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A note on some new classes of constitutive relations for elastic bodies

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The class of elastic bodies, that is bodies incapable of dissipation in whatever motion that they undergo, has been significantly enlarged recently (see Rajagopal 2003, On implicit constitutive theories. *Appl. Math.*, **48**, 279–319; Rajagopal 2007, The elasticity of elasticity. *Z. Angew. Math. Phys.* **58**, 309–317; Rajagopal, K. R. & Srinivasa, A. R. 2007, On the response of non-dissipative solids. *Proc. R. Soc. Lond. A*, **463**, 357–367). The new classes include fully implicit constitutive relations for the stress and the deformation gradient, and the interesting sub-class wherein the Cauchy–Green tensor or the linearized strain tensor bears a non-linear relationship to the stress. While a fully thermodynamic treatment of such elastic bodies, when defined through implicit constitutive relations between the Piola stress and the Green–St. Venant strain, within a 3D framework has been carried out (see Rajagopal, K. R. & Srinivasa, A. R. 2007, On the response of non-dissipative solids, *Proc. R. Soc. Lond. A*, **463**, 357–367), other possible implicit relationships between other stress and kinematic measures have not been investigated. This paper is devoted to the determination of the consequences of thermodynamics on the new class of elastic bodies, when they are expressed through implicit relations for different stress and stretch/strain measures.

Keywords: implicit constitutive theories; stored elastic energy function; large elastic deformations.

1. Introduction

Within the context of the classical approach to elasticity, a body is said to be a Cauchy elastic body (see Truesdell & Noll, 2004; see also Cauchy, 1823, 1828) if the stress depends upon the deformation gradient of the body. Green (1837, 1839–1842) assumed the existence of a potential (stored energy) that depends on the deformation gradient as his starting point, and realized that unless the stress in an elastic body is derivable from such a potential, the body in question could be a source of infinite energy, that is one could fashion a perpetual motion machine out of them. Despite Green's remark that such would indeed be the case, the use of Cauchy elasticity went along unfettered as Green had not clearly demonstrated his belief. Later, Rivlin (see the comments at the end of the communication presented by Truesdell, 1964) raised the same objection to the use of Cauchy elasticity, and it was only recently that Carroll (2009) came up with a clear example which showed that out of a Cauchy elastic body that is not Green elastic, one could fashion a body that is capable of providing inexhaustible energy. The result of Green presupposes that the stored energy depends only on the deformation gradient, an important fact to bear in mind.

The quintessential feature of an elastic body is that it is incapable of dissipating energy, that is converting mechanical working into energy in thermal form (heat). Elastic bodies such as those considered by Green are referred to as Green elastic bodies and they presuppose that the stored energy depends only on the deformation gradient that then leads to an explicit expression for the stress in terms of the deformation gradient. It would be perfectly legitimate to inquire into the possibility whether one could have bodies incapable of dissipating energy, that is elastic bodies, which are defined by means of an implicit relationship between the stress and the deformation gradient, or an associated stored energy that depends both on the stress and the deformation gradient. Recently, Rajagopal (2003, 2007, 2011a) addressed this question and showed that one could indeed have elastic bodies, wherein one has an implicit relationship between the stress and the deformation gradient, and also a stored energy that depends both on the stress and the deformation gradient. Rajagopal & Srinivasa (2007); Rajagopal & Srinivasa (2009) studied the consequences of thermodynamic requirements on the constitutive structure of such bodies. The studies of Rajagopal (2003, 2007) have been followed by several studies wherein the more general class of elastic bodies have been analysed (see Bustamante, 2009; Bustamante & Rajagopal, 2010; Rajagopal, 2010, 2011a,b). Also, numerous boundary value problems have been investigated within the context of such implicit constitutive theories (see, for example, Bustamante & Rajagopal, 2011, 2012; Kulvait *et al.*, 2013; Ortiz *et al.*, 2012; Ortiz-Bernardin *et al.*, 2014; Rajagopal & Walton, 2011; Rajagopal & Saravanan, 2011a). Unsteady problems have also been studied within the context of such theories by Kannan *et al.* (2014) and Kambapalli *et al.* (2014).

It is important to recognize that Green elastic bodies are a special sub-class of the new class of elastic materials introduced in Rajagopal & Srinivasa (2009) and Rajagopal (2010). Another interesting sub-class is the class of isotropic bodies for which constitutive relations are given for the Cauchy–Green tensor as a non-linear function of the stress (one could also obtain constitutive relations for anisotropic bodies wherein an appropriate kinematic quantity is a function of the stress). When one linearizes a model belonging to such a class by requiring that the displacement gradient is small, one obtains a model wherein the linearized strain bears a non-linear relationship to the stress. Such a model has the potential to resolve a long standing open problem, that of describing the stresses and strains at crack tips. Classical linearized elasticity predicts that the strains tends to blow up as $1/\sqrt{r}$, where r denotes the radial distance from the crack tip. We shall not get into a detailed discussion of this here but refer the reader to Rajagopal & Walton (2011), Kulvait *et al.* (2013) and Bulíček *et al.* (2013) for the same. This new class of models wherein the linearized strain is non-linearly related to the stress can also be used to describe a large body of recent experimental literature starting with the work of Saito *et al.* (2003) who observe that even for very small strains, strains which would imply that the classical non-linear models of elasticity collapse to the linearized elastic model, the relationship between the linearized strain and stress is non-linear (see also the experiments of Talling *et al.*, 2008; Li *et al.*, 2007; Withey *et al.*, 2008; Zhang *et al.*, 2009). Such a non-linear relationship between the linearized strain and the stress cannot be described within the context of the classical theory, while it can indeed be explained within the context of the new class of elastic bodies, and this point cannot be overemphasized.

We have discussed above two important classes of problems, which cannot be described within the context of the classical linearized theory or non-linear theory of elasticity, but can be explained adequately within the context of the new class of implicit models and their sub-classes. Thus, it is worthwhile to look into this class of implicit constitutive relations in more detail. In the studies on the new class of implicit constitutive relations, only the dependence of the stored energy on the Piola stress and the Green–St. Venant strain was considered within the context of a 3D thermodynamic framework (see Rajagopal & Srinivasa, 2007; Rajagopal & Srinivasa, 2009); within the context of one dimension Rajagopal (2003, 2007, 2011a) considered the possibility of the stored energy depending on the Cauchy stress and the linearized strain. It would be interesting to consider other conjugate pairs of measures of stresses and strains, and the constitutive relations that they give rise to, to describe the response of elastic bodies; and it is to this aspect of the development of implicit constitutive theories for elastic bodies that this paper is devoted.

If $\mathbf{S}$ and $\mathbf{E}$ denote the second Piola–Kirchhoff stress tensor and the Lagrangian Green–St. Venant strain tensor (to be defined in the next section), respectively, then if by an elastic body we mean a body that is not capable of dissipation of energy, that is no mechanical working can be converted into energy in thermal form (heat), in any process that the body is subject to, then in virtue of the inability to dissipate energy, we obtain in isothermal processes the condition that:

$$\mathrm{tr}(\mathbf{S}\dot{\mathbf{E}}) = \dot{W}, \tag{1.1}$$

where W is the stored energy accumulated by the body. The above equation is a mathematical expression of the rate of work, which is being performed on the body being equal to the rate of the energy being accumulated by the elastic body.

Let us consider an implicit constitutive relation of the form:[1]

$$\boldsymbol{f}(\mathbf{S},\mathbf{E}) = \mathbf{0}. \tag{1.2}$$

Rajagopal & Srinivasa (2007) considered a constitutive relation of the form (1.2) and obtained restrictions on $\boldsymbol{f}$ such that (1.2) holds in all thermodynamic processes, thereby guaranteeing that the body in question is an elastic body. The key point in their analysis was the assumption that:

$$W = W(\mathbf{S},\mathbf{E}). \tag{1.3}$$

By taking the time derivative of (1.2) and by considering (1.1) and (1.2) written in index notation (for simplicity in Cartesian coordinates, Greek index will be used to indicate the reference configuration, while Italian characters for the index will denote the current configuration), two relations can be obtained:

$$\frac{\partial f_{\alpha\beta}}{\partial S_{\gamma\delta}}\dot{S}_{\gamma\delta} + \frac{\partial f_{\alpha\beta}}{\partial E_{\gamma\delta}}\dot{E}_{\gamma\delta} = 0, \quad S_{\gamma\delta}\dot{E}_{\gamma\delta} = \frac{\partial W}{\partial S_{\gamma\delta}}\dot{S}_{\gamma\delta} + \frac{\partial W}{\partial E_{\gamma\delta}}\dot{E}_{\gamma\delta}. \tag{1.4}$$

If one considers an implicit relation of the form (1.2), a body is elastic if $\boldsymbol{f}$ and W satisfy (1.4) for any $\dot{\mathbf{S}}$ and $\dot{\mathbf{E}}$ (see Rajagopal & Srinivasa, 2007).

In the present note, we are interested in studying other constitutive relations, which have been proposed in the literature (see Bustamante, 2009; Rajagopal & Saravanan, 2011a; Steigmann *et al.*, 2014). Some of these constitutive relations take the form: $\boldsymbol{\varepsilon} = \partial\tilde{W}/\partial\mathbf{T}$, where $\tilde{W} = \tilde{W}(\mathbf{T})$; $\mathbf{U} = \boldsymbol{h}(\boldsymbol{\sigma})$ and $\mathbf{B} = \boldsymbol{g}(\mathbf{T})$, where $\boldsymbol{\varepsilon}$, $\mathbf{U}$ and $\mathbf{B}$ are the linearized strain tensor, the right stretch tensor and the left Cauchy–Green deformation tensor, respectively, and $\mathbf{T}$ and $\boldsymbol{\sigma}$ are the Cauchy stress tensor and the Biot stress tensor, respectively. These constitutive relations are special sub-classes of the general implicit relations proposed in Rajagopal (2003, 2007, 2011a), and as noted, for example, in Rajagopal (2014), can be of great utility for a number of applications in fracture mechanics and biomechanics.

2. Basic equations

Let $\mathbf{X}$, where $\mathbf{X} = \boldsymbol{\kappa}_r(X)$, denote the position of a particle X of a body $\mathscr{B}$ in the reference configuration $\kappa_r(\mathscr{B})$. It is assumed that there exists a one to one mapping $\boldsymbol{\chi}$ such that at any time t it assigns the

[1] Rajagopal (2003) considered implicit constitutive relations of the form $\boldsymbol{G}(\mathbf{T},\mathbf{F}) = \mathbf{0}$, where $\mathbf{T}$ is the Cauchy stress and $\mathbf{F}$ the deformation gradient, to describe the response of elastic bodies. Later, Rajagopal & Srinivasa (2007) considered implicit rate equations of the form $\boldsymbol{A}(\mathbf{S},\mathbf{E})\dot{\mathbf{S}} + \boldsymbol{B}(\mathbf{S},\mathbf{E})\dot{\mathbf{E}} = \mathbf{0}$.

position $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ to the particle X, in the current configuration $\kappa_t(\mathscr{B})$. The displacement field $\mathbf{u}$ is defined as:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}. \tag{2.1}$$

The deformation gradient $\mathbf{F}$, the right and left Cauchy–Green deformation tensors $\mathbf{C}$, $\mathbf{B}$, the right stretch tensor $\mathbf{U}$, the Lagrangian Green–St. Venant strain tensor $\mathbf{E}$ and the linearized strain tensor $\boldsymbol{\varepsilon}$ are defined through:

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} = \mathbf{R}\mathbf{U}, \quad \mathbf{C} = \mathbf{F}^{\top}\mathbf{F}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\top}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \quad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}\mathbf{u}^{\top}), \tag{2.2}$$

where $\nabla_{\mathbf{X}}$ is the gradient operator defined with respect to the reference configuration and $\mathbf{R}^{\top}\mathbf{R} = \mathbf{I} = \mathbf{R}\mathbf{R}^{\top}$. We shall henceforth assume that $J = \det \mathbf{F} > 0$.

If $\mathbf{T}$ is the Cauchy stress tensor, we have the following relationships with the nominal stress tensor $\boldsymbol{\tau}$, the second Piola–Kirchhoff stress tensor $\mathbf{S}$ and the Biot stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\tau} = J\mathbf{F}^{-1}\mathbf{T}, \quad \mathbf{S} = \boldsymbol{\tau}\mathbf{F}^{-\top}, \quad \boldsymbol{\sigma} = \mathbf{R}^{\top}\boldsymbol{\tau}^{\top}. \tag{2.3}$$

The equilibrium equation in terms of the Cauchy stress tensor is

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{2.4}$$

where ρ and $\mathbf{b}$ are the density of the body and the body forces in the current configuration, respectively.

More details concerning the above tensors and their definitions can be found, for example, in Chadwick (1999) and Truesdell & Toupin (1960).

The rate of mechanical work, the stored energy W and the implicit relation (1.2) can be expressed alternatively in terms of the conjugate pairs $(\boldsymbol{\tau}, \mathbf{F})$, $(\boldsymbol{\sigma}, \mathbf{U})$ and $(\mathbf{T}, \boldsymbol{\varepsilon})$, respectively. For example, in the case of $(\boldsymbol{\tau}, \mathbf{F})$ we have $W = W(\boldsymbol{\tau}, \mathbf{F}), \boldsymbol{f}(\boldsymbol{\tau}, \mathbf{F}) = \mathbf{0}$ and (1.4) becomes:

$$\frac{\partial f_{\alpha i}}{\partial \tau_{\beta i}}\dot{\tau}_{\beta i} + \frac{\partial f_{\alpha i}}{\partial F_{i\beta}}\dot{F}_{i\beta} = 0, \quad \tau_{\beta i}\dot{F}_{i\beta} = \frac{\partial W}{\partial \tau_{\beta i}}\dot{\tau}_{\beta i} + \frac{\partial W}{\partial F_{i\beta}}\dot{F}_{i\beta}. \tag{2.5}$$

Similar expressions can be found if one uses $(\boldsymbol{\sigma}, \mathbf{U})$ and $(\mathbf{T}, \boldsymbol{\varepsilon})$ as the variables of interest, which for the sake of brevity are not presented in this section. One can obtain the appropriate approximation for (2.5) in terms of the pair $(\mathbf{T}, \boldsymbol{\varepsilon})$ by using the classical linearization that the displacement gradient is appropriately small.

More details concerning the definitions presented in this section can be found in Rajagopal & Srinivasa (2007).

3. Restrictions for some constitutive relations

3.1 *A constitutive equation of the form* $\boldsymbol{\varepsilon} = \partial \tilde{W}/\partial \mathbf{T}$

If in (2.5) $\boldsymbol{f}$ and W depend on $\mathbf{T}$ and $\boldsymbol{\varepsilon}$ we have

$$\frac{\partial f_{ij}}{\partial T_{kl}}\dot{T}_{kl} + \frac{\partial f_{ij}}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} = 0, \quad T_{kl}\dot{\varepsilon}_{kl} = \frac{\partial W}{\partial T_{kl}}\dot{T}_{kl} + \frac{\partial W}{\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl}. \tag{3.1}$$

Bustamante & Rajagopal (2011) considered the constitutive equation (see also Bustamante, 2009)

$$\boldsymbol{\varepsilon} = \frac{\partial \tilde{W}}{\partial \mathbf{T}}, \tag{3.2}$$

where $\tilde{W} = \tilde{W}(\mathbf{T})$. Such a constitutive relation is particularly useful in describing bodies which are undergoing very small displacement gradients and hence very small strains, wherein the non-linear part of the strain can be ignored, but the strain yet depends non-linearly on the stress and the stress can be arbitrarily large. Since the stress has dimensions, one needs to really consider an appropriate non-dimensional version of the same. The important point to be recognized is that the relationship between the linearized strain and the stress can be non-linear. As mentioned in the introduction, such an approach is particularly well suited to describe the response of materials such as gum metal and titanium alloys (see Saito *et al.*, 2003; Talling *et al.*, 2008; Li *et al.*, 2007; Withey *et al.*, 2008; Zhang *et al.*, 2009). It is also particularly well suited to describe the problem of fracture in brittle materials (see Rajagopal & Walton, 2011) and in the state of stresses and strains in the vicinity of notches, etc. (see Kulvait *et al.*, 2013). Considering the expression (3.2) and assuming that $W = W(\mathbf{T})$, from (3.1) we obtain the following restriction for $\tilde{W}$ and W in order that the response is non-dissipative and the constitutive relation describes an elastic body:

$$T_{ij}\dot{\varepsilon}_{ij} = T_{ij}\frac{\partial^2 \tilde{W}}{\partial T_{ij}\partial T_{kl}}\dot{T}_{kl} = \frac{\partial W}{\partial T_{kl}}\dot{T}_{kl}, \tag{3.3}$$

which is equivalent to $(T_{ij}(\partial^2\tilde{W}/\partial T_{ij}\partial T_{kl}) - \partial W/\partial T_{kl})\dot{T}_{kl} = 0$. Usually, one assumes that since the condition needs to hold for all $\dot{\mathbf{T}}$, the term within the parenthesis has to be identically zero. We shall not appeal to such an argument as the constitutive relation itself may not be valid for arbitrary $\dot{\mathbf{T}}$. While it is tacitly assumed while postulating the constitutive relation (3.2) that it holds for all $\dot{\mathbf{T}}$, it is a mathematical idealization that might not hold for real physical systems. We merely appeal to a sufficient condition for the equality to hold. A sufficient condition that guarantees that such a condition will be met is $T_{ij}(\partial^2\tilde{W}/\partial T_{ij}\partial T_{kl}) - \partial W/\partial T_{kl} = 0$, which is equivalent to

$$\frac{\partial}{\partial T_{kl}}\left(T_{ij}\frac{\partial \tilde{W}}{\partial T_{ij}} - \tilde{W} - W\right) = 0, \tag{3.4}$$

which leads to the solution $T_{ij}(\partial\tilde{W}/\partial T_{ij}) - \tilde{W} - W = c_o$, where c_o is a constant (in the stress). If $c_o = 0$ we obtain the relation

$$W = T_{ij}\frac{\partial \tilde{W}}{\partial T_{ij}} - \tilde{W}. \tag{3.5}$$

This is the relation that $\tilde{W}$ and the elastic energy W have to satisfy in order for (3.2) to be a constitutive equation for an elastic body.

If we could find the form of $W = W(\mathbf{T})$, say on the basis of experiments, then $\tilde{W}$ can be found by appealing to (3.5) as:

$$\begin{aligned}\tilde{W}(\mathbf{T}) = T_{11}\Bigg[&\int_1^{T_{11}} \frac{1}{\xi^2} W\left(\xi, \frac{T_{22}}{T_{11}}\xi, \frac{T_{33}}{T_{11}}\xi, \frac{T_{12}}{T_{11}^2}\xi^2, \frac{T_{13}}{T_{11}^2}\xi^2, \frac{T_{23}}{T_{11}^2}\xi^2\right) \mathrm{d}\xi \\ &+ W_o\left(\frac{T_{22}}{T_{11}}, \frac{T_{33}}{T_{11}}, \frac{T_{12}}{T_{11}^2}, \frac{T_{13}}{T_{11}^2}, \frac{T_{23}}{T_{11}^2}\right)\Bigg],\end{aligned} \tag{3.6}$$

provided that $T_{11} \neq 0$, where W_o is an arbitrary function that depends on the vector $(T_{22}/T_{11}, T_{33}/T_{11}, T_{12}/T_{11}^2, T_{13}/T_{11}^2, T_{23}/T_{11}^2)$. If $T_{11} = 0$, a similar solution can be found using T_{22} or T_{33} instead, if $T_{22} \neq 0$ or $T_{33} \neq 0$.

3.2 *A constitutive equation of the form* $\mathbf{U} = \boldsymbol{h}(\boldsymbol{\sigma})$

Suppose $\boldsymbol{f}$ and W depend on $\mathbf{U}$ and $\boldsymbol{\sigma}$. In this case, we obtain two equations similar to (1.4).

Recently, a constitutive relation of the form (see Steigmann *et al.*, 2014):

$$\mathbf{U} = \boldsymbol{h}(\boldsymbol{\sigma}), \tag{3.7}$$

has been considered. For this particular form of $\boldsymbol{f}$, assuming that $W = W(\boldsymbol{\sigma})$, from (1.4), (3.7) (written in terms of $\mathbf{U}$ and $\boldsymbol{\sigma}$) we obtain $\sigma_{\alpha\beta}(\partial h_{\alpha\beta}/\partial\sigma_{\gamma\delta})\dot{\sigma}_{\gamma\delta} = (\partial W/\partial\sigma_{\gamma\delta})\dot{\sigma}_{\gamma\delta}$, and a sufficient condition that will guarantee that this holds is the partial differential equation:

$$\sigma_{\alpha\beta}\frac{\partial h_{\alpha\beta}}{\partial\sigma_{\gamma\delta}} - \frac{\partial W}{\partial\sigma_{\gamma\delta}} = 0. \tag{3.8}$$

This is the restriction that $\boldsymbol{h}(\boldsymbol{\sigma})$ and $W = W(\boldsymbol{\sigma})$ have to satisfy in order for them to describe the response of an elastic body. Equation (3.8) can be solved easily for W in terms of $\boldsymbol{h}$ by integrating in $\boldsymbol{\sigma}$. On the other hand, if there exists a scalar function $\breve{W} = \breve{W}(\boldsymbol{\sigma})$ such that $\boldsymbol{h}(\boldsymbol{\sigma}) = \partial\breve{W}/\partial\boldsymbol{\sigma}$, then from (3.8) it is possible to obtain a solution for $\breve{W}$ in terms of W, which is similar to (3.6).

Conditions equivalent to (3.5) and (3.6) can be found when we consider non-linear relations similar to (3.2) or (3.7) for $\mathbf{E}$ *and* $\mathbf{S}$*, and for* $\mathbf{F}$ *and* $\mathbf{T}$*. For brevity such conditions are not presented here.*

3.3 *A constitutive equation of the form* $\mathbf{B} = \boldsymbol{g}(\mathbf{T})$

Rajagopal (2003, 2007) proposed an implicit constitutive relation of the form:

$$\boldsymbol{G}(\mathbf{T}, \mathbf{B}) = \mathbf{0} \tag{3.9}$$

to describe the response of isotropic elastic bodies. Representation theorems then lead to (see Spencer, 1971)

$$\begin{aligned}\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{B} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{B}^2 + \alpha_5(\mathbf{TB} + \mathbf{BT}) + \alpha_6(\mathbf{T}^2\mathbf{B} + \mathbf{BT}^2)\\ + \alpha_7(\mathbf{B}^2\mathbf{T} + \mathbf{TB}^2) + \alpha_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) = \mathbf{0},\end{aligned} \tag{3.10}$$

where the material moduli α_i, $i = 0, 1, \ldots, 8$ depend upon

$$\operatorname{tr}\mathbf{T}, \quad \operatorname{tr}\mathbf{B}, \quad \operatorname{tr}\mathbf{T}^2, \quad \operatorname{tr}\mathbf{B}^2, \quad \operatorname{tr}\mathbf{T}^3, \quad \operatorname{tr}\mathbf{B}^3, \quad \operatorname{tr}(\mathbf{TB}), \quad \operatorname{tr}(\mathbf{T}^2\mathbf{B}), \quad \operatorname{tr}(\mathbf{B}^2\mathbf{T}), \quad \operatorname{tr}(\mathbf{T}^2\mathbf{B}^2). \tag{3.11}$$

A special sub-class of the above model is the following:

$$\mathbf{B} = \bar{\alpha}_0\mathbf{I} + \bar{\alpha}_1\mathbf{T} + \bar{\alpha}_2\mathbf{T}^2, \tag{3.12}$$

where the $\bar{\alpha}_i$, $i = 0, 1, 2$ depend on ρ, $\operatorname{tr}\mathbf{T}$, $\operatorname{tr}\mathbf{T}^2$ and $\operatorname{tr}\mathbf{T}^3$.

We shall investigate the possibility if akin to classical Green elasticity there exists a potential, which depends on some measure of the stress from which we can derive the kinematical quantity $\mathbf{B}$ (or in general $\mathbf{C}$). We first try to find if there exist potentials which only depend on the Piola stress $\mathbf{S}$ or the nominal stress $\boldsymbol{\tau}$, and we find in what follows that it is not possible to obtain a relation of the form (3.12). We, however, find that a more complicated choice wherein the potential depends on both $\mathbf{S}$ and $\mathbf{C}$ does lead to a constitutive relationship of the form (3.12).

To begin with we recognize that the pair $(\mathbf{T}, \mathbf{B})$ is not conjugated, therefore unlike the procedure followed in the previous cases we cannot rewrite (1.2) and (1.3) in terms of $\mathbf{B}$ and $\mathbf{T}$; instead we need to start with another pair of conjugated tensors and functions that satisfy conditions similar to (1.4), and by using the connections (2.2), (2.3) to obtain something similar to (3.12).

To understand the procedure to be presented below, let us recall briefly some elements of the classical non-linear theory of elasticity. In the theoretical development of non-linear elasticity a central assumption is that there exists a function $\boldsymbol{L} = \boldsymbol{L}(\mathbf{F})$, which can be associated with the nominal stress tensor, i.e. $\boldsymbol{\tau} = \boldsymbol{L}(\mathbf{F})$ (which is the definition of a Cauchy elastic body Truesdell & Noll (2004)). In the case of a hyperelastic or Green elastic body, it is assumed that the elastic energy W depends on the deformation gradient and $\boldsymbol{L}(\mathbf{F}) = \partial W/\partial \mathbf{F}$. As shown by Carroll (2009) only Cauchy elastic bodies that are hyperleastic satisfy (1.1). Now, for an isotropic body we have that $W = W(I_1, I_2, I_3)$, where $I_1 = \operatorname{tr}\mathbf{C}$, $I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)]$ and $I_3 = \det\mathbf{C}$ are the principal invariants of $\mathbf{C}$. From $\boldsymbol{\tau} = \partial W/\partial\mathbf{F}$ and the relation $(2.3)_1$ it is possible to find the constitutive equation $\mathbf{T} = 2J(\partial W/\partial I_3)\mathbf{I} + 2J^{-1}(\partial W/\partial I_1 + I_1(\partial W/\partial I_2))\mathbf{B} - 2J^{-1}(\partial W/\partial I_2)\mathbf{B}^2$, which can be rewritten in the form

$$\mathbf{T} = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2, \tag{3.13}$$

where the scalar functions α_i, $i = 0, 1, 2$ depend now on the invariants of $\mathbf{B}$.

The question now is: Can we obtain (3.12) from a scalar function W following similar steps as in the classical non-linear theory of elasticity? Let us consider a couple of cases, where the answer is no, before presenting the particular expressions for W from which (3.12) can be obtained.

3.3.1 *Two examples where the energy function depends on the second Piola–Kirchhoff stress tensor and the nominal stress tensor* Let us consider the pair $\mathbf{S}$, $\mathbf{E}$ and a scalar function $\bar{W}$ that depends on $\mathbf{S}$ such that $\mathbf{E} = \partial\bar{W}/\partial\mathbf{S}$. For an isotropic body we have $\bar{W}(\mathbf{S}) = \bar{W}(I_1, I_2, I_3)$, where $I_1 = \operatorname{tr}\mathbf{S}$, $I_2 = \frac{1}{2}\operatorname{tr}\mathbf{S}^2$ and $I_3 = \frac{1}{3}\operatorname{tr}\mathbf{S}^3$. Considering the definitions for the invariants and the chain rule for the derivative we obtain for isotropic bodies $\mathbf{C} = 2\mathbf{E} + \mathbf{I} = (1 + 2(\partial\bar{W}/\partial I_1))\mathbf{I} + 2(\partial\bar{W}/\partial I_2)\mathbf{S} + 2(\partial\bar{W}/\partial I_3)\mathbf{S}^2$. But $\mathbf{B} = \mathbf{F}\mathbf{C}\mathbf{F}^{-1}$ or $\mathbf{B} = \mathbf{F}^{-\top}\mathbf{C}\mathbf{F}^{\top}$, then using these relations and $(2.3)_{(1,2)}$ we obtain

$$\mathbf{B} = \left(1 + 2\frac{\partial\bar{W}}{\partial I_1}\right)\mathbf{I} + 2\frac{\partial\bar{W}}{\partial I_2}J\mathbf{T}\mathbf{B}^{-1} + 2\frac{\partial\bar{W}}{\partial I_3}J^2\mathbf{T}\mathbf{B}^{-1}\mathbf{T}\mathbf{B}^{-1}, \tag{3.14}$$

$$\mathbf{B} = \left(1 + 2\frac{\partial\bar{W}}{\partial I_1}\right)\mathbf{I} + 2\frac{\partial\bar{W}}{\partial I_2}J\mathbf{B}^{-1}\mathbf{T} + 2\frac{\partial\bar{W}}{\partial I_3}J^2\mathbf{B}^{-1}\mathbf{T}\mathbf{B}^{-1}\mathbf{T}, \tag{3.15}$$

respectively. We can see that (3.14) and (3.15) do not have the structure of (3.12), in fact they are implicit relations between $\mathbf{T}$ and $\mathbf{B}$.

Let us consider the pair $\boldsymbol{\tau}$, $\mathbf{F}$ and a scalar function $\hat{W} = \hat{W}(\boldsymbol{\tau})$ such that $\mathbf{F} = \partial\hat{W}/\partial\boldsymbol{\tau}$. It is not difficult to find a relation between the functions $\bar{W}$ and $\hat{W}$ and the elastic energy W such that (1.4) is satisfied, as presented in Section 3.1. For an isotropic body, we have $\hat{W}(\boldsymbol{\tau}) = \hat{W}(I_1, I_2, I_3)$, where in this case the invariants are defined as $I_1 = \operatorname{tr}(\boldsymbol{\tau}\boldsymbol{\tau}^{\top})$, $I_2 = \frac{1}{2}[I_1^2 - \operatorname{tr}((\boldsymbol{\tau}\boldsymbol{\tau}^{\top})^2)]$ and $I_3 = \det(\boldsymbol{\tau}\boldsymbol{\tau}^{\top})$. In this case using the chain rule and the convention $(\partial\hat{W}/\partial\boldsymbol{\tau})_{i\alpha} = \partial\hat{W}/\partial\tau_{\alpha i}$, after some manipulations we obtain

$$\mathbf{F} = 2\left[\frac{\partial\hat{W}}{\partial I_1}\boldsymbol{\tau}^\top + \frac{\partial\hat{W}}{\partial I_2}(I_1\boldsymbol{\tau}^\top - \boldsymbol{\tau}^\top\boldsymbol{\tau}\boldsymbol{\tau}^\top) + \frac{\partial\hat{W}}{\partial I_3}I_3\boldsymbol{\tau}^{-1}\right]. \tag{3.16}$$

If we use $(2.2)_3$, considering (3.16), it is possible to show that we do not obtain an expression of the form (3.12).

3.3.2 *An implicit expression* A relation of the form (3.12) can be obtained in the following way. Let us consider the pair $\mathbf{S}$ and $\mathbf{C}$, then (1.1) can be written as $\mathrm{tr}(\mathbf{S}\dot{\mathbf{C}}) = \dot{W}$. Let us assume the following implicit constitutive relation:

$$\boldsymbol{H}(\mathbf{S}, \mathbf{C}, \rho) = \mathbf{0}, \tag{3.17}$$

where ρ is the density of the body. For the elastic energy W we assume a similar dependence, i.e.:

$$W = W(\mathbf{S}, \mathbf{C}, \rho). \tag{3.18}$$

Recall the balance of mass implies that $\rho = \rho_r J^{-1}$ (where ρ_r is the density in the reference configuration); since ρ_r is constant, we replace ρ by J^{-1} in (3.17) and (3.18). Let us consider the following special expressions for $\boldsymbol{H}(\mathbf{S}, \mathbf{C}, \rho)$ and W:

$$\mathbf{C} = \tilde{\boldsymbol{H}}(J^{-1}\mathbf{SC}), \quad W = W(J^{-1}\mathbf{SC}). \tag{3.19}$$

We shall find it convenient to express $\boldsymbol{H}$ in the form shown above, as we shall find such a form more convenient for the manipulations that are carried out (not that $\boldsymbol{H}$ can always be expressed in such a manner). Let us define the tensor $\mathbf{G}$ as

$$\mathbf{G} = J^{-1}\mathbf{SC}, \tag{3.20}$$

therefore, considering (3.20), $(1.4)_2$ in index notation becomes

$$\left(S_{\alpha\beta}\frac{\partial\tilde{H}_{\alpha\beta}}{\partial G_{\gamma\delta}} - \frac{\partial W}{\partial G_{\gamma\delta}}\right)\dot{G}_{\gamma\delta} = 0,$$

which will be met if the following equation holds:

$$S_{\alpha\beta}\frac{\partial\tilde{H}_{\alpha\beta}}{\partial G_{\gamma\delta}} - \frac{\partial W}{\partial G_{\gamma\delta}} = 0. \tag{3.21}$$

For the particular case in which there exist a scalar function $\check{W}(\mathbf{G})$ such that $\tilde{\boldsymbol{H}} = \partial\check{W}/\partial\mathbf{G}$, the previous equation becomes

$$S_{\alpha\beta}\frac{\partial^2\check{W}}{\partial G_{\alpha\gamma}\partial G_{\gamma\delta}} - \frac{\partial W}{\partial G_{\gamma\delta}} = 0. \tag{3.22}$$

If this relation is satisfied for W and $\check{W}$ the body will display non-dissipative response and hence can be used to describe an elastic body.

Let us consider an isotropic body, in which case $\check{W} = \check{W}(I_1, I_2, I_3)$, where

$$I_1 = \operatorname{tr}\mathbf{G}, \quad I_2 = \tfrac{1}{2}\operatorname{tr}(\mathbf{G}^2), \quad I_3 = \tfrac{1}{3}\operatorname{tr}(\mathbf{G}^3). \tag{3.23}$$

From the implicit relation $\mathbf{C} = \tilde{\boldsymbol{H}}(\mathbf{G}) = \partial\check{W}/\partial\mathbf{G}$, using the chain rule for the derivative we obtain $\mathbf{C} = (\partial\check{W}/\partial I_1)\mathbf{I} + (\partial\check{W}/\partial I_2)\mathbf{G} + (\partial\check{W}/\partial I_3)\mathbf{G}^2$, which from (2.2)$_2$ is equivalent to

$$\mathbf{F}^\top\mathbf{F} = \frac{\partial\check{W}}{\partial I_1}\mathbf{I} + \frac{\partial\check{W}}{\partial I_2}\mathbf{F}^{-1}\mathbf{T}\mathbf{F} + \frac{\partial\check{W}}{\partial I_3}\mathbf{F}^{-1}\mathbf{T}^2\mathbf{F}. \tag{3.24}$$

Multiplying the previous equation from the left by $\mathbf{F}$ and from the right by $\mathbf{F}^{-1}$ we obtain

$$\mathbf{B} = \mathbf{F}\mathbf{F}^\top\mathbf{F}\mathbf{F}^{-1} = \frac{\partial\check{W}}{\partial I_1}\mathbf{I} + \frac{\partial\check{W}}{\partial I_2}\mathbf{T} + \frac{\partial\check{W}}{\partial I_3}\mathbf{T}^2. \tag{3.25}$$

In (3.25), the functions $\partial\check{W}/\partial I_i$, $i = 1, 2, 3$ depend on the invariants (3.23). But from (3.20), (2.2)$_2$ and (2.3)$_{(1,2)}$ those invariants become $I_1 = \operatorname{tr}\mathbf{G} = \operatorname{tr}\mathbf{T}$, $I_2 = \frac{1}{2}\operatorname{tr}\mathbf{G}^2 = \frac{1}{2}\operatorname{tr}\mathbf{T}^2$ and $I_3 = \frac{1}{3}\operatorname{tr}\mathbf{G} = \frac{1}{3}\operatorname{tr}\mathbf{T}^3$. As a result we see that (3.25) is of the form (3.12).

Exactly, the same result is obtained if the transpose of (3.24) is multiplied from the left by $\mathbf{F}^{-\top}$ and from the right by $\mathbf{F}^\top$.

Equation (3.25) satisfies the universal relation[2] $\mathbf{BT} = \mathbf{TB}$ and the same happens with (3.13).

4. Solutions of some simple boundary value problems: homogeneous distributions of strains produced by homogeneous distributions of stresses

Before adopting these new classes of constitutive relations (1.2), (3.2), (3.7), (3.9) and (3.12), it is necessary to solve several simple boundary value problems and compare the predictions of the solutions with the response of real bodies, to study the efficacy of the use of such models. In the case of the model (3.2) several solutions have been documented in the literature, see, for example, Bustamante & Rajagopal (2011, 2012), Ortiz *et al.* (2012) and Ortiz-Bernardin *et al.* (2014).

In this section, as an illustration, we shall study some simple solutions for boundary value problems within the context of the model (3.12). A detailed discussion of the methodology adopted for solving boundary value problems when working with (3.9) and (3.12) is presented in Rajagopal (2014). Also Rajagopal & Saravanan (2011a,b) some simple boundary value problems are solved for (3.12), with regard to non-homogeneous distribution of stresses and strains for a cylindrical tube and a hollow sphere.

In this work, we present some simple solutions assuming that the bodies deform uniformly with homogeneous stress distributions within the body, using (3.25).

Let us assume that $\boldsymbol{\chi}(\mathbf{X})$ is of the form (see, for example, Section 42 of Truesdell & Toupin, 1960)

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \mathbf{A}\mathbf{X} + \mathbf{x}_0, \tag{4.1}$$

where $\mathbf{A}$ is a second-order tensor that is constant in $\mathbf{X}$ such that $\det\mathbf{A} > 0$ and $\mathbf{x}_0$ is a constant vector. From (2.2)$_1$ and (2.2)$_3$ we find that $\mathbf{B} = \mathbf{A}\mathbf{A}^\top$ is constant. Let us assume further that this stretch field is produced by the stress tensor field $\mathbf{T} = \mathbf{T}_0$, where $\mathbf{T}_0$ does not depend on $\mathbf{x}$. Under these assumptions, we have a solution for the boundary value problem, since (2.4) is satisfied automatically (if there are no

[2] See, for example, Beatty (1987).

body forces), and for a given stress field $\mathbf{T}_0$ we can find an unique (up to a rigid body motion) vector field $\boldsymbol{\chi}(\mathbf{X})$ from (3.25) as $\mathbf{B}=\mathbf{A}\mathbf{A}^{\top}=(\partial\breve{W}/\partial I_1)(\mathbf{T}_0)\mathbf{I}+(\partial\breve{W}/\partial I_2)(\mathbf{T}_0)\mathbf{T}_0+(\partial\breve{W}/\partial I_3)(\mathbf{T}_0)\mathbf{T}_0^2$.

Let us study three special cases for (4.1).

4.1 *The uniform stretching (shortening) of a bar*

In this problem let us assume that a cylindrical bar, which in the reference configuration occupies the region (described in cylindrical coordinates) $0\leqslant R\leqslant R_o$, $0\leqslant\Theta\leqslant 2\pi$, $0\leqslant Z\leqslant L$, is deformed in the following manner:

$$r=\zeta R,\quad \theta=\Theta,\quad z=\lambda_z Z, \tag{4.2}$$

where $\zeta>0$ and $\lambda_z>0$ are constants. In this case, the deformation gradient $(2.2)_1$ is of the form $\mathbf{F}=\zeta\mathbf{e}_r\otimes\mathbf{E}_R+\zeta\mathbf{e}_\theta\otimes\mathbf{E}_\Theta+\lambda_z\mathbf{e}_z\otimes\mathbf{E}_Z$. Let us assume that this deformation field is produced by the uniform stress field $\mathbf{T}=\sigma_z\mathbf{e}_z\otimes\mathbf{e}_z$, where σ_z does not depend on $\mathbf{x}$ and where we have assumed that there is no normal stress either in the radial or in the θ directions. From (4.2), (3.25) and $(2.2)_{1,3}$ we obtain the relations:

$$\zeta^2=\breve{W}_1,\quad \lambda_z^2=\breve{W}_1+\breve{W}_2\sigma_z+\breve{W}_3\sigma_z^2, \tag{4.3}$$

where $\breve{W}_i=\partial\breve{W}/\partial I_i$, $i=1,2,3$ and where $I_1=\sigma_z$, $I_2=\sigma_z^2/2$ and $I_3=\sigma_z^3/3$.

We recognize that in virtue of (4.3), it would be very simple to study the experimental problem of the uniform tension of a bar. The stress σ_z is 'causing' the deformation of the bar, and the elongations ζ and λ_z, which are positive and unique in virtue of (4.3). Moreover, from (4.3) we obtain a direct physical restriction on $\breve{W}$, since for any σ_z for which the model (3.25) is valid we have the restrictions[3] $\breve{W}_1>0$ and $\breve{W}_1+\breve{W}_2\sigma_z+\breve{W}_3\sigma_z^2>0$. Compare this with the same problem in the context of the classical non-linear theory of elasticity using (3.13), where in general ζ and λ_z must be found indirectly (for each σ_z) by solving the (in general non-linear) algebraic equations:

$$0=\alpha_0(\zeta,\lambda_z)+\alpha_1(\zeta,\lambda_z)\zeta+\alpha_2(\zeta,\lambda_z)\zeta^2,\quad \sigma_z=\alpha_0(\zeta,\lambda_z)+\alpha_1(\zeta,\lambda_z)\lambda_z+\alpha_2(\zeta,\lambda_z)\lambda_z^2. \tag{4.4}$$

4.2 *The biaxial stretching (shortening) of a thin plate*

A problem which is closely connected with the previous one corresponds to the case of biaxial stretching (shortening) of a thin plate. Let us assume we have a plate defined in the reference configuration through $-L_1/2\leqslant X_1\leqslant L_1/2$, $-L_2/2\leqslant X_2\leqslant L_2/2$, $-H/2\leqslant X_3\leqslant H/2$, where $H\ll L_1$, $H\ll H_2$. This plate is deformed uniformly in the manner

$$x_1=\lambda_1 X_1,\quad x_2=\lambda_2 X_2,\quad x_3=\lambda_3 X_3, \tag{4.5}$$

where $\lambda_i>0$, $i=1,2,3$ are constants. We assume this deformation is produced due to the application of the homogeneous stress distribution $\mathbf{T}=\sigma_1\mathbf{e}_1\otimes\mathbf{e}_1+\sigma_2\mathbf{e}_2\otimes\mathbf{e}_2$, where σ_1, σ_2 do not depend on $\mathbf{x}$. In the direction 3, there is no stress. From $(2.2)_1$ on using (4.5) we obtain $\mathbf{F}=\lambda_1\mathbf{e}_1\otimes\mathbf{E}_1+\lambda_2\mathbf{e}_2\otimes\mathbf{E}_2+\lambda_3\mathbf{e}_3\otimes\mathbf{E}_3$ and from $(2.2)_3$ and (3.25) we have

$$\lambda_1^2=\breve{W}_1+\breve{W}_2\sigma_1+\breve{W}_3\sigma_1^2,\quad \lambda_2^2=\breve{W}_1+\breve{W}_2\sigma_2+\breve{W}_3\sigma_2^2,\quad \lambda_3^2=\breve{W}_1, \tag{4.6}$$

where $\breve{W}$ depends on the invariants $I_1=\sigma_1+\sigma_2$, $I_2=(\sigma_1^2+\sigma_2^2)/2$, $I_3=(\sigma_1^3+\sigma_2^3)/3$. Equation $(4.6)_3$ can be used to find the stretching (shortening) of the plate in the direction 3 directly from the constitutive

[3] See Rajagopal & Saravanan (2011b) for a deeper discussion of this issue.

equation. It is interesting to note that in the special case when $\sigma_1 = \sigma_2 = \sigma$, there is an unique solution for λ_1, λ_2 from $(4.6)_{1,2}$ (the solutions would be equal). That is not the case when working with the classical constitutive equation (3.13), where for such a problem we have

$$\sigma = \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2, \quad \sigma = \alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2, \quad 0 = \alpha_0 + \alpha_1\lambda_3 + \alpha_2\lambda_3^2, \tag{4.7}$$

where $\alpha_i = \alpha_i(\lambda_1, \lambda_2, \lambda_3)$, $i = 1, 2, 3$. It has been shown that from (4.7) for a given value of σ we can have more than one solution for λ_1, λ_2 (see, for example, Section 6 of Rivlin, 1948a,b). The stretching λ_3 has to be found from (in general non-linear) $(4.7)_3$.

4.3 *The simple shear of a slab*

Regarding the shear of the slab $-L_i/2 \leqslant X_i \leqslant L_i/2$, $i = 1, 2, 3$, in the classical theory of non-linear elasticity if we assume a deformation $x_1 = X_1 + \kappa X_2$, $x_2 = X_2$, $x_3 = X_3$, where $\kappa \geqslant 0$ is a constant, then from $(2.2)_{1,3}$ we can show that in general working with the classical model (3.13) such a deformation can only be produced by a combination of uniform shear stress plus some normal stresses.[4]

Let us study a different problem within the context of (3.25). Let us assume that the stress tensor $\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$ (with τ not depending on $\mathbf{x}$) is applied on the slab $-L_i/2 \leqslant X_i \leqslant L_i/2$, $i = 1, 2, 3$, and let us assume that such stress produces a deformation field of the form (see Section 46 of Truesdell & Toupin, 1960)

$$x_1 = \lambda_1 X_1 + \kappa X_2, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{4.8}$$

where $\kappa \geqslant 0$ and $\lambda_i > 0$, $i = 1, 2, 3$ are constants. From $(2.2)_{1,3}$ and (3.25), we obtain the relations:

$$\kappa^2 + \lambda_1^2 = \breve{W}_1 + \breve{W}_3\tau^2, \quad \lambda_2^2 = \breve{W}_1 + \breve{W}_3\tau^2, \quad \lambda_3^2 = \breve{W}_1, \quad \kappa\lambda_2 = \breve{W}_2\tau, \tag{4.9}$$

where $\breve{W} = \breve{W}(\tau)$ since for this stress field we have $I_1 = I_3 = 0$ and $I_2 = \tau^2$. These equations can be interpreted in the following way: given a uniform shear stress τ, from (4.9) we can find the corresponding λ_i, $i = 1, 2, 3$ and κ that are being 'produced' by such load. We see that this is a different version of the shear problem proposed by Rivlin (1948c) (see also Destrade *et al.*, 2012 for an interesting discussion of the shear problem for Green elastic bodies).

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References

Beatty, M. F. (1987) Topics in finite elasticity: hyperelasticity of rubber, elastomers and biological tissue-with examples. *Appl. Mech. Rev.*, **40**, 1699–1734.

Bulíček, M., Málek, J., Rajagopal, K. R. & Walton, J. R. (2013) Existence of solutions for the anti-plane stress for a new class of 'strain-limiting' elastic bodies (submitted for publication).

Bustamante, R. (2009) Some topics on a new class of elastic bodies. *Proc. R. Soc. Lond. A*, **465**, 1377–1392.

[4] See the comments in Section 54 of Truesdell & Noll (2004) on the original work of Rivlin (1948c) (Sections 12 and 13 therein).

Bustamante, R. & Rajagopal, K. R. (2010) A note on plain strain and stress problems for a new class of elastic bodies. *Math. Mech. Solids*, **15**, 229–238.

Bustamante, R. & Rajagopal, K. R. (2011) Solutions of some simple boundary value problems within the context of a new class of elastic materials. *Int. J. Nonlinear Mech.*, **46**, 376–386.

Bustamante, R. & Rajagopal, K. R. (2012) On the inhomogeneous shearing of a new class of elastic bodies. *Math. Mech. Solids*, **17**, 762–778.

Carroll, M. M. (2009) Must elastic materials be hyperelastic? *Math. Mech. Solids*, **14**, 369–376.

Cauchy, A. L. (1823) Recherches sur lequilibre et le mouvement interieur des corps solides ou fluids, elastiques ou non elastiques. *Bull. Soc. Philomath.*, 9–13.

Cauchy, A. L. (1828) Sur les equations qui experiments des conditions de equilibre ou le lois du mouvement interieur, d' un corps solide, elastique un non elastique. *Oeuvres*, **2**, 195–226.

Chadwick, P. (1999) *Continuum Mechanics Consice Theory and Problems*. Mineola, NY: Dover Publications, Inc.

Destrade, M., Murphy, J. G. & Saccomandi, G. (2012) Simple shear is not so simple. *Int. J. Nonlinear Mech.*, **47**, 210–214.

Green, G. (1839–1842) On the propagation of light in crystallized media. *Trans. Camb. Phil. Soc.*, **7**, 121–140 (see also (1871) Mathematical papers of the late George Green (N. M. Ferris ed.). London: MacMillan and Company, pp. 293–311).

Green, G. (1837) On the laws of reflexion and refraction of light at the common surface of two non-crystallized media. *Trans. Cambr. Phil. Soc.*, **7**, 1–24 (see also (1871) Mathematical papers of the late George Green (N. M. Ferris ed.). London: MacMillan and Company, pp. 245–269).

Kambapalli, M., Kannan, K. & Rajagopal, K. R. (2014) Circumferential stress waves in a non-linear cylindrical annulus in a new class of elastic materials. *Q. J. Mech. Appl. Math.*, **67**, 193–203.

Kannan, K., Rajagopal, K. R. & Saccomandi, G. (2014) Unsteady motions of a new class of elastic solids. *Wave Motion*, **51**, 833–843.

Kulvait, V., Malek, J. & Rajagopal, K. R. (2013) Anti-plane stress state of a plate with a V-notch for a new class of elastic solids. *Int. J. Fract.*, **179**, 59–73.

Li, T., Morris Jr, J. W., Nagasako, N., Kuramoto, S. & Chrzan, D. C. (2007) 'Ideal' engineering alloys. *Phys. Rev. Lett.*, **98**, 105503.

Ortiz, A., Bustamante, R. & Rajagopal, K. R. (2012) A numerical study of a plate with a hole for a new class of elastic bodies. *Acta Mech.* **223**, 1971–1981.

Ortiz-Bernardin, A., Bustamante, R. & Rajagopal, K. R. (2014) A numerical study of elastic bodies that are described by constitutive equations that exhibit limited strains. *Int. J. Solids Struct*., **51**, 875–885.

Rajagopal, K. R. (2003) On implicit constitutive theories. *Appl. Math.*, **48**, 279–319.

Rajagopal, K. R. (2007) The elasticity of elasticity. *Z. Angew. Math. Phys.*, **58**, 309–317.

Rajagopal, K. R. (2010) On a new class of models in elasticity. *Math. Comput. Appl.*, **15**, 506–528.

Rajagopal, K. R. (2011a) Conspectus of concepts of elasticity. *Math. Mech. Solids*, **16**, 536–562.

Rajagopal, K. R. (2011b) Non-linear elastic bodies exhibiting limiting small strain. *Math. Mech. Solids*, **16**, 122–139.

Rajagopal, K. R. (2014) Rethinking constitutive theories (in preparation).

Rajagopal, K. R. & Saravanan, U. (2011a) Spherical inflation of a class of compressible elastic bodies. *Int. J. Nonlinear Mech.*, **46**, 1167–1176.

Rajagopal, K. R. & Saravanan, U. (2011b) Extension, inflation and circumferential shearing of an annular cylinder for a class of compressible elastic bodies. *Math. Mech. Solids*. doi: 10.1177/1081286511423125.

Rajagopal, K. R. & Srinivasa, A. R. (2007) On the response of non-dissipative solids. *Proc. R. Soc. Lond. A*, **463**, 357–367.

Rajagopal, K. R. & Srinivasa, A. R. (2009) On a class of non-dissipative solids that are not hyperelastic. *Proc. R. Soc. Lond. A*, **465**, 493–500.

Rajagopal, K. R. & Walton, J. R. (2011) Modeling fracture in the context of strain-limiting theory of elasticity: a single anti-plane crack. *Int. J. Fracture*, **169**, 39–48.

Rivlin, R. S. (1948a) Large elastic deformations of isotropic materials. I. Fundamental concepts. *Phil. Trans. R. Soc. Lond. A*, **240**, 459–490 (Reprinted in: Barenblatt, G. I. & Joseph, D. D. (eds) (1997) *Collected Papers of R.S. Rivlin*, vol. I. Springer, pp. 23–54).

Rivlin, R. S. (1948b) Large elastic deformations of isotropic materials. II. Some uniqueness theorems for pure homogeneous deformations. *Phil. Trans. R. Soc. Lond. A*, **240**, 491–508 (Reprinted in: Barenblatt, G. I. & Joseph, D. D. (eds) (1997) *Collected Papers of R.S. Rivlin*, vol. I. Springer, pp. 55–72).

Rivlin, R. S. (1948c) Large elastic deformations of isotropic materials. IV. Further developments of the general theory. *Phil. Trans. R. Soc. Lond. A*, **241**, 379–397 (Reprinted in: Barenblatt, G. I. & Joseph, D. D. (eds) (1997) *Collected Papers of R.S. Rivlin*, vol. I. Springer, pp. 90–108).

Saito, T., Furuta, T., Hwang, J. H., Kuramoto, S., Nishino, K., Susuki, N., Chen, R., Yamada, A., Ito, K., Seno, Y., Nonaka, T., Ikehata, H., Nagasako, N., Iwamoto, C., Ikuhara, Y. & Sakuma, T. (2003) Multifunctional alloys obtained via a dislocation-free plastic deformation mechanism. *Science*, **300**, 464–467.

Spencer, A. J. M. (1971) Theory of invariants. *Continuum Physics* (A. C. Eringen ed.), vol. 1. New York: Academic Press, pp. 239–353.

Steigmann, D. J., Bustamante, R. & Rajagopal, K. R. (2014) A new class of elastic bodies considering large strains. (in preparation).

Talling, R. J., Dashwood, R. J., Jackson, M., Kuramoto, S. & Dye, D. (2008) Determination of C_{11}–C_{12} in Ti–36Nb–2Ta–3Zr–0.3O (xt.%) (Gum metal). *Scr. Mater.*, **59**, 669–672.

Truesdell, C. (1964) Second-order theory of wave propagation in isotropic elastic materials. *Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics* (M. Reiner & D. Abir eds). New York: MacMillan & Co, pp. 187–199.

Truesdell, C. A. & Noll, W. (2004) The non-linear field theories of mechanics (S. S. Antman ed.). Berlin: Springer.

Truesdell, C. A. & Toupin, R. (1960) The classical field theories. *Handbuch der Physik* (S. Flügge ed.), vol. III/1. Berlin: Springer, pp. 226–793.

Withey, E., Jim, M., Minor, A., Kuramoto, S., Chrzan, D. C. & Morris Jr, J. W. (2008) The deformation of ‘Gum Metal’ in nanoindentation. *Mater. Sci. Eng. A*, **493**, 26–32.

Zhang, S. Q., Li, S. J., Jia, M. T., Hao, Y. L. & Yang, R. (2009) Fatigue properties of a multifunctional titanium alloy exhibiting nonlinear elastic deformation behavior. *Scr. Mater.*, **60**, 733–736.