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Joint location-inventory problem with differentiated service levels using critical level policy

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A B S T R A C T

This paper analyzes the design of a distribution network for fast-moving items able to provide differentiated service levels in terms of product availability for two demand classes (high and low priority) using a critical level policy. The model is formulated as a MINLP with chance constraints for which we propose a heuristic to solve it. Although the heuristic does not guarantee an optimal solution, our computational experiments have shown that it provides good-quality solutions that are on average 0.8% and at worst 2.7% from the optimal solution.

1. Introduction

Fast moving-items are products with high demand volume or items with high inventory turnover. Examples include non-perishable food, toiletries, over-the-counter drugs, cleaning supplies, building supplies and office supplies. The distribution channels of these products have been concentrated in large retails chains requiring high service level in terms of product availability at the supplier's expense. Therefore, many wholesalers segment their customers based on service level. The simplest segmentation is to classify customers into two demand classes, (i) high priority class will correspond to large retail chains that require high service levels and (ii) low priority class corresponding to small retailers which can be provided lower service levels.

An efficient way of providing differentiated service levels is through a *critical level policy*. This policy is a inventory control model for rationing the inventory between different classes of customers and its main application is in inventory systems that must provide differentiated service levels to two or more classes of demand. Deshpande et al. (2003) and Escalona and Ordóñez (2015) have provided evidence of the efficiency of the critical level policy compared to other traditional inventory control policies that allow providing differentiated service levels as *round-up* or *separate stock* policies.

Let us now consider the design of a distribution network for fast-moving items able to provide differentiated service levels using a critical level policy. From literature review we observe that (i) models that integrate inventory and location decisions (Daskin et al., 2002; Shen et al., 2003; Miranda and Garrido, 2004; Shen, 2005; Shen and Daskin, 2005; Snyder et al., 2007; Ozsen et al., 2008; You and Grossmann, 2008; Atamtürk et al., 2012) considers that the entire distribution network provides the same service level which is equivalent to considering that all customers require the same service level or

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there is only one class of demand (customer category), (ii) for fast-moving items it is usually more convenient and efficient to model the demand over a time period by a continuous distribution, e.g., normal or gamma distributions (Peterson and Silver, 1979; Axsäter, 2006; Ramaekers and Janssens, 2008), and (iii) the critical level policy when demand volume is large, only it has been analyzed by Escalona and Ordóñez (2015) for a single DC. Therefore, to the best of our knowledge, there does not exist previous works that integrates differentiated service levels in the optimal configuration of a network distribution for fast-moving items using critical level policy.

The objective of this paper is determine the optimal configuration of a distribution network for fast moving items where a rationing inventory policy is used to provided differentiated service levels in terms of product availability to two demand classes (high and low priority). The optimal design of the distribution network should determine the number and location of distribution centers and the allocation of demand to DC, while meeting required service levels, so that fixed installation costs, transportation costs, and storage costs are minimized. We assume at each DC: (i) a continuous review (Q, r) policy, with a critical threshold value C, where Q is the fixed lot size, r is the reorder point and C denote the critical level for rationing the low priority class; (ii) normally distributed demand as a approximation to fast-moving items demand; and (iii) service level type I as service level measure. We formulate the location-inventory model with differentiated service levels, denoted (**P0**), as an MINLP problem with chance constraints and nonlinear objective function. The chance constraints of (**P0**) correspond to the service levels constraints. We observe that the location-inventory model with a single service level is a relaxation of (**P0**). We reformulate the location-inventory model with a single service level as a conic quadratic mixed integer program from which we obtain a lower bound of (**P0**). Using the resulting configuration of the relaxation of (**P0**) in terms of location and allocation variables we obtain the optimal control parameters of the critical level policy at each DC. The result is an upper bound (feasible solution) for the problem (**P0**). Furthermore, we propose a method to improve the solution based in the risk pooling effect. Computational results show that the best feasible solution is a good-quality solution, in which the maximum gap is 2.7%, and that the benefit of using a critical level policy in the configuration of a distribution network is greater when the holding cost per unit and unit time is high, and/or when the difference between the preset service levels for high and low priority class is high. The main contributions in this paper can be summarized as follows: (i) we address for the first time the modeling and solution of a supply chain design problem of fast moving items that considers the ability of the distribution network to provide and fulfill different service levels in term of product availability, (ii) we demonstrate that under no demand for one of the classes, the (Q, r, C) policy is equivalent to the traditional (Q, r) policy and (iii) the service level constraints, under rationing, remain valid under no demand for one of the classes.

The rest of this paper is structured as follows. In Section 2 we discuss relevant results in the literature. In Section 3 we formulate the service level constraints, the cost function and the model that integrates location, inventory and service levels. In Section 4, we describe the solution approach. We present our numerical experiments to evaluate the quality of the proposed solutions in Section 5. Section 6 presents our conclusions and future extensions to this work.

2. Related work

The traditional structure of Facility Location Problem (Erlenkotter, 1978) does not consider the relationship between location and inventory control decisions, nor its impact on the distribution network configuration. This is because the distribution network design is solved sequentially, by first solving the location problem and then the inventory problem. This is related to the natural separation between strategic and tactical decision making. However, when these decisions are addressed separately, it often results in suboptimal solutions. In the last decade there has been a strong move towards integrated models of inventories and location. These models simultaneously determine the location of the DCs that will be opened, the allocation of customers to DCs and the optimal parameters of the inventory policy so as to minimize the total system cost. A comprehensive characterization in location-inventory models can be found in Sadjadi et al. (2015).

Our work focuses on location inventory models that integrate the service level, in terms of product availability, in its formulation. In this sense, Daskin et al. (2002) study a location-inventory model that incorporates fixed facility location cost, ordering, holding and safety-stock inventory cost at the DCs, transportation costs from the supplier to the DCs, and local delivery costs from the DCs to the customers. The main difficulty of this model is that the inventory costs at each DC are not linear respect to customer assignments. The model is formulated as a nonlinear integer program and solved by Lagrangian relaxation for a special case in which the ratio between the variance and expected demand is constant for all customers. Shen et al. (2003) analyze the same problem as Daskin et al. (2002). Their work restructures the model into a set-covering integer programming model and use column generation to solve the LP-relaxation of the set covering model.

The model of Daskin et al. (2002) and Shen et al. (2003) has been generalized in different directions. For example: Shen (2005) generalizes the model to a multi-commodity case with a general cost function and proposes a Lagrangian-relaxation solution algorithm. Shen (2005) also relaxes the assumption that the variance of the demand is proportional to the mean for all customers and proposes a Lagrangian-relaxation approach using an algorithm proposed by Shu et al. (2005). Shen and Daskin (2005) introduce a service level element in the model through the distance coverage and propose a weighting method and a heuristic solution approach based on genetic algorithms. Snyder et al. (2007) present a stochastic version of the model. Ozsen et al. (2008) study a capacitated version of the model. Miranda and Garrido (2004) also study a capacitated version of the model and propose a Lagrangian-relaxation solution algorithm. You and Grossmann (2008) relax the assumption that each customer has identical variance-to-mean ratio, reformulating the INLP model as a MINLP problem and solve it with different solution approaches, including a heuristic method and a Lagrangean relaxation algorithm. Atamtürk et al. (2012) also relax the assumption that each customer has identical variance to mean ratio and reformulate the INLP model of Daskin et al. (2002) as conic quadratic mixed-integer problem and added cuts to improve the computational results. They consider cases with uncapacitated facilities, capacitated facilities, correlated retailer demand, stochastic lead times, and multi commodities. Atamtürk et al. (2012) show, through a computational study, that the conic formulation outperforms the column generation and Lagrangian based methods considered up to now. Shahabi et al. (2014) study a capacitated version with correlated retailer demand and propose a solution approach based on an outer approximation strategy.

All of the above authors assume that the inventory system at each DC operates under a continuous review (Q, r) policy with type I service level and fullbackorder. Under this policy, a replenishment order Q is emitted when the inventory level falls below the reorder point r. Based on the results of Axsäter (1996) and Zheng (1992), previous work has approximated the (Q, r) model assuming that each DC determines the replenishment batch Q using an EOQ model and determines the reorder point r (the safety stock) ensuring that the probability of a stockout at each DC is less than or equal to some preset service level. This preset service level is the same for all the distribution network. Further, a normally distributed demand is assumed at each DC as an approximation for a high volume Poisson demand process. With these approximations, the parameters for the continuous review (Q, r) policy are the result of the optimal allocation of customers to DCs.

Our work focuses on situations where customers may require different service levels or that there are different demand classes, which can be more realistic in many cases. To the best of our knowledge, there does not exist previous works that integrates differentiated service levels in the optimal configuration of the network distribution.

Several types of inventory control policies can be implemented in a distribution network to deal with different service requirements. We propose classifying this policies into two types. The first group of policies imposes general service conditions over the entire network distribution. The simplest mechanism is that each DC serves a single demand class, to which we refer as *single class allocation*. This policy tends to increase the number of DCs in the network and not to take advantage of the risk pooling benefits (Eppen, 1979). Another mechanism is to set the service level of the entire distribution network based on a preset level corresponding to the highest priority class, to which we refer as *global round-up policy*. This policy tends to provide too much inventory for classes that require less service level than the maximum. The second type of policy imposes conditions on the operation of the inventory system at each DC. In this case, the simplest mechanism is to impose that each DC serves the demand assigned to it from a common stockpile and uses separate safety stocks for each class (*Separate Stock Policy*). However, separating the safety stocks in each DC does not take advantage of the benefit of centralized inventories. The separate stock policy can be outperformed imposing at each DC, a mechanism that serves all demand assigned to it from a common stockpile and sets the safety stock as the maximum required between the sets of classes assigned to it (*Round-up Policy*). In this case, although all demand of a DC is centralized and the variability reduced, this policy may provide too much inventory for classes that require less service level than the maximum. A third mechanism of this type consists in each DC serving the demand assigned to it from a common stockpile, but using a critical level policy for rationing the inventory between different classes. With this policy, as soon as the inventory level falls below a critical level, the low priority demands are not attended.

In the current study we focus on critical level policy because as well as using the advantage of the pooling effect, it has the flexibility of providing different service levels to different customer classes without provide too much inventory for classes that require less service level than the maximum or increase the number of DCs in the network.

A comprehensive review of inventory rationing can be found at Kleijn and Dekker (1999) and a classification at Teunter and Haneveld (2008). Recent works are those of Möllering and Thonemann (2010), Wang et al. (2013a,b) and Escalona and Ordóñez (2015). From the literature review we conclude that literature on inventory systems with multiple demand classes is extensive, but to the best of our knowledge, only Escalona and Ordóñez (2015) analyzed the constant critical level policy when demand volume is large – which is our concern – while previous works has only consider the case of discrete demand, in particular Poisson distributed demand, which is the form to model demand for slow-moving items. Escalona and Ordóñez (2015) analyzed the constant critical level policy in a single DC when the rationing is due to the presence of two classes of demand (high and low priority) and the inventory system operates under a continuous review (Q, r, C) policy with type I service level, full-backorder, deterministic lead time and continuous demand distribution. Our work extends results from Escalona and Ordóñez (2015) to a network setting, where we not only decide optimal parameter of the critical level policy at a DC, but also the optimal number of DCs, their location and customer allocation.

3. Model formulation

Our location-inventory model with differentiated service levels can be stated as follows. Consider the design of a distribution network consisting of an external supplier and a set of J candidate sites for locating DCs which must supply a set I of retailers. These retailers could be customers or markets, but for convenience we denote them as retailers in the rest of this paper. We assume that the location of the external supplier, site candidates and retailers locations are known and that the supplier and DCs are uncapacitated. In this distribution network there are two categories of retailers or demand classes (high and low priority). The high priority retailers (class 1) require high service level and the low priority retailers (class 2) require lower service level. A retailer can be assigned to a single demand class, and we define $N_k = \{i \in I \mid i \text{ is class } k\}$, with $k = 1, 2,$

as the set of retailers of class k. We also assume that the class of each retailer is known. The demands per unit time at each retailer are independent and normally distributed with mean $\mu_i > 0$ and variance $\sigma_i^2 > 0$. The problem is to determine the optimal number of DCs, their locations, the retailers assigned to each DC, how much inventory to keep at each of them and how to meet the preset service level for each demand class so as to minimize the total system cost.

To provide differentiated service levels we assume at each DC a continuous review (Q, r, C) policy with full-backorder and deterministic lead time, operating as follows. When the inventory position at DC j falls below a reorder level r_j, a replenishment order for Q_j units is placed and arrives $L_j > 0$ time units later. Demand from both classes is satisfied as long as the inventory level is greater than the critical level C_j, otherwise only high priority demand is satisfied from on-hand inventory and low priority demand is backordered. If on-hand inventory level reaches zero, both demands are backordered. If a DC j provides only retailers belonging to one class demand, the critical level is zero, i.e., $C_j = 0$, and therefore, the rationing policy becomes the traditional continuous review (Q, r) policy.

As Daskin et al. (2002) and Shen et al. (2003) do, we assume the replenishment order Q_j is determined using an economic order quantity model (EOQ) and the steady-state backorders are negligible. Hence, there are four types of decision variables in our model: the reorder point in a candidate DC j, r_j; the critical level in the candidate DC j, C_j; the location variable, X_j; and the assignment variable, Y_{ij}. In particular, the last two variables are defined as:

$$X_j = \begin{cases} 1 & \text{if we locate a DC at candidate site } j \\ 0 & \sim \end{cases}$$

$$Y_{ij} = \begin{cases} 1 & \text{if demand at retailer } i \text{ is assigned to a DC at candidate site } j \\ 0 & \sim \end{cases}$$

Once we define the allocation variable Y_{ij}, we can characterize the demand at candidate DC j. Let $D_{kj}(\tau)$ be the total demand of class k during an interval of length τ at DC j and $D_j(\tau) = D_{1j}(\tau) + D_{2j}(\tau)$ be the total demand of both classes during an interval of length τ at DC j. As retailers, demand per unit time are independent and normally distributed, $D_{kj}(\tau)$ is also normally distributed with mean $\tau\mu_{kj}$ and variance $\tau\sigma_{kj}^2$, where $\mu_{kj} = \sum_{i \in N_k} \mu_i Y_{ij} \geqslant 0$ and $\sigma_{kj}^2 = \sum_{i \in N_k} \sigma_i^2 Y_{ij} \geqslant 0$. Furthermore, $D_j(\tau)$ is normally distributed with mean $\tau\mu_j$ and variance $\tau\sigma_j^2$, where $\mu_j = \mu_{1j} + \mu_{2j} = \sum_i \mu_i Y_{ij}$ and $\sigma_j^2 = \sigma_{1j}^2 + \sigma_{2j}^2 = \sum_i \sigma_i^2 Y_{ij}$. In Appendix A, we present a glossary of terms.

3.1. Service level type I under rationing

In this paper, the service level provided by a DC j to the class k is measured by the probability of satisfying the entire demand of class k assigned to him during a replenishment cycle from it on hand inventory, i.e., service level type I, which does not depend of the replenishment batch quantity Q_j.

For a single class demand, service level type I is defined as the probability of no stockout per order cycle (Axsäter, 2006, page 94), i.e., the probability to satisfying the entire demand during a replenishment cycle. Mathematical formulation of these measure depends of the type of the inventory system. In a traditional (Q, r) policy, the service level type I is the probability that total demand during the lead time is less than or equal to the reorder point (Axsäter, 2006, page 97), equivalent to the probability of no-stock out or satisfy the total demand during lead time. In a continuous review (Q, r, C) policy, Escalona and Ordóñez (2015) developed expressions for the service level type I for high and low priority class under strictly increasing Lévy process using normally distributed demand, i.e., they formulate expressions for service level type I of each class demand considering non-negative demand, and then imposing the normally distributed demand as an approximation. In order to determine the operational characteristics of the inventory system they assume a hitting time approach. In our work, the hitting time at DC j, $\tau_{H,D_j}^{r_j - C_j}$, is defined as the amount of time that elapses from the moment an order is placed in DC j until the time at which the critical level C_j is reached for the first time, i.e., $\tau_{H,D_j}^{r_j - C_j} = \inf\{\tau > 0 \mid D_j(\tau) > r_j - C_j\}$. The subscript H is used to remind the reader we refer to a hitting time, in this case the first time that demand D_j accumulates an amount of $r_j - C_j$.

Let $sl_k^j(r_j, C_j, Y_{ij})$ be the service level type I provided by the DC j to class k, and β_k the preset service level for class k, where $\beta_1 > \beta_2$. Using the expressions developed by Escalona and Ordóñez (2015), the service level type I provided by the DC j to the high and low priority class, under strictly increasing Lévy process, are:

$$sl_1^j(r_j, C_j, Y_{ij}) = \mathbb{P}(D_j(L_j) \leqslant r_j - C_j) + \mathbb{P}\Big(D_{1j}(L_j - \tau_{H,D_j}^{r_j - C_j}) \leqslant C_j \ \cap \ \tau_{H,D_j}^{r_j - C_j} < L_j\Big), \tag{1}$$

$$sl_2^j(r_j, C_j, Y_{ij}) = \mathbb{P}(D_j(L_j) \leqslant r_j - C_j), \tag{2}$$

where the first term of Eq. (1) is the probability that rationing does not exist in the lead time of DC j; and second term of Eq. (1) is the probability of rationing occurs in DC j and the class 1 demand during this period not reach the critical level C_j.

Escalona and Ordóñez (2015) assume that a single DC serves both types of demand. In our case, a DC j could serve both types of demand or only one type of demand. Therefore it is necessary to verify that Eqs. (1) and (2) make sense in the case that the DC j is assigned a single type of demand.

Proposition 1. *Under strictly increasing Lévy process, Eqs. (1) and (2) are general expressions for the service level type I provided to the high and low priority classes respectively.*

Proof. The proof is detailed in Appendix B. □

Using normally distributed demand and conditioning on the hitting time, the service levels provided by the candidate DC j to the high and low priority class are:

$$sl_1^j(r_j, C_j, Y_{ij}) = \int_0^{L_j} \Phi\left(\frac{C_j - (L_j - \tau)\sum_{i\in N_1}\mu_i Y_{ij}}{\sqrt{(L_j - \tau)\sum_{i\in N_1}\sigma_i^2 Y_{ij}}}\right) f_{H,D_j}^{r_j - C_j, Y_{ij}}(\tau)\, d\tau + \Phi\left(\frac{r_j - C_j - L_j\sum_i \mu_i Y_{ij}}{\sqrt{L_j\sum_i \sigma_i^2 Y_{ij}}}\right), \tag{3}$$

$$sl_2^j(r_j, C_j, Y_{ij}) = \Phi\left(\frac{r_j - C_j - L_j\sum_i \mu_i Y_{ij}}{\sqrt{L_j\sum_i \sigma_i^2 Y_{ij}}}\right), \tag{4}$$

where $\Phi(x)$ is the distribution function of the standard normal distribution,

$$f_{H,D_j}^{r_j - C_j, Y_{ij}}(\tau) = \frac{1}{\sqrt{\tau\sum_i \sigma_i^2 Y_{ij}}}\left(\frac{r_j - C_j + \tau\sum_i \mu_i Y_{ij}}{2\tau}\right)\varphi\left(\frac{r_j - C_j - \tau\sum_i \mu_i Y_{ij}}{\sqrt{\tau\sum_i \sigma_i^2 Y_{ij}}}\right), \tag{5}$$

is the density distribution of the hitting time $\tau_{H,D_j}^{r_j - C_j}$ using normally distributed demand and $\varphi(x)$ is the density function of the standard normal distribution. The superscripts indicates the dependence of the density distribution of the hitting time with respect to $r_j - C_j$ and assignment variable Y_{ij}, and the subscript D_j represent the total demand of both classes and H the hitting time.

3.2. Cost function

There is a fixed setup cost f_j of opening each distribution center. Each DC can serve more than one retailer, but each retailer should be only assigned to exactly one DC. The ordering cost from distribution center j is S_j. Linear transportation costs are incurred for shipment from the external supplier to distribution center j with unit cost a_j and from distribution center j to retailer i with unit cost d_{ij}. With this notation, the average cost per unit time at DC j is:

$$AC_j(r_j, C_j, Y_{ij}) = S_j\frac{\sum_i \mu_i Y_{ij}}{Q_j} + a_j\sum_i \mu_i Y_{ij} + \sum_i d_{ij}\mu_i Y_{ij} + h_j\left(\frac{Q_j}{2} + r_j - L_j\sum_i \mu_i Y_{ij}\right). \tag{6}$$

The first term of Eq. (6) is the ordering cost per unit time. The second and third term are the supply and distribution costs per unit time respectively. As we assume negligible backorders, the fourth term is approximated the holding cost per unit time. Each distribution center determines the replenishment order Q_j using an EOQ model, i.e.,

$$Q_j = \sqrt{\frac{2S_j}{h_j}\sum_i \mu_i Y_{ij}}. \tag{7}$$

Then, replacing Eq. (7) into Eq. (6), the average cost per unit time at DC j is:

$$AC_j(r_j, C_j, Y_{ij}) = \sum_i \hat{d}_{ij} Y_{ij} + k_j\sqrt{\sum_i \mu_i Y_{ij}} + h_j r_j, \tag{8}$$

where $k_j = \sqrt{2h_j S_j}$ and $\hat{d}_{ij} = (a_j + d_{ij} - h_j L_j)\mu_i$.

3.3. Problem formulation

We can formulate an integrated location-inventory model with differentiated service levels using normally distributed demand as an MINLP problem with constraints on service probability (non-convex) and nonlinear objective function including non-convexity in the assignments variables, denoted (**P0**), as follows.

Problem (**P0**):

$$\min_{X,Y,r,C} \quad \sum_{j\in J}\left\{f_jX_j + \sum_i \hat{d}_{ij}Y_{ij} + k_j\sqrt{\sum\nolimits_i \mu_i Y_{ij}} + h_j r_j\right\} \tag{9}$$

$$\text{s.t}: \quad \int_0^{L_j} \Phi\left(\frac{C_j - (L_j-\tau)\sum_{i\in N_1}\mu_i Y_{ij}}{\sqrt{(L_j-\tau)\sum_{i\in N_1}\sigma_i^2 Y_{ij}}}\right) f_{H,D_j}^{r_j-C_j,Y_{ij}}(\tau)\, d\tau + \Phi\left(\frac{r_j - C_j - L_j\sum_i \mu_i Y_{ij}}{\sqrt{L_j \sum_i \sigma_i^2 Y_{ij}}}\right) \geqslant \beta_1 \quad \forall j\in J, \tag{10}$$

$$r_j - C_j \geqslant L_j\sum_i \mu_i Y_{ij} + z_{\beta_2}\sqrt{L_j\sum\nolimits_i \sigma_i^2 Y_{ij}} \quad \forall j \in J, \tag{11}$$

$$\sum_{j\in J} Y_{ij} = 1 \qquad \forall i \in I, \tag{12}$$

$$Y_{ij} \leqslant X_j \qquad \forall i\in I, \forall j \in J, \tag{13}$$

$$r_j \geqslant C_j \geqslant 0 \qquad \forall j \in J, \tag{14}$$

$$X_j,\ Y_{ij} \in \{0,1\} \qquad \forall i\in I, \forall j\in J, \tag{15}$$

where z_{β_2} is the inverse standard normal distribution for a preset service level β_2 and the $f_{H,D_j}^{r_j-C_j,Y_{ij}}(\tau)$ is given by Eq. (5).

The objective is to minimize the total steady state cost per unit time including location fixed cost, ordering costs, supply cost from supplier to DCs, distribution cost from DCs to retailers, and holding cost at each DC. Constraints (10) and (11) ensure, at each DC, the fulfillment of the preset service level for the high and low priority class respectively. Constraint (12) ensures that each retailer is assigned to exactly one DC. Constraint (13) stipulates that retailers can only be assigned to open DCs. Constraint (14) ensures, at each DC, that the replenishment order be placed before the lower priority class is no longer served. Finally, constraint (15) is an integrality constraint.

4. Solution approach

Consider the joint location-inventory problem described by Shen et al. (2003), but without relying on the assumption that each retail has identical variance-to-mean ratio. This problem assumes that the distribution network is dominated by a continuous review (Q, r) policy, deterministic lead time and full-backorders. These authors considered a single demand class, i.e., to the distribution network providing a unique service level. Let β be this unique preset service level. Using the notation above the model of Shen et al. (2003) is expressed as:

Problem (**P1**):

$$\min_{X,Y} \quad \sum_{j\in J}\left\{f_jX_j + \sum_i (\hat{d}_{ij} + h_jL_j\mu_i)Y_{ij} + k_j\sqrt{\sum\nolimits_i \mu_i Y_{ij}} + z_\beta h_j\sqrt{L_j}\sqrt{\sum\nolimits_i \sigma_i^2 Y_{ij}}\right\} \tag{16}$$

$$\text{s.t}: \quad (12),\ (13),\ (15).$$

It is easy to show that the problem (**P1**) is a relaxation of (**P0**) when $\beta = \beta_2$. Therefore, the optimal solution of the problem (**P1**) when the level of service is the lowest, is a lower bound (LB) of problem (**P0**).

Also, if we solve (**P1**), and then use the resulting configuration in terms of location (**X**-variables) and allocation (**Y**-variables), we can obtain the optimal remaining feasible variables corresponding to this configuration, for each DC, always considering the actual proportion of customers requiring each class of service (β_2 and β_1). The result is a feasible solution and, hence, an upper bound (UB) for the problem (**P0**). Thus, we now have a lower bound and an upper bound of the original problem with two classes of service, obtained by solving two problems with a single class of service, and completing the solution of the second one by including both service classes.

Note that for fixed **X**-variables and **Y**-variables that satisfy the constraint (12), (13) and (15), the problem (**P0**) reduces to determine the optimal parameters of the critical level policy at all installed DCs, i.e., for all $X_j = 1$ we must solve the following service level problem **SLP**(j) using normally distributed demand:

Problem **SLP**(j):

$$\min_{r_j, C_j} \quad r_j \tag{17}$$

$$\text{s.t}: \quad (10),\ (11),\ (14).$$

Escalona and Ordóñez (2015) characterize under mild assumptions, the optimal solution of the problem **SLP**(**j**) using normally distributed demand, when a single DC satisfies the demand of both demand classes, i.e., when $\mu_{kj},\ \sigma_{kj}^2 > 0$.

Consider now that we solve the same two problems (**P1**) and **SLP**(**j**) for an increasing value of β, starting at β_2. Let $X_\beta^*,\ Y_\beta^*$ be the optimal location and assignments variables of the problem (**P1**) given a service level β; $FO_{P1}(\beta)$ be the objective function of problem (**P1**) given a service level β; and $FO_{P0}(X_\beta^*, Y_\beta^*)$ be the objective function of problem (**P0**) given the optimal network configuration of the problem (**P1**) and a service level β. We propose a simple heuristic to find a better upper bound of the problem (**P0**) based on the following four properties:

(1) $LB = FO_{P1}(\beta_2),\ UB = FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2})$ and $UB > LB$;
(2) $FO_{P1}(\beta)$ is strictly increasing in β for all $\beta \in [\beta_2, \beta_1]$ with $\beta_2 \geqslant 0.5$ because increasing the requirements in quality of service tightens the feasible space. However, as β increases, there is no guarantee that $FO_{P1}(\beta)$ is a lower bound anymore;
(3) Recall that the feasible solution $FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$ is computed by finding variables X and Y as if there were only customers requiring service at level β_1, and completed using the actual distribution of classes of customers. Then, $FO_{P1}(\beta_1) \geqslant FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$, because a *global round-up policy* induces a higher cost than a critical level policy. In other words, providing all customers with the highest class of service β_1 is costlier than having some of them with the lowest class of service β_2.
(4) As we increase the service level β and solve the problem (**P1**) one or more of the following events may occur:
  (i) the configuration $(X^*_{\beta_2}, Y^*_{\beta_2})$ remains constant for any $\beta \in [\beta_2, \beta_1]$;
  (ii) the optimal network configuration $(X^*_{\beta}, Y^*_{\beta})$ changes due to the reallocation of demand without changing the number of open DCs; and
  (iii) the optimal network configuration $(X^*_{\beta}, Y^*_{\beta})$ tends to make pooling, closing one or more DCs and reassigning demand.

Our improvement heuristic exploits the risk pooling effect to increase the service level β in the interval $[\beta_2, \beta_1]$. If pooling happens at some $\hat{\beta} \in [\beta_2, \beta_1]$ and the sum of the savings on holding and fixed costs are greater than the increase in transportation costs at that point, then $FO_{P0}(X^*_{\hat{\beta}-\varepsilon}, Y^*_{\hat{\beta}-\varepsilon}) > FO_{P0}(X^*_{\hat{\beta}}, Y^*_{\hat{\beta}})$. A heuristic, to find this $\hat{\beta}$ (if exist) based on the systematic increase in the service level β can be costly in terms of computational time because each increased service level β means solving the problems (**P0**) and **SLP**(**j**). We propose to evaluate $FO_{P0}(X^*_{\beta}, Y^*_{\beta})$ only in $\beta = \beta_1$ and $\beta = \beta_2$. Let FO^*_{P0} the objective value of the best solution found to (**P0**), then:

- if $FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2}) \leqslant FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$, an increase in the service level β in the interval $[\beta_2, \beta_1]$ produces no improvement in the initial solution (UB), which becomes the best solution found, i.e., $FO^*_{P0} = FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2})$;
- if $FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2}) > FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$, there is improvement in the initial solution and the second solution becomes the best solution found, i.e., $FO^*_{P0} = FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$;

i.e., $FO^*_{P0} = \min\{FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2}), FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})\}$.

4.1. Solution characterization for the problem (**P1**)

The square root term in the objective function of problem (**P1**) can give rise to difficulties in the optimization procedure. When the DC j is not selected, both square root terms would take a value of 0, which leads to unbounded gradients in the NLP optimization and hence numerical difficulties. Thus, we reformulate the problem (**P1**) in order to eliminate the square root terms. We first introduce two sets of nonnegative continuous variables, $Z1_j$ and $Z2_j$, to represent the square root terms in the objective function:

$$Z1_j^2 = \sum_i \mu_i Y_{ij}^2, \quad \forall j \in J, \tag{18}$$

$$Z2_j^2 = \sum_i (\sigma_i Y_{ij})^2, \quad \forall j \in J, \tag{19}$$

$$Z1_j, Z2_j \geqslant 0, \quad \forall j \in J. \tag{20}$$

Because the nonnegative variables $Z1_j$ and $Z2_j$ are introduced in the objective function with positive coefficients, and this problem is a minimization problem, Eqs. (18) and (19) can be further relaxed as the following inequalities:

$$Z1_j^2 \geqslant \sum_i \mu_i Y_{ij}^2, \quad \forall j \in J, \tag{21}$$

$$Z2_j^2 \geqslant \sum_i (\sigma_i Y_{ij})^2, \quad \forall j \in J. \tag{22}$$

Note that constraints (21) and (22) with constraint (20) define second-order cone constraints. Thus, the reformulated problem (**P1**) can be expressed as the following MINLP problem with second-order cone constraints denoted as (**P2**):
Problem (**P2**):

$$\min_{X,Y,Z1,Z2} \sum_{j\in J} \left\{ f_j X_j + \sum_i (\hat{d}_{ij} + h_j L_j \mu_i) Y_{ij} + k_j Z1_j + z_\beta h_j \sqrt{L_j} Z2_j \right\} \tag{23}$$
$$\text{s.t}: \quad (21),\ (22),\ (12),\ (13),\ (15),\ (20).$$

Problem (**P2**) can be trivially shown to be equivalent to problem (**P1**) but it has a linear objective function and second-order cone constraints (21) and (22). We can solve this problem using CPLEX 12.4., which handles second-order cone constraints in an efficient way.

4.2. Solution characterization for **SLP**(**j**) *using normally distributed demand*

In this section we suppress the j subscript in order to simplify the notation. Escalona and Ordóñez (2015) characterized under mild assumptions, the optimal solution of the problem **SLP**(**j**) under normally distributed demand, when a single DC satisfies demand of both classes as follows:

For, $\mu_1,\ \sigma_1^2,\ \mu_2,\ \sigma_2^2 > 0$ and $\beta_2 \in [0.5, 1)$, the optimal parameters of the critical level policy are obtained from the following system of equations:

(a) If $sl_1(r_2^0, 0) < \beta_1$:

$$r^* - C^* = \mu L + z_{\beta_2}\ \sigma\sqrt{L}, \tag{24}$$

$$\int_0^L \Phi\left(\frac{C^* - \mu_1(L-\tau)}{\sqrt{(L-\tau)\sigma_1^2}}\right) f_{H,D}^{r^*-C^*}(\tau)\ d\tau = \beta_1 - \beta_2, \tag{25}$$

where $r_2^0 = \mu L + z_{\beta_2}\ \sigma\sqrt{L}$ and the service levels provided to each class are equal to their preset levels, i.e., $sl_i(r^*, C^*) = \beta_i,\ i = 1, 2$.

(b) If $sl_1(r_2^0, 0) \geqslant \beta_1$:

$$C^* = 0, \tag{26}$$

$$r^* = \mu L + z_{\beta_2}\ \sigma\sqrt{L}, \tag{27}$$

and service levels provided to each class are: $sl_1(r^*, 0) \geqslant \beta_1$ and $sl_2(r^*, 0) = \beta_2$ for high and low priority class respectively.

In our network design, a candidate DC j can provide both demand classes, one or none. The following proposition indicates the optimal parameters of rationing policy when a DC provides only a single demand class.

Proposition 2. *Under normally distributed demand and single class demand, the optimal parameters of the critical level policy are:*

(a) *If $\mu_1 = \sigma_1^2 = 0$ and $\mu_2,\ \sigma_2^2 > 0$, the optimal parameters of the critical level policy are: $C^* = 0, r^* = r_2^0 = \mu_2 L + z_{\beta_2}\ \sigma_2\sqrt{L}$, and the service levels provided to the high and low priority class are $sl_1(r^*, C^*) = 1$ and $sl_2(r^*, C^*) = \beta_2$ respectively.*

(b) *If $\mu_1, \sigma_1^2 > 0$ and $\mu_2 = \sigma_2^2 = 0$, the optimal parameters of the critical level policy are $C^* = 0$ and $r^* = r_1^0$ solution of $sl_1(r^*, 0) = \beta_1$, i.e.,*

$$\int_0^L \Phi\left(\frac{-\mu_1(L-\tau)}{\sqrt{(L-\tau)\sigma_1^2}}\right) f_{H,D_1}^{r^*}(\tau)\ d\tau + \Phi\left(\frac{r^* - \mu_1 L}{\sqrt{\sigma_1^2 L}}\right) = \beta_1, \tag{28}$$

and the service levels provided to the high and low priority class are $sl_1(r^*, C^*) = \beta_1$ and $sl_2(r^*, C^*) > \beta_2$ respectively.

Proof. The proof is detailed in Appendix C. □

Note that under no demand for one of the classes $C^* = 0$. Therefore, the (Q, r, C) policy is equivalent to the traditional (Q, r) policy.

5. Computational study

In order to illustrate the applicability and evaluate the performance of our solution approach in terms of quality solution (optimality gap) and computational time, we carried out computational experiments for instances with 49 and 88 nodes from Daskin (2011). We generated several test problems for each data set in which we compare our solution approach with the Global Round-up policy. We denote these instances as *test sets*. In all cases, each retailer location is also a candidate DC location, i.e., there are as many candidate DC locations as retailer locations for each instance.

The test problems were generated around a base case with the following parameters: service level requirements $\beta_1 = 0.975$ and $\beta_2 = 0.75$; cost per unit to ship between retailer i and candidate DC site j, d_{ij} equal to the distance between retailer i and candidate DC j multiplied by a transport rate $c_{ij} = 0.01,\ \forall i,j$; and holding cost per unit and unit time at candidate DC site j, $h_j = 0.25,\ \forall j \in J$. Furthermore, all the test problems used the following common criterion and parameters: demand per unit time at each retailer is normally distributed with mean $\mu_i = U[10, 50]$ and coefficient of variation $CV_i = U[0.1, 0.4]$; class of the retail $i, n_i = \{1, 2\}$ with discrete uniform distribution; fixed (per unit time) cost of locating a DC at candidate site j, $f_j = U[200, 300]$; cost per unit to ship between external supplier and candidate DC site j, $a_j = 0.5,\ \forall j \in J$; ordering cost from candidate DC site j, $S_j = 1000,\ \forall j \in J$; and lead time, $L_j = \{2, 3, 4\}$ with discrete uniform distribution.

To illustrate the industrial applicability of the location-inventory model with differentiated service levels we also consider a illustrative example of a company that manufactures products derived from fruits and vegetables. The supply chain consisting of one plant, 38 potencial DCs, and 38 customers.

Problem (**P2**) was modeled with AMPL and solved with CPLEX 12.4. The equation systems (24) and (25) and Proposition 2, solutions of problem **SLP**(j), were programmed and solved by a C code. We integrate both codes through an AMPL script and the shell command to execute the C code. The time limit was set for 10,800 s. All test were carried on a PC with Intel Core i7 2.3 GHz processor and 16 GB RAM.

5.1. Test sets

We solved 63 problems, 30 for 49 nodes and 33 for 88 nodes. In each problem, we changed parameters relative to the base case. In particular the modified parameters were: the preset service level for low priority class (β_2), the holding cost per unit and unit time (h_j), and the transport rate (c_{ij}). Table D.3 shows the data-set used, the parameter modified from the base case; results of location-inventory model with differentiated service levels using critical level policy; results of the global round-up policy; and the relative difference between the total costs induced by critical level and global round-up policies.

Regarding the difficult to solve the problem we have the following comments derived from the computational experiments in Table D.3:

- as expected, as the problems grow larger, it becomes more difficult to solve them;
- as the holding cost per unit and unit time and/or the preset service level increases, the problem (**P2**) becomes harder to solve. This is because, higher values of h_j and z_β assign more weight on the nonlinear terms of the objective function of (**P2**).

We measure, for our approach, the relative optimally gap between the lower bound (LB) and the best solution found, i.e., $Gap(\%) = 100 \times (FO^*_{P0} - LB)/LB$. Fig. 1 show for each data set, how the relative optimally gap change as the holding cost per unit and unit time (h_j) and the preset service level of the low priority class (β_2) change.

From Fig. 1, we can see that relative optimality gap increase when the holding cost per unit and unit time and $\beta_1 - \beta_2$ increases. This is because, the absolute gap is less than or equal to $FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2}) - FO_{P1}(\beta_2) = \sum_j h_j(r^*_j - (L_j \sum_i \mu_i Y^*_{ij} + z_{\beta_2}\sqrt{L_j \sum_i \sigma_i^2 Y^*_{ij}}))$. From Table D.3 the maximum relative gap is 2.7%.

Regarding the number of DC installed we have the following comments derived from the computational experiments in Table D.3:

- as expected, as the holding cost per unit and unit time increases, the number of DCs decreases;
- as expected, as the transport rate increases, the number of DCs increases;
- no effect of $\beta_1 - \beta_2$ on the number of DC is observed.

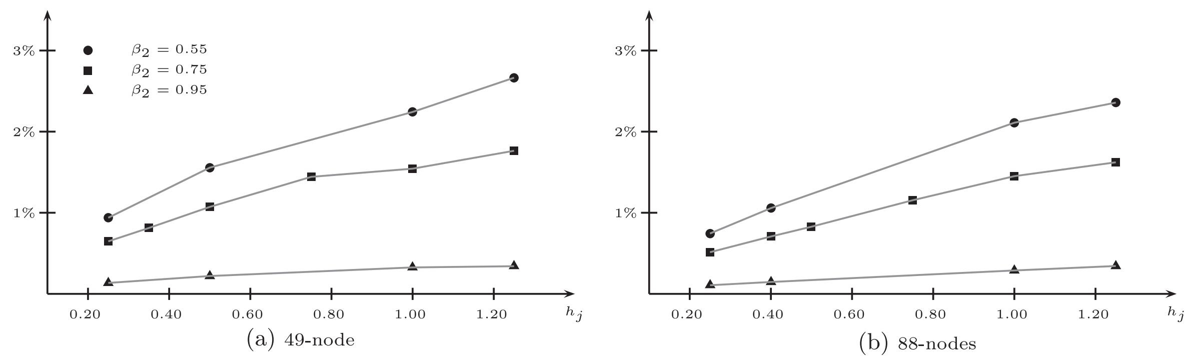

Fig. 1. Relative gap with $\beta_1 = 0.75,\ c_{ij} = 0.01$.

For all instances in Table D.3, the total cost of the location-inventory model with differentiated service levels using critical level policy is less than the total cost induced by the global round-up policy. We measure the relative benefit induced by the critical level policy regarding global round-up policy as $\Delta cost(\%) = 100 \times (FO_{P1}(\beta_1) - FO^*_{P0})/FO_{P1}(\beta_1)$. Fig. 2 show for each data set, how the relative benefit induced by the critical level policy change as the holding cost per unit and unit time (h_j) and the preset service level of the low priority class (β_2) change.

From Fig. 2, we can see that benefit induced by the critical level policy increase when the holding cost per unit and unit time and $\beta_1 - \beta_2$ increases. This is because, the absolute benefit is greater than or equal to $FO_{P1}(\beta_1) - FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1}) = \sum_j h_j((L_j \sum_i \mu_i Y^*_{ij} + z_{\beta_1} \sqrt{L_j \sum_i \sigma_i^2 Y^*_{ij}}) - r^*_j)$. Therefore, we concluded that our location-inventory model with differentiable service levels using critical level policy is useful when the difference between the preset service levels for high and low priority class is high and/or the holding cost per unit and time unit is high. From Table D.3 the maximum benefit induced by the critical level policy is 2.33%.

5.2. Illustrative example for industrial application

Consider the case of a company that manufactures products derived from fruits and vegetables which requires determining the number of distribution centers (DC) to locate in Santiago (Chile), where to locate, what kinds of customers should be assigned to each DC, how much inventory to keep each of them, and how to meet the required service level of their customer. The supply chain consisting of one plant, 38 potencial DCs, and 38 customers. The production plant is located 200 km south of Santiago (Chile). The company segments its customers by volume of annual demand. Thus, customers who demand more than the average annual demand are classified as high priority and its preset service level is $\beta_1 = 0.98$. Customers who require less than the average annual demand are classified as low priority and its preset service level is $\beta_2 = 0.70$.

The products manufactured by the company are derivative of fruits and vegetables, with holding cost per unit and unit time at candidate DC site j, $h_j = 0.005$ (US$/Kg day), $\forall j \in J$. The location, class, demand, and coefficient of variance for each customers is show in Table E.4. From Table E.4, note that 24% of customers are class 1 and they demand 72% of daily demand.

Ordering cost from candidate DC site j, $S_j = 250$ (US$/order), $\forall j \in J$; cost per unit to ship between plant and candidate DC site j, $a_j = 0.0069$ (US$/day), $\forall j \in J$; and lead time, $L_j = 4$ (day), $\forall j \in J$. The location and fixed cost to the 40 potential DCs is show in Table E.4. For class 1 customers the company uses medium goods vehicles and for class 2 customers the company uses light goods vehicle. Each vehicle uses a driver and an assistant. The unit cost of transport (US$/Kg) between candidate DC j and retail i, d_{ij}, is calculated as the fixed cost of loading and unloading (including labor and depreciation), plus variable cost that depends on the distance between candidate DC j and retail i (including labor, fuel and depreciation). Then, unit cost of transport are $d_{ij} = 0.0025 + 0.00012\ \theta_{ij}$ and $d_{ij} = 0.0021 + 0.00015\ \theta_{ij}$ for class 1 and 2 respectively, where θ_{ij} is the Euclidean distance between candidate DC j and retail i.

We analyze the location-inventory problem with differentiated service levels using critical level and global round-up policies. Table 1 shows the inventory policy, number of opened DC, the objective function and the cost components (FC: fixed cost; OC: ordering cost; SC: supply cost; CD: distribution cost; HC: holding cost).

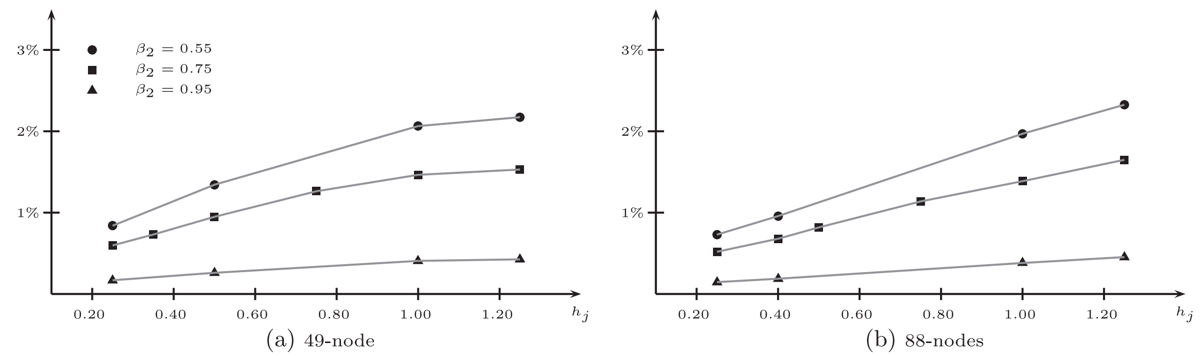

Fig. 2. Relative benefit with $\beta_1 = 0.75$, $c_{ij} = 0.01$.

Table 1
Illustrative example: results.

Policy	Result		Cost component				
	♯ DC	Objective function	FC	OC	SC	DC	HC
Critical level	1	805.73	195	122.84	166.58	115.21	206.10
Global round-up	1	825.18	195	122.84	166.58	115.21	225.56

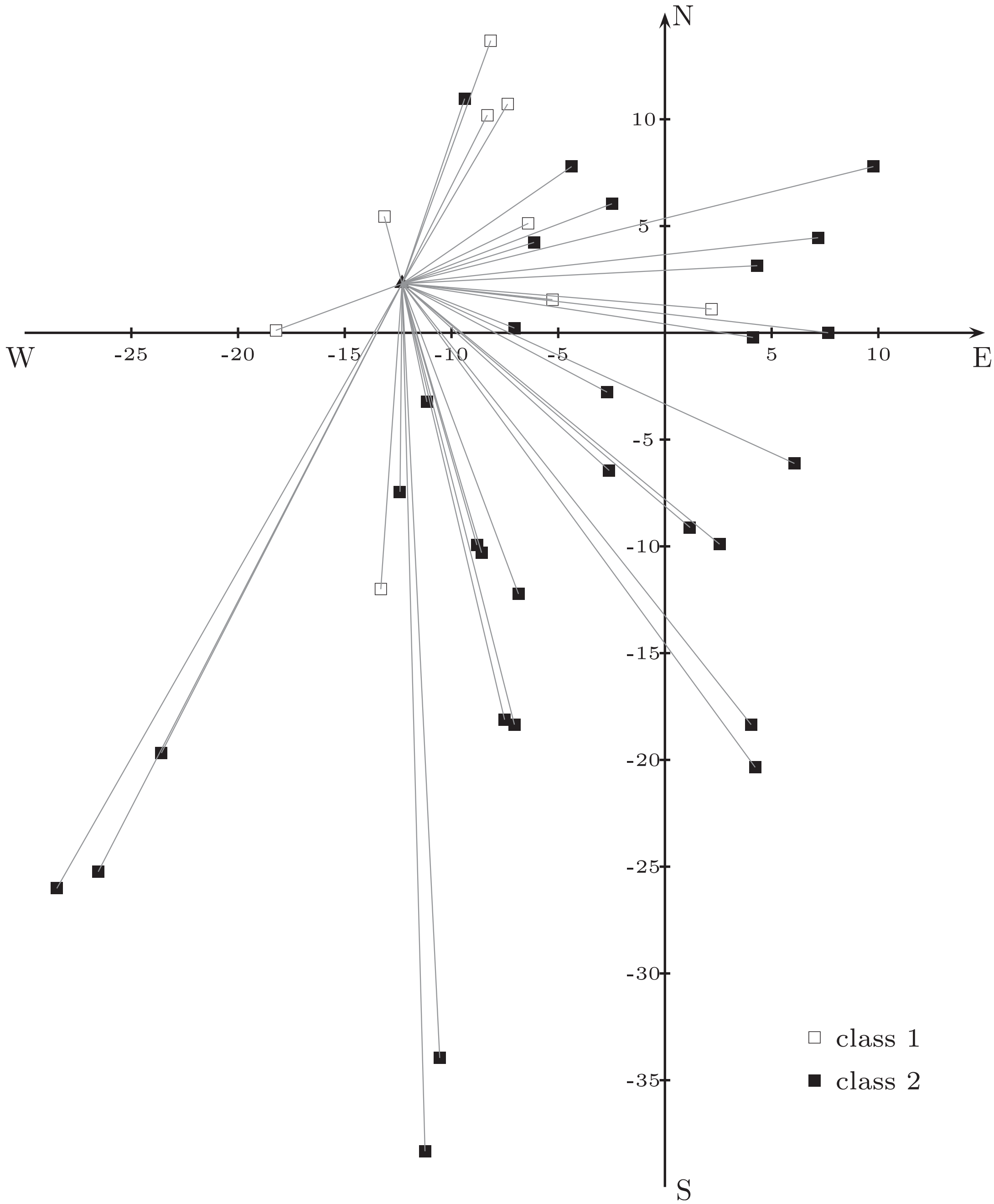

Fig. 3. Location-inventory model using critical level policy: $FO^* = 805.73$.

Table 1 indicates that the lower cost configuration is achieved with the critical level policy with a saving of 2.4% per day. Note that this saving is produced by lower holding cost induced by the critical level policy. The resulting network configuration is the same for critical level and global round-up policies. Fig. 3 show the network configuration.

6. Conclusions

This paper consider a location-inventory model for fast-moving items in which the distribution centers observe demand from two classes of customers, high and low priority. To provide differentiated service levels we assume, at each DC, a continuous review (Q, r, C) inventory policy. If a DC provides only one class of demand, the critical level policy becomes the traditional continuous review (Q, r) policy. In this paper the service level is measured by service level type I.

We formulate the location-inventory model with differentiated service levels as an MINLP problem with chance constraints, corresponding to the service levels constraints, and nonlinear objective function. We propose optimally solve a relaxation of the location-inventory model with differentiated service levels which allows us to obtain good-quality bounds.

The computational results show that our proposed heuristic able to find good-quality solutions because for test set problems, the maximum optimality gap is 2.7%, a very good solution in itself, which provides us the configuration of the network, including location of the CD's, allocation of demands and the required stock everywhere. We also consider a Illustrative example for industrial application, for which the total cost induced by the location-inventory model with differentiated service levels using critical level policy is 2.4% lower than the total cost induced by the global round-up policy.

The computational result also provides managerial insight: the benefit of using a critical level policy in the configuration of a distribution network is greater when the holding cost per unit and unit time is high, and/or when the difference between the preset service levels for high and low priority class is high.

There are a number of questions and issues left for future research. The first one, is to consider other policies to provide differentiated service levels in a distribution network, e.g., separate stock policy, single class allocation or round-up policy, so we can determine the best policy to provide differentiated service levels in a distribution network. The second one is related with the fact that our solution approach uses normally distributed demands. We believe that since the problem formulation is valid for any strictly increasing Levy process, similar solution approaches could be developed for other distributions in future research. Another possible extensions are: (i) consider other service levels measure, e.g., fill-rate; and (ii) use penalty cost as an alternative way of the service levels.

Acknowledgements

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Appendix A. Glossary of terms

See Table A.2.

Table A.2
Glossary of terms.

Sets	Definition
I	Set of retailers indexed by i
J	Set of candidate DC sites indexed by j
K	Set of class demand indexed by k, with $k = 1, 2$
N_k	Set of retailers that belong to the class k, with $k = 1, 2$
Parameters	
μ_i	Mean demand per unit time at retailer i
σ_i	Standard deviation of demand per unit time at retailer i
β_k	Preset service level for class k, with $\beta_1 > \beta_2$
f_j	Fixed (per unit time) cost of locating a DC at candidate site j
d_{ij}	Cost per unit to ship between retailer i and candidate DC site j
c_{ij}	Transport rate between retailer i and candidate DC j
a_j	Cost per unit to ship between external supplier and candidate DC site j
S_j	Ordering cost from candidate DC site j
h_j	Holding cost per unit time at candidate DC site j
L_j	Constant replenishment lead time at candidate DC site j
n_i	Class of retail i
Variables	
X_j	1 if we locate a DC in candidate site j, and 0 otherwise
Y_{ij}	1 if retailer i is served by the DC at candidate site j, and 0 otherwise
r_j	Reorder point at candidate DC site j
C_j	Critical level at candidate DC site j
Variable functions	
$\mu_j = \sum_i \mu_i Y_{ij} \geqslant 0$	Mean demand per unit time at candidate DC j
$\sigma_j = \sqrt{\sum_i \sigma_i^2 Y_{ij}} \geqslant 0$	Standard deviation of demand per unit time at candidate DC j
$\mu_{kj} = \sum_{i \in N_k} \mu_i Y_{ij} \geqslant 0$	Mean demand per unit time of class k at candidate DC j
$\sigma_{kj} = \sqrt{\sum_{i \in N_k} \sigma_i^2 Y_{ij}} \geqslant 0$	Standard deviation of demand per unit time of class k at candidate DC j
$Q_j = \sqrt{\frac{2S_j}{h_j} \sum_i \mu_i Y_{ij}}$	Replenishment order at candidate DC j

Appendix B. Proof of Proposition 1

To prove that Eqs. (1) and (2) are general expressions for the service level type I under rationing we must show that these equations remain valid in the absence of one of the two class demand. In order to simplify the notation, we suppress the subscript j.

Proof. The conditions to fully meet the demand of class 1 in a replenishment cycle when the DC only provides class 1 is that demand of class 1 during the lead time is less or equal to the reorder point r, i.e., $D_1(L) \leqslant r$. Therefore, the service level provided to the high priority class when the DC only provides service to class 1 is:

$$sl_1(r, C) = \mathbb{P}(D_1(L) \leqslant r). \tag{B.1}$$

Eq. (B.1) is equal to Eq. (1) when $D_2(\tau) = 0$ for any $\tau > 0$, because

$$\begin{aligned} P(D_1(L) \leqslant r) &= \mathbb{P}(D_1(L) \leqslant r \mid D_1(L) \leqslant r - C)\mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L) \leqslant r \mid D_1(L) > r - C)\mathbb{P}(D_1(L) > r - C) \\ &= \mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L) \leqslant r \mid D_1(L) > r - C)\mathbb{P}(D_1(L) > r - C) \\ &= \mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L) \leqslant r - C + C \mid \tau_{H,D_1}^{r-C} < L)\mathbb{P}(\tau_{H,D_1}^{r-C} < L) \\ &= \mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L) \leqslant D_1(\tau_{H,D_1}^{r-C}) + C \mid \tau_{H,D_1}^{r-C} < L)\mathbb{P}(\tau_{H,D_1}^{r-C} < L) \\ &= \mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L - \tau_{H,D_1}^{r-C}) \leqslant C \mid \tau_{H,D_1}^{r-C} < L)\mathbb{P}(\tau_{H,D_1}^{r-C} < L) \\ &= \mathbb{P}(D_1(L) \leqslant r - C) + \mathbb{P}(D_1(L - \tau_{H,D_1}^{r-C}) \leqslant C \, \cap \, \tau_{H,D_1}^{r-C} < L). \end{aligned}$$

Furthermore, in Section 4.2 we show that in the absence of demand for class 1, $sl_1(r^*, C^*) = 1 > \beta_1$, where (r^*, C^*) are the optimal reorder point and critical level respectively. Therefore, Eq. (1) is a general expression for the service level provided to the high priority class.

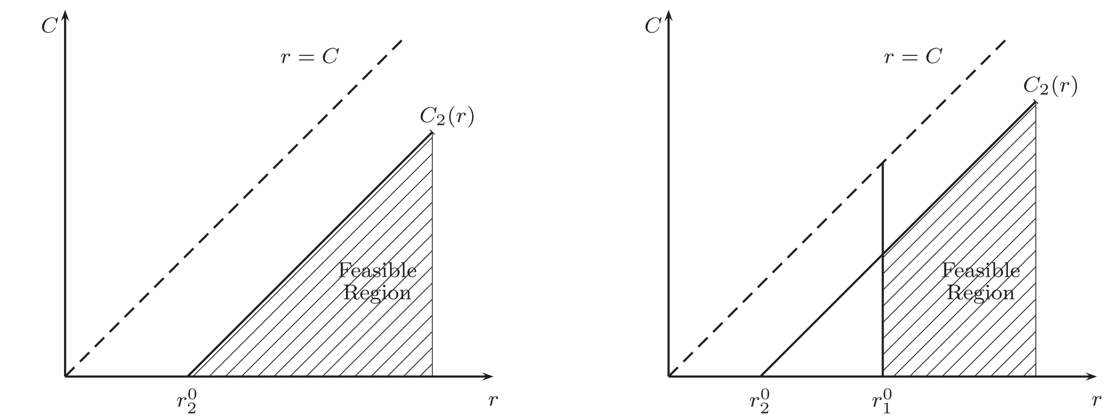
The conditions to fully meet the demand of class 2 in a replenishment cycle, when the DC only provides service to class 2 is that demand of class 2 during the lead time ($D_1(\tau) = 0$ for any $\tau > 0$) is less or equal to the reorder point r, i.e., $D_2(L) \leqslant r$. Therefore, the service level provided to the high priority class when the DC only provides service to class 2 is:

$$sl_2(r, C) = \mathbb{P}(D_2(L) \leqslant r). \tag{B.2}$$

Eq. (2) is equal to Eq. (B.2) when $D_2(\tau) > 0$ for any $\tau > 0, D_1(\tau) = 0$ for any $\tau > 0$ and $C = 0$. In Section 4.2 we show that when $D_1(\tau) = 0$ for any $\tau > 0$, then the optimal critical level is equal to zero, i.e., $C^* = 0$. Furthermore, in Section 4.2 we show that in the absence of demand for class 2, $sl_2(r^*, C^*) > \beta_2$. Therefore, Eq. (2) is a general expression for the service level provided to the low priority class. □

Appendix C. Proof of Proposition 2

Proof. Let $C_2(r)$ be the maximum critical level, given a reorder point r, that ensures a service level β_2, i.e., $C_2(r) = \max\{C \mid sl_2(r, C) \geqslant \beta_2\}$. Escalona and Ordóñez (2015) showed that $sl_2(r, C)$ is increasing in r (and decreasing in



(a) Feasible region of SLP problem when $\mu_1 = \sigma_1^2 = 0$ and $\mu_2, \sigma_2^2 > 0$

(b) Feasible region of SLP problem when $\mu_2 = \sigma_2^2 = 0$ and $\mu_1, \sigma_1^2 > 0$

Fig. C.4. Feasible region of SLP problem under normally distributed demand and single class demand.

Table D.3
Results for test set.

Data set	Vary	Parameters			Solution approach											Global round-up			$\Delta cost$ (%)
					First solution			Second solution			Critical level policy								
		β_2	h_j	c_{ij}	$\sharp$ *DC*	$FO_{P0}(X^*_{\beta_2}, Y^*_{\beta_2})$	Time(s)	$\sharp$ *DC*	$FO_{P0}(X^*_{\beta_1}, Y^*_{\beta_1})$	Time(s)	LB	FO^*	*Gap*(%)	$\sharp$ *DC*	Time(s)	$FO_{P1}(\beta_1)$	$\sharp$ *DC*	Time(s)	
49-nodes	Case base	0.75	0.25	0.01	12	8766.65	1.40	12	8766.65	1.78	8709.91	8766.65	0.65	12	3.18	8819.29	12	1.79	0.60
	β_2	**0.55**	0.25	0.01	12	8745.29	1.35	12	8745.29	1.77	8663.21	8745.29	0.94	12	3.11	8819.29	12	1.79	0.84
		0.65	0.25	0.01	12	8755.40	1.52	12	8755.40	1.79	8685.30	8755.40	0.80	12	3.32	8819.29	12	1.79	0.72
		0.85	0.25	0.01	12	8780.73	1.42	12	8780.73	1.77	8740.71	8780.73	0.46	12	3.19	8819.29	12	1.79	0.44
		0.95	0.25	0.01	12	8804.37	1.64	12	8804.37	1.76	8792.48	8804.37	0.14	12	3.41	8819.29	12	1.79	0.17
	h_j	0.75	**0.005**	0.01	12	6188.71	0.38	12	6188.71	0.31	6187.53	6188.71	0.02	12	0.69	6189.77	12	0.32	0.02
		0.75	**0.1**	0.01	12	7642.65	0.47	12	7642.65	0.50	7619.95	7642.65	0.30	12	0.97	7663.70	12	0.50	0.27
		0.75	**0.35**	0.01	11	9320.38	2.76	10	9318.37	3.04	9242.65	9318.37	0.81	10	5.80	9386.98	10	3.03	0.73
		0.75	**0.50**	0.01	10	9992.74	10.15	9	9995.60	16.86	9885.64	9992.74	1.07	10	27.02	10088.43	9	16.72	0.95
		0.75	**0.75**	0.01	9	10890.68	43.02	9	10890.68	67.19	10733.56	10890.68	1.44	9	110.22	11029.92	9	68.04	1.26
		0.75	**1.00**	0.01	7	11589.59	163.53	7	11589.59	141.39	11410.71	11589.59	1.54	7	304.92	11761.83	7	140.84	1.46
		0.75	**1.25**	0.01	7	12206.26	233.89	6	12197.85	145.95	11982.67	12197.85	1.76	6	379.85	12387.38	6	147.86	1.53
	c_{ij}	0.75	0.25	**0.005**	6	6558.11	6.30	6	6558.11	7.74	6517.31	6558.11	0.62	6	14.03	6597.14	6	7.76	0.59
		0.75	0.25	**0.015**	13	10476.10	0.81	12	10475.07	0.83	10415.98	10475.07	0.56	12	1.64	10527.71	12	0.82	0.50
		0.75	0.25	**0.02**	18	11552.20	1.16	18	11552.20	1.27	11487.46	11552.20	0.56	18	2.43	11620.44	18	1.30	0.59
	β_2, h_j	**0.55**	**0.5**	0.01	10	9952.95	9.61	9	9957.92	16.44	9798.06	9952.95	1.56	10	26.05	10088.43	9	16.72	1.34
		0.55	**1.00**	0.01	7	11519.14	238.56	7	11519.14	140.71	11260.81	11519.14	2.24	7	379.26	11761.83	7	139.37	2.06
		0.55	**1.25**	0.01	7	12118.20	450.61	6	12120.36	147.49	11795.28	12118.20	2.66	7	598.10	12387.38	6	147.16	2.17
		0.95	**0.5**	0.01	9	10062.13	14.57	9	10062.13	16.57	10040.00	10062.13	0.22	9	31.14	10088.43	9	16.71	0.26
		0.95	**1.00**	0.01	7	11713.94	201.50	7	11713.94	140.23	11675.76	11713.94	0.33	7	341.72	11761.83	7	139.37	0.41
		0.95	**1.25**	0.01	6	12334.61	226.86	6	12334.61	146.32	12292.57	12334.61	0.34	6	373.17	12387.38	6	147.41	0.43
	β_2, c_{ij}	**0.55**	0.25	**0.005**	6	6542.14	5.57	6	6542.14	7.75	6483.23	6542.14	0.90	6	13.32	6597.14	6	7.73	0.83
		0.55	0.25	**0.015**	12	10243.91	0.82	12	10243.91	0.71	10161.83	10243.91	0.80	12	1.53	10317.91	12	0.73	0.72
		0.55	0.25	**0.02**	19	11531.42	1.33	18	11524.42	1.34	11429.38	11524.42	0.82	18	2.67	11620.44	18	1.28	0.83
		0.95	0.25	**0.005**	6	6586.31	6.75	6	6586.31	7.69	6577.57	6586.31	0.13	6	14.44	6597.14	6	7.76	0.16
		0.95	0.25	**0.015**	12	10302.99	0.77	12	10302.99	0.71	10291.10	10302.99	0.12	12	1.48	10317.91	12	0.71	0.14
		0.95	0.25	**0.02**	18	11601.27	1.18	18	11601.27	1.26	11587.85	11601.27	0.12	18	2.44	11620.44	18	1.29	0.16
	h_j, c_{ij}	0.75	**1.25**	**0.005**	4	9023.39	1477.19	4	9023.39	3508.15	8858.55	9023.39	1.83	4	4985.34	9193.62	4	3537.09	1.85
		0.75	**1.25**	**0.015**	12	14271.58	21.546	11	14264.62	24.368	13987.86	14264.62	1.94	11	45.91	14521.85	11	24.973	1.77
		0.75	**1.25**	**0.02**	12	15770.25	2.16	12	15770.25	1.71	15486.53	15770.25	1.80	12	3.88	16033.43	12	1.796	1.64
88-nodes	Case base	0.75	0.25	0.01	15	15831.84	14.80	15	15831.84	23.98	15750.57	15831.84	0.51	15	38.78	15914.42	15	24.45	0.52
	β_2	**0.55**	0.25	0.01	15	15798.22	20.80	15	15798.22	24.31	15680.62	15798.22	0.74	15	45.10	15914.42	15	24.45	0.73
		0.65	0.25	0.01	15	15814.13	15.64	15	15814.13	24.16	15713.71	15814.13	0.63	15	39.80	15914.42	15	24.45	0.63
		0.85	0.25	0.01	15	15853.98	16.64	15	15853.98	24.22	15796.70	15853.98	0.36	15	40.86	15914.42	15	24.45	0.38
		0.95	0.25	0.01	15	15891.18	18.49	15	15891.18	24.03	15874.25	15891.18	0.11	15	42.52	15914.42	15	24.45	0.15
	h_j	0.75	**0.005**	0.01	21	11549.66	1.49	21	11549.66	1.42	11547.71	11549.66	0.02	21	2.91	11551.56	21	1.40	0.02
		0.75	**0.05**	0.01	20	13105.64	2.24	20	13105.64	2.23	13087.70	13105.64	0.14	20	4.47	13123.96	20	2.24	0.14
		0.75	**0.1**	0.01	19	14052.27	3.71	19	14052.27	4.04	14016.88	14052.27	0.25	19	7.75	14087.74	19	3.97	0.25
		0.75	**0.40**	0.01	13	17015.56	47.11	11	17010.60	31.30	16890.26	17010.60	0.71	11	78.42	17126.94	11	31.345	0.68
		0.75	**0.5**	0.01	11	17623.92	38.89	11	17623.92	53.30	17477.97	17623.92	0.83	11	92.19	17769.34	11	54.37	0.82
		0.75	**0.75**	0.01	11	18951.74	117.66	11	18951.74	220.41	18732.82	18951.74	1.16	11	338.07	19169.86	11	220.11	1.14
		0.75	**1.00**	0.01	11	20094.59	771.74	10	20094.23	4414.24	19802.70	20094.23	1.45	10	5185.98	20376.72	10	4320.96	1.39
		0.75	**1.25**	0.01	10	21059.36	2608.21	10	21059.36	10812.15[a]	20717.83	21059.36	1.62	10	13420.36	21412.47	10	10809.80[a]	1.65

c_{ij}	0.75	0.25	**0.005**	10	11719.49	72.59	10	11719.49	116.68	11651.18	11719.49	0.58	10	189.27	11790.11	10	118.12	0.60
	0.75	0.25	**0.015**	23	18449.29	8.43	22	18442.41	10.19	18346.98	18442.41	0.52	22	18.62	18538.83	22	10.16	0.52
	0.75	0.25	**0.02**	28	20274.77	2.49	28	20274.77	2.72	20164.00	20274.77	0.55	28	5.21	20382.88	28	2.67	0.53
β_2, h_j	**0.55**	**0.05**	0.01	20	13098.21	2.12	20	13098.21	2.23	13072.22	13098.21	0.20	20	4.36	13123.96	20	2.21	0.20
	0.55	**0.4**	0.01	13	16964.89	27.29	11	16962.91	30.51	16783.70	16962.91	1.06	11	57.81	17126.94	11	31.08	0.96
	0.55	**1.00**	0.01	11	19975.36	693.47	10	19978.26	4419.15	19553.91	19975.36	2.11	11	5112.62	20376.72	10	4385.18	1.97
	0.55	**1.25**	0.01	10	20914.40	2908.11	10	20914.40	10810.67[a]	20421.25	20914.40	2.36	10	13718.78	21412.47	10	10810.03[a]	2.33
	0.95	**0.05**	0.01	20	13118.74	2.21	20	13118.74	2.24	13115.07	13118.74	0.03	20	4.45	13123.96	20	2.21	0.04
	0.95	**0.4**	0.01	11	17094.85	34.68	11	17094.85	31.34	17069.80	17094.85	0.15	11	66.02	17126.94	11	31.38	0.19
	0.95	**1.00**	0.01	10	20299.07	2073.77	10	20299.07	4225.10	20240.49	20299.07	0.29	10	6298.87	20376.72	10	4330.09	0.38
	0.95	**1.25**	0.01	10	21315.41	10809.19[a]	10	21315.41	10809.53[a]	21242.19	21315.41	0.34	10	21618.73	21412.47	10	10809.46[a]	0.45
β_2, c_{ij}	**0.55**	0.25	**0.005**	10	11690.49	48.93	10	11690.49	115.82	11591.86	11690.49	0.84	10	164.75	11790.11	10	117.62	0.84
	0.55	0.25	**0.015**	23	18409.14	6.99	22	18403.29	10.33	18260.95	18403.29	0.77	22	17.31	18538.83	22	10.32	0.73
	0.55	0.25	**0.02**	28	20230.91	2.49	28	20230.91	2.66	20070.54	20230.91	0.79	28	5.15	20382.88	28	2.67	0.75
	0.95	0.25	**0.005**	10	11770.70	200.16	10	11770.70	117.83	11756.05	11770.70	0.12	10	317.99	11790.11	10	116.97	0.16
	0.95	0.25	**0.015**	22	18511.46	9.17	22	18511.46	10.29	18492.58	18511.46	0.10	22	19.46	18538.83	22	10.27	0.15
	0.95	0.25	**0.02**	28	20352.21	2.89	28	20352.21	2.76	20329.23	20352.21	0.11	28	5.66	20382.88	28	2.72	0.15
h_j, c_{ij}	0.75	**1.0**	**0.005**	7	15100.30	10812.81[a]	7	15100.30	10808.06[a]	14875.43	15100.30	1.49	7	22110.88	15337.88	7	10816.931[a]	1.55
	0.75	**1.0**	**0.015**	17	23868.85	3949.97	16	23862.61	10520.11	23535.58	23862.61	1.37	16	14470.09	24189.39	16	10274.19	1.35
	0.75	**1.0**	**0.02**	22	26387.26	370.98	22	26387.26	311.39	26018.35	26387.26	1.40	22	682.37	26772.94	22	311.59	1.44

[a] Suboptimal solution obtained for 3 h limit.

C), therefore, $C_2(r)$ is solution of $sl_2(r, C) = \beta_2$. Then, $C_2(r) = r - r_2^0 = r - \mu L - z_{\beta_2}\sigma\sqrt{L}$. In the same way we define $C_1(r)$ as the minimum critical level, given a reorder point r, that ensures a service level β_1, i.e., $C_1(r) = \min\{C \mid sl_1(r, C) \geqslant \beta_1\}$. Let r_i^0 be the minimum reorder point r such that the service level provided to the class i, given a critical level $C = 0$, is greater than or equal to his preset service level β_i, i.e., $r_i^0 = \min\{r \mid sl_i(r, 0) \geqslant \beta_i\}$, with $i = 1, 2$.

(a) If $\mu_1 = \sigma_1^2 = 0$, from Eq. (1) we have for any $r > C \geqslant 0$ that $sl_1(r, C) = \mathbb{P}(D_2(L) \leqslant r - C) + \mathbb{P}(\tau_{H,D_2}^{r-C} < L) = \mathbb{P}(D_2(L) \leqslant r - C) + 1 - \mathbb{P}(D_2(L) \leqslant r - C) = 1$. Therefore, $sl_1(r, C) = 1$, $\forall C \geqslant 0$ and $C_1(r) = 0$, $\forall r$. On the other hand, $C_2(r) = r - \mu_2 L - z_{\beta_2}\sigma_2\sqrt{L}$. Once $C_1(r)$ and $C_2(r)$ are defined, the feasible region of SLP problem where all (r, C) satisfies $sl_1(r, C) \geqslant \beta_1$, $sl_2(r, C) \geqslant \beta_2$ and $r > C \geqslant 0$, when $\mu_1 = \sigma_1^2 = 0$, is the intersection of the areas above $C_1(r)$, below $C_2(r)$ and strictly below $r = C$. The feasible region is shown in Fig. C.4a.
From Fig. C.4a we conclude that the minimum reorder point that guarantees a service level β_2 provided to the low priority class is r_2^0. Therefore, $r^* = r_2^0 = \mu_2 L + z_{\beta_2}\sigma_2\sqrt{L}$ and $C^* = 0$.

(b) If $\mu_2 = \sigma_2^2 = 0$, it valid that $sl_1(r, C) = sl_1(r, 0)$, $\forall C \geqslant 0$. Therefore, $C_1(r) = 0$ for any $r \geqslant r_1^0$. On the other hand, $C_2(r) = r - r_2^0 = r - \mu_1 L - z_{\beta_2}\sigma_1\sqrt{L}$. The feasible region of SLP problem where all (r, C) satisfies $sl_1(r, C) \geqslant \beta_1, sl_2(r, C) \geqslant \beta_2$ and $r > C \geqslant 0$, when $\mu_2 = \sigma_2^2 = 0$, is the intersection of the areas above $C_1(r)$, below $C_2(r)$ and strictly below $r = C$. The feasible region is shown in Fig. C.4b.
From Fig. C.4b we conclude that the minimum reorder point that guarantees a service level β_1 provided to the high priority class is r_1^0. Therefore, $r^* = r_1^0$ solution of $sl_1(r, 0) = \beta_1$ and for convenience $C^* = 0$. Note that $sl_2(r^*, 0) > \beta_2$ because $r_2^0 < r_1^0$. □

Table E.4
Data for illustrative example.

Candidate DCs				Customers					
DC	Location		f_j	Customer	Class	Location		Demand	
	X	Y	(US$/day)			X	Y	μ_i (kg/day)	CV_i
1	−9.47	21.02	191	1	1	−7.37	10.71	4647.33	0.53
2	−6.22	15.53	200	2	1	−18.23	0.11	1616.67	0.42
3	−6.67	15.92	195	3	1	−13.31	−12.01	1275.33	0.85
4	−7.98	16.39	206	4	2	−9.38	10.96	504.67	0.63
5	−8.60	16.71	206	5	2	−11.15	−3.24	430.00	0.59
6	−7.59	16.18	202	6	1	−5.27	1.55	882.00	0.93
7	−7.56	15.23	211	7	2	−6.13	4.23	248.00	0.36
8	−7.10	14.90	206	8	1	2.19	1.11	4880.00	0.78
9	−6.45	14.93	211	9	2	4.13	−0.22	386.67	0.78
10	−6.89	14.03	206	10	1	−8.16	13.68	1256.67	0.44
11	−5.97	13.62	244	11	1	−13.15	5.45	1476.67	0.61
12	−4.91	11.62	237	12	2	−2.48	6.04	176.00	0.17
13	−3.56	12.01	233	13	1	−8.32	10.19	700.67	0.15
14	−0.89	10.31	233	14	2	7.18	4.45	230.00	0.58
15	−4.49	9.56	255	15	1	−6.40	5.12	872.67	0.80
16	−4.35	9.29	255	16	2	−8.81	−9.94	196.00	0.94
17	−3.64	5.38	233	17	2	−7.05	0.22	388.00	0.22
18	−6.87	10.81	211	18	2	1.18	−9.12	176.00	0.61
19	−9.55	10.16	222	19	2	−11.24	−38.33	456.67	0.52
20	−10.27	10.08	211	20	2	−6.86	−12.23	114.00	0.11
21	−10.64	10.15	206	21	2	−2.61	−6.45	156.00	0.40
22	−10.65	9.79	217	22	2	−7.05	−18.35	250.67	0.25
23	−11.08	9.89	211	23	2	7.65	0.00	144.67	0.81
24	−11.87	9.55	203	24	2	−7.51	−18.12	321.33	0.38
25	−12.34	8.76	208	25	2	−10.55	−33.96	194.67	0.58
26	−22.13	2.04	200	26	2	4.04	−18.35	174.67	0.25
27	−12.88	2.72	222	27	2	−23.59	−19.68	168.00	0.64
28	−13.07	2.92	222	28	2	−2.71	−2.78	198.00	0.34
29	−13.10	2.53	211	29	2	−28.49	−26.02	150.67	0.69
30	−13.09	2.14	195	30	2	−12.41	−7.45	111.33	0.72
31	−12.32	2.31	211	31	2	9.77	7.78	100.00	0.77
32	−9.31	1.59	228	32	2	6.07	−6.12	88.67	0.51
33	−8.72	1.51	228	33	2	−8.59	−10.30	186.00	0.18
34	−6.93	−6.22	228	34	2	−4.37	7.78	154.00	0.31
35	−5.07	−3.42	217	35	2	−26.55	−25.24	264.67	0.92
36	−6.47	−8.86	222	36	2	2.56	−9.90	233.33	0.24
37	−6.98	−9.36	208	37	2	4.33	3.14	222.00	0.84
38	−6.49	−9.47	208	38	2	4.23	−20.35	109.33	0.58

Appendix D. Numerical results for test sets

See Table D.3.

Appendix E. Data for illustrative example

See Table E.4.

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