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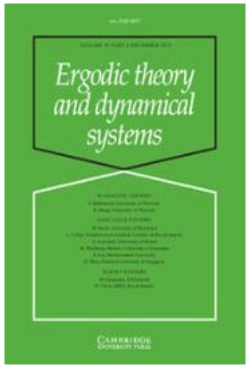

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Eigenvalues of Toeplitz minimal systems of finite topological rank

FABIEN DURAND†, ALEXANDER FRANK‡ and ALEJANDRO MAASS§

† *Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS-UMR 7352, Université de Picardie Jules Verne, 33 rue Saint Leu, 80039 Amiens cedex 1, France*
(e-mail: fabien.durand@u-picardie.fr)

‡ *Departamento de Ingeniería Matemática, Universidad de Chile, Avenida Blanco Encalada 2120, Santiago, Chile*
(e-mail: afrank@dim.uchile.cl)

§ *Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático, CNRS-UMI 2807, Universidad de Chile, Avenida Blanco Encalada 2120, Santiago, Chile*
(e-mail: amaass@dim.uchile.cl)

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Abstract. In this paper we characterize measure-theoretical eigenvalues of Toeplitz Bratteli–Vershik minimal systems of finite topological rank which are not associated to a continuous eigenfunction. Several examples are provided to illustrate the different situations that can occur.

1. *Introduction*

Seminal results by Dekking [**Dek78**] and Host [**Hos86**] state that eigenvalues of primitive substitution dynamical systems are always associated to continuous eigenfunctions. Thus the topological and measure-theoretical Kronecker factors coincide. It is natural to ask whether this phenomenon is still true for other classes of minimal Cantor systems. Most of the answers we have are negative.

Substitution dynamical systems correspond to expansive minimal Cantor systems having a periodic or stationary Bratteli–Vershik representation [**DHS99**]. A natural class to explore extending the former one are linearly recurrent minimal Cantor systems, which correspond to those systems having a Bratteli–Vershik representation with a bounded number of incidence matrices. In [**CDHM03**] and [**BDM05**] necessary and sufficient conditions based only on the combinatorial structure of the Bratteli diagrams are given for this class of systems, allowing us to differentiate continuous and measure-theoretical but non-continuous eigenvalues. The more general class of topological finite-rank minimal

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Cantor systems is explored in [**BDM10**], providing new examples and conditions to differentiate the topological and measure-theoretical Kronecker factors.

It is known that any countable subgroup of the torus $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ containing infinitely many rationals can be the set of eigenvalues of a Toeplitz system [**Iwa96**, **DL96**]. Nevertheless, in the class of finite-rank systems, Toeplitz systems exhibit a completely different behavior. Indeed, if a Toeplitz system is linearly recurrent then all its eigenvalues are associated to continuous eigenfunctions and if it has finite topological rank just a few extra non-continuous eigenvalues can appear and they are rational [**BDM10**]. So the assumption of finite topological rank restricts the possibilities of non-continuous eigenvalues to some particular ones. The purpose of this work is to study the nature of these particular non-continuous eigenvalues of finite-rank Toeplitz systems.

Our main result (Theorem 3) states a necessary and sufficient condition for $\lambda = \exp(2i\pi a/b)$, where a, b are integers with $(a, b) = 1$, to be a non-continuous eigenvalue of a finite topological rank Toeplitz system. This condition shows that non-continuous eigenvalues are very rare and impose particular local orders on the associated Bratteli–Vershik representations. In addition, even if this condition looks abstract, it is easily computable and allows us to produce concrete examples, showing particular behaviors of the group of eigenvalues in relation to the set of ergodic measures.

This paper is organized as follows. Section 2 contains the main definitions concerning eigenvalues of dynamical systems and Bratteli–Vershik representations, in particular the concept of Toeplitz minimal Cantor system of finite topological rank. In §3 we give the main result of the paper and its corollaries. In particular, we exhibit a relation between the number of ergodic measures and the number of non-continuous eigenvalues in the class of Toeplitz minimal Cantor systems of finite topological rank. Main technical lemmas used in the proofs are given in §4 and the proofs of the main result and its corollaries in §5. Finally, in §6 we provide several examples to illustrate the main result, its consequences and the fact that our condition is computable.

2. *Basic definitions*

2.1. *Dynamical systems and eigenvalues.* A *topological dynamical system*, or just dynamical system, is a compact Hausdorff space X together with a homeomorphism $T : X \to X$. We use the notation (X, T). If X is a Cantor set (i.e. X has a countable basis of closed and open sets and has no isolated points) we say that the system is Cantor. A dynamical system is *minimal* if all orbits are dense in X, or equivalently the only non-empty closed invariant set is X.

A complex number λ is a *continuous eigenvalue* of (X, T) if there exists a continuous function $f : X \to \mathbb{C}$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called a *continuous eigenfunction* (associated to λ). Let μ be a T-invariant probability measure, i.e. $T\mu = \mu$, defined on the Borel σ-algebra of X. A complex number λ is an *eigenvalue* of the dynamical system (X, T) with respect to μ if there exists $f \in L^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated to λ). If μ is ergodic, then every eigenvalue has modulus 1 and every eigenfunction has a constant modulus μ-almost surely. Of course, continuous eigenvalues are eigenvalues.

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2.2. *Bratteli–Vershik representations.* Let (X, T) be a minimal Cantor system. It can be represented by an ordered Bratteli diagram together with the Vershik transformation acting on it. For details on this theory see [**HPS92**] or [**Dur10**]. This couple is called a Bratteli–Vershik representation of the system. We give a brief outline of this construction, emphasizing the notation in this paper.

2.2.1. *Bratteli diagrams.* A Bratteli diagram is an infinite graph (V, E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \cdots$, $E = E_1 \cup E_2 \cup \cdots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and, for all $n \geq 1$, edges in E_n join vertices in V_{n-1} to vertices in V_n. It is also required that every vertex in V_n is the 'end-point' of some edge in E_n for $n \geq 1$ and an 'initial-point' of some edge in E_{n+1} for $n \geq 0$. We set $\#V_n = d_n$ for all $n \geq 1$.

Fix $n \geq 1$. We call *level n* of the diagram the subgraph consisting of the vertices in $V_{n-1} \cup V_n$ and the edges E_n between these vertices. Level 1 is called the *hat* of the Bratteli diagram. We describe the edge set E_n using a $V_{n-1} \times V_n$ incidence matrix M_n for which its (t_1, t_2) entry is the number of edges in E_n joining vertex $t_1 \in V_{n-1}$ with vertex $t_2 \in V_n$. We also set $P_n = M_2 \cdots M_n$ with the convention that $P_1 = I$, where I denotes the identity matrix. The number of paths joining $v_0 \in V_0$ and a vertex $t \in V_n$ is given by coordinate t of the *height row vector* $h_n = (h_n(t); t \in V_n) \in \mathbb{N}^{d_n}$. Notice that $h_1 = M_1$ and $h_n = h_1 P_n$.

We also consider several levels at the same time. For integers $0 \leq m < n$ we denote by $E_{m,n}$ the set of all paths in the graph joining vertices of V_m with vertices of V_n. We define matrices $P_{m,n} = M_{m+1} \ldots M_n$ with the convention that $P_{n,n} = I$ for $1 \leq m \leq n$. Clearly, coordinate $P_{m,n}(t_1, t_2)$ of matrix $P_{m,n}$ is the number of paths in $E_{m,n}$ from vertex $t_1 \in V_m$ to vertex $t_2 \in V_n$. It can be verified that $h_n = h_m P_{m,n}$.

We observe that the incidence matrices defined above correspond to the transpose of the matrices defined in the classical reference on this theory [**HPS92**]. This choice is done to simplify the reading and understanding of the paper.

2.2.2. *Ordered Bratteli diagrams and Bratteli–Vershik representations.* An *ordered* Bratteli diagram is a triple $B = (V, E, \preceq)$, where (V, E) is a Bratteli diagram and $\preceq$ is a partial ordering on E such that edges e and e' are comparable if and only if they have the same end-point. This partial ordering naturally defines maximal and minimal edges and paths. Also, the partial ordering of E induces another one on paths of $E_{m,n}$, where $0 \leq m < n$, $(e_{m+1}, \ldots, e_n) \preceq (f_{m+1}, \ldots, f_n)$ if and only if there is $m + 1 \leq i \leq n$ such that $e_j = f_j$ for $i < j \leq n$ and $e_i \preceq f_i$.

Given a strictly increasing sequence of integers $(n_k)_{k\geq 0}$ with $n_0 = 0$, one defines the *contraction* or *telescoping* of $B = (V, E, \preceq)$ with respect to $(n_k)_{k\geq 0}$ as

$$((V_{n_k})_{k\geq 0}, (E_{n_k,n_{k+1}})_{k\geq 0}, \preceq),$$

where $\preceq$ is the order induced in each set of edges $E_{n_k,n_{k+1}}$. The converse operation is called *microscoping* (see [**HPS92**] for more details).

Given an ordered Bratteli diagram $B = (V, E, \preceq)$, one defines X_B as the set of infinite paths $(x_1, x_2, \ldots)$ starting in v_0 such that for all $n \geq 1$ the end-point of $x_n \in E_n$ is the initial-point of $x_{n+1} \in E_{n+1}$. We topologize X_B by postulating a basis of open sets, namely

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the family of *cylinder sets*

$$[e_1, e_2, \ldots, e_n] = \{(x_1, x_2, \ldots) \in X_B \mid x_i = e_i, \text{ for } 1 \le i \le n\}.$$

Each $[e_1, e_2, \ldots, e_n]$ is also closed, as is easily seen, and so X_B is a compact, totally disconnected metrizable space.

When there is a unique $(x_1, x_2, \ldots) \in X_B$ such that x_n is (locally) maximal for any $n \ge 1$ and a unique $(y_1, y_2, \ldots) \in X_B$ such that y_n is (locally) minimal for any $n \ge 1$, one says that $B = (V, E, \preceq)$ is a *properly ordered* Bratteli diagram. Call these particular points $x_{\max}$ and $x_{\min}$, respectively. In this case one defines the dynamic V_B over X_B called the *Vershik map*. Let $x = (x_1, x_2, \ldots) \in X_B \setminus \{x_{\max}\}$ and let $n \ge 1$ be the smallest integer so that x_n is not a maximal edge. Let y_n be the successor of x_n for the local order and $(y_1, \ldots, y_{n-1})$ be the unique minimal path in $E_{0,n-1}$ connecting v_0 with the initial vertex of y_n. One sets $V_B(x) = (y_1, \ldots, y_{n-1}, y_n, x_{n+1}, \ldots)$ and $V_B(x_{\max}) = x_{\min}$.

The dynamical system (X_B, V_B) is minimal. It is called the *Bratteli–Vershik system* generated by $B = (V, E, \preceq)$. The dynamical system induced by any telescoping of B is topologically conjugate to (X_B, V_B). In [**HPS92**] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli–Vershik system (X_B, V_B). One says that (X_B, V_B) is a *Bratteli–Vershik representation* of (X, T). In what follows we identify (X, T) with any of its Bratteli–Vershik representations.

2.2.3. *Minimal Cantor systems of finite topological rank.* A minimal Cantor system is of finite (topological) rank if it admits a Bratteli–Vershik representation such that the number of vertices per level is uniformly bounded by some integer d. The minimum possible value of d is called the *topological rank* of the system. We observe that topological and measure-theoretical finite-rank notions are completely different. For instance, systems of topological rank one correspond to odometers, whereas in the measure-theoretical sense there are rank-one systems that are subshifts as classical Chacon's example.

To have a better understanding of the dynamics of a minimal Cantor system, and in particular to understand its group of eigenvalues, one needs to work with a 'good' Bratteli–Vershik representation. In the context of minimal Cantor systems of finite rank d we will consider representations such that:

(H1) The entries of h_1 are all equal to 1.
(H2) For every $n \ge 2$, $M_n > 0$.
(H3) For every $n \ge 2$, d_n is equal to d.
(H4) For every $n \ge 2$, all maximal edges of E_n start in the same vertex of V_{n-1}.

A Bratteli–Vershik representation of a minimal Cantor system (X, T) satisfying (H1), (H2), (H3) and (H4) will be called *proper*. In this case, to simplify notation and avoid the excessive use of indexes, we will identify V_n with $\{1, \ldots, d\}$ for all $n \ge 1$. The level n will be clear from the context.

It is not difficult to prove that a minimal Cantor system of finite rank d has a proper representation. We give a brief outline for completeness. We start from a given Bratteli–Vershik representation that we transform by telescoping. Condition (H1) follows by splitting the first level to separate all arrows in the hat and then duplicating accordingly

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the arrows of the second level. By minimality there is a telescoping of the diagram such that (H2) holds [**HPS92**]. Another telescoping to the levels where $\#V_n = d$ produces (H3). Property (H4) follows from a compactness argument and a series of telescopings: if this is not possible, then we can construct two disjoint maximal points and we get a contradiction.

A minimal Cantor system is *linearly recurrent* if it admits a proper Bratteli–Vershik representation such that the set $\{M_n \mid n \geq 1\}$ is finite. Clearly, linearly recurrent minimal Cantor systems are of finite rank (see [**DHS99**], [**Dur00**], [**Dur03**] and [**CDHM03**] for more details on this class of systems).

2.2.4. *Associated Kakutani–Rohlin partitions.* Let $B = (V, E, \preceq)$ be a properly ordered Bratteli diagram and (X, T) the associated minimal Cantor system. This diagram defines for each $n \geq 0$ a clopen *Kakutani–Rohlin* partition of X: for $n = 0$, $\mathcal{P}_0 = \{B_0(v_0)\}$, where $B_0(v_0) = X$, and for $n \geq 1$,

$$\mathcal{P}_n = \{T^{-j} B_n(t) \mid t \in V_n,\ 0 \leq j < h_n(t)\},$$

where $B_n(t) = [e_1, \ldots, e_n]$ and $(e_1, \ldots, e_n)$ is the unique maximal path from v_0 to vertex $t \in V_n$. For each $t \in V_n$ the set $\{T^{-j} B_n(t) \mid 0 \leq j < h_n(t)\}$ is called the *tower* t of $\mathcal{P}_n$. It corresponds to the set of all paths from v_0 to $t \in V_n$ (there are exactly $h_n(t)$ such paths). Denote by $\mathcal{T}_n$ the σ-algebra generated by the partition $\mathcal{P}_n$. The map $\tau_n : X \to V_n$ is given by $\tau_n(x) = t$ if x belongs to tower t of $\mathcal{P}_n$. The entrance time of x to $B_n(\tau_n(x))$ is given by $r_n(x) = \min\{j \geq 0 \mid T^j x \in B_n(\tau_n(x))\}$.

For each $x = (x_1, x_2, \ldots) \in X$ and $n \geq 0$ define the row vector $s_n(x) \in \mathbb{N}^{d_n}$, called the *suffix vector of order* n of x, by

$$s_n(x, t) = \#\{e \in E_{n+1} \mid x_{n+1} \preceq e,\ x_{n+1} \neq e,\ t \text{ is the initial vertex of } e\}$$

at each coordinate $t \in V_n$. A classical computation gives for all $n \geq 1$ (see for example [**BDM05**])

$$r_n(x) = s_0(x) + \sum_{i=1}^{n-1} \langle s_i(x), h_1 P_i \rangle = s_0(x) + \sum_{i=1}^{n-1} \langle s_i(x), h_i \rangle, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Observe that under the hypothesis (H1), i.e. $h_1 = (1, \ldots, 1)$, we have $s_0(x) = 0$.

2.2.5. *Invariant measures.* Let μ be an invariant probability measure of the system (X, T) associated to a properly ordered Bratteli diagram B, as in the previous subsection. It is determined by the values assigned to $B_n(t)$ for all $n \geq 0$ and $t \in V_n$. Define the column vector $\mu_n = (\mu_n(t)\ ;\ t \in V_n)$ with $\mu_n(t) = \mu(B_n(t))$. A simple computation allows us to prove the following useful relation:

$$\mu_m = P_{m,n} \mu_n \tag{2.2}$$

for integers $0 \leq m < n$. Also, $\mu(\tau_n = t) = h_n(t) \mu_n(t)$ for all $n \geq 1$ and $t \in V_n$.

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2.2.6. *Clean Bratteli–Vershik representations.* Let B be a proper ordered Bratteli diagram of finite rank d and (X, T) the corresponding minimal Cantor system. Recall that in this case we identify V_n with $\{1, \ldots, d\}$ for all $n \geq 1$. Then, by Theorem 3.3 in [**BKMS13**], there exist a telescoping of the diagram (which keeps the diagram proper) and $\delta > 0$ such that:

(1) For any ergodic measure μ there exists $I_\mu \subseteq \{1, \ldots, d\}$ satisfying:
 (a) $\mu(\tau_n = t) \geq \delta$ for every $t \in I_\mu$ and $n \geq 1$; and
 (b) $\lim_{n\to\infty} \mu(\tau_n = t) = 0$ for every $t \notin I_\mu$.

(2) If μ and ν are different ergodic measures then $I_\mu \cap I_\nu = \emptyset$.

When an ordered Bratteli diagram satisfies the previous properties we say it is *clean*. We remark that this is a modified version of the notion of *clean* Bratteli diagram given in [**BDM10**] that is inspired by the results of [**BKMS13**]. This property will be very relevant for formulating our main result. In [**BKMS13**], systems such that $I_\mu = \{1, \ldots, d\}$ for some ergodic measure μ are called of *exact finite rank*. Those systems are uniquely ergodic.

Let $\lambda \in \mathbb{S}^1$ be an eigenvalue of the system (X, T) associated to B for an ergodic measure μ. Let $f \in L^2(X, \mu)$ be an associated eigenfunction with $|f| = 1$. For $n \geq 1$ define $c_n : V_n \to \mathbb{R}_0^+$ and $\rho_n : V_n \to [0, 1)$ by the relation

$$\frac{1}{\mu_n(t)} \int_{B_n(t)} f \, d\mu = c_n(t)\lambda^{-\rho_n(t)} \quad \text{for } t \in V_n. \tag{2.3}$$

Notice that $0 \leq c_n(t) \leq 1$.

The sequence $(f_n \mid n \geq 1)$ of conditional expectations of f with respect to the sigma algebras $(\mathcal{T}_n \mid n \geq 1)$ generated by the Kakutani–Rohlin partitions satisfies

$$f_n(x) = \mathbb{E}(f|\mathcal{T}_n)(x) = c_n(\tau_n(x))\lambda^{-r_n(x)-\rho_n(\tau_n(x))}.$$

It can be proved that $\lambda^{-(r_n+\rho_n\circ\tau_n)}$ converges μ-a.e. (for a slightly deeper discussion we refer the reader to [**BDM05**]). Also, rephrasing a known result from [**BDM10**], we have the following lemma.

LEMMA 1. *If B is a clean Bratteli diagram and μ an ergodic measure for the associated minimal Cantor system, then*

(1) *for any* $t \in \{1, \ldots, d\}$, $\lim_{n\to\infty} \mu(\tau_n = t)(c_n(t) - 1) \to 0$*,*

(2) *for* $t \in I_\mu$, $\lim_{n\to\infty} c_n(t) \to 1$.

2.3. *Bratteli–Vershik systems of Toeplitz type.* A properly ordered Bratteli diagram $B = (V, E, \preceq)$ is of *Toeplitz type* if for all $n \geq 1$ the number of edges in E_n finishing at a fixed vertex of V_n is constant independent of the vertex. Denote this number by q_n and set $p_n = q_1 q_2 \cdots q_n$. Observe that p_n is the number of paths from v_0 to any vertex of V_n. Thus $h_n(t) = p_n$ for any $t \in V_n$. We say that $(q_n \mid n \geq 1)$ is the *characteristic sequence* of the diagram. This class was obtained in [**GJ00**] when characterizing Toeplitz subshifts.

The main object in this study are eigenvalues of minimal Cantor systems of finite rank d, having a proper Bratteli–Vershik representation of Toeplitz type. It is known that finite-rank minimal Cantor systems are either odometers or subshifts [**DM08**], so in our study we will be dealing only with Toeplitz subshifts or odometers.

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To state our main results we will need some extra notation. Fix a minimal Cantor system (X, T) with a Toeplitz type proper Bratteli–Vershik representation of rank d and characteristic sequence $(q_n \mid n \geq 1)$.

For $0 \leq m < n$ define $q_{m,n} = q_{m+1} \cdots q_n$, the number of paths in $E_{m,n}$ finishing in any fixed vertex $t \in V_n$. Clearly $q_{\ell,n} = q_{\ell,m} q_{m,n}$ if $0 \leq \ell < m < n$. Also, for $x = (x_1, x_2, \ldots) \in X$ define the integer $\overline{s}_{m,n}(x)$ as the number of paths in $E_{m,n}$ which end at $\tau_n(x)$ that are strictly bigger than $(x_{m+1}, \ldots, x_n)$ with respect to the induced partial order in $E_{m,n}$. Finally, define the set $\overline{S}_{m,n}(t_1, t_2)$ for $t_1 \in V_m$ and $t_2 \in V_n$ by

$$\overline{S}_{m,n}(t_1, t_2) = \{\overline{s}_{m,n}(x) \mid \tau_m(x) = t_1 \text{ and } \tau_n(x) = t_2\}.$$

It is not difficult to prove that the cardinality of $\overline{S}_{m,n}(t_1, t_2)$ is equal to $P_{m,n}(t_1, t_2)$, that is, the number of paths from $t_1 \in V_m$ to $t_2 \in V_n$.

If necessary, to simplify notation we will denote $\overline{S}_{n,n+1}(t_1, t_2)$ by $\overline{S}_n(t_1, t_2)$ and $\overline{s}_{n,n+1}$ by $\overline{s}_n$. Notice that $\overline{s}_n(x) = \langle s_n(x), (1, \ldots, 1)\rangle = \sum_{t \in V_n} s_n(x, t)$ for any $x \in X$.

We will need the following simple relations. For $0 \leq \ell < m < n$, $t_1 \in V_\ell$ and $x \in X$ the following equalities hold:

$$r_\ell(x) = \overline{s}_0(x) + \sum_{i=1}^{\ell-1} p_i \overline{s}_i(x), \tag{2.4}$$

$$\begin{aligned}\overline{s}_{\ell,m}(x) &= \overline{s}_\ell(x) + \sum_{i=\ell+1}^{m-1} q_{\ell+1} q_{\ell+2} \cdots q_i \overline{s}_i(x) \\ &= \frac{r_m(x) - r_\ell(x)}{p_\ell},\end{aligned} \tag{2.5}$$

$$\overline{s}_{\ell,n}(x) = \overline{s}_{\ell,m}(x) + q_{\ell,m} \overline{s}_{m,n}(x), \tag{2.6}$$

$$B_\ell(t_1) = \bigcup_{t_2 \in V_m} \bigcup_{s \in \overline{S}_{\ell,m}(t_1,t_2)} T^{-p_\ell s} B_m(t_2), \tag{2.7}$$

where the union on the right is disjoint.

3. *Eigenvalues of Toeplitz systems of finite rank*

As was mentioned in the introduction, any countable subgroup of $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ containing infinitely many rationals can be the set of eigenvalues of a Toeplitz subshift for a given invariant measure [**Iwa96**, **DL96**]. Also, $\exp(2i\pi\,\alpha) \in \mathbb{S}^1$ is a continuous eigenvalue of a minimal Cantor system with a Toeplitz type proper Bratteli–Vershik representation if and only if $\alpha = a/p_m$ for some $a \in \mathbb{Z}$ and $m \geq 1$ [**Wil84**, **JK69**]. A direct proof can be given using the particular combinatorial structure of the Bratteli–Vershik representation of a minimal Cantor system of Toeplitz type. We sketch it here. Using (2.4) and the fact that p_m divides p_n when $m \leq n$, one gets that $r_n(x)/p_m = (\bar{s}_0(x) + \sum_{i=1}^{m-1} p_i \bar{s}_i(x))/p_m \bmod \mathbb{Z}$, for all $n \geq m$. Hence, $\exp(2i\pi r_n(x)/p_m)$ converges uniformly when $n \to \infty$, which is a necessary and sufficient condition for $\exp(2i\pi/p_m)$, and thus $\exp(2i\pi\ a/p_m)$ for every $a \in \mathbb{Z}$, to be continuous eigenvalues in this context (see [**BDM05**, Proposition 12]).

In the opposite direction, using the same criterion, if $\exp(2i\pi/b)$ with $b \in \mathbb{Z}$ is a continuous eigenvalue, then $(r_{n+1}(x) - r_n(x))/b = p_n \bar{s}_n(x)/b \bmod \mathbb{Z}$ is close to 0 for any

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large enough $n \geq 1$ and uniformly in x. Taking a point x such that $\bar{s}_n(x) = 1$ allows us to conclude that $1/b = a/p_n$ for some large $n \geq 1$ and $a \in \mathbb{Z}$. More details about continuous eigenvalues of Toeplitz type Bratteli–Vershik systems can be found in [**BDM10**].

In the class of minimal Cantor systems with a Toeplitz type representation, the assumption of finite topological rank restricts the possibilities for non-continuous eigenvalues. But, importantly, all are rational. In addition, if the characteristic sequence of a proper representation is bounded (or equivalently, a proper representation gives a linearly recurrent system), then all the eigenvalues are continuous. The following theorem gives a very restrictive condition satisfied by non-continuous eigenvalues of Toeplitz systems in the finite-rank case that are not linearly recurrent.

THEOREM 2. [**BDM10**] *Let (X, T) be a minimal Cantor system with a Toeplitz type proper Bratteli–Vershik representation of rank d and characteristic sequence $(q_n \mid n \geq 1)$. Let μ be an ergodic probability measure. If $\exp(2i\pi\, a/b)$, with $(a, b) = 1$, is a non-continuous rational eigenvalue of (X, T) for μ, then $b/(b, p_n) \leq d$ for all n large enough.*

Let $\lambda = \exp(2i\pi\, a/b)$, with a, b integers such that $(a, b) = 1$, be a non-continuous rational eigenvalue as in the previous theorem. We notice that $b/(b, p_n) > 1$ for all n large enough. Indeed, if $b/(b, p_n) = 1$ for some $n \geq 1$, then $1/b = a'/p_n$ for some $a' \in \mathbb{Z}$, which by the discussion above implies that $\exp(2i\pi\, a/b)$ is a continuous eigenvalue. Also, observe that (b, p_n) is a non-decreasing sequence of integers bounded by b, so $b/(b, p_n)$ is eventually constant, say equal to $\mathbf{b}$. Since we are considering proper representations, the fact that $\mathbf{b} > 1$ implies that $(q_n \mid n \geq 1)$ tends to infinity with n. Otherwise, the system will be linearly recurrent, and thus all eigenvalues will be continuous, which implies that $b/(b, p_n) = 1$ for some $n > 1$.

We now state our main result.

THEOREM 3. *Let (X, T) be a minimal Cantor system with a Toeplitz type proper and clean Bratteli–Vershik representation of rank d and characteristic sequence $(q_n \mid n \geq 1)$. Let μ be an ergodic probability measure. Then $\lambda = \exp(2i\pi a/b)$, with a, b integers such that $(a, b) = 1$, is a non-continuous eigenvalue of (X, T) for μ if and only if*

(1) *$b/(b, p_n) = \boldsymbol{b}$ for all n large enough and some $1 < \boldsymbol{b} \leq d$, and,*

(2) *for all $t_2 \in I_\mu$,*

$$\sum_{t_1 \in V_m} \frac{|\sum_{s \in \overline{S}_{m,n}(t_1, t_2)} \lambda^{-p_m s}|}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 1,$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$.

As was mentioned in the introduction, even if this condition looks ‘heavy’ to check, in fact it is easy to verify and construct examples fulfilling it. This will be illustrated in §6. The main tool is provided by the following corollary that follows from the construction in the proof of Theorem 3.

COROLLARY 4. *Let (X, T) be a minimal Cantor system with a Toeplitz type proper and clean Bratteli–Vershik representation of rank d and characteristic sequence $(q_n \mid n \geq 1)$. Let μ be an ergodic probability measure. Let $(q_n \mid n \geq 1)$ be its characteristic sequence.*

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Then $\lambda = \exp(2i\pi a/b)$*, with* a, b *integers such that* $(a, b) = 1$*, is a non-continuous eigenvalue of* (X, T) *for* μ *if and only if up to a telescoping of the diagram we have*

(1) $p_n = p \bmod b$ *for some* $p \in \{0, \ldots, b-1\}$*, and, for all* $n \geq 2$*,*

(2) $b/(b, p_n) = \boldsymbol{b}$ *for all* n *large enough and some* $1 < \boldsymbol{b} \leq d$*,*

(3) *there exists a map* $k(\cdot,\cdot) : \{1, \ldots, d\} \times \{1, \ldots, d\} \to \{0, \ldots, \boldsymbol{b}-1\}$ *such that*

$$p \cdot k(t_1, t_3) = p \cdot k(t_1, t_2) + p \cdot k(t_2, t_3) \bmod b,$$
$$p \cdot k(t_1, t_1) = 0 \bmod b, \qquad p \cdot k(t_1, t_2) = -p \cdot k(t_2, t_1) \bmod b,$$

for all $t_1, t_2, t_3 \in I_\mu$*,*

(4) *for* μ*-almost every point* $x \in X$ *the equality* $\bar{s}_n(x) = k(\tau_n(x), \tau_{n+1}(x)) \bmod \boldsymbol{b}$ *holds for all large enough* $n \in \mathbb{N}$*.*

In what follows we provide a number of reformulations and corollaries of the main theorem. Some proofs are left to the reader since they can be easily deduced from a direct computation or Lemmas 12 and 13 provided below; others will be proved near the end of §5 after proving the main theorem.

We start with a natural reformulation of Theorem 3. It says that we can replace V_m by I_μ in the sum of statement (2) of the theorem. In other words, we only need to consider the vertices of the diagram determining the measure μ. We will need the following observation: for $t_1 \notin I_\mu$ and $t_2 \in I_\mu$ one has

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0 \tag{3.1}$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$. Indeed, since the diagram is clean, $\mu(\tau_n = t_2) \geq \delta > 0$ and $\lim_{m\to\infty} \mu(\tau_m = t_1) = 0$. These facts, together with the inequalities

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \cdot \delta \leq \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \mu(\tau_n = t_2) = \mu(\tau_m = t_1, \tau_n = t_2) \leq \mu(\tau_m = t_1),$$

allow us to deduce (3.1). Since the cardinality of $\overline{S}_{m,n}(t_1, t_2)$ is equal to $P_{m,n}(t_1, t_2)$, we also deduce that

$$\sum_{t_1 \in V_m \setminus I_\mu} \frac{|\sum_{s \in \overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}|}{q_{m,n}} \leq \sum_{t_1 \in V_m \setminus I_\mu} \frac{P_{m,n}(t_1, t_2)}{q_{m,n}}.$$

Therefore, a direct application of (3.1) in the last inequality allows us to reformulate Theorem 3 as follows.

COROLLARY 5. (Variation on Theorem 3) *The complex number* $\lambda = \exp(2i\pi a/b)$*, with* a, b *integers such that* $(a, b) = 1$*, is a non-continuous eigenvalue of* (X, T) *for* μ *if and only if*

(1) $b/(b, p_n) = \boldsymbol{b}$ *for all* n *large enough and some* $1 < \boldsymbol{b} \leq d$*, and*

(2) *for all* $t_2 \in I_\mu$*,*

$$\sum_{t_1 \in I_\mu} \frac{|\sum_{s \in \overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}|}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 1,$$

uniformly in $m, n \in \mathbb{N}$ *with* $m < n$*.*

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The following corollary is a reformulation of the main condition of Theorem 3 and the corresponding one in Corollary 5. It follows almost directly by combining Lemmas 12 and 13 in the next section, so its proof is left to the reader.

COROLLARY 6. *The main condition in Theorem 3 (Corollary 5) is equivalent to: for all* $t_2 \in I_\mu$ *and* $m \geq 1$ *there exists a sequence of partitions* $(\mathcal{H}_{m,n,t_2} \mid m < n)$ *of* V_m *(of* I_μ*) with* $\#\mathcal{H}_{m,n,t_2} = \boldsymbol{b}$ *such that*

$$\sum_{t_1 \in A} \frac{|\sum_{s \in \overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}|}{q_{m,n}} \xrightarrow[m,n\to\infty]{} \frac{1}{\boldsymbol{b}},$$

uniformly in m, $n \in \mathbb{N}$ *with* $m < n$ *for any* $A \in \mathcal{H}_{m,n,t_2}$.

This formulation gives information about the possible local orders that accept a Bratteli–Vershik representation to have non-continuous eigenvalues. Part (3) of Lemma 12 states that the main condition of Theorem 3 (or its equivalent formulations) implies that the local order of most of the arrows from a vertex in an atom $A \in \mathcal{H}_{m,n,t_2}$ to $t_2 \in I_\mu$ at level n must be congruent modulo $\mathbf{b}$. This condition is one of the main tools for exploring non-continuous rational eigenvalues of Toeplitz systems.

Another interesting fact is that we can relate non-continuous eigenvalues with the number of ergodic invariant measures of a Toeplitz system. Let (X, T) be a minimal Cantor system and μ an ergodic measure as in Theorem 3. Define

$$\mathbf{B}_\mu = \{\lim_{m\to\infty} b/(b, p_m) \mid b \in \mathbb{N},\ \exp(2i\pi/b) \text{ is a non-continuous eigenvalue for } \mu\}$$

and endow it with the divisibility (partial) order. Recall that $\lim_{m\to\infty} b/(b, p_m)$ is equal to $\mathbf{b} = b/(b, p_n)$ for a large $n \in \mathbb{N}$. Denote by $\mathcal{M}_{\mathrm{erg}}(X, T)$ the set of ergodic measures of (X, T) and consider the set $\mathcal{M}$ defined by

$$\mathcal{M} = \{\mu \in \mathcal{M}_{\mathrm{erg}}(X, T) \mid \mathbf{B}_\mu \neq \emptyset\}.$$

COROLLARY 7. *The following properties hold:*

(1) *For any* $\mu \in \mathcal{M}$ *and* $\boldsymbol{b} \in \boldsymbol{B}_\mu$, $\boldsymbol{b} \leq \#I_\mu$.

(2) *For any* $\mu \in \mathcal{M}$, $\boldsymbol{B}_\mu$ *has a unique divisibility-maximal element* $\boldsymbol{b}_\mu$.

(3) $\sum_{\mu\in\mathcal{M}} \boldsymbol{b}_\mu \leq d$.

(4) $\#\mathcal{M} \leq \#\mathcal{M}_{erg}(X, T) \leq d - \sum_{\mu\in\mathcal{M}}(\boldsymbol{b}_\mu - 1)$.

The proof of this corollary will be given at the end of §5.

Fix an ergodic measure μ. To understand better the last corollary let us suppose that the p_n are powers of the same prime number. In this case, for all integers b such that $\lambda = \exp(2i\pi/b)$ is a non-continuous eigenvalue for μ one has $(b, p_n) = 1$ and parts (1) and (2) of last corollary tell us that there is a unique $b = \mathbf{b}_\mu \leq \#I_\mu \leq d$ which is maximal in $\mathbf{B}_\mu$. All other non-continuous eigenvalues for μ are powers of λ. If $\mathbf{B}_\mu$ is empty, no non-continuous eigenvalues exist for μ. Notice that property (1) implies that we need at least $\mathbf{b}_\mu$ vertices to have the non-continuous eigenvalue λ. Since $I_\mu \cap I_\nu = \emptyset$ for different ergodic measures $\mathbf{b}_\nu \leq d - \#I_\mu \leq d - \mathbf{b}_\mu$. We will see in some examples of §6 that these inequalities can be strict.

In the particular case where $\mathbf{b}_\mu = d$ for some ergodic measure μ we get the following corollary.

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COROLLARY 8. *Consider $\lambda = \exp(2i\pi a/b)$, with a, b integers such that $(a, b) = 1$ and $b/(b, p_n) = d$ for all n large enough. Then λ is a non-continuous eigenvalue of (X, T) for the invariant measure μ if and only if for all $t_1, t_2 \in \{1, \ldots, d\}$,*

$$\frac{|\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}|}{q_{m,n}} \xrightarrow[m,n\to\infty]{} \frac{1}{d} \tag{3.2}$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$. If λ is an eigenvalue, then:

(1) *the system (X, T) is uniquely ergodic and μ is the unique invariant measure;*

(2) *for all $t \in \{1, \ldots, d\}$, $\lim_{n\to\infty} \mu(\tau_n = t) = 1/d$.*

Condition (3.2) and statement (1) follow almost directly from Corollaries 6 and 7. Nevertheless, we provide a complete proof of the corollary at the end of §5.

A result analogous to Corollary 8 can be obtained when the system is uniquely ergodic and $b/(b, p_n) = \#I_\mu$ for all n large enough. The statement is obtained by replacing d by $\#I_\mu$ and the set $\{1, \ldots, d\}$ by I_μ in the last corollary.

4. *Main technical lemmas*

In this section we will provide the main ingredients we need to prove Theorem 3 and its corollaries.

4.1. *A geometric lemma.*

The next lemma can be stated in a much more general situation and its proof follows from general facts of convex analysis. Nevertheless, since we consider a particular case, we provide a simple self-contained proof.

LEMMA 9. *Let N be a positive integer. Then there exists a constant C such that for any convex combination $w = \sum_{j=0}^{N-1} \alpha_j \xi^j$ of the Nth roots of unity $1, \xi, \ldots, \xi^{N-1}$ satisfying $1 - \varepsilon < |w| \leq 1$ for some $\varepsilon > 0$ one has*

$$1 - C\varepsilon < \alpha_i \leq 1$$

for some $0 \leq i \leq N - 1$.

Proof. A proof is given only in the case where $|w| \neq 1$. Write w as

$$w = \alpha_i \xi^i + \beta\zeta,$$

where $\alpha_i \geq 1/N$ and $\alpha_i + \beta = 1$ (note that ζ belongs to the convex hull of the Nth roots of unity different from ξ^i). The function $F(z) = \alpha_i \xi^i + \beta z$ has maximal absolute value at $z \in \{\xi^{i-1}, \xi^{i+1}\}$ when restricted to the convex hull of the Nth roots of unity different from ξ^i. Hence

$$\begin{aligned} 1 - \varepsilon < |w| \quad &(= |F(\zeta)|) \\ &\leq |F(\xi^{i+1})| \\ &= |1 + \beta(\xi - 1)| \\ &= \sqrt{1 - 2\beta(1-\beta)(1 - \cos 2\pi/N)} \\ &\leq 1 - \beta(1-\beta)(1 - \cos 2\pi/N) \\ &\leq 1 - \beta\bigg(\frac{1 - \cos(2\pi/N)}{N}\bigg) \end{aligned}$$

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and

$$\alpha_\ell > 1 - \left(\frac{N}{1-\cos(2\pi/N)}\right)\varepsilon. \qquad \square$$

4.2. *Special telescoping of a Bratteli–Vershik system.* At some point of the proof of Theorem 3 we will need to telescope an ordered Bratteli diagram in the following particular way.

LEMMA 10. *Let $B = (V, E, \preceq)$ be an ordered Bratteli diagram such that $\#V_n = d$ for all $n \geq 1$ and identify V_n with $\{1, \ldots, d\}$. For all $1 \leq m < n$ and $t \in \{1, \ldots, d\}$ consider $(\mathcal{G}_{m,n,t}, \leq_{m,n,t})$, where $\mathcal{G}_{m,n,t}$ is a partition of V_m and $\leq_{m,n,t}$ is a total ordering on the atoms of $\mathcal{G}_{m,n,t}$. Then there exists a strictly increasing sequence $(n_k)_{k\geq 0}$ in $\mathbb{N}$ such that for all $k_0 \geq 0$, $k > k_0$ and $t \in \{1, \ldots, d\}$ we have*

$$(\mathcal{G}_{n_{k_0},n_k,t}, \leq_{n_{k_0},n_k,t}) = (\mathcal{G}_{n_{k_0},n_{k+1},t}, \leq_{n_{k_0},n_{k+1},t}).$$

Proof. It suffices to remark that there are finitely many such structures on $\{1, \ldots, d\}$ (partitions endowed with total orderings). Then one proceeds by induction using the pigeonhole principle.

Let us give some details. Take $n_0 = 1$. By the pigeonhole principle, there exists a strictly increasing sequence $(n_k^{(0)})_{k\geq 0}$, with $n_0^{(0)} > n_0$, such that for all $k \geq 0$ and $t \in \{1, \ldots, d\}$ we have

$$(\mathcal{G}_{n_0,n_k^{(0)},t}, \leq_{n_0,n_k^{(0)},t}) = (\mathcal{G}_{n_0,n_{k+1}^{(0)},t}, \leq_{n_0,n_{k+1}^{(0)},t}).$$

Now, let $n_1 = n_0^{(0)}$. Using the same argument, there exists a strictly increasing subsequence $(n_k^{(1)})_{k\geq 0}$ of $(n_k^{(0)})_{k\geq 0}$, with $n_0^{(1)} > n_0^{(0)}$, such that for all $k \geq 0$ and $t \in \{1, \ldots, d\}$ we have $(\mathcal{G}_{n_1,n_k^{(1)},t}, \leq_{n_1,n_k^{(1)},t}) = (\mathcal{G}_{n_1,n_{k+1}^{(1)},t}, \leq_{n_1,n_{k+1}^{(1)},t})$. Observe that we also have $(\mathcal{G}_{n_0,n_k^{(1)},t}, \leq_{n_0,n_k^{(1)},t}) = (\mathcal{G}_{n_0,n_{k+1}^{(1)},t}, \leq_{n_0,n_{k+1}^{(1)},t})$ for all $k \geq 0$ and $t \in \{1, \ldots, d\}$ by construction. Proceeding in this way, we obtain the desired sequence $(n_k)_{k\geq 0}$. $\square$

4.3. *Uniform lower bound for consecutive towers in I_μ.*

LEMMA 11. *Let (X, T) be a minimal Cantor system with a Toeplitz type proper and clean Bratteli–Vershik representation of rank d and μ be an ergodic probability measure. Let $(q_n \mid n \geq 1)$ be its characteristic sequence. For all $m \geq 1$, there exists $n_0 > m$ such that for all $n \geq n_0$ and $t_1, t_2 \in I_\mu$,*

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \geq \frac{\delta}{3},$$

where $\delta > 0$ is such that $\mu(\tau_n = t) \geq \delta$ for any $t \in I_\mu$ and $n \in \mathbb{N}$ (coming from the cleanliness property of the diagram).

Proof. Fix $m \geq 1$ and $0 < \epsilon < \delta^2/3$. From Egorov's theorem and the ergodic theorem, there exists a measurable subset A_ϵ with $\mu(A_\epsilon) \geq 1 - \epsilon$ and a positive integer M_0 such that for all $x \in A_\epsilon$ and $M \geq M_0$ we have

$$\left|\frac{1}{M}\sum_{k=0}^{M-1} 1_{\{\tau_m = t_1\}}(T^k x) - \mu(\tau_m = t_1)\right| < \epsilon. \tag{4.1}$$

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Let $n > m$ be such that $p_n \geq M_0$ (recall that p_n is the number of paths from v_0 to any vertex of V_n). There exists $x \in A_\epsilon \cap T^{-p_n - j + 1} B_n(t_2)$ for some $0 \leq j \leq \lfloor \epsilon p_n / \delta \rfloor < p_n$. Indeed,

$$\mu\left(\bigcup_{j=0}^{\lfloor \epsilon p_n/\delta \rfloor} T^{-(p_n+j-1)} B_n(t_2) \right) = \left(\left\lfloor \frac{\epsilon p_n}{\delta} \right\rfloor + 1 \right) \mu(B_n(t_2)) > \frac{\epsilon}{\delta} \mu(\tau_n = t_2) \geq \epsilon,$$

since $\mu(\tau_n = t_2) = p_n \mu(B_n(t_2))$ and $t_2 \in I_\mu$. Hence, $\bigcup_{j=0}^{\lfloor \epsilon p_n/\delta \rfloor} T^{-(p_n+j-1)} B_n(t_2)$ must intersect A_ϵ. Notice that the iterates $T^j x, \ldots, T^{j+p_n-1} x$ cross completely tower $t_2 \in V_n$, from the lowest to the highest level. So those iterates enter tower $t_1 \in V_m$ exactly $P_{m,n}(t_1, t_2) p_m$ times.

Then, since $t_1 \in I_\mu$, $p_n + j \geq M_0$ and $x \in A_\epsilon$, we can use (4.1) to get

$$\begin{aligned}
\delta - \epsilon \leq \mu(\tau_m = t_1) - \epsilon &\leq \frac{1}{p_n + j} \sum_{k=0}^{p_n+j-1} 1_{\{\tau_m = t_1\}}(T^k x) \\
&\leq \frac{j}{p_n + j} + \frac{1}{p_n + j} \sum_{k=0}^{p_n - 1} 1_{\{\tau_m = t_1\}}(T^k(T^j x)) \\
&\leq \frac{\epsilon}{\delta} + \frac{P_{m,n}(t_1, t_2) p_m}{p_n + j} \leq \frac{\epsilon}{\delta} + \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \leq \frac{\delta}{3} + \frac{P_{m,n}(t_1, t_2)}{q_{m,n}},
\end{aligned}$$

which ends the proof. □

4.4. *Equivalent conditions for Theorem 3.* We follow the same notation as in Theorem 3: $\lambda = \exp(2i\pi a/b)$, with $(a, b) = 1$, and $\mathbf{b}$ is the limit in n of $b/(b, p_n)$, which is attained from some large $n \in \mathbb{N}$. In the sequel, equality modulo $\mathbf{b}$ and b will be written $\equiv_{\mathbf{b}}$ and $\equiv_b$ respectively.

To make the text lighter, we need to introduce some extra notation. For $t_1, t_2 \in \{1, \ldots, d\}$, $k \in \{0, \ldots, \mathbf{b} - 1\}$ and integers $1 \leq m < n$, set

$$\sigma_{m,n}(t_1, t_2) = \sum_{s \in \overline{S}_{m,n}(t_1, t_2)} \lambda^{-p_m s}, \tag{4.2}$$

$$\boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2) = \#\{s \in \overline{S}_{m,n}(t_1, t_2) \mid s \equiv_{\mathbf{b}} k\}. \tag{4.3}$$

Notice that for $s, s' \in \overline{S}_{m,n}(t_1, t_2)$, $\lambda^{-p_m s} = \lambda^{-p_m s'}$ if and only if $s \equiv_{\mathbf{b}} s'$. Then

$$\sigma_{m,n}(t_1, t_2) = \sum_{k=0}^{\mathbf{b}-1} \lambda^{-p_m k} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2), \tag{4.4}$$

$$P_{m,n}(t_1, t_2) = \sum_{k=0}^{\mathbf{b}-1} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2), \tag{4.5}$$

$$|\sigma_{m,n}(t_1, t_2)| \leq P_{m,n}(t_1, t_2), \tag{4.6}$$

$$q_{m,n} = \sum_{k=0}^{\mathbf{b}-1} \sum_{t_1 \in V_m} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2), \tag{4.7}$$

$$\sum_{t_1 \in V_m} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2) = \left\lfloor \frac{q_{m,n}}{\mathbf{b}} \right\rfloor \text{ or } \left\lfloor \frac{q_{m,n}}{\mathbf{b}} \right\rfloor + 1. \tag{4.8}$$

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LEMMA 12. *For any $t_2 \in \{1, \ldots, d\}$ the following conditions are equivalent:*

(1) $\sum_{t_1 \in V_m} |\sigma_{m,n}(t_1, t_2)|/q_{m,n} \xrightarrow[m,n\to\infty]{} 1$ *uniformly in $m, n \in \mathbb{N}$ with $m < n$ (this is condition* (2) *of Theorem 3 stated for any t_2).*

(2) *For all $t_1 \in \{1, \ldots, d\}$,*

$$\frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} - \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$.

(3) *For all integers $1 \le m < n$ and $t_1 \in \{1, \ldots, d\}$, there exists $k_{m,n}(t_1, t_2)$ in $\{0, \ldots, \boldsymbol{b} - 1\}$ such that*

$$\frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{q_{m,n}} - \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$.

Proof. (1)$\Longrightarrow$(2). We proceed by contradiction. Suppose that there exists $\overline{t_1} \in \{1, \ldots, d\}$ such that for infinitely many positive integers m, n with $m < n$,

$$\frac{P_{m,n}(\overline{t_1}, t_2)}{q_{m,n}} - \frac{|\sigma_{m,n}(\bar{t}_1, t_2)|}{q_{m,n}} \ge 2\varepsilon > 0, \tag{4.9}$$

where ε is a positive real.

From (1) we have that for any large enough positive integers m, n with $m < n$,

$$1 - \varepsilon < \sum_{t_1 \in V_m} \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} < 1 + \varepsilon. \tag{4.10}$$

Consider a pair of large integers m, n with $m < n$ satisfying (4.9). Then, from (4.6), (4.9) and (4.10), we get

$$1 = \sum_{t_1 \in V_m} \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \ge 2\varepsilon + \sum_{t_1 \in V_m} \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \ge 1 + \varepsilon,$$

which is impossible. Condition (2) follows.

(2)$\Longrightarrow$(3). Take $\varepsilon > 0$. By hypothesis and (4.6), there exists a positive integer N such that for all $n > m > N$ and $t_1 \in \{1, \ldots, d\}$,

$$0 \le \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} < \varepsilon.$$

Alternatively, the last inequality can be written as

$$1 - \frac{\varepsilon q_{m,n}}{P_{m,n}(t_1, t_2)} < \left| \sum_{k=0}^{\mathbf{b}-1} \frac{\boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2)}{P_{m,n}(t_1, t_2)} \lambda^{-kp_m} \right| \le 1.$$

Notice that $\{1, \lambda^{-p_m}, \ldots, \lambda^{-(\mathbf{b}-1)p_m}\}$ is the complete set of $\mathbf{b}$th roots of unity if m is large enough, and we have a convex combination of them. Applying Lemma 9, we deduce that there exists $k_{m,n}(t_1, t_2) \in \{0, \ldots, \mathbf{b} - 1\}$ such that

$$1 - \frac{C\varepsilon q_{m,n}}{P_{m,n}(t_1, t_2)} < \frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{P_{m,n}(t_1, t_2)} \le 1,$$

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or equivalently,

$$0 \le \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{q_{m,n}} < C\varepsilon.$$

The sequence constructed depends on ε. Taking a sequence $(\varepsilon_\ell \mid \ell \in \mathbb{N})$ tending to zero and using a diagonal process, one obtains the desired sequence

$$(k_{m,n}(t_1, t_2) \mid m, n \in \mathbb{N}, m < n).$$

(3)$\Longrightarrow$(1). Fix $\varepsilon > 0$. There exists a positive integer N large enough such that for any $n > m > N$ and $t_1 \in \{1, \ldots, d\}$,

$$0 \le \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{q_{m,n}} = \sum_{\substack{k=0 \\ k \neq k_{m,n}(t_1,t_2)}}^{\mathbf{b}-1} \frac{\boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2)}{q_{m,n}} < \varepsilon. \tag{4.11}$$

So, using relations (4.4) and (4.11), we deduce that

$$\begin{aligned} |\sigma_{m,n}(t_1, t_2)| &= \left| \boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)\lambda^{-p_m k_{m,n}(t_1,t_2)} + \sum_{\substack{k=0 \\ k \neq k_{m,n}(t_1,t_2)}}^{\mathbf{b}-1} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2)\lambda^{-p_m k} \right| \\ &\ge \boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2) - \sum_{\substack{k=0 \\ k \neq k_{m,n}(t_1,t_2)}}^{\mathbf{b}-1} \boldsymbol{\sigma}_{m,n}^{(k)}(t_1, t_2) \\ &\ge \boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2) - \epsilon q_{m,n}. \end{aligned}$$

From this inequality, (4.6) and (4.11), we get

$$\frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{q_{m,n}} - \varepsilon \le \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \le \frac{\boldsymbol{\sigma}_{m,n}^{(k_{m,n}(t_1,t_2))}(t_1, t_2)}{q_{m,n}} + \varepsilon.$$

Finally, from (4.5) and (4.11) applied to these last inequalities we deduce that

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - 2\varepsilon \le \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \le \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} + \varepsilon.$$

Adding over $t_1 \in V_m$ we get

$$\left| \sum_{t_1 \in V_m} \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} - 1 \right| \le 2d\varepsilon.$$

Property (1) follows since this inequality is valid for any $n > m > N$ given $\varepsilon > 0$. □

Notice that the sequence $(k_{m,n}(t_1, t_2) \mid m, n \in \mathbb{N}, m < n)$ in statement (3) of Lemma 12 is not necessarily uniquely defined.

4.5. *Constructing a partition from Theorem 3.* The next lemma allows us to construct several partitions of the vertices in a level of the Bratteli diagram such that the local order of most of the arrows starting in a vertex of an atom of such partition ending in the same vertex of a further level must be congruent modulo **b**. This is crucial in obtaining Corollary 6.

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LEMMA 13. *For $t_2 \in \{1, \dots, d\}$ assume that any of the equivalent conditions in Lemma 12 holds. For each $t_1 \in \{1, \dots, d\}$ fix a sequence $(k_{m,n}(t_1, t_2) \mid m, n \in \mathbb{N}, m < n)$ as in statement (3) of Lemma 12. Consider the map*

$$\Psi_{m,n,t_2} : \{1, \dots, d\} \to \{0, \dots, \boldsymbol{b} - 1\}$$
$$t_1 \mapsto k_{m,n}(t_1, t_2).$$

Then,

(1) *for any large enough $m, n \in \mathbb{N}$ with $m < n$, Ψ_{m,n,t_2} is onto,*

(2) *for any $k \in \{0, \dots, \boldsymbol{b} - 1\}$,*

$$\sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} \frac{1}{\boldsymbol{b}}$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$.

Proof. (1) Fix $0 < \varepsilon < 1/(d + 1)^2$. For any $t_1 \in \{1, \dots, d\}$ we have

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{\boldsymbol{\sigma}^{(k_{m,n}(t_1,t_2))}_{m,n}(t_1, t_2)}{q_{m,n}} = \sum_{\substack{k=1 \\ k\neq k_{m,n}(t_1,t_2)}}^{\mathbf{b}-1} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}}.$$

Then, since by hypothesis t_2 satisfies condition (3) of Lemma 12, for any $m, n \in \mathbb{N}$ with $m < n$ large enough $\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)/q_{m,n} < \varepsilon$ for all $t_1 \in \{1, \dots, d\}$ and $k \neq k_{m,n}(t_1, t_2)$. Since q_n goes to infinity with n, considering larger values of m, n we can also assume that $1/q_{m,n} < \varepsilon$.

If assertion (1) of the lemma is not true, then for some large m, n with $m < n$, there is $k \in \{0, \dots, \mathbf{b} - 1\} \setminus \operatorname{Im} \Psi_{m,n,t_2}$. Hence, by the previous considerations and equality (4.8),

$$\frac{1}{\mathbf{b}} - \varepsilon < \sum_{t_1 \in V_m} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} < d\varepsilon,$$

which, by the choice of ε, contradicts the fact that $\mathbf{b} \leq d$.

(2) Fix $\varepsilon > 0$. By part (1), there exists $N \in \mathbb{N}$ such that for all $n > m > N$, Ψ_{m,n,t_2} is surjective. Taking a larger N if necessary we can also assume that $1/q_{m,n}$ and $\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)/q_{m,n}$ are less than ε for all $t_1 \in \{1, \dots, d\}$ and $k \neq k_{m,n}(t_1, t_2)$.

Let k be an element in $\{0, \dots, \mathbf{b} - 1\}$. By (4.8) the following inequalities hold for all $n > m > N$:

$$\frac{1}{\mathbf{b}} - \varepsilon < \sum_{t_1 \in V_m} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \leq \frac{1}{\mathbf{b}} + \varepsilon,$$

$$\frac{1}{\mathbf{b}} - \varepsilon < \sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} + \sum_{t_1 \notin \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \leq \frac{1}{\mathbf{b}} + \varepsilon,$$

$$\frac{1}{\mathbf{b}} - d\varepsilon < \sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \leq \frac{1}{\mathbf{b}} + \varepsilon.$$

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We have proved that

$$\sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} \frac{1}{\mathbf{b}}$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$, which ends the proof. □

From the proof of the previous lemma one can deduce that the values of $k_{m,n}(t_1, t_2)$ are ultimately uniquely defined if $\liminf_{m,n\to\infty, m<n} (P_{m,n}(t_1, t_2)/q_{m,n}) > 0$.

5. *Proof of Theorem 3*

Throughout this section (X, T), μ and $(q_n; n \geq 1)$ are set as in Theorem 3.

5.1. *Proof that the technical condition is necessary.* It is enough to consider a non-continuous eigenvalue $\lambda = \exp(2i\pi/b)$ of (X, T) for the ergodic measure μ. Let $f \in L^2(X, \mu)$ be an associated eigenfunction with $|f| = 1$.

Proof. [Proof that the technical condition is necessary] Recall that $b/(b, p_n)$ is equal to $\mathbf{b}$ for all n large enough. We know from Theorem 2 that $2 \leq \mathbf{b} \leq d$. Otherwise, if $\mathbf{b} = 1$ the system would be linearly recurrent and λ a continuous eigenvalue, as was discussed before stating Theorem 3. Thus we only need to prove statement (2) of the theorem.

It is enough to prove that for all $t_1 \in \{1, \ldots, d\}$ and $t_2 \in I_\mu$,

$$\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{|\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}|}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0 \tag{5.1}$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$. From here, we finish the proof by adding over $t_1 \in \{1, \ldots, d\}$.

First, we integrate f over $B_m(t_1)$ and use the decomposition given in (2.7):

$$\begin{aligned}
\int_{B_m(t_1)} f\,d\mu &= \sum_{t_2\in V_n} \sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \int_{T^{-p_m s} B_n(t_2)} f\,d\mu \\
&= \sum_{t_2\in V_n} \sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \int_{B_n(t_2)} f \circ T^{-p_m s}\,d\mu \\
&= \sum_{t_2\in V_n} \bigg(\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s} \bigg) \int_{B_n(t_2)} f\,d\mu.
\end{aligned}$$

But, from (2.3), we have that

$$\int_{B_m(t_1)} f\,d\mu = \mu_m(t_1)c_m(t_1)\lambda^{-\rho_m(t_1)}, \quad \int_{B_n(t_2)} f\,d\mu = \mu_n(t_2)c_n(t_2)\lambda^{-\rho_n(t_2)}.$$

Thus, substituting the corresponding expressions in the previous deduction we get

$$\mu_m(t_1)c_m(t_1)\lambda^{-\rho_m(t_1)} = \sum_{t_2\in V_n} \bigg(\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s} \bigg) \mu_n(t_2)c_n(t_2)\lambda^{-\rho_n(t_2)},$$

$$\mu(\tau_m = t_1)c_m(t_1)\lambda^{-\rho_m(t_1)} = \sum_{t_2\in V_n} \frac{\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s}}{q_{m,n}} \mu(\tau_n = t_2)c_n(t_2)\lambda^{-\rho_n(t_2)},$$

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where in the last equality we have used the relations $\mu(\tau_m = t_1) = p_m \mu_m(t_1)$, $\mu(\tau_n = t_2) = p_n \mu_n(t_2)$ and $p_n/p_m = q_{m,n}$. Using (4.2), we get the expression

$$\mu(\tau_m = t_1) c_m(t_1) \lambda^{-\rho_m(t_1)} = \sum_{t_2 \in V_n} \frac{\sigma_{m,n}(t_1, t_2)}{q_{m,n}} \mu(\tau_n = t_2) c_n(t_2) \lambda^{-\rho_n(t_2)}. \tag{5.2}$$

From (2.2) we have that for $0 < m < n$ and $t_1 \in \{1, \ldots, d\}$,

$$\mu(\tau_m = t_1) = \sum_{t_2 \in V_n} \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \mu(\tau_n = t_2). \tag{5.3}$$

Then, taking absolute value in (5.2) and using (4.6) and (5.3), we deduce that

$$\begin{aligned} \mu(\tau_m = t_1) c_m(t_1) &\le \sum_{t_2 \in V_n} \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \mu(\tau_n = t_2) c_n(t_2) \\ &\le \sum_{t_2 \in V_n} \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \mu(\tau_n = t_2) \\ &\le \sum_{t_2 \in V_n} \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} \mu(\tau_n = t_2) \\ &= \mu(\tau_m = t_1). \end{aligned}$$

Notice that in the second inequality we have used the fact that $c_n(t_2) \le 1$ for any $n \in \mathbb{N}$ and $t_2 \in V_n$.

Finally, applying Lemma 1 in the preceding inequalities, we deduce that

$$\sum_{t_2 \in V_n} \left(\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{|\sigma_{m,n}(t_1, t_2)|}{q_{m,n}} \right) \mu(\tau_n = t_2) \xrightarrow[m,n \to \infty]{} 0$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$. If $t_2 \in I_\mu$, then $\mu(\tau_n = t_2) > \delta$ (recall that δ comes from the cleanliness property of the diagram). Therefore, the desired convergence in (5.1) holds. □

5.2. *Proof that the technical condition is sufficient.* For this proof we will need the following result from [**BDM05**] which we adapt to the language of Bratteli–Vershik systems.

THEOREM 14. *Let (X, T) be a minimal Cantor system given by a proper Bratteli–Vershik system. A complex number λ is an eigenvalue of (X, T) with respect to the ergodic probability measure μ if and only if there exists a sequence of real functions $\rho_n : V_n \to \mathbb{R}$, $n \in \mathbb{N}$, such that*

$$\lambda^{r_n(x) + \rho_n(\tau_n(x))} \text{ converges} \tag{5.4}$$

for μ-almost every $x \in X$ when n tends to infinity.

We recall that $r_n(x) = \bar{s}_0(x) + \sum_{i=1}^{n-1} p_i \bar{s}_i(x)$ is the entrance time of x to $B_n(\tau_n(x))$ (see (2.4)).

Proof that the technical condition is sufficient. We notice that condition (2) in Theorem 3 is stable under telescoping, so we will telescope our Bratteli–Vershik representation freely.

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5.2.1. *Constructing a partition.* Take $t_2 \in I_\mu$ and $m, n \in \mathbb{N}$ with $m < n$ enough large. Notice that our hypothesis is condition (1) in Lemma 12 with $t_2 \in I_\mu$. Thus, for any $t_1 \in \{1, \ldots, d\}$ there exist $k_{m,n}(t_1, t_2)$ given by condition (3) of Lemma 12 and the map $\Psi_{m,n,t_2} : \{1, \ldots, d\} \to \{0, \ldots, \mathbf{b} - 1\}$ given by Lemma 13. Define

$$\mathcal{H}_{m,n,t_2} = \{A^{(0)}_{m,n,t_2}, A^{(1)}_{m,n,t_2}, \ldots, A^{(\mathbf{b}-1)}_{m,n,t_2}\},$$

where $A^{(k)}_{m,n,t_2} = \Psi^{-1}_{m,n,t_2}(k)$ for $k \in \{0, \ldots, \mathbf{b} - 1\}$.

From Lemma 10 we can suppose after telescoping that $\mathcal{H}_{m,n,t_2} = \mathcal{H}_{m,m+1,t_2}$ for all $m, n \in \mathbb{N}$ with $m < n$ and $t_2 \in I_\mu$. Thus we set $\mathcal{H}_{m,n,t_2} = \mathcal{H}_{m,t_2}$ and $A^{(k)}_{m,n,t_2} = A^{(k)}_{m,t_2}$ for $k \in \{0, \ldots, \mathbf{b} - 1\}$. In addition, after another telescoping, we can suppose that $A^{(k)}_{m,t_2} = A^{(k)}_{m',t_2}$ for all $m, m' \geq 1$ and $k \in \{0, \ldots, \mathbf{b} - 1\}$. We set $\mathcal{H}_{t_2} = \mathcal{H}_{m,t_2}$, $A^{(k)}_{t_2} = A^{(k)}_{m,t_2}$ and thus $k(t_1, t_2) = k_{m,n}(t_1, t_2)$ for any $m, n \in \mathbb{N}$ with $m < n$, $t_1 \in \{1, \ldots, d\}$ and $t_2 \in I_\mu$.

5.2.2. *Constructing a good set of full measure.* For $m, n \in \mathbb{N}$ with $m < n$ consider the set

$$\mathcal{C}_{m,n} = \{\tau_n \in I_\mu,\ \bar{s}_{m,n} \not\equiv_{\mathbf{b}} k(\tau_m, \tau_n)\} \cup \{\tau_n \notin I_\mu\}.$$

Recall that the map $k(t_1, t_2)$ has been defined only for $t_2 \in I_\mu$. Let us compute the measure of $\mathcal{C}_{m,n}$:

$$\begin{aligned}
\mu(\mathcal{C}_{m,n}) &= \sum_{t_2 \in I_\mu} \sum_{t_1 \in V_m} \mu(\tau_m = t_1, \tau_n = t_2, \bar{s}_{m,n} \not\equiv_{\mathbf{b}} k(t_1, t_2)) + \mu(\tau_n \notin I_\mu) \\
&= \sum_{t_2 \in I_\mu} \sum_{t_1 \in V_m} (P_{m,n}(t_1, t_2) - \boldsymbol{\sigma}^{(k(t_1,t_2))}_{m,n}(t_1, t_2)) p_m\, \mu_n(t_2) + \mu(\tau_n \notin I_\mu) \\
&= \sum_{t_2 \in I_\mu} \left(\sum_{t_1 \in V_m} \frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{\boldsymbol{\sigma}^{(k(t_1,t_2))}_{m,n}(t_1, t_2)}{q_{m,n}} \right) \mu(\tau_n = t_2) + \mu(\tau_n \notin I_\mu),
\end{aligned}$$

where we have used $q_{m,n}\, p_m = p_n$ and $\mu(\tau_n = t_2) = p_n\, \mu_n(t_2)$.

Since condition (3) of Lemma 12 holds for $t_2 \in I_\mu$ and $\mu(\tau_n \notin I_\mu)$ goes to 0 when n tends to ∞ (recall that the diagram is clean), $\mu(\mathcal{C}_{m,n}) \xrightarrow[m,n\to\infty]{} 0$ uniformly in $m, n \in \mathbb{N}$ with $m < n$.

Thus, we can telescope the diagram in order that

$$\sum_{n \in \mathbb{N}} \mu(\mathcal{C}_{n,n+1}) \text{ converges.} \tag{5.5}$$

Hence, from the Borel–Cantelli lemma we deduce that $\mu(\mathcal{C}) = 1$, where

$$\mathcal{C} = \liminf_{n\to\infty} \mathcal{C}^c_{n,n+1} = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{\tau_n \in I_\mu, \bar{s}_n \equiv_{\mathbf{b}} k(\tau_n, \tau_{n+1})\}.$$

5.2.3. *Constructing an eigenfunction.* After telescoping we can suppose that $p_n \equiv_b p$ for some $p \in \{0, \ldots, b - 1\}$ and for all $n \geq 1$. This will transform expressions of the form $\lambda^{-p_n s}$ below to λ^{-ps}, which is independent of n.

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For $m, n \in \mathbb{N}$ with $m < n$, $t_1 \in \{1, \ldots, d\}$ and $t_2 \in I_\mu$ we have

$$\sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s} = \sum_{\substack{s\in\overline{S}_{m,n}(t_1,t_2)\\ s\equiv_{\mathbf{b}} k(t_1,t_2)}} \lambda^{-pk(t_1,t_2)} + \sum_{\substack{s\in\overline{S}_{m,n}(t_1,t_2)\\ s\not\equiv_{\mathbf{b}} k(t_1,t_2)}} \lambda^{-ps}$$
$$= P_{m,n}(t_1, t_2)\lambda^{-pk(t_1,t_2)} + \sum_{\substack{s\in\overline{S}_{m,n}(t_1,t_2)\\ s\not\equiv_{\mathbf{b}} k(t_1,t_2)}} (\lambda^{-ps} - \lambda^{-pk(t_1,t_2)}),$$

where we have used the fact that $\#\overline{S}_{m,n}(t_1, t_2) = P_{m,n}(t_1, t_2)$. Also, since

$$\#\{s \in \overline{S}_{m,n}(t_1, t_2) \mid s \not\equiv_{\mathbf{b}} k(t_1, t_2)\} = P_{m,n}(t_1, t_2) - \boldsymbol{\sigma}_{m,n}^{(k(t_1,t_2))}(t_1, t_2),$$

we have that

$$\left|\sum_{\substack{s\in\overline{S}_{m,n}(t_1,t_2)\\ s\not\equiv_{\mathbf{b}} k(t_1,t_2)}} (\lambda^{-ps} - \lambda^{-pk(t_1,t_2)})\right| \le 2 \cdot (P_{m,n}(t_1, t_2) - \boldsymbol{\sigma}_{m,n}^{(k(t_1,t_2))}(t_1, t_2)).$$

As mentioned before, condition (2) of the main theorem using $t_2 \in I_\mu$ implies that the equivalent conditions in Lemma 12 hold. So, by Lemma 12 (3), for $t_1 \in \{1, \ldots, d\}$ and $t_2 \in I_\mu$ we have

$$\frac{P_{m,n}(t_1, t_2) - \boldsymbol{\sigma}_{m,n}^{(k(t_1,t_2))}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0$$

uniformly in $m, n \in \mathbb{N}$ with $m < n$.

We summarize the previous discussion. Fix a real number $\epsilon > 0$. Then, for all large enough $m, n \in \mathbb{N}$ with $m < n$, $t_1 \in \{1, \ldots, d\}$ and $t_2 \in I_\mu$, we can write

$$\frac{1}{q_{m,n}} \sum_{s\in\overline{S}_{m,n}(t_1,t_2)} \lambda^{-p_m s} = \frac{P_{m,n}(t_1, t_2)}{q_{m,n}}\lambda^{-pk(t_1,t_2)} + \epsilon_{m,n}(t_1, t_2), \tag{5.6}$$

where $\epsilon_{m,n}(t_1, t_2)$ is a complex number with $|\epsilon_{m,n}(t_1, t_2)| \le \epsilon$.

Now, consider $\ell, m, n \in \mathbb{N}$ with $\ell < m < n$ enough large (such that the different uses of (5.6) below are valid), $t_1 \in \{1, \ldots, d\}$ and $t_3 \in I_\mu$. Then, by using (2.6) to get the second equality and (5.6) three times, we get

$$\frac{P_{\ell,n}(t_1, t_3)}{q_{\ell,n}}\lambda^{-pk(t_1,t_3)} + \epsilon_{\ell,n}(t_1, t_3)$$
$$= \frac{1}{q_{\ell,n}} \sum_{s\in\overline{S}_{\ell,n}(t_1,t_3)} \lambda^{-p_\ell s}$$
$$= \frac{1}{q_{\ell,n}} \sum_{t_2\in V_m} \sum_{s_1\in\overline{S}_{l,m}(t_1,t_2)} \sum_{s_2\in\overline{S}_{m,n}(t_2,t_3)} \lambda^{-p_\ell s_1 - p_m s_2}$$
$$= \sum_{t_2\in V_m} \left(\frac{1}{q_{\ell,m}} \sum_{s_1\in\overline{S}_{\ell,m}(t_1,t_2)} \lambda^{-p_\ell s_1}\right)\left(\frac{1}{q_{m,n}} \sum_{s_2\in\overline{S}_{m,n}(t_2,t_3)} \lambda^{-p_m s_2}\right)$$
$$= \sum_{t_2\in I_\mu} \left(\frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}}\lambda^{-pk(t_1,t_2)} + \epsilon_{\ell,m}(t_1, t_2)\right)\left(\frac{P_{m,n}(t_2, t_3)}{q_{m,n}}\lambda^{-pk(t_2,t_3)} + \epsilon_{m,n}(t_2, t_3)\right)$$
$$+ \sum_{t_2\in V_m\setminus I_\mu} \left(\frac{1}{q_{\ell,m}} \sum_{s_1\in\overline{S}_{\ell,m}(t_1,t_2)} \lambda^{-p_\ell s_1}\right)\left(\frac{P_{m,n}(t_2, t_3)}{q_{m,n}}\lambda^{-pk(t_2,t_3)} + \epsilon_{m,n}(t_2, t_3)\right)$$

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Set $k(t_1, t_2) = 0$ for $t_1 \in \{1, \ldots, d\}$ and $t_2 \notin I_\mu$ (recall that this map is only defined for $t_2 \in I_\mu$). Adding and subtracting the terms $(P_{\ell,m}(t_1, t_2)/q_{\ell,m})\lambda^{-pk(t_1,t_2)}$ when $t_2 \notin I_\mu$ in the last equality of previous deduction gives

$$\begin{aligned}
&\frac{P_{\ell,n}(t_1, t_3)}{q_{\ell,n}}\lambda^{-pk(t_1,t_3)} + \epsilon_{\ell,n}(t_1, t_3) \\
&\quad = \sum_{t_2 \in I_\mu} \left(\frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}}\lambda^{-pk(t_1,t_2)} + \epsilon_{\ell,m}(t_1, t_2) \right) \\
&\qquad \times \left(\frac{P_{m,n}(t_2, t_3)}{q_{m,n}}\lambda^{-pk(t_2,t_3)} + \epsilon_{m,n}(t_2, t_3) \right) \\
&\qquad + \sum_{t_2 \in V_m \setminus I_\mu} \left(\frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}}\lambda^{-pk(t_1,t_2)} \right) \left(\frac{P_{m,n}(t_2, t_3)}{q_{m,n}}\lambda^{-pk(t_2,t_3)} + \epsilon_{m,n}(t_2, t_3) \right) \\
&\qquad + \sum_{t_2 \in V_m \setminus I_\mu} \left(\frac{1}{q_{\ell,m}} \sum_{s_1 \in \overline{S}_{\ell,m}(t_1,t_2)} \lambda^{-p_\ell s_1} - \frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}}\lambda^{-pk(t_1,t_2)} \right) \\
&\qquad \times \left(\frac{P_{m,n}(t_2, t_3)}{q_{m,n}}\lambda^{-pk(t_2,t_3)} + \epsilon_{m,n}(t_2, t_3) \right).
\end{aligned}$$

Finally, multiplying the terms, we get that

$$\begin{aligned}
&\frac{P_{\ell,n}(t_1, t_3)}{q_{\ell,n}}\lambda^{-pk(t_1,t_3)} + \epsilon_{\ell,n}(t_1, t_3) \\
&\quad = \epsilon' + \sum_{t_2 \in V_m} \frac{P_{\ell,m}(t_1, t_2) P_{m,n}(t_2, t_3)}{q_{\ell,n}} \lambda^{-p(k(t_1,t_2)+k(t_2,t_3))},
\end{aligned} \tag{5.7}$$

where

$$|\epsilon'| \leq 2d\epsilon + d\epsilon^2 + d\epsilon + \sum_{t_2 \in V_m \setminus I_\mu} 2 \cdot \frac{P_{m,n}(t_2, t_3)}{q_{m,n}} + 2d\epsilon. \tag{5.8}$$

But, for $t_2 \notin I_\mu$, $t_3 \in I_\mu$ and any large $m, n \in \mathbb{N}$ with $m < n$, we have that $\mu(\tau_n = t_3) \geq \delta$ and $\mu(\tau_m = t_2) \leq \delta\epsilon$, where δ comes from the definition of a clean Bratteli–Vershik representation. Consequently, using equality (5.3), we have that

$$\frac{P_{m,n}(t_2, t_3)}{q_{m,n}} \leq \frac{\mu(\tau_m = t_2)}{\mu(\tau_n = t_3)} \leq \frac{\mu(\tau_m = t_2)}{\delta} \leq \epsilon. \tag{5.9}$$

Thus, combining (5.9) in (5.8), we get

$$|\epsilon'| \leq 5d\epsilon + 2d\epsilon \leq 8d\epsilon.$$

Now, a simple reordering of terms in (5.7) gives

$$1 + \frac{q_{\ell,n}}{P_{\ell,n}(t_1, t_3)}(\epsilon_{\ell,n}(t_1, t_3) - \epsilon')\lambda^{pk(t_1,t_3)} \tag{5.10}$$

$$= \sum_{t_2 \in V_m} \frac{P_{\ell,m}(t_1, t_2) P_{m,n}(t_2, t_3)}{P_{\ell,n}(t_1, t_3)} \lambda^{p(k(t_1,t_3)-k(t_1,t_2)-k(t_2,t_3))}. \tag{5.11}$$

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Recall from Lemma 11 that for every $\ell \in \mathbb{N}$ enough large there exist integers m, n with $n > m > \ell$ such that for all $t_1, t_2, t_3 \in I_\mu$,

$$\frac{P_{\ell,n}(t_1, t_3)}{q_{\ell,n}} \geq \frac{\delta}{3}, \quad \frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}} \geq \frac{\delta}{3}, \quad \frac{P_{m,n}(t_2, t_3)}{q_{m,n}} \geq \frac{\delta}{3}. \tag{5.12}$$

Then, if considering $t_1, t_3 \in I_\mu$ and fixing integers $\ell, m, n \in \mathbb{N}$ with $\ell < m < n$ enough large to satisfy (5.12), and using (5.10), we get

$$1 + \epsilon'' = \sum_{t_2 \in V_m} \frac{P_{\ell,m}(t_1, t_2) P_{m,n}(t_2, t_3)}{P_{\ell,n}(t_1, t_3)} \lambda^{p(k(t_1,t_3)-k(t_1,t_2)-k(t_2,t_3))}, \tag{5.13}$$

where $|\epsilon''| \leq \hat{C}\epsilon$ and $\hat{C}$ is a positive constant only depending on the system.

Let us show that $pk(t_1, t_3) \equiv_b p(k(t_1, t_2) + k(t_2, t_3))$ for all $t_2 \in I_\mu$. We rewrite the right-hand side of (5.13), which is a convex sum, as $\sum_{i=0}^{b-1} \alpha_i \lambda^i$, where

$$\alpha_i = \sum_{\{t_2 \in V_m \mid p(k(t_1,t_3)-k(t_1,t_2)-k(t_2,t_3)) \equiv_b i\}} \frac{P_{\ell,m}(t_1, t_2) P_{m,n}(t_2, t_3)}{P_{\ell,n}(t_1, t_3)}.$$

By (5.13), we can use Lemma 9. Then there is $i_0 \in \{0, \ldots, b-1\}$ such that $\alpha_{i_0} > 1 - C\hat{C}\epsilon$ (C is the constant of Lemma 9 for the bth roots of unity). Moreover, if ϵ is taken small enough, we have that $i_0 = 0$ since the convex combination is close to 1. But, again using (5.12), for all $t_2 \in I_\mu$,

$$\frac{P_{\ell,m}(t_1, t_2) P_{m,n}(t_2, t_3)}{P_{\ell,n}(t_1, t_3)} = \frac{P_{\ell,m}(t_1, t_2)}{q_{\ell,m}} \frac{P_{m,n}(t_2, t_3)}{q_{m,n}} \frac{q_{\ell,n}}{P_{\ell,n}(t_1, t_3)} \geq \frac{\delta^2}{9} > C\hat{C}\epsilon$$

if ϵ is taken small enough. Since $\alpha_{i_0} > 1 - C\hat{C}\epsilon$, then for all $t_2 \in I_\mu$,

$$p(k(t_1, t_3) - k(t_1, t_2) - k(t_2, t_3)) \equiv_b i_0 = 0.$$

This proves our claim.

Summarising, we have proved that for all $t_1, t_2, t_3 \in I_\mu$,

$$p \cdot k(t_1, t_3) \equiv_b p \cdot k(t_1, t_2) + p \cdot k(t_2, t_3), \tag{5.14}$$

$$p \cdot k(t_1, t_1) \equiv_b 0, \; p \cdot k(t_1, t_2) \equiv_b -p \cdot k(t_2, t_1). \tag{5.15}$$

To finish we will verify the criterion of Theorem 14 for $\lambda = \exp(2i\pi/b)$. Fix an element $t_0 \in I_\mu$ and for each $n \geq 1$ define $\rho_n : V_n \to \mathbb{R}$ by $\rho_n(t) = -pk(t_0, t)$.

Let x be an element in $\mathcal{C}$. By definition of $\mathcal{C}$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\tau_n(x) \in I_\mu$ and $\bar{s}_n(x) \equiv_{\mathbf{b}} k(\tau_n(x), \tau_{n+1}(x))$. Notice that, since $p\mathbf{b}$ is divisible by b (recall that $\mathbf{b} = p/(b, p)$), after multiplying by p we get that $p\bar{s}_n(x) \equiv_b pk(\tau_n(x), \tau_{n+1}(x))$. Then for $n \geq N$, we have

$$\begin{aligned} |\lambda^{r_{n+1}(x)+\rho_{n+1}(\tau_{n+1}(x))} - \lambda^{r_n(x)+\rho_n(\tau_n(x))}| &= |\lambda^{r_{n+1}(x)-r_n(x)+\rho_{n+1}(\tau_{n+1}(x))-\rho_n(\tau_n(x))} - 1| \\ &= |\lambda^{p\bar{s}_n(x)-pk(t_0,\tau_{n+1}(x))+pk(t_0,\tau_n(x))} - 1| \\ &= |\lambda^{p\bar{s}_n(x)-(pk(\tau_n(x),t_0)+pk(t_0,\tau_{n+1}(x)))} - 1| \\ &= |\lambda^{p\bar{s}_n(x)-pk(\tau_n(x),\tau_{n+1}(x))} - 1| = 0, \end{aligned}$$

where to deduce the second equality we use (2.4) and to derive the last one we apply (5.14) and (5.15). This proves that $\lambda^{r_n(x)+\rho_n(\tau_n(x))}$ is eventually constant, so it converges. We finish the proof using Theorem 14. □

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Let us remark that from the previous proof Corollary 4 follows directly. In fact, it is just a reformulation of the last part of the proof.

5.3. *Proof of Corollary 7.* (1) Let μ be an ergodic measure such that $\mathbf{B}_\mu$ is non-empty. Let $\lambda = \exp(2i\pi a/b)$ be a non-continuous eigenvalue for μ such that $b/(b, p_n) = \mathbf{b} \in \mathbf{B}_\mu$ for all large enough integers $n \in \mathbb{N}$. The hypotheses of Lemma 13 hold for all $t_2 \in I_\mu$ using this value of λ. Then, from Lemma 13 (2), for every $t_2 \in I_\mu$ and $k \in \{0, \ldots, \mathbf{b} - 1\}$ the sum

$$\sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k)} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} = \sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k) \cap I_\mu} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} + \sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k) \cap I^c_\mu} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}}$$

converges uniformly in $m, n \in \mathbb{N}$ with $m < n$ to $1/\mathbf{b}$. But, since $(\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2))/q_{m,n} \leq (P_{m,n}(t_1, t_2))/q_{m,n}$, from (3.1) we deduce that

$$\sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k) \cap I_\mu} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} \frac{1}{\mathbf{b}}, \qquad \sum_{t_1 \in \Psi^{-1}_{m,n,t_2}(k) \cap I^c_\mu} \frac{\boldsymbol{\sigma}^{(k)}_{m,n}(t_1, t_2)}{q_{m,n}} \xrightarrow[m,n\to\infty]{} 0$$

converges uniformly in $m, n \in \mathbb{N}$ with $m < n$.

We deduce that for any large enough $m, n \in \mathbb{N}$ with $m < n$, $t_2 \in I_\mu$ and $k \in \{0, \ldots, \mathbf{b} - 1\}$ each set $\Psi^{-1}_{m,n,t_2}(k)$ must contain an element of I_μ. Thus $\#I_\mu \geq \mathbf{b}$.

(2) Let us consider $\mu \in \mathcal{M}_{\mathrm{erg}}(X, T)$ such that $\mathbf{B}_\mu \neq \emptyset$. Let $\exp(2i\pi/b_1)$ and $\exp(2i\pi/b_2)$ be two different non-continuous eigenvalues for μ. Then, by Bézout's identity, $\exp(2i\pi/\mathrm{lcm}(b_1, b_2))$ is also an eigenvalue for μ. Moreover, it is a non-continuous eigenvalue. Indeed, if this fact is not true, then for some $n \in \mathbb{N}$ and $a \in \mathbb{Z}$ we have that $1/\mathrm{lcm}(b_1, b_2) = a/p_n$. This implies that $1/b_1 = (a\mathrm{lcm}(b_1, b_2)/b_1)/p_n$ which is a contradiction since $\exp(2i\pi/b_1)$ is a non-continuous eigenvalue. This proves our claim.

Denote $\mathrm{lcm}(b_1, b_2)$ by b. Decomposing $b_1 = \mathfrak{b}_1 \cdot \mathfrak{b}_2 \cdot \mathfrak{b}_3 \cdot \mathfrak{b}_4$ and $b_2 = \mathfrak{b}_3 \cdot \mathfrak{b}_4 \cdot \mathfrak{b}_5 \cdot \mathfrak{b}_6$, where $(b_1, b_2) = \mathfrak{b}_3 \cdot \mathfrak{b}_4$, $(b_1, p_n) = \mathfrak{b}_2 \cdot \mathfrak{b}_3$ and $(b_2, p_n) = \mathfrak{b}_3 \cdot \mathfrak{b}_5$, we get the identity

$$\mathrm{lcm}\left(\frac{b_1}{(b_1, p_n)}, \frac{b_2}{(b_2, p_n)}\right) = \frac{b}{(b, p_n)}.$$

From this identity follows that it is not possible to have more than one divisibility-maximal element in $\mathbf{B}_\mu$.

(3) For different ergodic measures μ and ν we have $I_\mu \cap I_\nu = \emptyset$ (recall that the Bratteli–Vershik representation is clean). Then

$$\sum_{\mu \in \mathcal{M}_{\mathrm{erg}}(X,T)} \#I_\mu \leq d.$$

But (1) implies that $\mathbf{b}_\mu \leq \#I_\mu$ for each $\mu \in \mathcal{M}$, so (3) follows.

(4) As in the proof of (3), we use the fact that, for different ergodic measures μ and ν, $I_\mu \cap I_\nu = \emptyset$. Hence,

$$\begin{aligned} \#\mathcal{M} \leq \#\mathcal{M}_{\mathrm{erg}}(X, T) &= \sum_{\mu \in \mathcal{M}_{\mathrm{erg}}(X,T)} \#I_\mu - \sum_{\mu \in \mathcal{M}_{\mathrm{erg}}(X,T)} (\#I_\mu - 1) \\ &\leq d - \sum_{\mu \in \mathcal{M}} (\mathbf{b}_\mu - 1), \end{aligned}$$

where in the inequality we have used (1). This proves (4).

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5.4. *Proof of Corollary 8.* Consider $\lambda = \exp(2i\pi/b)$ with b an integer such that $b/(b, p_n) = d$ for all n large enough.

First, we prove the necessary and sufficient condition given by (3.2). If λ is a non-continuous eigenvalue, then $\mathbf{b}_\mu$ defined in Corollary 7 is equal to d. In addition, since $\mathbf{b}_\mu = d$, the partition of Corollary 6 is made of singletons and we get property (3.2) for any $t_2 \in I_\mu$. But, using statement (1) of Corollary 7, one deduces that $I_\mu = \{1, \ldots, d\}$. Thus property (3.2) is true for any $t_2 \in \{1, \ldots, d\}$. Clearly, property (3.2) implies that λ is a non-continuous eigenvalue by Corollary 6.

Now, assume that λ is a non-continuous eigenvalue. Using (3) in Corollary 7, we get that $\mathcal{M}_{\mathrm{erg}}(X, T)$ has a unique element, so the system is uniquely ergodic. This proves statement (1).

Finally we prove statement (2). Recall that under our hypothesis equivalent conditions of Lemma 12 hold for any $t_2 \in \{1, \ldots, d\}$. Then, from the equality

$$\left|\mu(\tau_m = t_1) - \frac{1}{d}\right| = \left|\sum_{t_2 \in V_n}\left(\frac{P_{m,n}(t_1, t_2)}{q_{m,n}} - \frac{1}{d}\right)\mu(\tau_n = t_2)\right|,$$

(3.2) and Lemma 12 (2) we get

$$\lim_{m\to\infty} \mu(\tau_m = t_1) = \frac{1}{d}.$$

This proves the desired statement.

6. *Examples*

6.1. *Example 1: A model example.* We start with a basic *model example* that will be used later to illustrate several behaviors of the eigenvalues with respect to the ergodic measures. We start with a general framework to construct a family of examples where the Bratteli–Vershik representations are not necessarily proper. Later we modify this family to obtain proper representations. Finally, we prove that in this family of examples all ergodic measures share the same non-continuous eigenvalue $\exp(2i\pi/6)$.

6.1.1. Define the sequence $q_1 = 1$, $q_2 = 2 \cdot 5^2$ and $q_n = 5^{2n}$ for $n > 2$. First, consider the (not necessarily proper) Toeplitz diagram with the characteristic sequence $(q_n \mid n \in \mathbb{N})$ such that $V_n = \{1, 2, 3, 4, 5, 6, 7\}$ for all $n \geq 1$ and the local order of the q_{n+1} arrows arriving at $t \in V_{n+1}$ is given by the following associated sequences of vertices in V_n:

$$t \to v_{n+1}(t) \quad \text{for all } 1 \leq t \leq 7,$$

where each $v_{n+1}(t)$ is a fixed word of length q_{n+1} on the alphabet V_n built in the following way:

(1) Set $W_1 = \{1, 4, 7\}$, $W_2 = \{2, 5\}$ and $W_3 = \{3, 6\}$.

(2) For $n \geq 2$ the words $v_{n+1}(1)$, $v_{n+1}(4)$ and $v_{n+1}(7)$ begin with an element of W_1, followed by an element of W_2 and then by an element of W_3. Then we restart from W_1 and so on. Because $q_{n+1} \equiv_3 1$ for $n \geq 2$, all these three words end with an element of W_1. The words $v_{n+1}(2)$ and $v_{n+1}(5)$ follow the same periodic scheme, starting with an element of W_2, then of W_3 and so on (and therefore ending with an element of W_2). And finally the words $v_{n+1}(3)$ and $v_{n+1}(6)$ follow the periodic scheme starting in W_3.

(3) Level 2 is built in any way.

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Define $k : \{1, \ldots, 7\} \times \{1, \ldots, 7\} \to \{0, 1, 2\}$ by $k(t_1, t_2) = j - i \bmod 3$ if $t_1 \in W_i$ and $t_2 \in W_j$. The following two properties are straightforward. First, for $t_1, t_2, t_3 \in \{1, \ldots, 7\}$, we have

$$k(t_1, t_3) \equiv_3 k(t_1, t_2) + k(t_2, t_3). \tag{6.1}$$

Second, let x be an infinite sequence in the ordered Bratteli diagram. For $n \geq 2$,

$$\overline{s}_n(x) \equiv_3 k(\tau_n(x), \tau_{n+1}(x)). \tag{6.2}$$

Now we modify slightly the previously defined local orders to get a proper Bratteli–Vershik representation for the system. To produce the new orders we change sequences $v_{n+1}(t)$ into $w_{n+1}(t)$ in such a way that: (1) $w_{n+1}(t) = v_{n+1}(t)$, except for at most a fixed number of letters, say L, independent of n; (2) $w_{n+1}(t)$ begins and ends with 1; and (3) $w_{n+1}(t)$ contains every element of $\{1, 2, 3, 4, 5, 6, 7\}$ at least once. This diagram is clearly proper and induces a Toeplitz system of finite rank (X, T).

Consider any invariant measure μ on the system. We prove that $\exp(2\pi i/6)$ is a non-continuous eigenvalue of (X, T) for μ, and this fact is independent of the measure μ we choose. In order to do that, we verify conditions (1)–(4) of Corollary 4.

By construction $p_n \equiv_6 2$ and $\mathbf{b} = 6/(6, p_n) = 3$ for all $n \geq 2$, so conditions (1) and (2) hold. Condition (3) follows directly from (6.1). To prove condition (4) we need to find a set of full measure where $\overline{s}_n(x) \equiv_3 k(\tau_n(x), \tau_{n+1}(x))$ for all large enough $n \in \mathbb{N}$.

Similarly as in the proof of Theorem 3, for $n \in \mathbb{N}$ consider $\mathcal{C}_n = \{x \in X \mid \overline{s}_n(x) \not\equiv_3 k(\tau_n(x), \tau_{n+1}(x))\}$. Since (6.2) holds before modifying the orders and those modifications alter no more than L letters per level, we easily check that

$$\begin{aligned}
\mu(\mathcal{C}_n) &= \sum_{t_1=1}^{7} \sum_{t_2=1}^{7} \mu(\tau_n = t_1, \tau_{n+1} = t_2, \overline{s}_n(x) \not\equiv_3 k(t_1, t_2)) \\
&\leq \sum_{t_1=1}^{7} \sum_{t_2=1}^{7} L h_n(t_1) \mu_{n+1}(t_2) \\
&\leq \sum_{t_1=1}^{7} \sum_{t_2=1}^{7} \frac{L}{q_{n+1}} \mu(\tau_{n+1} = t_2) \\
&\leq \frac{7L}{q_{n+1}}.
\end{aligned}$$

So $\sum_{n \geq 1} \mu(\mathcal{C}_n)$ converges. Hence, from the Borel–Cantelli lemma, we get $\mu(\mathcal{C}) = 1$, where $\mathcal{C} = \liminf_{n \to \infty} \mathcal{C}_n^c$.

6.2. *Example 2: A first particular case of the model example.* In this example we make precise the construction of Example 1 in order to show that the model example can produce a uniquely ergodic system, where $\exp(2i\pi/6)$ is a non-continuous eigenvalue for the unique invariant measure. In addition, this will illustrate that inequalities in Corollary 7 can be strict and that Corollary 8 is not reversible since we can have $\mathbf{b}_\mu < d$ in the uniquely ergodic case.

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First, for $n \geq 3$ define c_n such that $q_n = 12c_n + 1$ and define words giving the order of the diagram by

$$\begin{aligned}
w_{n+1}(1) &= (123456723756)^{c_{n+1}}1,\\
w_{n+1}(2) &= 1(312645372675)^{c_{n+1}-1}(312)^3671,\\
w_{n+1}(3) &= 1(123456723756)^{c_{n+1}-1}(123)^3451,\\
w_{n+1}(4) &= (156423756723)^{c_{n+1}}1,\\
w_{n+1}(5) &= 1(345612375672)^{c_{n+1}-1}(645)^3311,\\
w_{n+1}(6) &= 1(156423756723)^{c_{n+1}-1}(723)^3121,\\
w_{n+1}(7) &= (153426753726)^{c_{n+1}}1.
\end{aligned}$$

It is straightforward that these orders fit the model construction in Example 1. Also, for any invariant measure μ, the system satisfies: $\mu(\tau_n = t) \xrightarrow[n\to\infty]{} \frac{1}{6}$ for $t = 2, 3, 5, 6, 7$ and $\mu(\tau_n = t) \xrightarrow[n\to\infty]{} \frac{1}{12}$ for $t = 1, 4$. The proof is a simple computation. For example,

$$\frac{c_{n+1}+1}{q_{n+1}} < \mu(\tau_n = 1) < \frac{c_{n+1}+4}{q_{n+1}},$$

and then we use the fact that $c_{n+1}/q_{n+1} \xrightarrow[n\to\infty]{} \frac{1}{12}$. Since $\#I_\mu = 7$, then we deduce that the system is uniquely ergodic. Also, since 3 divides $\mathbf{b}_\mu$, $\mathbf{b}_\mu < \#I_\mu$.

6.3. *Example 3: A second particular case of the model example.* Here we will use the model example to produce a Bratteli–Vershik system having exactly two ergodic measures. Then, for each one, $\exp(2i\pi/6)$ is a non-continuous eigenvalue. Let us take in the model example the following particular choice of $w_{n+1}(t)$ for $t \in \{1, \ldots, 7\}$ and $n \geq 2$. First define c_n so that $q_n = 3c_n + 1$, and then set:

$$\begin{aligned}
w_{n+1}(1) &= (123)^{c_{n+1}-2}4567231,\\
w_{n+1}(2) &= 13(123)^{c_{n+1}-2}45671,\\
w_{n+1}(3) &= 1(123)^{c_{n+1}-2}456721,\\
w_{n+1}(4) &= 146(456)^{c_{n+1}-2}7231,\\
w_{n+1}(5) &= 14(456)^{c_{n+1}-2}12371,\\
w_{n+1}(6) &= 1(456)^{c_{n+1}-2}123761,\\
w_{n+1}(7) &= 156(456)^{c_{n+1}-2}7231.
\end{aligned}$$

As was shown in [**BKMS13**, Theorem 3.3 (2)], any ergodic measure is obtained as an extension of a finite measure on a system defined on a *subdiagram*. A subdiagram is obtained fixing subsets of vertices at each level and considering only the paths which go along the vertices in such subsets. The order is defined naturally following the order of the complete diagram. Here we will fix a unique subset of $\{1, \ldots, 7\}$ for all levels.

Consider the subset $A_1 = \{1, 2, 3\}$ and construct the associated subdiagram. Using the same nomenclature as before, for levels $n \geq 2$ the corresponding subdiagram has the

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following induced local orders:

$$1 \to (123)^{c_{n+1}-2}231,$$
$$2 \to 13(123)^{c_{n+1}-2}1,$$
$$3 \to 1(123)^{c_{n+1}-2}21.$$

This order determines a proper diagram that is of Toeplitz type and has the characteristic sequence $(\overline{q}_n)_{n\in\mathbb{N}}$, with $\overline{q}_n = q_n - 4$ for $n > 2$. Analogously to Example 2, we can see that the system (Y, S) induced by this diagram is uniquely ergodic. Moreover, the unique invariant measure μ of this system can be naturally extended to a finite ergodic measure of (X, T). For a deeper discussion of this extension we refer the reader to [**BKMS13**, §3]. Let us call $\widehat{\mu}$ the normalized extension of μ. Then $\widehat{\mu}$ is an ergodic probability measure on (X, T).

Analogously, consider $A_2 = \{4, 5, 6\}$. In this case the corresponding subdiagram has the following local orders. For $n \geq 2$,

$$4 \to 46(456)^{c_{n+1}-2},$$
$$5 \to 4(456)^{c_{n+1}-2},$$
$$6 \to (456)^{c_{n+1}-2}6.$$

This diagram has unique maximal and minimal paths, and the words have lengths $q_{n+1} - 5$, $q_{n+1} - 6$ and $q_{n+1} - 6$, respectively. As before, one proves that the system (Z, R) associated to this diagram is uniquely ergodic and that the unique ergodic measure ν can be extended to a finite ergodic measure of (X, T). We call $\widehat{\nu}$ the normalized extension of ν.

From [**BKMS13**, Theorem 3.3 (4)] one deduces that (X, T) has no other ergodic probability measures than $\widehat{\mu}$ and $\widehat{\nu}$. Furthermore, one proves by simple computations that the diagram is clean and $I_{\widehat{\mu}} = \{1, 2, 3\}$ and $I_{\widehat{\nu}} = \{4, 5, 6\}$.

6.4. *Example 4: A small variation of the model example.* We provide an example of a finite-rank Toeplitz system with two ergodic measures. For one there is a non-continuous eigenvalue, while for the other all eigenvalues are continuous. We keep the values for q_n of Example 1 but we consider the following choice of $w_{n+1}(t)$ for $t \in \{1, \ldots, 7\}$ and $n \geq 2$, where c_n is such that $q_n = 12c_n + 1$:

$$w_{n+1}(1) = (123456423156)^{c_{n+1}-1}(123)^3 7561,$$
$$w_{n+1}(2) = 1(312645342615)^{c_{n+1}-1}(312)^3 671,$$
$$w_{n+1}(3) = 1(123456423156)^{c_{n+1}-2}(123)^3 751,$$
$$w_{n+1}(4) = (156423456123)^{c_{n+1}-1}(123)^3 7561,$$
$$w_{n+1}(5) = 1(345612315642)^{c_{n+1}-1}(645)^3 371,$$
$$w_{n+1}(6) = 1(156423456123)^{c_{n+1}-2}(123)^3 721.$$
$$w_{n+1}(7) = 1(7)^{q_{n+1}-7}654321$$

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This order does not fit conditions of Example 1, so we cannot ensure that $\exp(2i\pi/6)$ is a non-continuous eigenvalue for every ergodic measure μ on the system (X, T) induced by this diagram.

As in the previous example one proves that the subdiagrams associated to the subsets of vertices $\{1, 2, 3, 4, 5, 6\}$ and $\{7\}$ at all levels define systems (Y, S) and (Z, R) respectively, which are uniquely ergodic and the normalized extensions of their unique probability measures, $\widehat{\mu}$ and $\widehat{\nu}$, are ergodic measures on (X, T). Furthermore, a detailed computation allows one to prove that the diagram is clean with respect to these measures and that $I_{\widehat{\mu}} = \{1, 2, 3, 4, 5, 6\}$ and $I_{\widehat{\nu}} = \{7\}$. This implies there is no other ergodic probability measure on (X, T) aside from such extensions.

Now we prove that $\exp(2i\pi/6)$ is a non-continuous eigenvalue for $\widehat{\mu}$ and that $\widehat{\nu}$ does not have non-continuous eigenvalues. The only difference between the model example and this case is the measure of the set

$$\mathcal{C}_n = \{x \in X \mid \overline{s}_n(x) \not\equiv_3 k(\tau_n(x), \tau_{n+1}(x))\}.$$

Here, $\widehat{\mu}(\mathcal{C}_n) \leq (2/(q_{n+1})) + \widehat{\mu}(\tau_{n+1} = 7)$ and a simple computation allows us to prove that $\sum_{n\geq 1} \widehat{\mu}(\mathcal{C}_n)$ converges. We deduce by using Corollary 4 that $\exp(2i\pi/6)$ is a non-continuous eigenvalue for $\widehat{\mu}$.

The absence of non-continuous rational eigenvalues, say $\lambda = \exp(2i\pi/b)$, for $\widehat{\nu}$ follows from inequalities $1 < b/(b, p_n) \leq \#I_{\widehat{\nu}} = 1$, which is a contradiction.

6.5. *Example 5: A big variation of the model example.* Here we provide a Bratteli–Vershik system of Toeplitz type with rank 7 having two ergodic measures and different non-continuous eigenvalues associated to them. The first eigenvalue is $\exp(2i\pi/6)$ and the corresponding $\mathbf{b} = 3$, and the other eigenvalue is $\exp(2i\pi/8)$ with $\mathbf{b} = 4$. In particular, this example shows that all inequalities of Corollary 7 (4) can be equalities. We keep the values for q_n of Example 1 and for $t \in \{1, \ldots, 7\}$ and $n \geq 2$ we consider the following choice of $w_{n+1}(t)$, where c_n is such that $q_n = 12c_n + 1$:

$$\begin{aligned}
w_{n+1}(1) &= (123)^{4c_{n+1}-2}1245671,\\
w_{n+1}(2) &= 1(312)^{4c_{n+1}-2}345671,\\
w_{n+1}(3) &= 1(123)^{4c_{n+1}-2}145671,\\
w_{n+1}(4) &= 1(5674)^{3c_{n+1}-2}23745671,\\
w_{n+1}(5) &= 15(7456)^{3c_{n+1}-2}7452371,\\
w_{n+1}(6) &= 15(4567)^{3c_{n+1}-2}2367471,\\
w_{n+1}(7) &= 12(5674)^{3c_{n+1}-2}3674571.
\end{aligned}$$

As before, we prove that the subdiagrams associated to the sets $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$ define systems (Y, S) and (Z, R) respectively which are uniquely ergodic, and the extensions of their unique probability measures are ergodic measures on (X, T). Denote the ergodic measures on (X, T) by $\widehat{\mu}$ and $\widehat{\nu}$. One also has that the diagram is clean and $I_{\widehat{\mu}} = \{1, 2, 3\}$, $I_{\widehat{\nu}} = \{4, 5, 6, 7\}$. Thus, there is no other ergodic probability measure on (X, T) aside from $\widehat{\mu}$ and $\widehat{\nu}$.

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Now we sketch a proof that $\lambda = \exp(2i\pi/6)$ is a non-continuous eigenvalue for $\widehat{\mu}$. Similarly, one proves that $\lambda = \exp(2i\pi/8)$ is a non-continuous eigenvalue for $\widehat{\nu}$. This last case is left to the reader.

First, a direct computation (one can easily compute nine cases) serves to prove that for any $t_1, t_2 \in \{1, 2, 3\}$, up to a bounded number of elements $s \in \bar{S}_n(t_1, t_2)$, all are constant modulo 3. Denote such a constant by $k(t_1, t_2)$. Moreover, if $k(t_1, 1) \equiv_3 c$ then $k(t_1, 2) \equiv_3 c + 1$ and $k(t_1, 3) \equiv_3 c + 2$; and if $k(1, t_2) \equiv_3 c'$ then $k(2, t_2) \equiv_3 c' + 2$ and $k(3, t_2) \equiv_3 c' + 1$. A precise inspection of values of c and c' for all t_1 and t_2 allows us to prove that

$$k(t_1, t_2) \equiv_3 k(t_1, t) + k(t, t_2) \quad \text{for any } t \in \{1, 2, 3\}.$$

This additive map is the one required by Corollary 4. To finish the proof it is enough to produce a set $\mathcal{C}$ of full measure such that for any point $x \in \mathcal{C}$ we have $\bar{s}_n(x) \equiv_3 k(\tau_n(x), \tau_{n+1}(x))$ for all enough large $n \in \mathbb{N}$. As before, by considering for any $n \in \mathbb{N}$ the set

$$\mathcal{C}_n = \{x \in X \mid \bar{s}_n(x) \not\equiv_3 k(\tau_n(x), \tau_{n+1}(x))\}$$

and using the fact that any $s \in \bar{S}_n(t_1, t_2)$ up to a bounded number of elements, say L, is constant modulo 3, one gets that $\hat{\mu}(\mathcal{C}_n) \leq 3L/q_{n+1}$. We finish the proof of the claim by the Borel–Cantelli lemma, taking $\mathcal{C} = \liminf_{n\to\infty} \mathcal{C}_n^c$.

6.6. *Example 6: Another (similar) big variation of the model example.* Here we modify the previous example to provide a system with two ergodic measures and non-continuous eigenvalues $\exp(2i\pi/6)$ and $\exp(2i\pi/4)$, respectively. This example shows that the first inequality of Corollary 7 (4) is an equality and the second is a strict inequality. For $t \in \{1, \ldots, 7\}$ and $n \geq 2$, consider the following choice of $w_{n+1}(t)$ and write $q_n = 12c_n + 1$:

$$\begin{aligned}
w_{n+1}(1) &= (123)^{4c_{n+1}-2}1245671,\\
w_{n+1}(2) &= 1(312)^{4c_{n+1}-2}345671,\\
w_{n+1}(3) &= 1(123)^{4c_{n+1}-2}145671,\\
w_{n+1}(4) &= 1(647465)^{2c_{n+1}-1}237461,\\
w_{n+1}(5) &= 1(656574)^{2c_{n+1}-1}652361,\\
w_{n+1}(6) &= 16(646575)^{2c_{n+1}-1}72361,\\
w_{n+1}(7) &= 16(757564)^{2c_{n+1}-1}73261.
\end{aligned}$$

In this example the subdiagrams associated to $\{1, 2, 3\}$ and $\{4, 5, 6, 7\}$ define systems (Y, S) and (Z, R) respectively which are uniquely ergodic, and the extensions of these ergodic measures, $\widehat{\mu}$ and $\widehat{\nu}$, are ergodic probability measures in (X, T). As in the previous example there is no other ergodic probability measure on (X, T). Furthermore, the diagram is clean, $I_{\widehat{\mu}} = \{1, 2, 3\}$ and $I_{\widehat{\nu}} = \{4, 5, 6, 7\}$.

In relation to eigenvalues, similar computations to those in the previous example yield that $\exp(2\pi i/6)$ is a non-continuous eigenvalue for $\widehat{\mu}$ and that $\exp(2\pi i/4)$ is a non-continuous eigenvalue for $\widehat{\nu}$, while $\exp(2\pi i/8)$ is not.

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