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Continuous phase-space methods on discrete phase spaces

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Abstract – We show that discrete quasiprobability distributions defined via the discrete Heisenberg-Weyl group can be obtained as discretizations of the continuous $SU(N)$ quasiprobability distributions. This is done by identifying the phase-point operators with the continuous quantisation kernels evaluated at special points of the phase space. As an application we discuss the positive-P function and show that its discretization can be used to treat the problem of diverging trajectories. We study the dissipative long-range transverse-field Ising chain and show that the long-time dynamics of local observables is well described by a semiclassical approximation of the interactions.

Introduction. – Phase-space representations of quantum mechanics, usually called quasiprobability distributions, provide a natural language for the quantum-classical correspondence of non-relativistic quantum mechanics. They were first developed in the context of harmonic oscillators [1] and extensively applied in quantum optics [2,3]. It has been realised that the quantum-classical correspondence is intimately related to symmetry properties of the underlying classical phase-space. This led to an axiomatic approach to the phase-space formulation of quantum mechanics known as Stratonovich-Moyal-Weyl correspondence [4,5]. In order to apply the ideas of Stratonovich and Berezin to finite-dimensional Hilbert spaces (*e.g.*, spin systems), two main research directions are followed. The first is to construct a continuous phase space with a natural continuous symmetry of the system (*e.g.*, $SU(2)$ for spin systems) [6–10], which typically leads to evolution equations in terms of stochastic differential equations [11,12] or even ordinary differential equations [11,13] that can be efficiently simulated. The second direction is to exploit the discrete nature of the system and to define phase-space distributions on the basis of the discrete Heisenberg-Weyl group [14–22]. This formulation enables a simpler representation of quantum states and has been applied in quantum information [23,24] and quantum tomography [25,26].

We show that the two approaches to formulate phase-space quasiprobability distributions on finite-dimensional Hilbert spaces are equivalent, in the sense that discrete distributions can be obtained by evaluating the continuous ones at special points of the phase space. This identification provides a formal justification of Monte Carlo methods on finite-dimensional phase spaces proposed in [27,28], their systematic expansion beyond the semiclassical (truncated Wigner) approximation [12,29] and extension to open quantum systems. Hence, the revealed relation between continuous and discrete phase spaces should be relevant to further development of contemporary phase-space methods for simulation of long-range many-body quantum systems.

A related correspondence between the operators on a plane and on a torus has been established and studied in the case of continuous automorphisms of the torus [30,31]. Further, a similar discretisation was obtained in case of the $SU(2)$ group [32,33] by contracting a continuous $SU(2)$ kernel to a discrete one. Here we take the opposite route and extend the discrete kernel to the continuous one for any dimension of the Hilbert space N.

In order to prove the main result of the paper we first review the basic properties of the discrete and $SU(N)$ quasiprobability distributions on N-dimensional Hilbert space.

Discrete quasiprobability distributions. A one-parameter family of discrete Weyl symbols for an operator A acting on a N-dimensional Hilbert space is usually defined as

$$W_A^{(s)}(\alpha,\beta) = \mathrm{tr}\, A\Delta^{(s)}_{\alpha,\beta}, \tag{1}$$

with the parameter s denoting the ordering of the distribution; $s = 1, 0, -1$ for normal, symmetric, and anti-normal ordering, respectively. The phase-point operators $\Delta^{(s)}_{\alpha,\beta}$, $\alpha, \beta = 0, 1, 2, \ldots N-1$ satisfy the following properties [16,19,21,34]:

1a) Hermicity: $(\Delta^{(s)}_{\alpha,\beta})^\dagger = \Delta^{(s)}_{\alpha,\beta}$;

2a) Normalization: $\mathrm{tr}\,\Delta^{(s)}_{\alpha,\beta} = 1$;

3a) Covariance: $\Delta^{(s)}_{\alpha-\mu,\beta-\nu} = T_{\mu,\nu}\Delta^{(s)}_{\alpha,\beta}T^\dagger_{\mu,\nu}$, where $T_{\mu,\nu}$ denotes a unitary irrep of the discrete Heisenberg-Weyl group of order N;

4a) Traciality: $\mathrm{tr}\,\Delta^{(s)}_{\alpha,\beta}\Delta^{(-s)}_{\alpha',\beta'} = N\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$.

The quasiprobability distribution is given by the Weyl symbol of the density matrix. For each $\Delta_{\alpha,\beta}$ satisfying the properties 1a)–4a) the Weyl symbol (1) defines a one-to-one mapping between operators on a finite-dimensional Hilbert space and functions on a discrete phase space. The explicit form of the phase-point operators is not relevant for our purpose and will be omitted; we refer the interested reader to refs. [14,15,19] —a particular example is given in the next section (11). The kernel $\Delta^{(s)}_{\alpha,\beta}$ can be rewritten in the following useful form [19]:

$$\Delta^{(s)}_{\alpha,\beta} = T_{\alpha,\beta}\Delta^{(s)}_{0,0}T^\dagger_{\alpha,\beta}. \tag{2}$$

The decomposition (2) is valid for any unitary realisation $T_{\alpha,\beta}$ of the discrete Heisenberg-Weyl group of order N. The constant matrix $\Delta^{(s)}_{0,0}$ is determined by demanding that for $s = -1$ the quasiprobability distribution represents the discrete Q function. This gives the boundary condition

$$\Delta^{(-1)}_{\alpha,\beta} = T_{\alpha,\beta}|0\rangle\langle 0|T^\dagger_{\alpha,\beta}, \tag{3}$$

where $|0\rangle$ denotes a chosen vacuum state. Other $\Delta^{(s)}_{0,0}$ can be determined by using the traciality condition [21]. Phase-point operators satisfying the properties 1a)–4a) can be defined and calculated for any Hilbert space dimension N. If, in addition, N is a prime number or a power of a prime number, the quasiprobability distribution is defined on a discrete phase space with a well-defined geometry [14,18,35]. For such systems (sometimes called Galois quantum systems) phase-space methods based on discrete symplectic transformations were developed that are similar to the ones used for harmonic-oscillator systems [18]. Phase spaces with non-prime number dimensions N do not have a well-defined geometry. As a consequence the standard construction of the mutually unbiased basis does not work, leaving this fundamental and interesting problem unsolved in the Hilbert spaces of non-prime number dimension. In addition, on even-dimensional phase spaces the discrete kernels are not uniquely determined [16,24]. This, however, does not affect our results.

$SU(N)$ quasiprobability distributions. The $SU(N)$ quasiprobability distributions on a manifold $\mathcal{M}$ are constructed in a similar manner as the discrete quasiprobability distributions through the quantisation kernel $\Delta^{(s)}(\Omega)$. The Weyl symbol is given by

$$W^{(s)}_A(\Omega) = \mathrm{tr}\,A\Delta^{(s)}(\Omega), \tag{4}$$

with the parameter $s = -1, 0, 1$ denoting the normal, symmetric, and anti-normal ordering. The quantisation kernel has to satisfy similar properties as in the discrete case, namely [4]

1b) Hermicity: $(\Delta^{(s)}(\Omega))^\dagger = \Delta^{(s)}(\Omega)$;

2b) Normalization: $\mathrm{tr}\,\Delta^{(s)}(\Omega) = 1$;

3b) Covariance: $\Delta^{(s)}_{g^{-1}\circ\Omega} = \Lambda(g)\Delta^{(s)}(\Omega)\Lambda^\dagger(g)$, where $\Lambda(g)$ denotes a unitary irrep of a coset element $g \in \mathcal{G} = SU(N)/U(N-1)$;

4b) Traciality: $\int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\mathrm{tr}\left(\Delta^{(s)}(\Omega)\Delta^{(-s)}(\Omega')\right) f^{(s)}(\Omega) = f^{(s)}(\Omega')$, where $\mathrm{d}\mu(\Omega)$ is the invariant measure, $f^{(s)}(\Omega)$ denotes a differentiable s-ordered function on the manifold $\mathcal{M}$ being isomorphic to $\mathcal{G}$.

For each quantisation kernel $\Delta^{(s)}(\Omega)$ satisfying the properties 1b)–4b) the Weyl symbol (4) represents a one-to-one mapping between operators on the N-dimensional Hilbert space and smooth functions on the classical manifold $\mathcal{M}$. There are several constructions of the quantisation kernel for the $SU(N)$ group [6,9,10]. Here we focus on the fundamental representation for which the most explicit expression for the quantisation kernel was given in [10]. The explicit form of the $SU(N)$ kernel shall be omitted; the interested reader is referred to [6,9,10] —a particular example is given in the next section (12). As in the discrete case all realisations of the $SU(N)$ group admit the following decomposition of the kernel [9]:

$$\Delta^{(s)}(\Omega) = \Lambda(\Omega)D^{(s)}\Lambda(\Omega)^\dagger, \tag{5}$$

where $D^{(s)}$ is a constant diagonal matrix containing the essential information about the quasiprobability distribution. It can be determined by demanding that for $s = -1$ the usual Q function should be recovered, namely

$$\Delta^{(-1)}(\Omega) = \Lambda(\Omega)|0\rangle\langle 0|\Lambda(\Omega)^\dagger. \tag{6}$$

All other $D^{(s)}$ are determined by the traciality condition [9,10].

Construction of the $SU(N)$ kernel from the phase-point operators. – In the following we shall prove that continuous $SU(N)$ quasiprobability distributions for the fundamental representation of the $SU(N)$ group can be constructed from the discrete phase-point operators. We start by realising that the discrete Heisenberg-Weyl group of order N is a subgroup of the coset $\mathcal{G}$. Hence, the realisations $T_{\alpha,\beta}$ associated to the N-dimensional Hilbert space can be seen as fundamental realisations of particular group elements $\Omega_{\alpha,\beta} \in \mathcal{G}$, namely $T_{\alpha,\beta} = \Lambda(\Omega_{\alpha,\beta})$, suggesting that the phase-point operators can be regarded as the $SU(N)$ kernels evaluated at phase-space points $\Omega_{\alpha,\beta}$. This is immediately clear for the Q function, however, it can be shown for any s without knowing the precise form of the kernels $\Delta^{(s)}_{\alpha,\beta}$ and $\Delta^{(s)}(\Omega)$.

The discrete phase-point operators can be extended to the $SU(N)$ group by defining

$$\Delta^{(s)}_{\alpha,\beta}(\Omega) = \Lambda(\Omega)\Delta^{(s)}_{\alpha,\beta}\Lambda^{\dagger}(\Omega). \tag{7}$$

The extended Weyl symbol for an operator A is then defined as

$$W^{(s)}_A(\alpha,\beta,\Omega) = \mathrm{tr}\left(A\Delta^{(s)}_{\alpha,\beta}(\Omega)\right). \tag{8}$$

In the following we shall show that for any fixed Ω the matrix $W^{(s)}_A(\alpha,\beta,\Omega)$ represents a discrete Weyl symbol and that for any fixed pair α, β the function $W^{(s)}_A(\alpha,\beta,\Omega)$ represents a continuous Weyl symbol. The properties 1a), 2a), and 4a) of the phase-point operators are trivially satisfied. The covariance property 3a) follows by using a rotated realisation of the discrete Heisenberg-Weyl group, namely $T_{\alpha,\beta}(\Omega) = \Lambda(\Omega)T_{\alpha,\beta}\Lambda^{\dagger}(\Omega)$. Further, the properties 1b)–3b) of the continuous quantisation kernel are evident. Finally, the continuous traciality property 4b) follows from the invariance of the Haar measure and from the fact that $\Delta_{\alpha,\beta}(\Omega)$ are also discrete phase-point operators for any but fixed $\Omega \in \mathcal{G}$. By using these two properties any operator A acting on the Hilbert space can be decomposed as

$$\begin{aligned} A &= \sum_{\alpha,\beta} \mathrm{tr}\,(\Delta^{(-s)}_{\alpha,\beta}(\Omega)A)\Delta^{(s)}_{\alpha,\beta}(\Omega) \\ &= \frac{1}{\Omega_N}\int_{\mathcal{M}} \mathrm{d}\mu(\Omega) \sum_{\alpha,\beta} \mathrm{tr}\,(\Delta^{(-s)}_{\alpha,\beta}(\Omega)A)\Delta^{(s)}_{\alpha,\beta}(\Omega) \\ &= \frac{N^2}{\Omega_N}\int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\,\mathrm{tr}\,(\Delta^{(-s)}_{\alpha,\beta}(\Omega)A)\Delta^{(s)}_{\alpha,\beta}(\Omega), \end{aligned} \tag{9}$$

where Ω_N denotes the invariant volume of the phase space $\mathcal{M}$. From above equality (9) the continuous traciality condition follows:

$$\begin{aligned} &f^{(s)}_{\alpha,\beta}(\Omega) = \mathrm{tr}\,(F\Delta^{(-s)}_{\alpha,\beta}(\Omega)) \\ &= \frac{N^2}{\Omega_N}\int_{\mathcal{M}} \mathrm{d}\mu(\Omega) f^{(s)}_{\alpha,\beta}(\Omega')\,\mathrm{tr}\left(\Delta^{(s)}_{\alpha,\beta}(\Omega')\Delta^{(-s)}_{\alpha,\beta}(\Omega)\right). \end{aligned} \tag{10}$$

Hence, the kernel $\Delta_{\alpha,\beta}(\Omega)$ can be used to generate a discrete or a continuous phase space.

As an example we consider a spin-(1/2) case, where the phase space is a sphere, which can be useful to gain some intuition on the mapping between the discrete to the continuous phase space. We start by choosing the following phase-point operators:

$$\begin{aligned} A_{0,0} &= \begin{pmatrix} \frac{1}{2}\left(1+\sqrt{3}\right) & 0 \\ 0 & \frac{1}{2}\left(1-\sqrt{3}\right) \end{pmatrix}, \\ A_{0,1} &= \begin{pmatrix} \frac{1}{6}\left(3-\sqrt{3}\right) & \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}, \\ A_{1,0} &= \begin{pmatrix} \frac{1}{6}\left(3-\sqrt{3}\right) & (-1)^{2/3}\sqrt{\frac{2}{3}} \\ -\sqrt[3]{-1}\sqrt{\frac{2}{3}} & \frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}, \\ A_{1,1} &= \begin{pmatrix} \frac{1}{6}\left(3-\sqrt{3}\right) & -\sqrt[3]{-1}\sqrt{\frac{2}{3}} \\ (-1)^{2/3}\sqrt{\frac{2}{3}} & \frac{1}{6}\left(3+\sqrt{3}\right) \end{pmatrix}. \end{aligned} \tag{11}$$

Using the $SU(2)$ kernel for the Wigner function

$$\Delta^{(0)}(z) = \begin{pmatrix} \frac{\sqrt{3}(1-|z|^2)}{2(|z|^2+1)} + \frac{1}{2} & \frac{\sqrt{3}z}{|z|^2+1} \\ \frac{\sqrt{3}\bar{z}}{|z|^2+1} & \frac{\sqrt{3}(|z|^2-1)}{2(|z|^2+1)} + \frac{1}{2} \end{pmatrix} \tag{12}$$

one can easily check that the phase-point operators (11) correspond to the continuous kernel (12) evaluated at special points of the phase space, namely $A_{i,j} = \Delta^{(0)}(z_{i,j})$ with $z_{0,0} = 0$, $z_{0,1} = \sqrt{2}$, $z_{1,0} = (-1)^{2/3}\sqrt{2}$ and $z_{1,1} = -(-1)^{2/3}\sqrt{2}$. Applying the stereographic projection $z = \tan(\theta/2)\exp(\mathrm{i}\phi)$ we find that the points corresponding to the phase-point operators form vertices of a regular tetrahedron embedded in the continuous $SU(2)$ phase space (see fig. 1).

We have established a connection between discrete quasiprobability distributions and $SU(N)$ quasiprobability distributions on finite-dimensional Hilbert spaces. This identification enables to combine sampling of the initial state in the discrete phase-space and continuous simulation in the $SU(N)$ phase space [27]. Further, continuous phase-space methods enable a systematic expansion in the quantum noise parameter [11,12] and extension to the dissipative models [11]. However, in order to use the phase-space methods one needs to find a differential representation for the operators on the local Hilbert space. This is in general a difficult task, which was accomplished in for any $SU(N)$ group only in the $s = 1$ (P representation) case [36], and for any s only in the two-dimensional case [11] (*i.e.*, for the $SU(2)$ group). Once the equations of motion in the phase space are obtained the main obstacles for their efficient simulation are divergence of stochastic trajectories and negativity of the diffusion kernel. The later can be avoided by using the positive-P

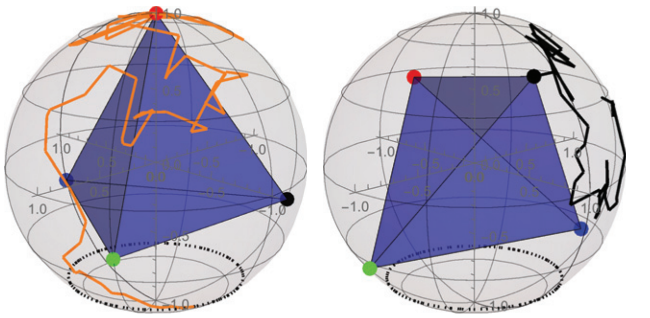

Fig. 1: (Colour on-line) Embedding of the discrete phase space into the condinuous phase space: each sphere represents the continuous $SU(2)$ phase space and the vertices of the tetrahedron correspond to discrete phase-space points (11). The left and right panels show a different realisation of the discrete phase space (*e.g.*, on the left the red, green, blue and black vertex points correspond to $z_{0,0}$, $z_{0,1}$, $z_{1,0}$ and $z_{1,1}$ defined in the text, respectively). Each rotation of this tetrahedron is a valid discrete probability distribution. An example of stochastic trajectory with discrete phase-space projection: the line in the left panel corresponds to a stochastic evolution before the projection. When $z(t) = z_{\max} = 3$ (*i.e.*, when the line in the left panel reaches the black dashed circle) it is represented by a discrete probability distribution shown by the vertices of the tetrahedron in the right panel. In the next step, this discrete probability distribution is randomly projected onto one of its phase space points (in the presented case this is the the black vertex). After that the stochastic evolution continues (shown by the black line in the right panel).

representation (positive-P function), whereas the former can be to some extent circumvented by exploiting a particular gauge freedom [37].

Discrete positive-P distribution, diverging trajectories and noise reduction. – In the rest of the paper we discuss the applicability of the positive-P function on discrete phase spaces and propose a discrete phase-space method to treat the problem of diverging trajectories. The normalised positive-P function is usually defined through the off-diagonal coherent state expansion of the density matrix [36]

$$\rho = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\mathrm{d}\mu(\Omega')P^{+}(\Omega,\Omega')\frac{|\Omega\rangle\langle\Omega'|}{\langle\Omega'|\Omega\rangle}. \qquad (13)$$

where $|\Omega\rangle := \Lambda(\Omega)|0\rangle$ denotes a normalized coherent state. We can write the above equation (13) in terms of the quantisation kernel (7) as

$$\rho = \int_{\mathcal{M}} \mathrm{d}\mu(\Omega)\mathrm{d}\mu(\Omega')P^{(s)}(\Omega,\Omega')\Delta^{(-s)}(\Omega,\Omega'), \qquad (14)$$

with the kernel

$$\Delta^{(s)}(\Omega,\Omega') = \frac{\Lambda(\Omega)\Delta^{(s)}\Lambda^{\dagger}(\Omega')}{\mathrm{tr}\,(\Lambda(\Omega)\Delta^{(s)}\Lambda^{\dagger}(\Omega'))}. \qquad (15)$$

By choosing $s = 1$ in eq. (14) we obtain the density matrix expansion given in (13)[1]. The positive-P function $P^{(+1)}(\Omega,\Omega')$ always exists and can be expressed in terms of the discrete phase-space values as

$$P^{(+1)}(\Omega,\Omega') = P^{(+1)}(\Omega)\delta(\Omega^{-1}_{\alpha,\beta}\Omega)\delta(\Omega'), \qquad (16)$$

where $P^{(+1)}(\Omega_{\alpha,\beta}) = \mathrm{tr}\,(\rho\Delta^{(+1)}_{\alpha,\beta})$ and $\delta(\Omega)$ is the delta function for the measure $\mu(\Omega)$. The above equation (16) means that the discrete-P distribution can be interpreted as a continuous positive-P distribution. Hence, the discrete-P representation can be used to sample the initial condition in simulations of the positive-P function. The importance of the initial condition sampling and a comparison of continuous and discrete distributions was discussed in [27]. However, the derivation in [27] is valid only for Wigner functions of spin-(1/2) systems and is based on a product-probability assumption, which results in a truncated Wigner-type approximation not permitting a systematic expansion or application to open quantum systems. The approach presented here is valid for any N, any s-ordered quasiprobability distribution and the generalised positive-P function, enables a systematic expansion in the noise terms, and is applicable to open quantum systems.

Using the phase-space correspondence one can express the action of the elements of the $su(N)$ algebra on the density matrix by using only first derivatives with respect to the phase-space variables [36]. Hence, any evolution equation for the density matrix which contains at most linear elements in the $su(N)$ generators can be expressed as an ordinary differential equation on the extended phase space. On the other hand, any evolution equation which contains at most quadratic elements in the $su(N)$ generators can be expressed as a stochastic differential equation on the extended phase space. Stochastic terms arise from the interaction between different sites of a many-body system or dissipation and make the simulation of evolution equations inefficient for longer times. One procedure to reduce the noise is to add a gauge or to experiment with different decompositions of the diffusion matrix [37]. Another possibility is to enlarge the local Hilbert space with dimension N by including k nearest neighbours. Inside the block the evolution in the $SU(N^k)$ generalised phase space is described by ordinary differential equations, whereas the interaction between the blocks is still treated stochastically. The decomposition into larger blocks is still exact but reduces the number of stochastic terms at the expense of enlarging the local phase space[2].

[1] In the cases $s = 0, -1$ the kernel (15) seizes to have appropriate analyticity properties with respect to the phase-space variables, hence, the positive-P distributions cannot be generalised to the $s = 0,\ -1$ cases in a straightforward way.

[2] A similar method was proposed in [13], where the interaction between the blocks was treated in the first-order approximation (TWA). In our case the evolution equation is still exact and the interaction is treated by inclusion of stochastic terms.

Reduction of noise does not necessary solve the problem of diverging stochastic trajectories, which can be avoided by using discrete distributions as follows. Whenever the phase-space variable becomes too large we expand the corresponding kernel in terms of an equivalent discrete distribution. Then we randomly choose (according to the obtained discrete distribution) one of the discrete phase-point operators (kernels) and continue the simulation with the chosen kernel. For a large number of simulated trajectories this procedure converges to the exact result. An example trajectory for the model considered in the next section is shown in fig. 1.

Example: dissipative long-range transverse Ising chain. – As an example of the discrete phase space projection method described in the previous section we consider a long-range dissipative Ising chain in a transverse field with the Hamiltonian

$$H = \sum_{j,k=1}^{n} \frac{J(\alpha,n)}{2|j-k|^{\alpha}} \sigma_j^{\mathrm{x}} \sigma_k^{\mathrm{x}} + h \sum_j \sigma_j^{\mathrm{z}}, \qquad (17)$$

boundary Lindblad operators $L_1 = \sqrt{\gamma_1}\sigma_1^+$, $L_2 = \sqrt{\gamma_2}\sigma_1^-$, $L_3 = \sqrt{\gamma_3}\sigma_n^+$, $L_4 = \sqrt{\gamma_4}\sigma_n^+$, and bulk dephasing $L_{4+j} = \sqrt{\gamma_{\mathrm{D}}\sigma_j^{\mathrm{z}}}$, $j = 1, 2, \ldots n$. We are using the Kac normalization $J(\alpha,n) = (\sum_{j=1}^{n} j^{-\alpha})^{-1}$. The time evolution of the density matrix is given by the Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}[H,\rho] + \sum_{\mu=1}^{n+4} L_\mu \rho L_\mu^\dagger - \frac{1}{2}\{L_\mu^\dagger L_\mu, \rho\} \qquad (18)$$

and can be expressed as a partial differential equation on the extended phase space [2]. This problem cannot be solved exactly by any known method to treat open quantum systems [38–42] and it is hard to simulate with existing numerical methods [43]. Usually we are interested in the steady state of open quantum systems. In order to estimate the long-time behaviour we consider the interaction semiclassically, whereas the dissipation is treated fully quantum mechanically. The semiclassical limit is obtained by taking into account only first-order derivatives in the evolution equation, *i.e.* omitting nonlinear (in the $gl(N)$ generators) terms of the Hamiltonian [4]. For small system sizes ($n = 5$) we compare the results obtained by the proposed method with exact results and observe good agreement for short and long times. It is surprising that semiclassical approximation of the interaction provides an accurate estimation of the long-time behaviour even in the strongly interacting regime (see fig. 2). We also compute the long-time averages for a larger system with $n = 20$. Parameters used in presented simulations are $\alpha = 1.5$, $h = 1$, $\gamma_1 = 0.2$, $\gamma_2 = 0.02$, $\gamma_3 = 0.1$, $\gamma_4 = 0.05$, $\gamma_{\mathrm{D}} = 0.001$.

Conclusions and discussion. – We have shown that the phase-point operators defining discrete quasiprobability distributions are equivalent to the continuous quantisation kernel evaluated at spacial points of the phase space. This identification justifies the use the of continuous phase-space methods on discrete phase spaces. Further, we demonstrated that discrete phase-space sampling can be used to treat the problem of diverging trajectories. We applied the procedure on a dissipative transverse Ising chain and calculated the long-time expectation values of local observables up to system size $n = 20$. We found that long-time behaviour of local observables is well described by a semiclasical approximation of the interaction even in the strongly interacting regime. It is an interesting question if this generalises to other models. For example to the integrable XXZ chain, where the steady state is described by a quasilocal operator [44]. The discrete phase space can be regarded as a special basis in the

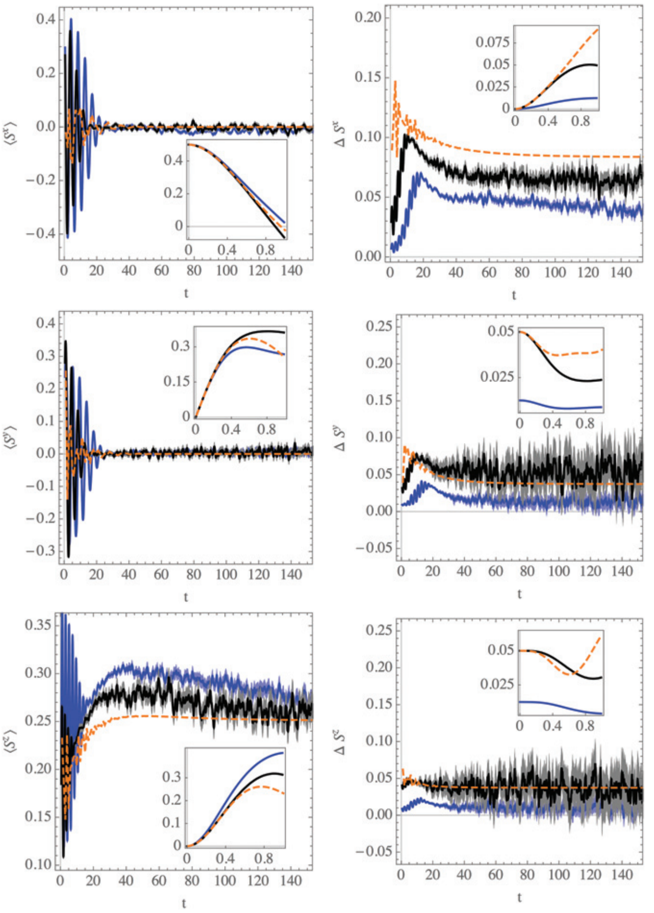

Fig. 2: (Colour on-line) The left panels show the average spin per particle in the x-, y-, z-direction (from top to bottom, respectively), $S^\alpha = \frac{1}{2n}\sum_{k=1}^{n} \sigma_j^\alpha$, $\alpha = x, y, z$. The right panels show the two point correlations $\Delta S^\alpha = \langle S^\alpha S^\alpha \rangle - \langle S^\alpha \rangle^2$. The orange dashed line represents the exact result for $n = 5$, the black and blue lines show the stochastic simulation result for $n = 5$ and $n = 20$, respectively. The grey/light blue regions denote the statistical error of the stochastic simulation with 10^3 trajectories. The initial state is a product state with all spins pointing in the x-direction. The insets show the short-time dynamics. The parameters used in the simulation are $\alpha = 1.5$, $h = 1$, $\gamma_1 = 0.2$, $\gamma_2 = 0.02$, $\gamma_3 = 0.1$, $\gamma_4 = 0.05$, $\gamma_{\mathrm{D}} = 0.001$.

space of operators. Thus, the established connection between discrete and continuous phase space may enable a construction of mixed Hilbert-space–phase-space methods for simulation of many-body quantum systems. From a more theoretical point of view it would be interesting to explore its implications on the problem of mutually unbiased bases in non-prime number dimensions.

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