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On the independence of strain invariants of two preferred direction nonlinear elasticity

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ABSTRACT

It is often assumed in the literature that the nine classical strain invariants, which are used to characterize the strain energy of a compressible anisotropic elastic solid with two preferred non-orthogonal directions are independent. In this paper, it is shown that only six of the classical strain invariants are independent, and syzygies exist between the classical invariants. Alternatively, using principal axis techniques, it is simply proven that, only six of the classical strain invariants are independent and syzygies exist between the principal axis strain invariants. Consequently, all other sets of strain invariants, proposed in the literature, which are uniquely related to the set of principal axis strain invariants, have only six independent invariants. Due to syzygies, it is shown that the number of ground state constants required to fully describe the quadratic linear strain energy function of two-fibre solids is fourteen, not thirteen, as assumed in the literature.

1. Introduction

Following the work of Spencer (1984), a strain energy function W_F of a compressible elastic material with two preferred unit directions $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed as

$$W_F = W(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}), \tag{1}$$

where $\boldsymbol{C}$ is the right Cauchy–Green deformation tensor and $\otimes$ denotes the dyadic product. W is an isotropic invariant function of $\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}$ and $\boldsymbol{b}\otimes\boldsymbol{b}$, i.e.,

$$W(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}) = W\left(\boldsymbol{QCQ}^T, \boldsymbol{Q}(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{b}\otimes\boldsymbol{b})\boldsymbol{Q}^T\right) \tag{2}$$

must be satisfied for all proper orthogonal tensors $\boldsymbol{Q}$. It follows that the strain energy function W_e can be expressed in terms of a set of invariants

$$\mathcal{S}_B = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}\}, \tag{3}$$

where

$$I_1 = \mathrm{tr}(\boldsymbol{C}), \quad I_2 = \frac{I_1^2 - \mathrm{tr}\left(\boldsymbol{C}^2\right)}{2}, \quad I_3 = det(\boldsymbol{C}), \quad I_4 = \boldsymbol{a}\bullet\boldsymbol{Ca}, \tag{4}$$

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$$I_5 = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a}, \quad I_6 = \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b}, \quad I_7 = \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b}, \quad I_8 = (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{b}, \tag{5}$$

$$I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2 \neq 0, \quad I_{10} = (\boldsymbol{a} \bullet \boldsymbol{b})\boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{b} \tag{6}$$

and tr denotes the trace of a second order tensor. The invariant I_9 is independent of strain and hence the set

$$\mathcal{S}_C = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_{10}\} \tag{7}$$

of 9 invariants is commonly used to describe the strain energy function (see Spencer, 1984). In this paper, we show that only seven of the the 10 invariants in (3) are independent or six of the nine invariants in (7) are independent. In the case when the preferred directions are orthogonal, $I_8 = I_9 = I_{10} = 0$, Shariff (2013) has shown that only six of the seven invariants I_j, $j = 1, 2, 3, \ldots, 7$ are independent. In Section 2, the proof is presented using a set of principal axis invariants, while in Section 3 the proof is done directly using the definition of the invariants given in (4)–(6). In Section 4, the consequences of the syzygies on the number of ground state constants are discussed via linear elasticity theory.

Preliminary concepts: Functional and integrity bases, syzygy

Let us review some concepts given, for example, in Zheng (1994), Spencer (1971) and Xiao (1996). Consider a set of isotropic invariants $I_1, \ldots, I_k$ of the tensors $\boldsymbol{C}, \boldsymbol{a} \otimes \boldsymbol{a}$ and $\boldsymbol{b} \otimes \boldsymbol{b}$ (denoted by $\mathcal{S}$).

1. Any single-valued function of $I_1, \ldots, I_B$

$$f(\mathcal{S}) = g(I_1, , \ldots I_B) \tag{8}$$

   is called a *representation* for isotropic scalar-valued functions of $\mathcal{S}$. If one of the invariants in the set $\{I_1, \ldots, I_B\}$ is expressible as a single-valued function of the remainders, the invariant is said to be *functionally reducible*. The representation is said to be *complete*, if any isotropic scalar-valued function of $\mathcal{S}$ can be expressed in the form (8). A functional basis for isotropic scalar-valued functions of $\mathcal{S}$ is the set of invariants in a complete representation for isotropic scalar-valued functions of $\mathcal{S}$. A functional basis is said to be *irreducible*, if none of its proper subsets is a functional basis.
2. If the function $f(\mathcal{S})$ is restricted to polynomial functions, then integrity bases are dealt with. A polynomial invariant is said to be *reducible* if it can be expressed as a *polynomial* in other invariants; otherwise, it is said to be *irreducible*. A set $\mathcal{S}_P$ of polynomial invariants which has the property that any polynomial scalar function can be expressed as a polynomial in members of the given set, is called an *integrity basis*. The integrity basis is said to be *minimal*, if none of its proper subset is an integrity basis. It frequently happens that polynomial relations exist between invariants which do not permit any one invariant to be expressed as a polynomial in the remainder. Such relations are called *syzygies*.
3. An minimal integrity basis is not necessarily an irreducible functional basis, and the later, in general, contains fewer elements than the former.

2. Proof using principal axis invariants that only seven(six) of the ten(nine) invariants are independent

In this paper all subscripts i and j take the values of 1, 2 and 3, unless stated otherwise. If we write

$$\boldsymbol{C} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{e}_i \otimes \boldsymbol{e}_i \tag{9}$$

where λ_i and $\boldsymbol{e}_i$, $i = 1, 2, 3$ are the principal values and the principal directions of the right stretch tensor $\boldsymbol{U}$, respectively, and substitute (9) in (4)–(6), we have the expressions:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2, \tag{10}$$

$$I_4 = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3, \quad I_5 = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3, \tag{11}$$

$$I_6 = \lambda_1^2\xi_1 + \lambda_2^2\xi_2 + \lambda_3^2\xi_3, \quad I_7 = \lambda_1^4\xi_1 + \lambda_2^4\xi_2 + \lambda_3^4\xi_3, \tag{12}$$

$$I_8 = \sum_{i=1}^{3} \lambda_i^2 \chi_i, \quad I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2, \quad I_{10} = \sum_{i=1}^{3} \lambda_i^4 \chi_i, \tag{13}$$

where

$$\zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2, \quad \xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2, \quad \chi_i = (\boldsymbol{a} \bullet \boldsymbol{b})(\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i) \quad i = 1, 2, 3. \tag{14}$$

The thirteen terms

$$\lambda_i, \quad \zeta_i, \quad \xi_i, \quad \chi_i \quad (i = 1, 2, 3), \quad \alpha = I_9 = (\boldsymbol{a} \bullet \boldsymbol{b})^2 \tag{15}$$

are invariants with respect to all proper orthogonal tensors $\boldsymbol{Q}$. We note that if we write the strain energy function in the principal axis form, i.e.,

$$W_F = \overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{e}_1 \otimes \boldsymbol{e}_1, \boldsymbol{e}_2 \otimes \boldsymbol{e}_2, \boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \boldsymbol{a} \otimes \boldsymbol{a}, \boldsymbol{b} \otimes \boldsymbol{b}), \tag{16}$$

then it is shown in Appendix A that $\bar{W}$ can be expressed in terms of the thirteen principal axis invariants λ_i, ζ_i, ξ_i, χ_i $(i = 1, 2, 3)$, and α (see also, for example, Shariff (2011)). A subset of these invariants have been used by Shariff (2008, 2012) to describe the mechanical behavior of nonlinear anisotropic elastic solids.

It can be easily shown that

$$\zeta_3 = 1 - \zeta_1 - \zeta_2, \quad \xi_3 = 1 - \xi_1 - \xi_2, \quad \chi_3 = \alpha - \chi_1 - \chi_2, \tag{17}$$

and the syzygies

$$\chi_1^2 = \alpha\zeta_1\xi_1, \quad \chi_2^2 = \alpha\zeta_2\xi_2. \tag{18}$$

We note that from (17) and (18) $\zeta_1, \zeta_2, \zeta_3, \xi_3, \chi_3$ are functionally reducible and hence they can be omitted from the functional basis. As well as this, ζ_3, ξ_3 and χ_3 are reducible and hence can be omitted from the integrity basis.

In addition to the syzygies in (18), we show below that there exists another syzygy via the relation

$$\alpha = \left(\sum_{i=1}^{3} (\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i) \right)^2. \tag{19}$$

From (19) we get

$$\zeta_2\xi_2\zeta_3\xi_3(8\zeta_1\xi_1 + 4c)^2 = [c^2 + 4(\zeta_2\xi_2\zeta_3\xi_3 - \zeta_1\xi_1\zeta_2\xi_2 - \zeta_1\xi_1\zeta_3\xi_3)]^2, \tag{20}$$

where

$$c = \alpha - \sum_{i=1}^{3} \zeta_i\xi_i. \tag{21}$$

Hence, we have thirteen invariants and six independent relations (three in (17), two in (18) and one in (20)), which shows that only seven invariants are independent. If we omit the non-deformation invariant α in the list (15) then only six of the nine remaining invariants are independent. Since we have three syzygies, the minimal integrity basis

$$\mathcal{S}_A = \{\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1, \xi_2, \chi_1, \chi_2, \alpha\} \tag{22}$$

contains ten invariants. The irreducible functional basis

$$\mathcal{F}_A = \{\lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2, \chi_1, \chi_2, \alpha\} \tag{23}$$

contains eight invariants. In view of (13), seven independent principal axis invariants suggest that only seven of the classical invariants I_j, $j = 1, 2, 3, \ldots, 10$ are independent (in Section 3 we show this using only the classical invariants).

3. Proof using the classical invariants that only seven(six) of the ten(nine) invariants are independent

In this section we prove (without using the principal axis invariants) that only seven of the ten classical invariants are independent. To do this, we first consider the right-handed set of orthogonal unit vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

$$\boldsymbol{v}_1 = \boldsymbol{a}, \quad \boldsymbol{v}_2 = \frac{\boldsymbol{b} - (\boldsymbol{v}_1 \bullet \boldsymbol{b})\boldsymbol{v}_1}{\beta}, \quad \boldsymbol{v}_3 = \frac{1}{\beta}\boldsymbol{a} \times \boldsymbol{b}, \tag{24}$$

where $\beta = | \boldsymbol{b} - (\boldsymbol{v}_1 \bullet \boldsymbol{b})\boldsymbol{v}_1 |$. Let us define the invariants

$$I_{ij} = I_{ji} = \boldsymbol{v}_i \bullet \left(\boldsymbol{C}\boldsymbol{v}_j\right), \quad H_i = \boldsymbol{v}_i \bullet \left(\boldsymbol{C}^2\boldsymbol{v}_i\right) \quad (\text{no sum in } i), \quad i, j = 1, 2, 3, \tag{25}$$

taking note that

$$I_{11} = I_4, \quad H_1 = I_5, \quad \beta^2 = 1 - I_9. \tag{26}$$

Using Eq. (24) and from Eq. (B11) of Appendix B we have

$$I_{12}^2 = I_2 + I_4 I_{22} + I_5 + H_2 - I_1(I_4 + I_{22}), \tag{27}$$

where, in view of (24) we obtain

$$I_{22} = \frac{I_6 - 2I_8 + I_9 I_4}{1 - I_9}, \quad H_2 = \frac{I_7 - 2I_{10} + I_9 I_5}{1 - I_9}, \tag{28}$$

while

$$I_{12}^2 = (\boldsymbol{v}_1 \bullet \boldsymbol{C}\boldsymbol{v}_2)^2 = \frac{I_8^2 I_9^{-1} - 2I_8 I_4 + I_4^2 I_9}{1 - I_9}. \tag{29}$$

Hence, from (27)–(29), we have the syzygy

$$I_8^2 = I_9(1 - I_9)I_2 + 2I_1 I_8 I_9 - 2I_9 I_{10} + I_9 I_4 I_6 - I_1 I_9 (I_4 + I_6) + I_9(I_5 + I_7). \tag{30}$$

From Eq. (30), we have that I_{10} (say) is functionally reducible and hence can be omitted from the functional basis. An alternative derivation for the syzygy (30) can also be found in Bustamante (2007).

Using (24), we obtain

$$I_{10} = \beta(\boldsymbol{a} \bullet \boldsymbol{b})\left(\boldsymbol{v}_1 \bullet \boldsymbol{C}^2\boldsymbol{v}_2\right) + I_9 I_5, \tag{31}$$

where[1]

$$\boldsymbol{v}_1 \bullet \boldsymbol{C}^2\boldsymbol{v}_2 = I_{12}(I_4 + I_{22}) + I_{13}I_{32}. \tag{32}$$

In view of (31) and (32), we have

$$I_{10} - I_9 I_5 = \beta(\boldsymbol{a} \bullet \boldsymbol{b})(I_{12}(I_4 + I_{22}) + I_{13}I_{32}). \tag{33}$$

Using (33) we have the relation

$$\left\{(I_{10} - I_9 I_5)^2 - (1 - I_9)I_9\left(I_{12}^2(I_4 + I_{22})^2 + I_{13}^2 I_{32}^2\right)\right\}^2 = 4I_{13}^2 I_{32}^2 I_{12}^2 (I_4 + I_{22})^2, \tag{34}$$

where from Appendix B we have

$$I_{23}^2 = \frac{I_7 - 2I_{10} + I_9 I_5}{1 - I_9} - \left(\frac{I_6 - 2I_8 + I_9 I_4}{1 - I_9}\right)^2 - \left(\frac{I_8^2 I_9^{-1} - 2I_8 I_4 + I_4^2 I_9}{1 - I_9}\right), \tag{35}$$

$$I_{13}^2 = I_5 - I_4^2 - \left(\frac{I_8^2 I_9^{-1} - 2I_8 I_4 + I_4^2 I_9}{1 - I_9}\right). \tag{36}$$

The first relation was obtained from (B4) considering $(28)_{1,2}$ and (29), whereas the second relation above was obtained using (B3) considering (26) and (29). A syzygy can be easily obtained using (34)–(36), (29) and $(28)_1$.

Since I_{ij} are components of $\boldsymbol{C}$ relative to the orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, we have $I_3 = det(\boldsymbol{C}) = det(I_{ij})$, hence:

$$I_3 = I_4 I_{22} I_{33} - I_4 I_{23}^2 - I_{22} I_{13}^2 - I_{33} I_{12}^2 + 2I_{12} I_{13} I_{23}. \tag{37}$$

This relation can be also obtained from Eq. (5.5) of Holzapfel and Ogden (2009) taking (see the notation in that paper) $\boldsymbol{f}_0$ as $\boldsymbol{v}_1$, $\boldsymbol{s}_0$ as $\boldsymbol{v}_2$, $\boldsymbol{n}_0$ as $\boldsymbol{v}_3$ and considering $(26)_1$. From $(28)_1$ and (24) we also have

$$I_{33} = I_1 - I_4 - \left(\frac{I_6 - 2I_8 + I_9 I_4}{1 - I_9}\right). \tag{38}$$

Considering (37), we have the third relation

$$[I_3 + I_4(I_{23}^2 - I_{22}I_{33}) + I_{22}I_{13}^2 + I_{33}I_{12}^2]^2 = 4I_{12}^2 I_{13}^2 I_{23}^2, \tag{39}$$

Another syzygy can be easily obtained using (39),(38),(35), (36), (29) and $(28)_1$.

Hence, due to the three relations (30), (34) and (39), we only have seven independent invariants. If we omit the non-deformation invariant I_9, then only six of the nine remaining invariants are independent. The minimal integrity basis is $\mathcal{S}_B$ and the irreducible functional basis

$$\mathcal{F}_B = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9\} \tag{40}$$

contains only nine invariants.

3.1. Symmetry

It is clear that the syzygy (30) is symmetric with respect to an interchange of I_4 with I_6, and I_5 with I_7 (or equivalently an interchange of $\boldsymbol{a}$ and $\boldsymbol{b}$). Consider the right-handed set of orthogonal unit vectors $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, where

$$\boldsymbol{w}_1 = \boldsymbol{b}, \quad \boldsymbol{w}_2 = \frac{\boldsymbol{a} - (\boldsymbol{w}_1 \bullet \boldsymbol{a})\boldsymbol{w}_1}{\beta}, \quad \boldsymbol{w}_3 = \frac{1}{\beta}\boldsymbol{b} \times \boldsymbol{a} \tag{41}$$

and let $\bar{I}_{ij} = \boldsymbol{w}_i \bullet \boldsymbol{C}\boldsymbol{w}_j$. We note that $I_{ij} = f_{ij}(\boldsymbol{C}, \boldsymbol{a}, \boldsymbol{b})$ and $\bar{I}_{ij} = f_{ij}(\boldsymbol{C}, \boldsymbol{b}, \boldsymbol{a})$ are components of $\boldsymbol{C}$ with respect the orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$, respectively. Since I_{10} and I_3 are invariants and due to the fact that I_{ij} and $\bar{I}_{ij}$ have the same functional form, we have

[1] See Appendix B and Eqs. (B1) and (B2), using such expressions to calculate $\boldsymbol{C}\boldsymbol{v}_2$ and $\boldsymbol{v}_1 \cdot \boldsymbol{C}^2\boldsymbol{v}_2$, respectively.

$$\begin{aligned} I_{10} &= (\boldsymbol{a}\bullet\boldsymbol{b})\boldsymbol{a}\bullet\boldsymbol{C}^2\boldsymbol{b} = (\boldsymbol{a}\bullet\boldsymbol{b})\left\{\boldsymbol{a}\bullet\left[\sum_{i,j}^{3}\left(\sum_{k}^{3} I_{ik}I_{kj}\right)\boldsymbol{v}_i\otimes\boldsymbol{v}_j\right]\boldsymbol{b}\right\} = g(\boldsymbol{C},\boldsymbol{a},\boldsymbol{b}), \\ &= (\boldsymbol{b}\bullet\boldsymbol{a})\left\{\boldsymbol{b}\bullet\left[\sum_{i,j}^{3}\left(\sum_{k}^{3} \bar{I}_{ik}\bar{I}_{kj}\right)\boldsymbol{w}_i\otimes\boldsymbol{w}_j\right]\boldsymbol{a}\right\} = g(\boldsymbol{C},\boldsymbol{b},\boldsymbol{a}) \end{aligned} \tag{42}$$

and

$$I_3 = det\left(I_{ij}\right) = det\left(f_{ij}(\boldsymbol{C},\boldsymbol{a},\boldsymbol{b})\right) = f(\boldsymbol{C},\boldsymbol{a},\boldsymbol{b}) = det\left(\bar{I}_{ij}\right) = det\left(f_{ij}(\boldsymbol{C},\boldsymbol{b},\boldsymbol{a})\right) = f(\boldsymbol{C},\boldsymbol{b},\boldsymbol{a}). \tag{43}$$

In view of (24), and (41)–(43), relations that are symmetric with respect to interchange of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be constructed from (34) and (39), since they are derived from the symmetric relations (42) and (43).

4. On the number of ground state constants

Due to polynomial relations, a minimal integrity basis is useful in formulating a polynomial strain energy function. An infinitesimal strain strain energy function is a quadratic polynomial function and hence the appropriate minimal integrity basis is used to evaluate the number of constants needed to formulate a general quadratic strain energy function. In this section, we derive the number of constants needed for a general two preferred direction infinitesimal strain energy function W_e. We start by considering the invariants

$$J_1 = \text{tr}\boldsymbol{\varepsilon}, \quad J_2 = \text{tr}\boldsymbol{\varepsilon}^2, \quad J_3 = \text{tr}\boldsymbol{\varepsilon}^3, \quad J_4 = \boldsymbol{a}\bullet(\boldsymbol{\varepsilon}\boldsymbol{a}), \quad J_5 = \boldsymbol{a}\bullet\left(\boldsymbol{\varepsilon}^2\boldsymbol{a}\right), \tag{44}$$

$$J_6 = \boldsymbol{b}\bullet(\boldsymbol{\varepsilon}\boldsymbol{b}), \quad J_7 = \boldsymbol{b}\bullet\left(\boldsymbol{\varepsilon}^2\boldsymbol{b}\right), \quad J_8 = \cos(2\phi)\boldsymbol{a}\bullet(\boldsymbol{\varepsilon}\boldsymbol{b}), J_9 = (\boldsymbol{a}\bullet\boldsymbol{b})^2 = \cos^2(2\phi), \tag{45}$$

$$J_{10} = \cos(2\phi)\boldsymbol{a}\bullet\left(\boldsymbol{\varepsilon}^2\boldsymbol{b}\right), \tag{46}$$

which are suitable for an infinitesimal strain energy function W_e, where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor. Spencer (1984) omitted J_{10} from the set of invariants in W_e via the relation (which is shown later on that is incorrect):

$$(1-J_9)(J_1^2-J_2) + 2J_8J_1 - J_{10} - (J_4+J_6)J_1 + J_4J_6 - \frac{(J_8)^2}{J_9} + J_5 + J_7 = 0, \tag{47}$$

which is given by Eq. (33) in Spencer (1984). We note that Eq. (47) shows that J_{10} is not reducible since J_9 is considered an invariant although it is independent of the deformation. It is commonly believe that J_{10} is reducible[2] and hence it is often omitted from the strain energy function. We also note that the relation (47) is not correct. We prove this, simply, by letting $\boldsymbol{\varepsilon} = \epsilon\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity tensor, and we have, on the left hand side of (47),

$$3\epsilon^2\sin^2(2\phi) \neq 0. \tag{48}$$

The correct relation for J_{10} is obtained from (30), where I_j is replaced by J_j, i.e.:

$$J_{10} = \frac{1}{2}\left[J_5 + J_7 - \frac{J_8^2}{J_9} + J_4J_6 + J_1(2J_8 - J_4 - J_6) - J_2(J_9 - 1)\right]. \tag{49}$$

In view of the irreducible J_{10} in (49), we prove that the infinitesimal strain energy function W_e should have fourteen elastic constants for compressible materials, not thirteen as stated in Spencer (1984), and for incompressible materials the number of elastic constants is reduced to ten (not nine, as commonly assumed). However, we approach this proof, simply, using the principal axis invariants for infinitesimal deformations. Our final result indicates that this is equivalent to including J_{10} in the quadratic construction of W_e. Since most readers are not familiar with the principal axis formulations, for clarification, we start with isotropic materials and work our way through to two fibre materials. In this section, we let ν_i and $\boldsymbol{e}_i$ (we use the same notation for the eigenvector $\boldsymbol{e}_i$ corresponding to $\boldsymbol{U}$) to be the principal values and directions of $\boldsymbol{\varepsilon}$, respectively. The infinitesimal strain energy is assumed to be zero when $\boldsymbol{\varepsilon} = \boldsymbol{0}$, where the infinitesimal stress is also zero, except for an incompressible material, where it has an arbitrary value of p which is independent of $\boldsymbol{\varepsilon}$.

4.1. Isotropic elasticity

The strain energy function for an isotropic material is expressed by

$$W_e = W_{iso}(\nu_1, \nu_2, \nu_3) \tag{50}$$

with the symmetrical property

$$W_{iso}(\nu_1, \nu_2, \nu_3) = W_{iso}(\nu_2, \nu_1, \nu_3) = W_{iso}(\nu_1, \nu_3, \nu_2) = \text{etc}. \tag{51}$$

[2] The proof of I_{10} is reducible has not be found in the literature either.

The function W_{iso} must be quadratic in ν_i and possess the symmetrical property (51). To generate this type of quadratic strain energy function we use the following linear and quadratic invariants

$$J_1 = \nu_1 + \nu_2 + \nu_3 = \mathrm{tr}\boldsymbol{\varepsilon}, \quad J_2 = \nu_1^2 + \nu_2^2 + \nu_3^2 = \mathrm{tr}\boldsymbol{\varepsilon}^2, \tag{52}$$

$$K_1 = \nu_1\nu_2 + \nu_1\nu_3 + \nu_2\nu_3. \tag{53}$$

Since $2K_1 = J_1^2 - J_2$, we omit K_1 in the above list. We note that the above invariants satisfy the symmetrical property (51). A general quadratic strain energy function then takes the form

$$W_e = \mu\left(\nu_1^2 + \nu_2^2 + \nu_3^2\right) + \lambda(\nu_1 + \nu_2 + \nu_3)^2, \tag{54}$$

where μ and λ is the shear modulus and Lame's constant, respectively.

4.2. Transversely isotropic elasticity

Following the work of Shariff (2008), the strain energy function for a material with the preferred direction $\boldsymbol{a}$ can be written as

$$W_e = W_{trs}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3) \tag{55}$$

with the symmetrical property

$$\begin{aligned} &W_{trs}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3) \\ &\quad = W_{trs}(\nu_2, \nu_1, \nu_3, \zeta_2, \zeta_1, \zeta_3) \\ &\quad = W_{trs}(\nu_3, \nu_2, \nu_1, \zeta_3, \zeta_2, \zeta_1) = \text{etc.} \end{aligned} \tag{56}$$

It is *very important* to know that W_{trs} should be independent of ζ_i and ζ_j when $\nu_i = \nu_j$, $i \neq j$ in order W_{trs} to have a unique value due to the non-unique values of $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ when $\nu_i = \nu_j$. Similarly, W_{trs} should be independent of ζ_k, $k = 1, 2, 3$ when $\nu_1 = \nu_2 = \nu_3$. We call this independent property the *P-property*. To generate a general quadratic expression for W_{trs} with this property and the symmetrical property (56), we use the following linear and quadratic symmetrical invariants

$$J_4 \;=\; \zeta_1\nu_1 + \zeta_2\nu_2 + \zeta_3\nu_3 = \boldsymbol{a} \bullet (\boldsymbol{\varepsilon}\boldsymbol{a}), \quad J_5 = \zeta_1\nu_1^2 + \zeta_2\nu_2^2 + \zeta_3\nu_3^2 = \boldsymbol{a} \bullet \left(\boldsymbol{\varepsilon}^2\boldsymbol{a}\right), \tag{57}$$

$$K_2 = \zeta_1\nu_2\nu_3 + \zeta_2\nu_1\nu_3 + \zeta_3\nu_1\nu_2 \tag{58}$$

together with the invariants J_1 and J_2. However,

$$K_2 = J_5 + \frac{J_1^2 - J_2}{2} - J_4J_1, \tag{59}$$

hence, we omit K_2 from the above list. We note that, for example, when $\nu_1 = \nu_2 = \nu$, we have

$$J_4 = \nu(1 - \zeta_3) + \zeta_3\nu_3, \quad J_5 = \nu^2(1 - \zeta_3) + \zeta_3\nu_3^2, \tag{60}$$

and they are independent of ζ_1 and ζ_2. It is also clear that they are independent of ζ_k, $k = 1, 2, 3$ when all the principal variables have the same value. A general quadratic strain energy function takes the form

$$W_e = q_1J_1^2 + q_2J_1J_4 + q_3J_4^2 + q_4J_2 + q_5J_5 \tag{61}$$

with 5 material constants q_l, $l = 1, 2, 3, 4, 5$; this is consistent with existing theory. For an incompressible material, $J_1 = 0$ and W_e has only 3 material constants.

4.3. Orthotropic elasticity

Following Shariff (2011), we have

$$W_e = W_{ort}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3), \tag{62}$$

for materials with two preferred orthogonal directions $\boldsymbol{a}$ and $\boldsymbol{b}$, and having the symmetric property

$$\begin{aligned} W_{ort}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3) &= W_{ort}(\nu_2, \nu_1, \nu_3, \zeta_2, \zeta_1, \zeta_3, \xi_2, \xi_1, \xi_3) \\ &= W_{ort}(\nu_3, \nu_2, \nu_1, \zeta_3, \zeta_2, \zeta_1, \xi_3, \xi_2, \xi_1) = \text{etc} \end{aligned} \tag{63}$$

together with the *P-property* similar to that described in Section 4.2. To generate a general quadratic W_{ort}, we use the following symmetrical invariants

$$J_6 \;=\; \xi_1\nu_1 + \xi_2\nu_2 + \xi_3\nu_3 = \boldsymbol{b} \bullet (\boldsymbol{\varepsilon}\boldsymbol{b}), \quad J_7 = \xi_1\nu_1^2 + \xi_2\nu_2^2 + \xi_3\nu_3^2 = \boldsymbol{b} \bullet \left(\boldsymbol{\varepsilon}^2\boldsymbol{b}\right), \tag{64}$$

$$K_3 = \xi_1\nu_2\nu_3 + \xi_2\nu_1\nu_3 + \xi_3\nu_1\nu_2 \tag{65}$$

together with J_1, J_2, J_4 and J_5. The above invariants have similar *P-property* as J_4 and J_5. We omit K_3 from the above list since

$$K_3 = J_7 + \frac{J_1^2 - J_2}{2} - J_6 J_1. \tag{66}$$

A general quadratic strain energy function takes the form

$$W_e = q_1 J_1^2 + q_2 J_1 J_4 + q_3 J_4^2 + q_4 J_2 + q_5 J_5 + q_6 J_1 J_6 + q_7 J_6^2 + q_8 J_4 J_6 + q_9 J_7, \tag{67}$$

with 9 (6 for incompressible materials) material constants, as given in the literature.

4.4. Two-preferred-direction elasticity

This section deals with materials with two preferred non-orthogonal directions $\boldsymbol{a}$ and $\boldsymbol{b}$. In view of the results presented Section 2, we have

$$W_e = W_{two}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \chi_1, \chi_2, \chi_3) \tag{68}$$

with the symmetrical property

$$\begin{aligned} &W_{two}(\nu_1, \nu_2, \nu_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \chi_1, \chi_2, \chi_3) \\ &\quad = W_{two}(\nu_2, \nu_1, \nu_3, \zeta_2, \zeta_1, \zeta_3, \xi_3, \xi_1, \xi_3, \chi_2, \chi_1, \chi_3) \\ &\quad = W_{two}(\nu_1, \nu_3, \nu_2, \zeta_1, \zeta_3, \zeta_2, \xi_1, \xi_3, \xi_2, \chi_1, \chi_3, \chi_2) = \text{etc} \end{aligned} \tag{69}$$

and satisfying the *P-property*. The additional symmetrical invariants satisfying the *P-property*, needed for a quadratic function W_e, are:

$$J_8 = \sum_{i=1}^{3} \chi_i \nu_i, \quad J_{10} = \sum_{i=1}^{3} \chi_i \nu_i^2, \tag{70}$$

$$K_4 = \chi_1 \nu_2 \nu_3 + \chi_2 \nu_1 \nu_3 + \chi_3 \nu_1 \nu_2. \tag{71}$$

Where K_4 can be omitted from the above list since

$$K_4 = J_{10} + \alpha \frac{J_1^2 - J_2}{2} - J_8 J_1. \tag{72}$$

The quadratic strain energy function then takes the form

$$W_e = q_1 J_1^2 + q_2 J_1 J_4 + q_3 J_4^2 + q_4 J_2 + q_5 J_5 + q_6 J_1 J_6 + q_7 J_6^2 + q_8 J_4 J_6 + q_9 J_7 + q_{10} J_1 J_8 + q_{11} J_4 J_8 + q_{12} J_6 J_8 + q_{13} J_8^2 + q_{14} J_{10} \tag{73}$$

with 14 constants. For an incompressible material it reduces to 10 constants. We note that the number of constants is reduced to 13 if the invariant J_{10} is omitted from (73). For a mechanically equivalent material W_e is unchange if we interchange $\boldsymbol{a}$ with $\boldsymbol{b}$. For this type of material $q_2 = q_6$ and $q_3 = q_7$ and $q_5 = q_9$, and hence it requires only 11 constants to characterize its mechanical behavior; this number of constants seems to agree with that obtained by Murphy (2014).

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Appendix A

The strain energy function for two preferred direction solids can be represented by

$$W_F = W(\boldsymbol{C}, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}) = \overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{e}_1\otimes\boldsymbol{e}_1, \boldsymbol{e}_2\otimes\boldsymbol{e}_2, \boldsymbol{e}_3\otimes\boldsymbol{e}_3, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}). \tag{A1}$$

$\bar{W}$ is an isotropic invariant function of $\boldsymbol{E}_1 = \boldsymbol{e}_1\otimes\boldsymbol{e}_1$, $\boldsymbol{E}_2 = \boldsymbol{e}_2\otimes\boldsymbol{e}_2$, $\boldsymbol{E}_3 = \boldsymbol{e}_3\otimes\boldsymbol{e}_3$, $\boldsymbol{a}\otimes\boldsymbol{a}$ and $\boldsymbol{b}\otimes\boldsymbol{b}$, i.e.,

$$\begin{aligned} &\overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{a}\otimes\boldsymbol{a}, \boldsymbol{b}\otimes\boldsymbol{b}) \\ &\quad = \overline{W}\left(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{Q}\boldsymbol{E}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{E}_2\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{E}_3\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{a}\otimes\boldsymbol{a})\boldsymbol{Q}^T, \boldsymbol{Q}(\boldsymbol{b}\otimes\boldsymbol{b})\boldsymbol{Q}^T\right) \end{aligned} \tag{A2}$$

for all proper orthogonal tensors $\boldsymbol{Q}$. Taking note that $\operatorname{tr}\boldsymbol{E}_i = \operatorname{tr}(\boldsymbol{a}\otimes\boldsymbol{a}) = \operatorname{tr}(\boldsymbol{b}\otimes\boldsymbol{b}) = 1$, $\boldsymbol{E}_i = \boldsymbol{E}_i^2 = \boldsymbol{E}_i^3 = \ldots$, $\boldsymbol{a}\otimes\boldsymbol{a} = (\boldsymbol{a}\otimes\boldsymbol{a})^2 = (\boldsymbol{a}\otimes\boldsymbol{a})^3 = \ldots$, $\boldsymbol{b}\otimes\boldsymbol{b} = (\boldsymbol{b}\otimes\boldsymbol{b})^2 = (\boldsymbol{b}\otimes\boldsymbol{b})^3 = \ldots$ and $\boldsymbol{E}_i\boldsymbol{E}_j = 0$, $i \neq j$, and using the results of Spencer (1971) for five matrices, it follows that W_e can be expressed as

$$W_e = \bar{W}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3, \chi_1, \chi_2, \chi_3, \alpha) \tag{A3}$$

where we have the invariants $\zeta_i = \text{tr}(\boldsymbol{E}_i(\boldsymbol{a} \otimes \boldsymbol{a}))$, $\xi_i = \text{tr}(\boldsymbol{E}_i(\boldsymbol{b} \otimes \boldsymbol{b}))$, $\chi_i = \text{tr}(\boldsymbol{E}_i(\boldsymbol{a} \otimes \boldsymbol{a})(\boldsymbol{b} \otimes \boldsymbol{b}))$, $i = 1, 2, 3$ and $\alpha = \text{tr}((\boldsymbol{a} \otimes \boldsymbol{a})(\boldsymbol{b} \otimes \boldsymbol{b})) = (\boldsymbol{a} \bullet \boldsymbol{b})^2$. We note that the invariants ζ_i, ξ_i and χ_i do not have unique values if two or three eigenvalues of $\boldsymbol{U}$ have the same value. However,

$$\zeta_3 = 1 - \zeta_1 - \zeta_2, \quad \xi_3 = 1 - \xi_1 - \xi_2 \quad \text{and} \quad \chi_3 = \alpha - \chi_2 - \chi_3. \tag{A4}$$

Appendix B

Consider the right-hand set of orthonormal vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. Note that

$$\boldsymbol{C}\boldsymbol{v}_1 = (\boldsymbol{v}_1 \bullet \boldsymbol{C}\boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{v}_2 \bullet \boldsymbol{C}\boldsymbol{v}_1)\boldsymbol{v}_2 + (\boldsymbol{v}_3 \bullet \boldsymbol{C}\boldsymbol{v}_1)\boldsymbol{v}_3. \tag{B1}$$

Hence

$$\boldsymbol{v}_1 \bullet \boldsymbol{C}^2\boldsymbol{v}_1 = (\boldsymbol{v}_1 \bullet \boldsymbol{C}\boldsymbol{v}_1)^2 + (\boldsymbol{v}_2 \bullet \boldsymbol{C}\boldsymbol{v}_1)^2 + (\boldsymbol{v}_3 \bullet \boldsymbol{C}\boldsymbol{v}_1)^2. \tag{B2}$$

From Eq. (B2) we have

$$I_{13}^2 = H_1 - I_{11}^2 - I_{12}^2. \tag{B3}$$

Similarly, it can be easily shown that

$$I_{23}^2 = H_2 - I_{12}^2 - I_{22}^2 \tag{B4}$$

and

$$H_3 = I_{13}^2 + I_{23}^2 + I_{33}^2. \tag{B5}$$

From the relation

$$I_1 = \text{tr}(\boldsymbol{C}) = \boldsymbol{v}_1 \bullet \boldsymbol{C}\boldsymbol{v}_1 + \boldsymbol{v}_2 \bullet \boldsymbol{C}\boldsymbol{v}_2 + \boldsymbol{v}_3 \bullet \boldsymbol{C}\boldsymbol{v}_3, \tag{B6}$$

we have

$$I_{33} = I_1 - I_{11} - I_{22}. \tag{B7}$$

From the above equations, we have

$$H_3 = H_1 + H_2 - 2I_{12}^2 + I_1^2 - 2I_1(I_{11} + I_{22}) + 2I_{11}I_{22}. \tag{B8}$$

From the relation

$$\text{tr}\left(\boldsymbol{C}^2\right) = \boldsymbol{v}_1 \bullet \boldsymbol{C}^2\boldsymbol{v}_1 + \boldsymbol{v}_2 \bullet \boldsymbol{C}^2\boldsymbol{v}_2 + \boldsymbol{v}_3 \bullet \boldsymbol{C}^2\boldsymbol{v}_3 \tag{B9}$$

we get

$$I_1^2 - 2I_2 = \text{tr}\left(\boldsymbol{C}^2\right) = H_1 + H_2 + H_3. \tag{B10}$$

Substituting Eq. (B8) into (B10) we have the relation

$$\begin{aligned} I_{12}^2 &= I_2 + I_{11}I_{22} + H_1 + H_2 - I_1(I_{11} + I_{22}) \\ &= I_2 + I_4I_{22} + I_5 + H_2 - I_1(I_4 + I_{22}). \end{aligned} \tag{B11}$$

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