight-balanced* with respect to G and x^* if $\alpha_i \leq 0$ for all $i \notin B[x^*]$.

The following proposition describes an important characterization of maximally-violated inequalities that we will later see is very helpful for separation.

Proposition 3. *Let (G, a, b) define an instance of PCKP satisfying our working assumptions. Consider $x^* \in P(G, a, b)$ and an inequality $\alpha x \leq \beta$ valid for $P(G, a, b) \cap \mathbb{Z}^V$. There exists an inequality $\bar{\alpha} x \leq \beta$, valid for $P(G, a, b) \cap \mathbb{Z}^V$, that is weight-balanced with respect to x^*, and such that, $\alpha x^* = \bar{\alpha} x^*$. Moreover, $\bar{\alpha}$ can be obtained from α by the following recursive procedure: If α is not weight-balanced with respect to x^*, let $i \notin B[x^*]$ be such that $\alpha_i > 0$ and let $j \in B[x^*, i]$ such that $x^*_i = x^*_j$. Define $\bar{\alpha}$ such that $\bar{\alpha} x = \alpha x + \alpha_i(x_j - x_i)$. If $\bar{\alpha}$ is not weight-balanced, redefine α as $\bar{\alpha}$ and repeat.*

Proof. Suppose $\alpha x \leq \beta$ defines a valid inequality for $P(G, a, b)$ that is not weight-balanced with respect to x^*. By definition, because $\alpha x \leq \beta$ is not weight-balanced, there must exists $i \notin B[x^*]$ such that $\alpha_i > 0$. From Proposition 2 we know there exists $j \in B[x^*, i]$ such that $x^*_i = x^*_j$. Since $x_j \leq x_i$ is valid for $P(G, a, b)$, it follows that $\alpha x + \alpha_i(x_j - x_i) \leq \beta$ is valid for $P(G, a, b)$ as well. Also, it is clear that $\alpha x^* = \alpha x^* + \alpha_i(x^*_j - x^*_i) = \bar{\alpha} x^*$. That the recursion terminates follows from the fact that,

$$|\{i \notin B[x^*] : \bar{\alpha}_i > 0\}| < |\{i \notin B[x^*] : \alpha_i > 0\}|.$$

■

A point of concern regarding Proposition 3 is that the result is not necessarily closed to specific classes of inequalities. For example, if $x(C) \leq |C| - 1$ is a maximally violated MIC inequality that is not weight-balanced, it is possible that after applying the weight-balancing procedure described in Proposition 3, we might obtain an inequality that is not a MIC inequality. The same concern applies to induced clique inequalities. This is addressed in the following propositions.

Proposition 4. *Let (G, a, b) define an instance of PCKP satisfying our working assumptions, and consider $x^* \in P(G, a, b)$. Let C be a minimally induced cover. There exists another minimally induced cover $\bar{C}$ such that (a) constraint $x(\bar{C}) \leq |\bar{C}| - 1$ is weight-balanced and (b) $x(\bar{C}) \leq |\bar{C}| - 1$ is at least as violated as $x(C) \leq |C| - 1$.*

Proof. Since C is a minimal induced cover we know $a(cl(C)) > b$. From Proposition 2 we know that for all $i \in C$ there exists $j(i) \in B[x^*, i]$ such that $x^*_i = x^*_{j(i)}$. Let $C' = \{j(i) : i \in C\}$. Since $(j(i), i) \in A$ for all $i \in C$ we know that $cl(C) \subseteq cl(C')$. Since $a \geq 0$ it follows that C' is an induced cover. Let $\bar{C} \subseteq C'$ be a minimal induced cover. It follows that,

$$\sum_{i \in \bar{C}} (1 - x^*_i) \leq \sum_{i \in C'} (1 - x^*_i) \leq \sum_{i \in C} (1 - x^*_{j(i)}) = \sum_{i \in C} (1 - x^*_i).$$

Hence, $x^*(\bar{C}) - (|\bar{C}| - 1) \geq x^*(C) - (|C| - 1)$. Further, since $\bar{C} \subseteq B[x^*]$ and all coefficients of inequality $x(\bar{C}) \leq |\bar{C}| - 1$ are positive, we conclude that it weight-balanced. ■

Proposition 5. *Let* (G, a, b) *define an instance of PCKP satisfying our working assumptions, and consider* $x^* \in P(G, a, b)$*. Consider an induced clique* $C \subseteq V$*. There exists another induced clique* $\bar{C} \subseteq V$ *such that (a) constraint* $x(\bar{C}) \leq 1$ *is weight-balanced and (b) constraint* $x(\bar{C}) \leq 1$ *is at least as violated as* $x(C) \leq 1$*.*

Proof. Since $x(C) \leq 1$ is a valid clique inequality we know that if $i, j \in C$ are such that $i \neq j$, then $a(cl(\{i, j\})) > b$. From Proposition 2 we know that for all $i \in C$ there exists $j(i) \in B[x^*, i]$ such that $x^*_i = x^*_{j(i)}$. Let $\bar{C} = \{j(i) : i \in C\}$. Since $(j(i), i) \in A$ for all $i \in C$ we know that $cl(i) \subseteq cl(j(i))$. This implies that for $i, k \in \bar{C}$ such that $i \neq k$, we have $a(cl\{i, k\}) > b$. Thus $x(\bar{C}) \leq 1$ defines a clique inequality. To see that $x(\bar{C}) \leq 1$ is at least as violated, observe that by Proposition 1 it is not possible to have two vertices i and i' in C such that $j(i) = j(i')$. Hence,

$$\sum_{i \in C} x^*_i = \sum_{i \in C} x^*_{j(i)}$$

■

4 Shrinking and PCKP

In applying separation algorithms to combinatorial optimization problems defined over graphs it is very helpful to preprocess these graphs in order to reduce the optimization problem to one that is equivalent, but defined over a smaller graph. This serves to reduce the number of operations that must be performed. Given a graph G and a set of vertices $S \subseteq V$ let G/S denote the graph obtained by contracting the set S into a single node s and deleting any self-loop edges that appear. This operation is called *shrinking* S in G, and has been used with great success in solving large-scale integer programming problems such as the Traveling Salesman Problem [21, 11]. In this section we describe how shrinking can be defined in the context of PCKP. As we will see, the notion of weight-balanced inequalities plays a key role in shrinking.

Theorem 2. *Let* (G, a, b) *define an instance of PCKP satisfying our working assumptions, and consider* $x^* \in P(G, a, b)$*. Let* $\mathcal{S}$ *be a family of sets defining a partition of* V *such that if* $i, j \in S$ *then* $x^*_i = x^*_j$*. Let* $\bar{G} = (\bar{V}, \bar{A})$ *be the graph obtained from contracting the sets* $S \in \mathcal{S}$*, and let* $\bar{a} \in \mathbb{Z}^{\bar{V}}$ *defined by* $\bar{a}_S = a(S)$*. Note that* $(\bar{G}, \bar{a}, b)$ *defines an instance of PCKP.*

- *Let* $z^* \in \mathbb{R}^{\bar{V}}$ *be defined such that* $z^*_S = x^*_i$*,* $\forall i \in S$*,* $\forall S \in \mathcal{S}$*. Then,* $z^* \in P(\bar{G}, \bar{a}, b)$*. We will say that* $\bar{G}$*,* $\bar{a}$*, and* z^* *are obtained from* G*,* a*, and* x^* *by* shrinking *the partition* $\mathcal{S}$*.*
- *Consider an inequality* $\alpha x \leq \beta$ *valid for* $P(G, a, b) \cap \mathbb{Z}^V$*. Define* $\bar{\alpha}_S = \alpha(S)$ *for each* $S \in \mathcal{S}$*. Then,* $\bar{\alpha} z \leq \beta$ *is valid for* $P(\bar{G}, \bar{a}, b) \cap \mathbb{Z}^{\bar{V}}$*. Moreover,* $\alpha x^* = \bar{\alpha} z^*$*.*

Proof. It is clear that $z^* \in [0,1]^{\bar{V}}$. Furthermore,

$$\bar{a}z^* = \sum_{S\in\mathcal{S}} \bar{a}_S z_S^* = \sum_{S\in\mathcal{S}} \left(\sum_{i\in S} a_i\right) z_S^* = \sum_{S\in\mathcal{S}} \sum_{i\in S} a_i x_i^* = \sum_{i\in V} a_i x_i^* = ax^* \leq b \quad (4)$$

Now, consider $(S,T) \in \bar{A}$. Then, there exist $i \in S$ and $j \in T$ such that $(i,j) \in A$, which implies $z_S^* = x_i^* \leq x_j^* = z_T^*$. This proves that $z^* \in P(\bar{G}, \bar{a}, b)$.

Consider an inequality $\alpha x \leq \beta$ valid for $P(G,a,b) \cap \mathbb{Z}^V$. First, observe that $\alpha x^* = \bar{\alpha} z^*$. This can be proved replacing a and $\bar{a}$ by α and $\bar{\alpha}$ in equation (4).

Let $z \in P(\bar{G}, \bar{a}, b) \cap \mathbb{Z}^{\bar{V}}$ and define $y \in [0,1]^V$ as $y_i = z_S \ \forall i \in S, \forall S \in \mathcal{S}$. We first prove that $y \in P(G,a,b) \cap \mathbb{Z}^V$. In fact, replacing x^* and z^* by y and z in equation (4), we prove that $ay \leq b$. Finally, for each $(i,j) \in A$ either i and j are in the same set $S \in \mathcal{S}$, or in different sets. In the first case, $y_i = y_j = z_S$, so in particular it satisfies $y_i \leq y_j$. In the second case, suppose that $i \in S$ and $j \in T$. Since $(i,j) \in A$ then $(S,T) \in \bar{A}$, so $y_i = z_S \leq z_T = y_j$.

Now, since $y \in P(G,a,b)$ it follows that $\alpha y \leq \beta$. Thus, replacing $a, \bar{a}, x^*$ and z^* by $\alpha, \bar{\alpha}, y$ and z respectively in equation (4), we obtain:

$$\bar{\alpha}z = \sum_{S\in\mathcal{S}} \bar{\alpha}_S z_S = \sum_{S\in\mathcal{S}} \left(\sum_{i\in S} \alpha_i\right) z_S = \sum_{S\in\mathcal{S}} \sum_{i\in S} \alpha_i y_i = \sum_{i\in V} \alpha_i y_i = \alpha y \leq \beta$$

■

As shown in Theorem 2, if there exists a violated inequality for PCKP, then there exists a violated inequality for the shrunken graph. This result, however, does not imply that every violated constraint in the shrunken problem can be mapped back to a violated constraint in the original variable space. This issue is addressed in the following theorem. Before, however, we need to show how it is possible to define a partition of V such that it is safe to perform shrinking for PCKP.

Definition 10. Let (G,a,b) define an instance of PCKP satisfying our working assumptions, and consider $x^* \in P(G,a,b)$. For each $i \in V$ define,

$$\Pi_i = \{j \in V : B[x^*, i] = B[x^*, j]\}.$$

As immediately follows from Proposition 2, the family of sets $\mathcal{S} = \{\Pi_i\}_{i\in V}$ defines a partition of V. Moreover, every $S \in \mathcal{S}$ is such that $i,j \in S$ implies $x_i^* = x_j^*$. We say that the family of sets $\mathcal{S}$ defines the *canonical* partition of V implied by G and x^*.

Theorem 3. *Let (G,a,b) define an instance of PCKP satisfying our working assumptions, and consider $x^* \in P(G,a,b)$. Let $\mathcal{S}$ represent the canonical partition of V defined by G and x^*. Let $\bar{G} = (\bar{V}, \bar{A})$, $\bar{a}$ and z^* be obtained from G, a and x^* by shrinking partition $\mathcal{S}$ as described in Theorem 2. For each set $S \in \mathcal{S}$ let us choose $i_S \in S$ such that $(j, i_S) \in A$ implies $j \notin S$. Finally, let $\bar{\alpha}z \leq \beta$ be*

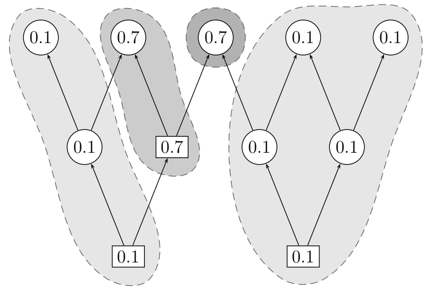

Figure 2: Example of a canonical partition of V implied by G and x^*. The values of x^* are written inside the vertices. The vertices in $B[x^*]$ are represented as rectangles.

a valid inequality for $P(\bar{G}, \bar{a}, b) \cap \mathbb{Z}^{\bar{V}}$ that is weight-balanced with respect to z^*. Define,

$$\alpha_i = \begin{cases} \bar{\alpha}_S & \text{if } i = i_S \text{ for some } S \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, inequality $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$ and weight-balanced with respect to x^*. Moreover, $\alpha x^* = \bar{\alpha} z^*$.

Proof. It is easy to see that $\alpha x^* = \bar{\alpha} z^*$. First, we prove that $\alpha x \leq \beta$ is weight-balanced with respect to x^*. We only need to prove that if $i \notin B[x^*]$ and $i = i_S$ for some $S \in \mathcal{S}$ then $\alpha_i \leq 0$. In fact, there must exists $j \in T \neq S$ such that $(j, i) \in A$ and $x_i^* = x_j^*$. But, this means that $(T, S) \in \bar{A}$ and $z_S^* = z_T^*$, so $S \notin B[z^*]$. Since $\bar{\alpha} z \leq b$ is weight-balanced with respect to z^*, then $\alpha_i = \bar{\alpha}_S \leq 0$.

Secondly, we focus on showing $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$. For this consider $x \in P(G, a, b) \cap \mathbb{Z}^V$ and let $F = \{i \in B[x^*] : x_i = 1\}$. Define:

$$\hat{x}_i = \begin{cases} 1 & \text{if } i \in cl(F), \\ 0 & \text{otherwise.} \end{cases}$$

Also define $\hat{z} \in \mathbb{Z}^{\hat{V}}$ so that $\hat{z}_S = \hat{x}_{i_S}$ for each $S \in \mathcal{S}$. An outline of this proof is as follows: We first prove (a) $\hat{x} \in P(G, a, b)$. From this follows (b) $\hat{z} \in P(\bar{G}, \bar{a}, b)$. We then show (c) $\alpha \hat{x} \geq \alpha x$ and (d) $\alpha \hat{x} = \bar{\alpha} \hat{z}$. From (c),(d) and (b) it follows that $\alpha x \leq \alpha \hat{x} = \bar{\alpha} \hat{z} \leq \beta$, with which we conclude.

(a) By definition $\hat{x}$ is the incidence vector of the smallest closure containing F. Since it is the incidence vector of a closure, $\hat{x}$ satisfies all precedence constraints. Further, since x is also the incidence vector of a closure containing F we know that $\hat{x} \leq x$. That $a\hat{x} \leq b$ follows from this and $a \geq 0$.

(b) We first show that $(S, T) \in \bar{A}$ and $S \neq T$ imply $\hat{z}_S \leq \hat{z}_T$. We only need concern ourselves with the case $\hat{z}_S = 1$. In this case $\hat{x}_{i_S} = 1$, and so, by how $\hat{x}$ was constructed, there exists $k \in B[x^*, i_S]$ such that $\hat{x}_k = 1$. On the other hand, $(S,T) \in \bar{A}$ implies there exists $(i,j) \in A$ such that $i \in S$ and $j \in T$. Since $S \neq T$ and $(i,j) \in A$ we know $B[x^*, i] \subsetneq B[x^*, j]$. This, however, implies that $k \in B[x^*, i_S] = B[x^*, i] \subsetneq B[x^*, j] = B[x^*, i_T]$ and so $\hat{x}_{i_T} = 1$. With this we conclude $\hat{z}_T = 1$. Second, we must show that $\bar{a}\hat{z} \leq b$. For this, it suffices to show that $\hat{x}_{i_S} \leq \hat{x}_i$ for all $i \in S$. In fact, this implies that,

$$\bar{a}\hat{z} = \sum_{S \in \mathcal{S}} \bar{a}_S \hat{z}_S = \sum_{S \in \mathcal{S}} \left(\sum_{i \in S} a_i \right) \hat{x}_{i_S} \leq \sum_{S \in \mathcal{S}} \sum_{i \in S} a_i \hat{x}_i = a\hat{x} \leq b$$

Consider $S \in \mathcal{S}$. To prove that $\hat{x}_{i_S} \leq \hat{x}_i$ for all $i \in S$ we only need to consider when $\hat{x}_{i_S} = 1$. From the definition of $\hat{x}$ we know $\hat{x}_{i_S} = 1$ implies there exists $k \in B[x^*, i_S]$ such that $\hat{x}_k = 1$. Since $B[x^*, i_S] = B[x^*, i]$ for all $i \in S$, it follows that $k \in B[x^*, i]$ for all $i \in S$. From this we conclude $\hat{x}_i = 1$ for all $i \in S$.

(c) Note that $D = \{i \in V : x_i > \hat{x}_i\} \subseteq V \setminus B[x^*]$. Since $\alpha x \leq \beta$ is weight-balanced with respect to x^*, it follows that $\alpha_i \leq 0$ for all $i \in D$. Thus, $\alpha\hat{x} \geq \alpha x$.

(d) In fact,

$$\sum_{i \in V} \alpha_i \hat{x}_i = \sum_{S \in \mathcal{S}} \alpha_{i_S} \hat{x}_{i_S} = \sum_{S \in \mathcal{S}} \bar{\alpha}_S \hat{z}_S$$

■

5 A simple mechanism for strengthening valid inequalities

Given a valid inequality $\alpha x \leq \beta$ for $P(G,a,b) \cap \mathbb{Z}^V$ with $\alpha \in \mathbb{Z}^V, \beta \in \mathbb{Z}_+$, in this section we are concerned with quickly finding a stronger inequality $\alpha' x \leq \beta'$ that is also valid for $P(G,a,b) \cap \mathbb{Z}^V$. The mechanism we introduce requires solving a single instance of a maximum closure problem, as described in the following theorem.

Theorem 4. *Let (G,a,b) define an instance of PCKP satisfying our working assumptions. Let $\alpha x \leq \beta$ be a valid inequality for $P(G,a,b) \cap \mathbb{Z}^V$, and let $S = \{i \in V : \alpha_i \neq 0\}$. Let $\hat{x}$ be an optimal solution to problem,*

$$\begin{array}{lll} \max & \alpha x & \\ s.t. & x_i \leq x_j & \forall (i,j) \in A \\ & x_i \in \{0,1\} & \forall i \in V, \end{array} \tag{5}$$

such that $I(\hat{x}) = \{i \in V : \hat{x}_i = 1\}$ is minimal inclusion-wise. Then, inequality

$$\sum_{i \in I(\hat{x})} \alpha_i x_i \leq \beta \tag{6}$$

is valid for $P(G,a,b) \cap \mathbb{Z}^V$. Moreover, if $I(\hat{x}) \subsetneq S$, then constraint (6) is stronger than constraint $\alpha x \leq \beta$.

Proof. We first show that (6) is valid for $P(G,a,b)\cap\mathbb{Z}^V$. In fact, consider $x\in P(G,a,b)\cap\mathbb{Z}^V$ and let $I(x)=\{i\in V : x_i=1\}$. Since $I(\hat{x})$ and $I(x)$ are both closures, we know that $I(x)\cap I(\hat{x})$ is also a closure. Define $x'\in\{0,1\}^V$ so that $x'_i=1$ iff $i\in I(x)\cap I(\hat{x})$. Since $x'\le x$ and $a\ge 0$ we know that $ax'\le b$ and so $x'\in P(G,a,b)\cap\mathbb{Z}^V$. From all this it follows that,

$$\sum_{i\in I(\hat{x})}\alpha_i x_i=\sum_{i\in I(\hat{x})\cap I(x)}\alpha_i x_i=\sum_{i\in V}\alpha_i x'_i\le\beta.$$

We now show that if $I(\hat{x})\subsetneq S$, then constraint (6) is stronger than constraint $\alpha x\le\beta$. For this we first show that there exists $z\in P(G,a,b)\cap\mathbb{Z}^V$ such that $\alpha z<\sum_{i\in I(\hat{x})}\alpha_i z_i$. Since $I(\hat{x})\subsetneq S$, we know that there exists $i\in S\setminus I(\hat{x})$ such that $\alpha_i\ne 0$. Since $\hat{x}$ is an optimal solution of (5), we know that $\alpha(cl(i)\setminus I(\hat{x}))\le 0$. Moreover, we can assume that this $i\in S\setminus I(\hat{x})$ is such that for all $(i,j)\in A$ then $j\in I(\hat{x})$ or $\alpha_j=0$. In this case, $\alpha(cl(i)\setminus I(\hat{x}))=a_i<0$. Define $z\in\{0,1\}^V$ so that $z_i=1$ iff $i\in cl(i)$. Since $cl(i)$ is by definition a closure, and since $az\le b$, we know that $z\in P(G,a,b)\cap\mathbb{Z}^V$. On the other hand,

$$\alpha z=\sum_{i\in cl(i)\cap I(\hat{x})}\alpha_i+\sum_{i\in cl(i)\setminus I(\hat{x})}\alpha_i<\sum_{i\in cl(i)\cap I(\hat{x})}\alpha_i=\sum_{i\in I(\hat{x})}\alpha_i z_i$$

Finally, we show that for any $x\in P(G,a,b)\cap\mathbb{Z}^V$ such that $\alpha x=\beta$ we have $\sum_{i\in I(\hat{x})}\alpha_i x_i=\beta$. Let $I(x)=\{i\in I : x_i=1\}$. Note that $\alpha(I(x)\setminus I(\hat{x}))\le 0$. Otherwise, we would have that $\alpha(I(x)\cup I(\hat{x}))>\alpha(I(\hat{x}))$, which, because $I(\hat{x})\cup I(x)$ is a closure, contradicts the optimality assumption of $\hat{x}$. From this we have that,

$$\alpha x=\sum_{i\in S\cap I(x)}\alpha_i x_i=\sum_{i\in I(\hat{x})}\alpha_i x_i+\sum_{i\in I(x)\setminus I(\hat{x})}\alpha_i x_i\le\sum_{i\in I(\hat{x})}\alpha_i x_i\le\beta$$

Thus $\alpha x=\beta$ and $\sum\limits_{i\in I(\hat{x})}\alpha_i x_i=\beta$, and we conclude our result. ■

6 Lifting and PCKP

Definition 11. Let $P(G,a,b)$ define an instance of PCKP satisfying our working assumptions. Consider disjoint sets $O,I\subseteq V$ and define,

$$P(G,a,b,O,I)=\{x\in P(G,a,b) : x_i=1\ \ \forall i\in I,\ x_i=0\ \ \forall i\in O\}.$$

Assume $I,O\subseteq V$ are such that $P(G,a,b,O,I)\cap\mathbb{Z}^V$ is non-empty, and consider an inequality $\alpha x\le\beta$ valid for $P(G,a,b,O,I)\cap\mathbb{Z}^V$. We say that *lifting* inequality $\alpha x\le\beta$ consists in computing coefficients $\gamma,\eta\ge 0$ such that $\gamma\ne 0$ or $\eta\ne 0$, and such that inequality

$$\alpha x+\sum_{i\in I}\gamma_i(1-x_i)+\sum_{j\in O}\eta_j x_j\le\beta,\tag{7}$$

is valid for $P(G,a,b)\cap\mathbb{Z}^V$.

For an introduction to lifting, see [24, 25, 15]. Lifting in the context of PCKP can be used to serve two important purposes:

- **Strengthening valid inequalities.** Consider an inequality $\alpha x \le \beta$ that is valid for $P(G,a,b) \cap \mathbb{Z}^V$, but that is not facet-defining. If $P(G,a,b,I,O) \cap \mathbb{Z}^V$ is non-empty for sets $I, O \subseteq V$, lifting can be used to *strengthen* inequality $\alpha x \le \beta$. Such a strengthening procedure would correspond to tilting [15] inequality $\alpha x \le b$ in directions $(1 - e_i)$ and e_j for $i \in I$ and $j \in O$. This use of lifting has been studied by Boyd [9], Park and Park [22], van de Leesel et al [18] and Boland et al. [8] in the context of induced cover inequalities, K-covers, $(1,K)$-configurations and induced clique inequalities.

- **Speeding up the computation of cutting planes.** Consider a non-integral solution $\bar{x} \in P(G,a,b)$. Let $I = \{i \in V : \bar{x}_i = 1\}$ and $O = \{i \in V : \bar{x}_i = 0\}$. It is well known that a cutting plane separating $\bar{x}$ from $P(G,a,b,O,I) \cap \mathbb{Z}^V$ exists if and only if a cutting plane separating $\bar{x}$ from $P(G,a,b)$ exists. From a computational point of view, it could be easier to compute a cutting plane separating $\bar{x}$ from $P(G,a,b,O,I) \cap \mathbb{Z}^V$, since this results in solving a separation problem of lower dimension. If the separation algorithm in $P(G,a,b,O,I)$ is successful we obtain an inequality $\alpha x \le b$ that is valid for $P(G,a,b,O,I) \cap \mathbb{Z}^V$, and violated by $\bar{x}$. Lifting, we obtain an inequality of form (7) that is valid for $P(G,a,b) \cap \mathbb{Z}^V$ and violated by $\bar{x}$.

In order for lifting to be computationally effective it should be quick. In this section we describe simple techniques for quick lifting in the context of PCKP. In Section 6.1 we describe simple relaxations of the lifting problem that are easy to solve. In Section 6.2 we characterize those variables that can be lifted and those that cannot. Finally, in Section 6.3 we present an argument for lifting in a greedy manner so as to obtain highly-violated inequalities.

6.1 Optimal lifting and relaxed lifting of a single variable

Definition 12. Let (G,a,b) define an instance of PCKP satisfying our working assumptions, and consider $i \in V$. Assume inequality $\alpha x \le \beta$ is valid for $P(G,a,b) \cap \{x : x_i = 1\} \cap \mathbb{Z}^V$. *Down-lifting* variable i in constraint $\alpha x \le \beta$ consists in computing a coefficient γ_i such that

$$\alpha x + \gamma_i (1 - x_i) \le \beta \qquad \forall x \in P(G,a,b) \cap \mathbb{Z}^V. \tag{8}$$

Optimally down-lifting variable i consists in computing the largest possible lifting coefficient γ_i. This can be done by solving,

$$\begin{aligned} z_i = \max \quad & \alpha x \\ s.t. \quad & x_i = 0 \\ & x \in P(G,a,b) \cap \mathbb{Z}^V \end{aligned} \tag{9}$$

and defining $\gamma_i = \beta - z_i$. Observe that if $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$, then $\gamma_i \geq 0$. Moreover, if $\gamma_i > 0$, then constraint (8) is strictly stronger than $\alpha x \leq \beta$.

The problem with optimal down-lifting is that solving each instance of (9) could potentially be very difficult. An alternative is to lift in a non-optimal way by solving a relaxation of (9). That is, by replacing constraint $x \in P(G, a, b) \cap \mathbb{Z}^V$ by a weaker constraint $x \in R(G, a, b)$ where $P(G, a, b) \cap \mathbb{Z}^V \subseteq R(G, a, b)$, and solving the *relaxed down-lifting problem* with respect to to $R(G, a, b)$,

$$\begin{aligned} w_i = \max \quad & \alpha x \\ s.t. \quad & x_i = 0 \\ & x \in R(G, a, b). \end{aligned} \tag{10}$$

If $w_i > 0$, then $\gamma_i = \beta_i - w_i$ is still a valid lifting coefficient. Although γ_i might be a weaker coefficient, if $R(G, a, b)$ is chosen appropriately, it should be computationally easier to obtain. The first relaxation $R(G, a, b)$ we consider is,

$$R_{LP}(G, a, b) = \left\{x \in [0, 1]^V : ax \leq b,\ x_i \leq x_j\ \forall (i, j) \in A\right\}. \tag{11}$$

In relaxation (11) we relax the integrality condition $x \in \{0, 1\}^V$. Solving (9) subject to this relaxation can be done using the Critical Multiplier Algorithm proposed by Chicoisne et al. [10], which runs in $\mathcal{O}(mn \log n)$.

Remark 2. *In some cases, the use of relaxation* (11) *may result in a coefficient γ that is not integral. If the original constraint $\alpha x \leq \beta$ is such that α and β are both integer (as is the case of induced cover and induced clique inequalities) this results in an opportunity to further strengthen the resulting inequality. In fact, if a coefficient γ_i is fractional, and all other lifted coefficients so far are integral, γ_i can be rounded up to obtain a stronger, integral coefficient. That this new coefficient is valid can be proved using a simple argument: Add a constraint $\epsilon x_i \leq \epsilon$ to the lifted inequality, choosing ϵ so that the fractional left-hand side coefficient becomes integer. Then, round the right-hand side down applying Gomory's Rounding Procedure to obtain the desired result.*

A second relaxation to consider is that proposed by Park and Park [22].

$$R_{\alpha,\beta}(G, a, b) = \left\{x \in \{0, 1\}^V : \alpha x \leq \beta,\ x_i \leq x_j \quad \forall (i, j) \in A,\ i, j \in cl(\alpha)\right\}, \tag{12}$$

where $\alpha x \leq \beta$ is the valid inequality for $P(G, a, b) \cap \mathbb{Z}^V$ to be lifted. Using this relaxation amounts to solving,

$$\begin{aligned} z_i = \max \quad & \alpha x \\ s.t. \quad & \alpha x \leq \beta \\ & x_k \leq x_j \quad \forall (k, j) \in A,\ i, j \in cl(\alpha) \\ & x \in \{0, 1\}^V. \end{aligned} \tag{13}$$

It is not difficult to see that the solution to (13) is of the form

$$z_i = \min\{\beta, \max\{\alpha x : x_k \leq x_j\ \forall (k, j) \in A,\ k, j \in cl(\alpha),\ x \in [0, 1]^V\}\},$$

thus it only requires solving a single max-closure problem.

Definition 13. Let $P(G, a, b)$ define an instance of PCKP satisfying our working assumptions, and consider $i \in V$. Assume inequality $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \{x : x_i = 0\} \cap \mathbb{Z}^V$. *Up-lifting* variable i in constraint $\alpha x \leq \beta$ consists in computing a coefficient η_i such that

$$\alpha x + \eta_i x_i \leq \beta \qquad \forall x \in P(G, a, b) \cap \mathbb{Z}^V. \tag{14}$$

Optimally up-lifting variable i consists in computing the largest possible lifting coefficient η_i. This can be done by solving,

$$\begin{array}{rl} w_i = \max & \alpha x \\ s.t. & x_i = 1 \\ & x \in P(G, a, b) \cap \mathbb{Z}^V \end{array} \tag{15}$$

and defining $\eta_i = \beta - w_i$. Observe that if $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$, then $\eta_i \geq 0$. Moreover, if $\eta_i > 0$, then constraint (14) is strictly stronger than $\alpha x \leq \beta$.

As in optimal down-lifting, solving each instance of (15) can be very difficult. As before, an alternative is to solve a relaxed lifting problem using $R_{LP}(G, a, b)$ or $R_{\alpha,\beta}(G, a, b)$.

6.2 Selecting which variables to lift.

The following Lemma says that if $\alpha x \leq \beta$ is a face-defining inequality, it only makes sense to down-lift a very specific set of variables.

Lemma 1. *Let $P(G, a, b)$ define an instance of PCKP satisfying our working assumptions, and consider $i \in V$. Assume inequality $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$ and that there exists $\bar{x} \in P(G, a, b) \cap \mathbb{Z}^V$ such that $\alpha \bar{x} = \beta$. Finally, let $S_+ = \{s \in V : \alpha_s > 0\}$. Let γ_i be the optimal down-lifting coefficient for variable i in constraint $\alpha x \leq \beta$. If $i \notin cl(S_+)$ then $\gamma_i = 0$.*

Proof. Define $\hat{x} \in \mathbb{Z}^V$ as follows,

$$\hat{x}_j = \begin{cases} 0 & \text{if } j \in rcl(i) \\ \bar{x}_j & \text{otherwise.} \end{cases}$$

First, note that the support of $\hat{x}$ defines a closure, since it is obtained by removing a reverse closure from a closure. Second, note that $a\hat{x} \leq b$ since $a \geq 0$ and $\hat{x} \leq \bar{x}$. These two facts imply that $\hat{x} \in P(G, a, b) \cap \mathbb{Z}^V$, and $\alpha\hat{x} \leq \beta$. Next, note that $\alpha\hat{x} = \beta$. In fact, let $S = \{s \in V : \alpha_s \neq 0\}$. It is easy to see that if $i \notin cl(S_+)$ then $rcl(i) \cap S_+ = \emptyset$ and $rcl(i) \cap S \subseteq S \setminus S_+$. Hence,

$$\alpha\hat{x} = \sum_{j \in S} \alpha_j \hat{x}_j = \sum_{j \in S \setminus rcl(i)} \alpha_j \hat{x}_j = \sum_{j \in S \setminus rcl(i)} \alpha_j \bar{x}_j \geq \sum_{j \in S} \alpha_j \bar{x}_j = \beta.$$

Thus, since $\hat{x} \in P(G, a, b) \cap \mathbb{Z}^V$ and since $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$, we conclude $\alpha\hat{x} = \beta$. Moreover, since $\hat{x}_i = 0$ and $\alpha\hat{x} = \beta$, we conclude that $\hat{x}$ is an optimal solution of Problem 9 with objective β, where we conclude that $\gamma_i = 0$. ■

The following Lemma gives a similar result for up-lifting.

Lemma 2. *Let $P(G, a, b)$ define an instance of PCKP satisfying our working assumptions, and consider $i \in V$. Let $S = \{s \in V : \alpha_s \neq 0\}$. Assume inequality $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \mathbb{Z}^V$ and that for all $j \in S$ there exists $x^j \in P(G, a, b) \cap \{x_j = 1\} \cap \mathbb{Z}^V$ such that $\alpha x^j = \beta$. Let η_i be the optimal up-lifting coefficient for variable i in constraint $\alpha x \leq \beta$. If $i \in cl(S)$ then $\eta_i = 0$.*

Proof. Since $i \in cl(S)$, there exists $j \in S$ such that $i \in cl(\{j\})$. Consider $x^j \in P(G, a, b) \cap \{x_j = 1\} \cap \mathbb{Z}^V$ such that $\alpha x^j = \beta$ (as in the hypothesis). Since $i \in cl(\{j\})$ we also have that x^j is in $P(G, a, b) \cap \{x_i = 1\} \cap \mathbb{Z}^V$. Thus, x^j is feasible for problem (15), and since it has objective function value β we conclude it is also optimal. Hence $\eta_i = 0$. ■

Note that induced cover inequalities and induced clique inequalities satisfy the conditions required by the two previous Lemmas.

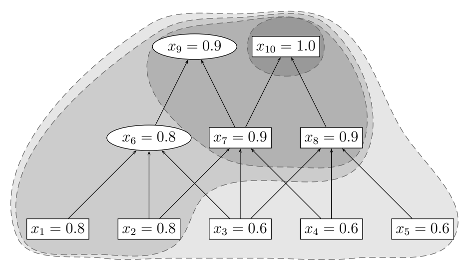

Figure 3: Example of a fractional solution for the PCKP instance with the graph shown in the figure and knapsack constraint $\sum\{x_i : i = 1, \ldots, 10\} \leq 8$.

6.3 Choosing a lifting order

We are now ready to address the following two important questions: Is the lifting order important when strengthening valid inequalities through lifting? If so, which is the *best* ordering of the variables in which to do lifting? To answer the first question consider the instance of PCKP and the corresponding fractional solution depicted in Figure 3. Suppose we would like to strengthen the following inequality that is valid for $P(G, a, b) \cap \mathbb{Z}^V$:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 3. \tag{16}$$

Note that this inequality is violated by the fractional solution by 0.4. Lemma 2 shows that we should only consider down-lifting variables x_6, x_7, x_8, x_9 and x_{10}. Moreover, optimally down-lifting variable x_k is the same as solving the problem

$$\gamma_k = 3 - \begin{array}{ll} \max & x_1 + x_2 + x_3 + x_4 + x_5 \\ s.t. & x \in P(G, a, b) \cap \{x_k = 0\}. \end{array}$$

If we optimally down-lift (in order) variables x_6, x_7, x_8, x_9 and x_{10}, we obtain the (facet-defining) inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_6 - x_7 - x_8 \leq 0,$$

which is violated by 0.8. If, instead, we optimally down-lift in the same variables in reversed order (e.g., $x_{10}, x_9, x_8, x_7, x_6$), we obtain the valid inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 - x_9 - 2x_{10} \leq 0 \tag{17}$$

which is violated by 0.5. What this shows is that, in general, the resulting inequality, and the amount by which it is violated, is dependent of the *lifting order*. So the question of choosing an appropriate lifting order is very relevant. Again, let use the same example as before. We know that if we optimally down-lift x_{10} and then x_9 we obtain inequality (17), which is violated by 0.5. If, instead, we optimally down-lift variables x_9 and x_{10} we obtain inequality,

$$x_1 + x_2 + x_3 + x_4 + x_5 - 2x_9 - x_{10} \leq 0. \tag{18}$$

This equality is violated by 0.6. That is, lifting x_9 before x_{10} results in an inequality that is more violated. This is because we are down-lifting and because $(1-x_{10}^*) < (1-x_9^*)$. It turns out that this (pairwise) greedy way of lifting always results in an inequality that is more violated. This is explained in the following Lemma.

Lemma 3. *Consider an inequality $\alpha x \leq \beta$ that is valid for $P(G, a, b) \cap \{0, 1\}^n$. Let R be an integral polyhedral relaxation of $P(G, a, b)$. That is, let R be a polyhedron such that $P \subseteq R$ and such that the extreme points of R are integral. Let $x^* \in P(G, a.b)$, and suppose we want to solve the relaxed lifting problem for variables x_i and x_j.*

1. *If $(1-x_i^*) \geq (1-x_j^*)$, down-lifting x_i and then down-lifting x_j results in an inequality with violation that is equal to or greater than the violation of the inequality that would be obtained by down-lifting x_j and then down-lifting x_i.*
2. *If $x_i^* \geq x_j^*$, up-lifting x_i and then up-lifting x_j results in an inequality with violation that is equal to or greater than the violation of the inequality that would be obtained by up-lifting x_j and then up-lifting x_i.*
3. *If $(1 - x_i^*) \geq x_j^*$, down-lifting x_i and then up-lifting x_j results in an inequality with violation that is equal to or greater than the violation of the inequality that would be obtained by down-lifting x_j and then up-lifting x_i.*

Proof. We assume that $\alpha x \leq \beta$ is valid for $P(G, a, b) \cap \{0, 1\}^n$ and that $i, j \in \{1, \ldots, n\}$ are two index of variables in $P(G, a, b)$. To prove our result, we need some previous definitions:

Let

$$Z_{pq} = \min \{\beta, \max\{\alpha x : x \in R, x_i = p, x_j = q\}\},$$

Note that Z_{pq} can be seen as a bound of the maximum value of αx attainable in $P(G, a, b) \cap \{0, 1\}^n \cap \{x_i = p, x_j = q\}$. With this, we prove each part of

Lemma 3 by showing that the sum of the lifting coefficients remains constant when we exchange the order of lifting between two consecutive variables.

1. We call γ_i, γ_j the lifting coefficients for x_i, x_j obtained when first down-lifting x_i, and then x_j, and δ_i, δ_j the lifting coefficients for x_i, x_j obtained when first down-lifting x_j and then x_i. We will prove that $\delta_i+\delta_j = \gamma_i+\gamma_j$.
   For this, note that

$$
\begin{aligned}
\gamma_i = &\quad (\beta - \max\{Z_{00}, Z_{01}\}) \\
\gamma_j = &\quad (\beta - \gamma_i - \max\{Z_{00}, Z_{10} - \gamma_i\}) \\
\delta_j = &\quad (\beta - \max\{Z_{00}, Z_{10}\}) \\
\delta_i = &\quad (\beta - \delta_j - \max\{Z_{00}, Z_{01} - \delta_j\})
\end{aligned}
$$

   Hence,

$$
\begin{aligned}
\gamma_i + \gamma_j &= \beta - \max\{Z_{00}, Z_{10} - \gamma_i\} \\
&= \beta - \max\{Z_{00}, Z_{10} - \beta + \max\{Z_{00}, Z_{01}\}\} \\
&= \beta - \max\{Z_{00}, Z_{10} - \beta + Z_{00}, Z_{10} - \beta + Z_{01}\}\} \\
&= \beta - \max\{Z_{00}, Z_{00} - (\beta - Z_{10}), Z_{10} + Z_{01} - \beta\}\}
\end{aligned}
$$

   But, since $\beta \geq Z_{10}$, we conclude that

$$
\gamma_i + \gamma_j = \beta - \max\{Z_{00}, Z_{10} + Z_{01} - \beta\}\}
$$

   Since δ_i and δ_j are obtained from γ_i and γ_j by swapping Z_{10} and Z_{01}, we conclude that $\delta_i + \delta_j = \gamma_i + \gamma_j$.
2. We call γ_i, γ_j the lifting coefficients for x_i, x_j obtained when first up-lifting x_i and then x_j, and δ_i, δ_j the up-lifting coefficients for x_i, x_j obtained when first lifting x_j and then x_i. We will prove that $\delta_i + \delta_j = \gamma_i + \gamma_j$.
   For this, note that

$$
\begin{aligned}
\gamma_i = &\quad (\beta - \max\{Z_{10}, Z_{11}\}) \\
\gamma_j = &\quad (\beta - \max\{Z_{01}, Z_{11} + \gamma_i\}) \\
\delta_j = &\quad (\beta - \max\{Z_{01}, Z_{11}\}) \\
\delta_i = &\quad (\beta - \max\{Z_{10}, Z_{11} + \delta_j\})\,.
\end{aligned}
$$

   Hence,

$$
\begin{aligned}
\gamma_i + \gamma_j &= 2\beta - \max\{Z_{10}, Z_{11}\} - \max\{Z_{01}, Z_{11} + \beta - \max\{Z_{10}, Z_{11}\}\} \\
&= 2\beta - \max\{Z_{01} + \max\{Z_{10}, Z_{11}\}, Z_{11} + \beta - \max\{Z_{10}, Z_{11}\} + \max\{Z_{10}, Z_{11}\}\} \\
&= 2\beta - \max\{Z_{01} + Z_{10}, Z_{01} + Z_{11}, Z_{11} + \beta\}
\end{aligned}
$$

   But, since $\beta \geq Z_{01}$, we conclude that

$$
\gamma_i + \gamma_j = 2\beta - \max\{Z_{01} + Z_{10}, Z_{11} + \beta\}
$$

   Again, since δ_i and δ_j are obtained from γ_i and γ_j by swapping Z_{10} and Z_{01}, we conclude that $\delta_i + \delta_j = \gamma_i + \gamma_j$.

3. We call γ_i, γ_j the lifting coefficients for x_i, x_j obtained when first down-lifting x_i, and then up-lifting x_j, and δ_i, δ_j the up-lifting coefficients for x_i, x_j obtained when first up-lifting x_j and then down-lifting x_i. We will prove that $\delta_i + \delta_j = \gamma_i + \gamma_j$.
   For this, note that

$$\begin{aligned} \gamma_i = &\quad (\beta - \max\{Z_{00}, Z_{01}\}) \\ \gamma_j = &\quad (\beta - \gamma_i - \max\{Z_{01}, Z_{11} - \gamma_i\}) \\ \delta_j = &\quad (\beta - \max\{Z_{01}, Z_{11}\}) \\ \delta_i = &\quad (\beta - \max\{Z_{00}, Z_{01} + \delta_j\})\,. \end{aligned}$$

   Hence,

$$\begin{aligned} \gamma_i + \gamma_j &= \beta - \max\{Z_{01}, Z_{11} - \beta + \max\{Z_{00}, Z_{01}\}\} \\ &= \beta - \max\{Z_{01}, Z_{11} - \beta + Z_{00}, Z_{11} - \beta + Z_{01}\} \\ &= \beta - \max\{Z_{01}, Z_{11} - \beta + Z_{00}, Z_{01} - (\beta - Z_{11})\} \end{aligned}$$

   But, since $\beta \geq Z_{11}$, we conclude that

$$\gamma_i + \gamma_j = \beta - \max\{Z_{01}, Z_{11} - \beta + Z_{00}\}\}$$

   On the other hand,

$$\begin{aligned} \delta_i + \delta_j &= \beta - \max\{Z_{01}, Z_{11}\} + \beta - \max\{Z_{00}, Z_{01} + \beta - \max\{Z_{01}, Z_{11}\}\} \\ &= 2\beta - \max\{Z_{00} + \max\{Z_{01}, Z_{11}\}, Z_{01} + \beta - \max\{Z_{01}, Z_{11}\} + \max\{Z_{01}, Z_{11}\}\} \\ &= 2\beta - \max\{Z_{00} + Z_{01}, Z_{00} + Z_{11}, Z_{01} + \beta\} \\ &= \beta - \max\{Z_{00} + Z_{01} - \beta, Z_{00} + Z_{11} - \beta, Z_{01}\} \\ &= \beta - \max\{Z_{01} - (\beta - Z_{00}), Z_{00} + Z_{11} - \beta, Z_{01}\} \end{aligned}$$

   But, since $\beta \geq Z_{00}$, we conclude that

$$\delta_i + \delta_j = \beta - \max\{Z_{00} + Z_{11} - \beta, Z_{01}\}$$

   from where $\delta_i + \delta_j = \gamma_i + \gamma_j$.

This proves our result. ■

An important limitation of Lemma 3 is that it is only true when it comes to lifting pairs of variables. That is, if we want to lift three or more variables, it is not clear if lifting in a greedy manner will result in an inequality that is most violated with respect to all possible permutations of lifting orders. Park and Park [22] prove that in the case of lifting Induced Cover Inequalities it is indeed true that greedy lifting orders (for down-lifting variables) result in maximally violated inequalities. However, it remains an open question to determine whether it is true in general or not.

7 Acknowledgements

This work was partially funded by FONDECYT grants 1110674 and 1110024, ANILLO grant ACT-88, Basal project CMM (Universidad de Chile) and the Complex Engineering Systems Institute (ICM: P-05-004-F, CONICYT: FBO16).

References

[1] Amaya, J., Espinoza, D., Goycoolea, M., Moreno, E., Prevost, T., Rubio, E.: A scalable approach to optimal block scheduling. In: Proceedings of APCOM 2009, pp. 567–575. The Canadian Institute of Mining, Metallurgy and Petroleum (2009)

[2] Applegate, D., Bixby, R.E., Chvátal, V., Cook, W.: The Traveling Salesman Problem: A Computational Study. Princeton University Press, Princeton, New Jersey (2006)

[3] Applegate, D.L., Bixby, R.E., Chvátal, V., Cook, W., Espinoza, D.G., Goycoolea, M., Helsgaun, K.: Certification of an optimal tsp tour through 85,900 cities. Operations Research Letters **37**(1), 11–15 (2009)

[4] Balas, E., Jerowslow, R.: Canonical cuts in the hypercube. Appl. Math. **23**, 661–669 (1972)

[5] Bienstock, D., Zuckerberg, M.: Solving lp relaxations of large-scale precedence constrained problems. Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) **6080 LNCS**, 1–14 (2010)

[6] Bley, A., Boland N.and Fricke, C., Froyland, G.: A strengthened formulation and cutting planes for the open pit mine production scheduling problem. Computers and Operations Research **37**(9), 1641–1647 (2010)

[7] Boland, N., Bley, A., Fricke, C., Froyland, G., Sotirov, R.: Clique-based facets for the precedence constrained knapsack problem. Mathematical Programming pp. 1–31 (2011)

[8] Boland, N., Dumitrescu, I., G.Froyland, Gleixner, A.: Lp-based disaggregation approaches to solving the open pit mining production scheduling problem with block processing selectivity. Computers and Operations Research **36**, 1064–1089 (2009)

[9] Boyd, E.A.: Polyhedral results for the precedence-constrained knapsack problem. Discrete Applied Mathematics **41**, 185–201 (1993)

[10] Chicoisne, R., Espinoza, D., Goycoolea, M., Moreno, E., Rubio, E.: A new algorithm for the open-pit mine scheduling problem. Operations Research **to appear** (2012)

[11] Cook, W., Espinoza, D., Goycoolea, M.: Computing with domino-parity inequalities for the tsp. INFORMS Journal on Computing **19**(3), 356–365 (2007)

[12] Correa, J., Schulz, A.: Single-machine scheduling with precedence constraints. Mathematics of Operations Research **30**(4), 1005–1021 (2005)

[13] Cullenbine, C., Wood, R., Newman, A.: A sliding time window heuristic for open pit mine block sequencing. Optimization Letters **5**(3), 365–377 (2011)

[14] Dagdelen, K., Johnson, T.: Optimum open pit mine production scheduling by lagrangian parameterization. 19th APCOM Symposium of the society of mining engineers (AIME) (1986)

[15] Espinoza, D., Fukasawa, R., Goycoolea, M.: Lifting, tilting and fractional programming revisited. Operations Research Letters (2010)

[16] Johnson, D., Niemi, K.: On knapsacks, partitions, and a new dynamic programming technique for trees. Mathematics of Operations Research **8**(1), 1–14 (1983)

[17] Johnson, T.: Optimum open-pit mine production scheduling. In: A. Weiss (ed.) A decade of digital computing in the mining industry, chap. 4. AIME, New York (1969)

[18] van de Leesel, R., van Hoesel, C., van de Kludert, J.: Lifting valid inequalities for the precedence constrained knapsack problem. Mathematical Programming **86**, 161–185 (1999)

[19] Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization, chap. III.3.6, pp. 688–690. Discrete Mathematics and Optimization. Wiley-Interscience (1999)

[20] Padberg, M.W.: (1, k)-Configurations and facets for packing problems. Mathematical Programming **18**(1), 94–99 (1980)

[21] Padberg, M.W., Rinaldi, G.: A branch-and-cut algorithm for the resolution of large-scale symmetric traveling salesman problems. SIAM Review **33**, 60–100 (1991)

[22] Park, K., Park, S.: Lifting cover inequalities for the precedence-constrained knapsack problem. Discrete Applied Mathematics **72**, 219–241 (1997)

[23] Queyranne, M., Schulz, A.: Approximation bounds for a general class of precedence constrained parallel machine scheduling problems. SIAM Journal on Computing **35**(5), 1241–1253 (2006)

[24] Wolsey, L.: Faces for a linear inequality in 0-1 variables. Mathematical Programming **8**(2), 165–178 (1975)

[25] Wolsey, L.: Facets and strong valid inequalities for integer programs. Operations Research **24**(2), 367–372 (1976)