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Construction of a stable periodic solution to a semilinear heat equation with a prescribed profile

Fethi Mahmoudi[a], Nejla Nouaili[b,*], Hatem Zaag[c]

[a] Centro de Modelamiento Matemático, Universidad de Chile, Chile
[b] CEREMADE, Université Paris Dauphine, Paris Sciences et Lettres, France
[c] Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France

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ABSTRACT

We construct a periodic solution to the semilinear heat equation with power nonlinearity, in one space dimension, which blows up in finite time T only at one blow-up point. We also give a sharp description of its blow-up profile. The proof relies on the reduction of the problem to a finite dimensional one and the use of index theory to conclude. Thanks to the geometrical interpretation of the finite-dimensional parameters in terms of the blow-up time and blow-up point, we derive the stability of the constructed solution with respect to initial data.

1. Introduction

The behavior of partial differential equations (PDEs) may depend on the space Ω where they are considered. As a matter of fact, it is classical in the literature to compare the case of $\Omega = \mathbb{R}^N$ with the case where Ω is a bounded domain, with boundary conditions.

However, in both cases, the space is "flat", which may prevent from seeing the effect of curvature, very important in many physical situations. Thus, considering the case of a new flat Ω appears to be relevant, at least for this reason, but not only. Indeed, that case appears also to be very challenging, from a mathematical point of view.

As a matter of fact, many authors considered this case in the literature, in recent years. Let us mention the case of the nonlinear Schrödinger equation on $\mathbb{S}^N$ by Pausader, Tzvetkov and Wang in [30] and also by Burq, Gérard and Tzvetkov in [3]. There is also the work of Méhats and Gérard [10] who consider the

* Corresponding author.
E-mail addresses: fmahmoudi@dim.uchile.cl (F. Mahmoudi), Nejla.Nouaili@dauphine.fr (N. Nouaili), Hatem.Zaag@univ-paris13.fr (H. Zaag).

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Schrödinger–Poisson system on $\mathbb{S}^2$. We cite also the case of the Navier–Stokes equation on the sphere $\mathbb{S}^2$ by Cao, Rammaha and Titi [4].

In the case of the nonlinear heat equation, we mention the work by Matano, Punzo and Tesei in [20], where they considered front propagation on the hyperbolic space $\mathbb{H}^N, N \geq 2$. We also refer to the work by Cao, Rammaha and Titi [5], where they considered the nonlinear parabolic equation on $\mathbb{S}^2$.

In this paper, we would like to see whether the curvature may influence the blow-up behavior, by considering the following semilinear heat equation.

$$\partial_t u = \Delta_{\mathbb{S}^N} u + |u|^{p-1}u, \tag{1}$$

where $u(t) : x \in \mathbb{S}^N \to \mathbb{R}$ and $\Delta_{\mathbb{S}^N}$ denotes the Laplace–Beltrami operator on $\mathbb{S}^N \subset \mathbb{R}^{N+1}$.

By using the localization in charts, we felt that the interaction between the singular regions (where blow-up occurs) and the regular region is highly challenging, and no easy solution seems to be available. For that reason, we restrict ourselves to the case where $N = 1$ which is easier, due to the one-dimensional character and the periodicity. Of course the case $N = 1$ conserves all the challenging characters of the interaction question.

The Cauchy problem for system (1) can be solved in $(L^\infty(\mathbb{S}^N))$, locally in time. We say that $u(t)$ blows up in finite time $T < \infty$, if $u(t)$ exists for all $t \in [0, T)$ and $\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty$. In that case, T is called the blow-up time of the solution. A point $x_0 \in \mathbb{S}^N$ is said to be a blow-up point if there is a sequence $\{(x_j, t_j)\}$, such that $x_j \to x_0, t_j \to T$ and $|u(x_j, t_j)| \to \infty$ as $j \to \infty$.

Two questions arise in the blow-up study:

- the classification for arbitrary blow-up solutions,
- the construction of examples obeying same prescribed behavior.

Let us first briefly mention some literature on blow-up in the case where $\Omega = \mathbb{R}^N$. Consider the equation

$$\partial_t u = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^N. \tag{2}$$

The blow-up question for Eq. (2), has been studied intensively by many authors and no list can be exhaustive.

Regarding the classification question, when it comes to deriving the blow-up profile, the situation is completely understood in one space dimension (however, less is understood in higher dimensions, see Velázquez [36–38] and Zaag [40–42] for partial results). In one space dimension, given a blow-up point a, we have the following alternative

- either

$$\sup_{|x-a|\leq K\sqrt{(T-t)\log(T-t)}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f\left(\frac{x-a}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \to 0, \tag{3}$$

- or for some $m \in \mathbb{N}, m \geq 2$, and $C_m > 0$

$$\sup_{|x-a|<K(T-t)^{1/2m}} \left|(T-t)^{\frac{1}{p-1}}u(x,t) - f_m\left(\frac{C_m(x-a)}{(T-t)^{1/2m}}\right)\right| \to 0, \tag{4}$$

as $t \to T$, for any $K > 0$, where

$$f(z) = \left(p - 1 + \frac{(p-1)^2}{4p}z^2\right)^{-\frac{1}{p-1}} \quad \text{and} \quad f_m(z) = \left(p - 1 + |z|^2 m\right)^{-\frac{1}{p-1}}. \tag{5}$$

Let us mention that when Ω is a subdomain of $\mathbb{R}^N$ with $C^{2,\alpha}$ boundary, Giga and Kohn proved in [13] that no blow-up occurs at the boundary and many of the above results hold.

Regarding the construction question, from Bricmont and Kupiainen [2] and Herrero and Velázquez [15], we have examples of initial data leading to each of the above-mentioned scenarios. Note that (3) corresponds to the fundamental mode of the harmonic oscillator in the leading order, whereas (4) corresponds to higher modes. Moreover, Herrero and Velázquez proved the genericity of the behavior (3) in one space dimension in [14,16], and only announced the result in the higher dimensional case (the result has never been published). Note also that the stability of such a profile with respect to initial data has been proved by Fermanian Kammerer, Merle and Zaag in [9,8]. For more results on Eq. (2), see [1,11–13,15,16,18,19,25–27,31].

In this paper, we address the construction question of Eq. (1) when $N = 1$, and we give the first example of a blow-up solution of Eq. (1) in $\mathbb{S} = \mathbb{S}^1$, with a full description of its blow-up profile, obeying behavior (3) (note that our method extends with no difficulty to the construction of an analogous solution obeying the behavior (4); however, the proof should be even more technical).

Making the change of variables $x = e^{i\theta} \in \mathbb{S}$, we can rewrite (1) as

$$\partial_t u = \partial_\theta^2 u + |u|^{p-1}u, \tag{6}$$

where $\theta \in \mathbb{R}$ and $u(\cdot, t)$ is 2π-periodic. As we said above, the first main purpose of this work is to show that behavior (3) does occur. More precisely, we prove the existence of a blow-up solution for Eq. (6) and we give a description of its profile. This can be summarized in the following result.

Theorem 1 (*Existence of a Blow-up Solution for (6) with Prescribed Profile*)**.** *There exists $T > 0$ such that Eq. (6) has a solution $u(\theta, t)$ in $\mathbb{S} \times [0, T)$ such that:*

(i) *the solution u blows up in finite time T only in $2\pi\mathbb{Z}$;*
(ii) *there holds that for all $R > 0$,*

$$\sup_{\Lambda_R} \left| (T-t)^{\frac{1}{p-1}} u(\theta + 2k\pi, t) - f\left(\frac{\theta + 2k\pi}{\sqrt{(T-t)|\log(T-t)|}} \right) \right| \longrightarrow 0 \quad \text{as } t \to T, \tag{7}$$

where $\Lambda_R := \left\{ |\theta - 2k\pi| \le R\sqrt{(T-t)|\log(T-t)|}, k \in \mathbb{Z} \right\}$ and where the function f is defined by (5).

(iii) *for all $\theta \notin 2\pi\mathbb{Z}, u(\theta, t) \to u(\theta, T)$ as $t \to T$, with $u(\theta, T) \sim u^*(\theta - 2k\pi)$ as $\theta \to 2k\pi$, for any $k \in \mathbb{Z}$, where*

$$u^*(\theta) = \left[\frac{(p-1)^2 |\log \theta|}{8p\theta^2} \right]^{-\frac{1}{p-1}}. \tag{8}$$

Remark 1.1. 1. A natural extension of our result would be to consider the heat equation in $\mathbb{R}^N$ in the periodic setting (or equivalently, in the torus $\mathbb{R}^N/\mathbb{Z}^N$ for example). We believe that our result also holds in this setting. However, understanding the interaction between the inner and the outer parts (or the solution in the regular and the blow-up parts; see Section 2) would be much more difficult, though the difficulty is only technical. For that reason, we focus on one space dimension in this paper to keep the paper within reasonable limits.

2. In the case of Eqs. (1) and (2), there do exist solutions which behave like (3) and others like (4) (see Bricmont–Kupiainen, [2], for the construction in both cases, when the equation is considered in $\mathbb{R}^N$). In our paper, we focus on the construction of solutions obeying (3), but we do believe that our method can be adapted to construct solutions obeying (4). However, we only construct a solution obeying (3), since this should be the only stable solution.

The proof of Theorem 1 uses some ideas developed by Bricmont and Kupiainen [2] and Merle and Zaag [24,23] to construct a blow-up solution for the semilinear heat Eq. (2) obeying the behavior (3). In [7], Ebde and Zaag use the same ideas to show the persistence of the profile (3) under perturbations of Eq. (1) in the real case by lower order terms involving u and ∇u. See also [28] by Nguyen and Zaag, where the authors consider stronger perturbation than the one considered in [7].

In [17], Masmoudi and Zaag adapted that method to the following complex Ginzburg–Landau equation with no gradient structure

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1}u, \quad \text{with } \beta, \delta \in \mathbb{R}.$$

Notice that the case $\beta = 0$ and δ small was first considered by Zaag, see [39]. We also mention the recent work by Nouaili and Zaag in [29] for a complex-valued equation with no gradient structure.

More precisely, the proof relies on the understanding of the dynamics of the self-similar version of (1) (see system (22)) around the profile (3). Moreover, we proceed in two steps:

- First, we reduce the question to a finite-dimensional problem: we show that it is enough to control a $(N + 1)$-dimensional variable in order to control the solution (which is infinite dimensional) near the profile.
- Second, we proceed by contradiction to solve the finite-dimensional problem and conclude using index theory.

Surprisingly enough, we would like to mention that this kind of methods has proved to be successful in various situations including hyperbolic and parabolic PDE, in particular with energy-critical exponents. This was the case for the construction of multi-solitons for the semilinear wave equation in one space dimension by Côte and Zaag [6], the wave maps by Raphaël and Rodnianski [32], the Schrödinger maps by Merle, Raphaël and Rodnianski [22], the critical harmonic heat flow by Schweyer [34] and the two-dimensional Keller–Segel equation by Raphaël and Schweyer [33].

The interpretation of the parameters of the finite dimensional problem in terms of the blow-up time and blow-up point allows us, as in [24,23], to derive the stability of the constructed solution as stated in the following result.

Proposition 2 (*Stability of the Constructed Solutions*)**.** *Denote by $\hat{u}$ the solution constructed in Theorem 1 and by $\hat{T}$ its blow-up time. Then, there exists $\varepsilon_0 > 0$ such that for any initial data $u_0 \in L^\infty(\mathbb{S})$, satisfying*

$$\|u_0 - \hat{u}(\cdot, 0)\|_{L^\infty} \le \varepsilon_0,$$

the solution of Eq. (6), with initial data u_0 blows up in finite time $T(u_0)$ at only one blow-up point $a(u_0) \in \mathbb{S}$.

Moreover, the function $u(\theta - a(u_0), \cdot)$ satisfies the same estimates as u with $\hat{T}$ replaced by $T(u_0)$.

Furthermore, it follows that

$$T(u_0) \to \hat{T}, \qquad a(u_0) \to a_0 \quad \text{as } u_0 \to \hat{u}(0).$$

The proof of this stability result follows exactly as in [24,23]. For that reason we skip it and refer the interested readers to those papers.

We proceed in 3 sections to prove Theorem 1. We first give in Section 2 an equivalent formulation of the problem in the scale of the well-known similarity variables. Section 3 is devoted to the proof of the similarity variables formulation (this is a central part in our argument). In the last section, we prove Theorem 1.

2. Formulation of the problem

We would like to find initial data u_0 such that the solution u of Eq. (1) blows up in time T with

$$\sup_{\Lambda_R}\left|(T-t)^{\frac{1}{p-1}}u(\theta+2k\pi,t)-f\left(\frac{\theta+2k\pi}{\sqrt{(T-t)|\log(T-t)|}}\right)\right|\longrightarrow 0\quad \text{as } t\to T \tag{9}$$

where $\Lambda_R := \left\{|\theta-2k\pi|\leq R\sqrt{(T-t)|\log(T-t)|}, k\in\mathbb{Z}\right\}$.

This is the main estimate and the other results of Theorem 1 will appear as by-products of the proof (see Section 4 for the proof of all the estimates of Theorem 1). From periodicity, we will consider θ in one period, which depends on the region we consider (see below in the definitions of the regular and the blow-up region).

First, we introduce the following cut-off function $\chi_0 \in C_0^\infty(\mathbb{R},[0,1])$,

$$\chi_0(\xi)=\begin{cases}1 & \text{if } |\xi|\leq 1,\\ 0 & \text{if } |\xi|\geq 2.\end{cases} \tag{10}$$

In the following, we will divide our work in two parts; the blow-up region and the regular region.

- In the regular region, we will study $\bar{u}$ defined by:

$$\bar{u}(\theta)=\begin{cases}u(\theta)\bar{\chi}(\theta) & \text{if } \theta\in[0,2\pi],\\ 0 & \text{if } \theta\in\mathbb{R}\setminus[0,2\pi],\end{cases} \tag{11}$$

  where the function $\bar{\chi}$ is 2π-periodic and defined for all $\xi\in[-\pi,\pi]$ by

$$\bar{\chi}(\xi)=1-\chi_0\left(\frac{4\xi}{\varepsilon_0}\right)$$

  with $\varepsilon_0>0$ will be fixed small enough later. Then, for all $\theta\in\mathbb{R}, \bar{u}(\theta)$ satisfies the following equation

$$\partial_t\bar{u}=\partial_\theta^2\bar{u}+|u|^{p-1}\bar{u}-2\bar{\chi}'\partial_\theta u-\bar{\chi}''u. \tag{12}$$

  We control $\bar{u}$ using classical parabolic estimates on u as we will see in Proposition 3.13.
- In the blow-up region of $u(\theta,t)$, we make the following self-similar transformation of problem (6)

$$\begin{aligned} &W(y,s)=(T-t)^{\frac{1}{p-1}}u(\theta,t),\\ &y=\frac{\theta}{\sqrt{T-t}},\qquad s=-\log(T-t),\end{aligned} \tag{13}$$

then $W(y,s)$, for $y\in\mathbb{R}$, satisfies the following equation

$$\partial_s W=\partial_y^2 W-\frac{1}{2}y\partial_y W-\frac{1}{p-1}W+|W|^{p-1}W. \tag{14}$$

We note that for all $s\in\mathbb{R}, W$ is $2\pi e^{s/2}$ periodic. Let us define

$$w(y,s)=\begin{cases}W(y,s)\chi(y,s) & \text{if } |y|\leq \pi e^{s/2},\\ 0 & \text{otherwise},\end{cases} \tag{15}$$

with

$$\chi(y,s) = \chi_0\left(\frac{ye^{-s/2}}{\varepsilon_0}\right), \tag{16}$$

where χ_0 is defined by (10) and ε_0 will be fixed small enough later in the proof.

Then we multiply Eq. (14) by $\chi(y,s)$ and we get

$$\chi\partial_s W = \chi\partial_y^2 W - \frac{1}{2}y\chi\partial_y W - \frac{1}{p-1}w + |W|^{p-1}w,$$

therefore

$$\partial_s w = \partial_y^2 w - \frac{1}{2}y\partial_y w - \frac{1}{p-1}w + |w|^{p-1}w + F(y,s), \tag{17}$$

where

$$F(y,s) = \begin{cases} W\partial_s\chi - 2\partial_y\chi\partial_y W - W\partial_y^2\chi + \frac{1}{2}yW\partial_y\chi + |W|^{p-1}W\left(\chi - \chi^p\right) & \text{if } |y| \le \pi e^{s/2}, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Remark 2.1. We note that w is not periodic, and that Eq. (17) is valid for all $y \in \mathbb{R}$.

Now, let us introduce

$$w = \varphi + q, \tag{19}$$

with

$$\varphi = f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa}{4ps}, \tag{20}$$

where

$$f(z) = \left(p - 1 + bz^2\right)^{-\frac{1}{p-1}}, \qquad \kappa = (p-1)^{-\frac{1}{p-1}} \quad \text{and} \quad b = \frac{(p-1)^2}{4p}. \tag{21}$$

The problem is then reduced to constructing a function q such that

$$\lim_{s\to\infty} \|q(y,s)\|_{L^\infty} = 0,$$

and q is a solution of the following equation for all $(y,s) \in \mathbb{R} \times [s_0(= -\log T), \infty)$,

$$\partial_s q = (\mathcal{L} + V)q + B(y,s) + R(y,s) + F(y,s), \tag{22}$$

where

$$\mathcal{L} = \partial_y^2 - \frac{1}{2}y\partial_y + 1, \qquad V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}, \tag{23}$$

$$B(y,s) = |\varphi + q|^{p-1}(\varphi + q) - \varphi^p - p\varphi^{p-1}q, \tag{24}$$

and

$$\begin{aligned} R(y,s) &= \partial_y^2\varphi - \frac{1}{2}y\partial_y\varphi - \frac{\varphi}{p-1} + \varphi^{p-1} - \partial_s\varphi, \\ F(y,s) &= H(y,s) + \partial_y G(y,s) \quad \text{with,} \\ H(y,s) &= W\left(\partial_s\chi + \partial_y^2\chi + \frac{1}{2}y\partial_y\chi\right) + |W|^{p-1}W(\chi - \chi^p), \\ G(y,s) &= -2\partial_y\chi W. \end{aligned} \tag{25}$$

The control of q near 0 obeys two facts:

- *Localization*: the fact that our profile $\varphi(y,s)$ dramatically changes its value from $1+\frac{1}{4s}$ in the region near 0 to $\frac{1}{4s}$ in the region near infinity, according to a free boundary moving at the speed $\sqrt{s}$. This will require different treatments in the regions $|y|<2K_0\sqrt{s}$ and $2K_0\sqrt{s}<|y|<\frac{\pi}{2}e^{s/2}$ for some K_0 to be chosen.
- *Spectral information*: the fact that the operator $\mathcal{L}$ is selfadjoint, B is quadratic in q and that

$$\|R(s)\|_{L^\infty}+\|V(s)\|_{L^2_\rho}\to 0 \quad \text{as } s\to\infty,$$

from (19) and (23), which shows that the dynamics of Eq. (22) near 0 are driven by the spectral properties of $\mathcal{L}$. This will require a decomposition of the solution according to the spectrum of $\mathcal{L}$. Note that the operator $\mathcal{L}$ is self-adjoint in the Hilbert space

$$L^2_\rho=\left\{g\in L^2_{loc}(\mathbb{R},\mathbb{C}),\|g\|^2_{L^2_\rho}\equiv\int_{\mathbb{R}}|g|^2e^{-\frac{|y|^2}{4}}dy<+\infty\right\} \quad \text{with } \rho(y)=\frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{1/2}}.$$

The spectrum of $\mathcal{L}$ is explicitly given by

$$\operatorname{spec}(\mathcal{L})=\left\{1-\frac{m}{2},\ m\in\mathbb{N}\right\}.$$

All the eigenvalues are simple, the eigenfunctions are dilations of Hermite's polynomial and given by

$$h_m(y)=\sum_{n=0}^{[\frac{m}{2}]}\frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{26}$$

Note that $\mathcal{L}$ has two positive (or expanding) directions ($\lambda=1$ and $\lambda=\frac{1}{2}$), and a zero direction ($\lambda=0$). Complying with the localization and spectral information facts, we will decompose q accordingly as stated above:

- First, let us introduce

$$\chi_1(y,s)=\chi_0\left(\frac{|y|}{K_0\sqrt{s}}\right), \tag{27}$$

where χ_0 is defined in (10), $K_0\geq 1$ will be chosen large enough so that various technical estimates hold. Then, we write $q=q_b+q_e$, where the inner part and the outer part are given by

$$q_b=q\chi_1, \qquad q_e=q(1-\chi_1). \tag{28}$$

Let us remark that

$$\operatorname{supp}(q_b(s))\subset B(0,2K_0\sqrt{s}) \quad \text{and} \quad \operatorname{supp}(q_e(s))\subset\mathbb{R}\setminus B(0,K_0\sqrt{s}).$$

- Second, we study q_b using the structure of $\mathcal{L}$, isolating the nonnegative directions. More precisely we decompose q_b as follows:

$$q_b(y,s)=\sum_0^2 q_m(s)h_m(y)+q_-(y,s), \tag{29}$$

where q_m is the projection of q_b on $h_m, q_-(y,s) = P_-(q_b)$ and P_- is the projection on $\{h_i,\ i \geq 3\}$ the negative subspace of the operator $\mathcal{L}$.

In summary, we can decompose q in 5 components as follows:

$$q(y,s) = \sum_{m=0}^{2} q_m(s)\, h_m(y) + q_-(y,s) + q_e(y,s). \tag{30}$$

Here and throughout the paper, we call $q_-(y,s)$ the negative part of q and q_2, the null mode of q.

3. The construction method in self similar variables

This section is devoted to the proof of the existence of a solution u of Eq. (1) satisfying $\|q(s)\|_{L^\infty} \to 0$. This is a central argument in our proof. Though we refer to the earlier work by Merle and Zaag [24] for purely technical details, we insist on the fact that we can completely split from that paper as long as ideas and arguments are considered. We hope that the explanation of the strategy we give in this section will be more reader friendly.
We proceed in 3 subsections:

- In the first subsection, we give all the arguments of the proof without the details, which are left for the following subsection (readers not interested in technical details may stop here).
- In the second subsection, we give various estimates concerning initial data.
- In the third subsection, we give the dynamics of system (22) near the zero solution, in accordance with the decomposition (30) and taking into account the interaction between the singular region and the regular region.

3.1. The proof without technical details

Given $T > 0$, we consider initial data for Eq. (1) 2π-periodic defined for all $\theta \in [-\pi,\pi]$ by:

$$u_0(\theta, d_0, d_1) = T^{-\frac{1}{p-1}} \left\{ \varphi(y,s_0)\chi(8y,s_0) + \frac{A}{s_0^2}(d_0 + d_1 y)\chi_1(2y,s_0) \right\}, \tag{31}$$

where $s_0 = -\log T, y = \frac{\theta}{\sqrt{T}}, \chi$ is defined in (16) and χ_1 is defined in (27).

Notice that u_0 depends also on K_0, ε_0, A and T, but we omit that dependence in (31) for simplicity.

Notice also that the transition at $-\pi + 2k\pi$ in u_0 is smooth, since $u_0 \equiv 0$ is some open interval around that number.

Thanks to Section 2, in order to control $u(s)$ near φ, it is enough to control it in some shrinking set defined as follows:

Definition 3.1 (*Definition of a Shrinking Set for the Components of q*). For all $K_0 > 0, \varepsilon_0 > 0, A > 0, 0 < \eta_0 \leq 1$ and $T > 0$, we define for all $t \in [0,T)$ the set $S^*(K_0, \varepsilon_0, A, \eta_0, T, t)$ as being the set of all functions $u \in L^\infty(\mathbb{R})$ satisfying:

(i) *Estimates in $\mathcal{R}_1$:* $q(s) \in V_{K_0,A}(s)$ where $s = -\log(T-t), q(s)$ is defined in (13), (15), (19) and (20) and $V_{K_0,A}(s)$ is the set of all functions $r \in L^\infty(\mathbb{R})$ such that

$$\begin{cases} |r_m(s)| \leq As^{-2} (m = 0,1), & |r_2(s)| \leq A^2 s^{-2} \log s, \\ |r_-(y,s)| \leq As^{-2}(1 + |y|^3), & |r_e(y,s)| \leq As^{-1/2}, \end{cases} \tag{32}$$

where

$$\begin{cases} r_e(y,s) = (1-\chi_1(y,s))r(y,s), & r_-(s) = P_-(\chi_1(s)r), \\ \text{for } m \in \mathbb{N}, \quad r_m(y,s) = \displaystyle\int d\rho k_m(y)\chi_1(y,s)r(y), & \end{cases} \tag{33}$$

χ_1 is defined in (27), P_- is the L^2_ρ projector on $\mathrm{Vect}\{h_m | m \geq 3\}$.

(ii) *Estimates in $\mathcal{R}_2$:* For all $\frac{\varepsilon_0}{2} \leq |\theta| \leq \pi, |u(\theta,t)| \leq \eta_0$.

Remark 3.1. For simplicity, we may write $S^*(t)$ instead of $S^*(K_0, \varepsilon_0, A, \eta_0, T, t)$. Note also that our arguments work with $\eta_0 = 1$.

Our aim becomes then to prove the following result.

Proposition 3.2 (*Existence of a Solution of (22) Trapped in $S^*(t)$*)**.** *There exists $K_{01} > 0$ such that for each $K_0 \geq K_{01}$, there exists $\delta_1(K_0)$, such that for any $\varepsilon_0 \leq \delta_1(K_0)$, there exists $A_1(K_0, \varepsilon_0)$, such that for any $A \geq A_1$ and $0 < \eta_0 \leq 1$, there exists $s_{0,0}(K_0, \varepsilon_0, A, \eta_0)$ such that for all $T \leq e^{-s_{0,0}}$, there exists $(d_0, d_1) \in \mathbb{R}^2$, such that:*

if u is a solution of (6) with initial data given by (31), then

$$\forall t \in [0,T), \quad u(t) \in S^*(K_0, \varepsilon_0, A, \eta_0, T, t).$$

To prove this proposition we need some intermediate lemmas.

In the following lemma, we find a set $D_{K_0,\varepsilon_0,A,T} = D_T$ such that $u(0) \in S^*(0)$, whenever $(d_0, d_1) \in D_T$. More precisely, we claim the following:

Lemma 3.3 (*Choice of Parameters d_0, d_1 to Have Initial Data in $S^*(0)$*)**.** *There exists $K_{02} > 0$ such that for each $K_0 \geq K_{02}$ there exist $\varepsilon_0 > 0, A \geq 1$, there exists $s_{0,1}(K_0, \varepsilon_0, A) \geq 0$ such that for all $s \geq s_{0,1}$:*

If initial data for Eq. (1) are given by (31): then, there exists a rectangle

$$D_{K_0,\varepsilon_0,A,T} = D_T \subset [-2,2]^2, \tag{34}$$

such that, for all $(d_0, d_1) \in D_T$, we have

$$u(K_0, T, A, d_0, d_1) \in S^*(0).$$

Proof. The proof is purely technical and follows as the analogous step in [24], for that reason we refer the reader to Lemma 3.5 page 156 and Lemma 3.9 page 160 in [24]. ■

Let us consider $(d_0, d_1) \in D_T$ and $s_0 = -\log T \geq s_{0,1}$ defined in Lemma 3.3. Since u_0 is 2π-periodic, from the local Cauchy theory, we define a maximal 2π-periodic solution u to Eq. (1) with initial data (31), and a maximal time $t_*(d_0, d_1) \in [0,T)$ such that,

$$\text{for all } t \in [0, t_*), \ u(t) \in S^*(t), \tag{35}$$

- either $t_* = T$,
- or $t_* < T$ and from continuity,

$$u(t_*) \in \partial S^*(t_*), \tag{36}$$

in the sense that when $t = t_*$, one 1 '$\leq$' symbol in the definition of $S^*(t_*)$ is replaced by the symbol '$=$'.

Our aim is to show that for all A and T small enough, one can find a parameter (d_0, d_1) in D_T such that

$$t_*(d_0, d_1) = T. \tag{37}$$

We argue by contradiction, and assume that for all $(d_0, d_1) \in D_T, t_*(d_0, d_1) < T$. As we have just stated, one of the symbols '$\leq$' in the definition of $S^*(t)$ should be replaced by '$=$' symbols when $t = t_*$.

In fact, no '$=$' sign occurs for q_2, q_-, q_e and the estimate in $\mathcal{R}_2$, as one sees in the following lemma.

Lemma 3.4 (*Reduction to a Finite Dimensional Problem*)**.** *For any $K_0 > 0$, there exists $\delta_2(K_0) > 0$, such that for any $\varepsilon_0 \leq \delta_2(K_0)$, there exists $A_2(K_0, \varepsilon_0)$, such that for $A \geq A_2$ and $0 < \eta_0 \leq 1$, there exists $s_{0,2}(K_0, \varepsilon_0, A, \eta_0)$ such that for any $s \geq s_{0,2}$, we have*

$$(q_0(s_*), q_1(s_*)) \in \partial\left(\left[-\frac{A}{s_*^2}, \frac{A}{s_*^2}\right]^2\right), \quad \text{where } s_* = -\log(T - t_*).$$

Remark 3.2. The choice of parameters cited below is particularly intricate. We explain that at conclusion of Part 1 and Part 2.

Proof. This is a direct consequence of the dynamics of Eq. (22), as we will show in Section 3.3.

Just to give a flavor of the argument, we invite the reader to look at Proposition 3.8, where we project Eq. (22) on the different components of q introduced in (30). There, one can see that the components q_2, q_- and q_e correspond to decreasing directions of the flow and since they are "small" at $s = s_0 = -\log T$ (see Lemma 3.7), they remain small for $s \in [s_0, s_*]$, and cannot touch the boundary of the intervals imposed by the definition of $S^*(t)$ in (32). Thus, only q_0 or q_1 may touch the boundary of the intervals in (32) at $s = s_*$.

For more details on the arguments, see Section 3.3. This ends the proof of Lemma 3.4. ∎

From Lemma 3.4, we may define the rescaled flow Φ at $s = s_*$ for the two expanding directions, namely q_0 and q_1, as follows:

$$\begin{aligned} \Phi : D_T &\longrightarrow \partial([-1,1]^2) \\ (d_0, d_1) &\longmapsto \left(\frac{s_*^2 q_0}{A}, \frac{s_*^2 q_1}{A}\right)_{d_0, d_1} (s_*). \end{aligned} \tag{38}$$

In particular,

$$\text{either } \omega q_0(s_*) = \frac{A}{s_*^2}, \quad \text{or} \quad \omega q_1(s_*) = \frac{A}{s_*^2} \tag{39}$$

and $\omega \in \{-1, 1\}$, both depending on (d_0, d_1). In the following lemma, we show that q_m actually crosses its boundary at $s = s_*$, resulting in the continuity of s_* and Φ. More precisely, we have the following result.

Lemma 3.5 (*Transverse Crossing*)**.** *For each $K_0 > 0, \varepsilon_0 > 0$, there exists $A_3(K_0, \varepsilon_0) > 0$ such that for all $A \geq A_3$, there exists $s_{0,3}(K_0, \varepsilon_0, A)$ such that if $s_0 \geq s_{0,3}, 0 < \eta_0 \leq 1$ and (39) holds, then*

$$\omega \frac{dq_m}{ds}(s_*) > 0. \tag{40}$$

Clearly, from the transverse crossing, we see that

$$(d_0, d_1) \mapsto s_*(d_0, d_1) \text{ is continuous,}$$

hence by definition (38), Φ is continuous. In order to find a contradiction and conclude, we crucially use the particular form we choose for initial data in (31). More precisely, we have the following:

Lemma 3.6 (*Degree 1 on the Boundary*)**.** *There exists $K_{05} > 0$ such that for each $K_0 \geq K_{05}, \varepsilon_0 > 0$ and $A \geq 1$, there exists $s_{0,4}(K_0, \varepsilon_0, A)$ such that if $s_0 \geq s_{0,4}$, then the mapping $(d_0, d_1) \to (q_0(s_0), q_1(s_0))$ maps ∂D_T into $\partial \left([-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2\right)$, and has degree one on the boundary.*

Indeed, from this lemma and the transverse crossing property of Lemma 3.5, we see that if $(d_0, d_1) \in \partial D_T$, then $s_*(d_0, d_1) = s_0$,

$$\Phi(s_*(s, d_0, d_1), d_0, d_1) = \left(\frac{s_*^2 q_0}{A}, \frac{s_*^2 q_1}{A}\right)(s_0)$$

and Φ defined in (38) is a continuous function from the rectangle $D_T \subset \mathbb{R}^2$ to $\partial[-1, 1]^2$, whose restriction to ∂D_T is of degree 1. This is a contradiction.

Thus, there exists $d_0, d_1 \in D_T$ such that

$$t_*(d_0, d_1) = T \text{ and } \forall t \in [0, T), \quad u(t) \in S^*(t), \tag{41}$$

and Proposition 3.2 follows at once.

Thus, Proposition 3.2 is proved, and from (i) of Definition 3.1, we have constructed a solution q to system (22), such that

$$\|q(s)\|_{L^\infty} \to 0 \quad \text{as } s \to \infty.$$

The next subsections will be devoted to the proofs of the technical Lemmas 3.3–3.6, referring to earlier works when the proof is the same.

3.2. Preparation of initial data

In this subsection, we study initial data given by (31). More precisely, we state a lemma which directly implies Lemmas 3.3 and 3.6. It also shows the (relative) smallness of the components q_2, q_- and q_e, an information which will be useful for the next subsection, dedicated to the dynamics of Eq. (22), crucial for the proofs of the reduction to a finite dimensional problem (Lemma 3.4) and the transverse crossing property (Lemma 3.5). More precisely, we claim the following:

Lemma 3.7 (*Decomposition of Initial Data in Different Components*)**.** *There exists $K_{06} > 0$ such that for each $K_0 \geq K_{06}, \varepsilon_0 > 0, A \geq 1$, there exists $s_{0,5}(K_0, \varepsilon_0 A) \geq e$ such that for all $s_0 \geq s_{0,5}$*

(i) *there exists a rectangle*

$$D_{K_0,\varepsilon_0,A,T} = D_T \subset [-2, 2]^2, \tag{42}$$

such that the mapping $(d_0, d_1) \to (q_0(s_0), q_1(s_0))$ is linear and one to one from D_T onto $[-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2$ and maps the boundary ∂D_T into the boundary $\partial\left([-\frac{A}{s_0^2}, \frac{A}{s_0^2}]^2\right)$. Moreover, it is of degree one on the boundary.

(ii) *For all* $(d_0, d_1) \in D_T$, *we have:*

$$|q_2(s_0)| \le CAe^{-s_0}, \qquad |q_-(y,s_0)| \le \frac{c}{s_0^2}(1+|y|^3) \quad \textit{and} \quad q_e(y,s_0) = 0,$$
$$|d_0| + |d_1| \le 1. \tag{43}$$

(iii) *For all* $(d_0, d_1) \in D_T$ *and* $\frac{\varepsilon_0}{4} \le |\theta| \le \pi$, *we have* $u(\theta, d_0, d_1) = 0$.

Proof. (i) and (ii) Since we have almost the same definition of the set $V_{K_0,A}$, and almost the same expression of initial data as in [24], we refer the reader to Lemma 3.5 page 156 and Lemma 3.9 page 160 from [24].

(iii) This follows by definition (31) of initial data for s_0 large enough. ■

3.3. Details on the dynamics of Eq. (22)

This subsection is dedicated to the proof of Lemmas 3.4 and 3.5. They both follow from the understanding of the flow of Eq. (22) in the set $S^*(t)$.

We proceed in two sections: We first prove Lemma 3.4, then Lemma 3.5.

3.3.1. Reduction to a finite-dimensional problem

Here we prove Lemma 3.4. Since the definition of $S^*(t)$ shows two different types of estimates, in the regions $\mathcal{R}_1$ and $\mathcal{R}_2$, accordingly, we need two different approaches to handle those estimates:

- In $\mathcal{R}_1$, we work in similarity variables (13), in particular we crucially use the projection of Eq. (22) with respect to the decomposition given in (30).
- In, $\mathcal{R}_2$, we directly work in the variables $u(x,t)$, using standard parabolic estimates.

Part 1: Estimates in $\mathcal{R}_1$.

In this part, we will show that

$$|q_2(s_*)| \le A^2 s_*^{-2} \log s_* - s_*^3, \qquad |q_-(y,s_*)| \le \frac{A}{2} s_*^{-2}(1+|y|^3),$$
$$|q_e(y,s_*)| \le \frac{A^2}{2} s_*^{-2}, \tag{44}$$

where $s_* = -\log(T-t_*), q$ is defined in (19) and the notation is given in (30), for a good choice of the parameters. In fact, this will follow from the projection of Eq. (22) on the components q_2, q_- and q_e as we will see at the end of Part 1. Let us first give the behavior of those components in the following.

Proposition 3.8 (*Control of the Null, Negative and Outer Mode of Eq.* (22)). *For all* $K_0 >, \varepsilon_0 > 0$, *there exists* $A_4 \ge 1$ *such that for all* $A \ge A_4$ *and* $\varsigma > 0$, *there exists* $s_{0,6}(K_0, \varepsilon_0, A, \eta)$ *such that the following holds for all* $s_0 \ge s_{0,6}$:

Assume that for some $\tau \ge s_0$ *and for all* $s \in [\tau, \tau + \varsigma]$,

$$u(t) \in S^*(t), \quad \textit{with } t = T - e^{-s}.$$

Then, the following holds for all $s \in [\tau, \tau+\varsigma]$:

$$|q_2(s)| \le \frac{\tau^2}{s^2}|q_2(\tau)| + \frac{CA(s-\tau)}{s^3},$$
$$\left\|\frac{q_-(s)}{1+|y|^3}\right\|_{L^\infty} \le Ce^{-\frac{(s-\tau)}{2}}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|q_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C(1+s-\tau)}{s^2},$$
$$\|q_e(s)\|_{L^\infty} \le Ce^{-\frac{(s-\tau)}{2}}\|q_e(\tau)\|_{L^\infty} + C\frac{e^{s-\tau}}{s^{3/2}}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C(1+s-\tau)}{s^{1/2}}.$$

Let us first insist on the fact that the derivation of (44) follows from Proposition 3.8, exactly as in the real case treated in [24] (see pages 163 to 166 and 158 to 159 in [24]). For that reason, we only focus in the following on the proof of Proposition 3.8.

Proof of Proposition 3.8. The proof of Proposition 3.8 consists in the projection of Eq. (22) on the different components of q defined in (30).

We note that the proof is already available from Lemma 3.13 page 167 and Lemma 3.8 page 158 from [24], in the case of the standard heat equation in $\mathbb{R}^N$ without truncation terms.

Since the equation satisfied by q in (22) shares the same linear part as the corresponding equation in [24], the proof is similar to the argument in [24], and the only novelty concerns the truncation term F in (22). For that reason, we only give the ideas here, focusing only on the new term F and kindly ask the interested reader to look at Lemma 3.13 page 167 and Lemma 3.8 page 158 in [24] for the technical details.

Let us first write Eq. (22) satisfied by q in its Duhamel formulation,

$$q(s) = K(s,\tau)q(\tau) + \int_\tau^s d\sigma K(s,\sigma)B(q(\sigma)) + \int_\tau^s d\sigma K(s,\sigma)R(\sigma) + \int_\tau^s d\sigma K(s,\sigma)(H+\partial_y G)(\sigma), \tag{45}$$

where K is the fundamental solution of the operator $\mathcal{L}+V$. We write $q = \alpha+\beta+\gamma+\delta+\tilde{\delta}$ with

$$\begin{aligned}
&\alpha(s) = K(s,\tau)q(\tau), \qquad \beta(s) = \int_\tau^s d\sigma K(s,\sigma)B(q(\sigma)),\\
&\gamma(s) = \int_\tau^s d\sigma K(s,\sigma)R(\sigma), \qquad \delta(s) = \int_\tau^s d\sigma K(s,\sigma)H(\sigma)\\
&\tilde{\delta}(s) = \int_\tau^s d\sigma K(s,\sigma)\partial_y G(\sigma),
\end{aligned} \tag{46}$$

where, for a function $F(y,\sigma)$, $K(s,\sigma)F(\sigma)$ is defined by

$$K(s,\sigma)F(\sigma) = \int_\tau^s d\sigma \int dx K(s,\sigma,y,x)F(x,\sigma).$$

We assume that $q(s) \in V_A(s)$ for each $s \in [\tau, \tau+\varsigma]$. Clearly, proceeding as the derivation of Lemma 3.13 page 167 in [24], Proposition 3.8 follows from the following:

Lemma 3.9 (*Projection of the Duhamel Formulation*)**.** *For all $K_0 > 0, \varepsilon_0 > 0$, there exists $A_5 \ge 1$ such that for all $A \ge A_5$ and $\varsigma > 0$ there exists $s_{0,7}(K_0,\varepsilon_0,A,\varsigma)$, such that for all $s_0 \ge s_{0,7}$, if $0 < \eta_0 \le 1$ and we assume that for some $\tau \ge s_0$ and for all $s \in [\tau,\tau+\varsigma], u(t) \in S^*(t)$ with $t = T - e^{-s}$, then*

(i) *(Linear terms)*

$$
\begin{aligned}
|\alpha_2(s)| &\le \frac{\tau^2}{s^2}|q_2(\tau)| + \frac{CA(s-\tau)}{s^3},\\
\left\|\frac{\alpha_-(s)}{1+|y|^3}\right\|_{L^\infty} &\le Ce^{-\frac{(s-\tau)}{2}}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + C\frac{e^{-(s-\tau)^2}\|q_e(\tau)\|_{L^\infty}}{s^{3/2}} + \frac{C}{s^2},\\
\|\alpha_e(s)\|_{L^\infty} &\le Ce^{-\frac{(s-\tau)}{2}}\|q_e(\tau)\|_{L^\infty} + Ce^{s-\tau}s^{3/2}\left\|\frac{q_-(\tau)}{1+|y|^3}\right\|_{L^\infty} + \frac{C}{\sqrt{s}},
\end{aligned}
\tag{47}
$$

(ii) *(Nonlinear terms)*

$$
\begin{aligned}
&|\beta_2(s)| \le \frac{(s-\tau)}{s^3}, && |\beta_-(y,s)| \le \frac{(s-\tau)}{s^2}(1+|y|^3), && \|\beta_e(s)\|_{L^\infty} \le \frac{(s-\tau)}{\sqrt{s}},\\
&|\delta_2(s)| \le C\frac{(s-\tau)}{s^3}, && |\delta_-(y,s)| \le C\frac{(s-\tau)}{s^2}(1+|y|^3), && \|\delta_e(s)\|_{L^\infty} \le C\frac{(s-\tau)}{\sqrt{s}},\\
&|\tilde\delta_2(s)| \le C\frac{(s-\tau)}{s^3}, && |\tilde\delta_-(y,s)| \le C\frac{(s-\tau)}{s^2}(1+|y|^3), && \|\tilde\delta_e(s)\|_{L^\infty} \le C\frac{(s-\tau)}{\sqrt{s}}.
\end{aligned}
$$

(iii) *(Source term)*

$$
|\gamma_2(s)| \le C(s-\tau)s^{-3}, \qquad |\gamma_-(y,s)| \le C(s-\tau)(1+|y|^3)s^{-2}, \qquad \|\gamma_e(s)\|_{L^\infty} \le (s-\tau)s^{-1/2}.
$$

Proof. We consider, $A \ge 1, \varsigma > 0$, and $s_0 \ge \varsigma$. The terms α, β and γ are already present in the case of the real-valued semilinear heat equation, so we refer to Lemma 3.1 page 167 in [24] for the estimates involving them. Thus, we only focus on the new terms $\delta(y,s)$ and $\tilde\delta(y,s)$.

Note that since $s_0 \ge \varsigma$, if we take $\tau \ge s_0$, then $\tau + \varsigma \le 2\tau$ and if $\tau \le \sigma \le s \le \tau + \varsigma$, then

$$
\frac{1}{2\tau} \le \frac{1}{s} \le \frac{1}{\sigma} \le \frac{1}{\tau}. \tag{48}
$$

Let us first derive the following bounds when $u(t) \in S^*(t)$:

Lemma 3.10. *For all $K_0 > 0, \varepsilon_0 > 0, A \ge 1$, there exists $s_0 \ge s_{0,8}(K_0, \varepsilon_0, A)$ such that if $s \ge s_{0,8}, 0 < \eta_0 \le 1$ and we assume that $u(t) \in S^*(t)$ defined in Definition 3.1, where $t = T - e^{-s}$. Then, we have*

(i) *for all $y \in \mathbb{R}$,* $\quad |q(y,s)| \le CA^2\dfrac{\log s}{s^2}(1+|y|^3)$,

(ii) $\|q(s)\|_{L^\infty} \le C\dfrac{A^2}{\sqrt{s}}$,

(iii) $\|W(s)\|_{L^\infty} \le \kappa + 2$.

Proof. (i) and (ii): Since $u(t) \in S^*(t)$, it follows by definition that $q(s) \in V_{K_0,A}(s)$, where $s = -\log(T-t)$, therefore, the proof is the same as the corresponding part in [24]. See Proposition 3.7 page 157 in [24] for details.

(iii) From (13), (15) and (19), we see that:

- If $|y| \le \varepsilon_0 e^{s/2}$, then $W(y,s) = w(y,s) = \varphi(y,s) + q(y,s)$. Since $\|\varphi\|_{L^\infty} \le \kappa + 1$ from (20), using (ii), we see that $\|W\|_{L^\infty} \le \kappa + 2$ for s large enough that is for T small enough.

- If $|y| \geq \varepsilon_0 e^{s/2}$, then $W(y,s) = e^{-\frac{s}{p-1}} u(\theta e^{-s/2}, t)$ with $|\theta| \geq \frac{\varepsilon_0}{2}$. By (ii) of Definition 3.1, we see that $|W(y,s)| \leq \eta_0 e^{-\frac{s}{p-1}} \leq \eta_0 T^{\frac{1}{p-1}} \leq 1$, if $\eta_0 \leq 1$ and $T \leq 1$.

This concludes the proof of Lemma 3.10. ■

Let us now recall from Bricmont and Kupiainen [2] the following estimates on $K(s,\sigma)$, the semigroup generated by $\mathcal{L} + V$.

Lemma 3.11 (*Properties of* $K(s,\sigma)$)**.** *For all* $s \geq \tau \geq 1$, *with* $s \leq 2\tau$, *we have the following:*

(i) *for all* $y, x \in \mathbb{R}$, *we have,*

$$|K(s,\sigma,y,x)| \leq C e^{(s-\sigma)\mathcal{L}}(y,x),$$

where $e^{\varrho\mathcal{L}}$ *is given explicitly by the Mehler's formula* [35]

$$e^{\varrho\mathcal{L}}(y,x) = \frac{e^{\varrho}}{\sqrt{4\pi(1-e^{-\varrho})}} \exp\left[-\frac{(ye^{-\varrho/2}-x)^2}{4(1-e^{-\varrho})}\right]. \tag{49}$$

(ii) *We have*

$$\left|\int K(s,\tau,y,x)(1+|x|^m)dx\right| \leq C\int e^{(s-\tau)\mathcal{L}}(y,x)(1+|x|^m)dx \leq e^{s-\tau}(1+|y|^m). \tag{50}$$

(iii) *For all* $g \in L^\infty$, *such that* $xg \in L^\infty$

$$\|K(s,\tau)\partial_x g\|_{L^\infty} \leq Ce^{s-\tau}\left\{\frac{\|g\|_{L^\infty}}{\sqrt{1-e^{-(s-\tau)}}} + \frac{(s-\tau)}{s}(1+s-\tau)\left((1+e^{\frac{s-\tau}{2}})\|xg\|_{L^\infty} + e^{\frac{s-\tau}{2}}\|g\|_{L^\infty}\right)\right\}.$$

Proof. (i) See page 181 in [24]
(ii) See Corollary 3.14 page 168 in [24].
(iii) See Appendix. ■

Now, with Lemmas 3.10 and 3.11 at hand, we are in position to finish the proof of Lemma 3.9. As we mentioned at the beginning of the proof, we only focus on the proof of the estimates on δ and $\tilde{\delta}$, and refer the readers to Lemma 3.13 page 167 in [24] for the estimate involving α, β and δ.

Estimates on δ *defined in* (46):

Consider $s \in [\tau, \tau+\varsigma]$ and recall that $0 < \eta_0 \leq 1$. Since $u(t) \in S^*(t)$ with $t = T - e^{-s}$, we see from the definition (13) of W that when

$$|y| \geq \frac{\varepsilon_0}{2} e^{s/2}, \qquad |W(y,s)| \leq \eta_0 e^{-\frac{s}{p-1}} \leq e^{-\frac{s}{p-1}}. \tag{51}$$

Moreover by definition (16) of χ, we see that

$$\begin{aligned}
&|\partial_y \chi| \leq \frac{C}{\varepsilon_0} e^{-s/2} \mathbf{I}_{\varepsilon_0 e^{s/2} < |y| < 2\varepsilon_0 s^{s/2}}, \\
&|y\partial_y \chi| \leq C\mathbf{I}_{\varepsilon_0 e^{s/2} < |y| < 2\varepsilon_0 s^{s/2}}, \\
&\text{and} \quad |\partial_s \chi| + (1+|y|)|\partial_y \chi| + |\partial_y^2 \chi| + (\chi - \chi^p) \leq \frac{C}{\varepsilon_0^2} \mathbf{I}_{\varepsilon_0 e^{s/2} < |y| < 2\varepsilon_0 s^{s/2}}.
\end{aligned} \tag{52}$$

Therefore, by definition (25), we see that

$$\|H(s)\|_{L^\infty} \le \frac{C}{\varepsilon_0^2}\eta_0 e^{-\frac{s}{p-1}} \le \frac{C}{\varepsilon_0^2}e^{-\frac{s}{p-1}}.$$

In particular, if $\tau \le \sigma \le s \le \tau + \varsigma$, we see from (48) that $\sigma \ge s/2$, hence

$$\|H(\sigma)\|_{L^\infty} \le \frac{C}{\varepsilon_0^2}e^{-\frac{\sigma}{p-1}} \le \frac{C}{\varepsilon_0^2}e^{-\frac{s}{2(p-1)}}. \tag{53}$$

Using Lemma 3.11 and the definition (46) of δ, we write

$$\begin{aligned}
|\delta(y,s)| &\le \int_\tau^s d\sigma \int_{\mathbb{R}} |K(s,\sigma,y,x)H(x,\sigma)|\,dx,\\
&\le \int_\tau^s d\sigma \int_{\mathbb{R}} e^{(s-\sigma)\mathcal{L}}(y,x)\frac{C}{\varepsilon_0^2}e^{-\frac{s}{2(p-1)}}\,dx,\\
&\le \frac{C}{\varepsilon_0^2}e^{-\frac{s}{2(p-1)}}\int_\tau^s d\sigma e^{(s-\sigma)},\\
&\le \frac{C}{\varepsilon_0^2}e^{-\frac{s}{2(p-1)}}(s-\tau),\\
&\le \frac{(s-\tau)}{s^2},
\end{aligned} \tag{54}$$

for s large enough depending on η_0.

By definition of q_m, q_- and q_e for $m \le 2$, we write

$$\begin{aligned}
|\delta_m(s)| &\le \left|\int_{\mathbb{R}} \chi(y,s)\delta(y,s)k_m(y)\rho(y)dy\right| \le C\int_{\mathbb{R}} |\delta(y,s)|(1+|y|^2)\rho(y)dy \le \frac{C(s-\tau)}{s^3},\\
|\delta_-(y,s)| &= \left|\chi(y,s)\delta(y,s) - \sum_{i=0}^{2}\delta_i(s)k_i(y)\right| \le (s-\tau)(1+|y|^3)\frac{C}{s^2}\\
\|\delta_e(y,s)\|_{L^\infty} &= \|(1-\chi(y,s))\delta(y,s)\|_{L^\infty} \le (s-\tau)\frac{C}{\sqrt{s}},
\end{aligned} \tag{55}$$

which are the desired estimations on δ in Lemma 3.9.

Estimates on $\tilde{\delta}$ defined in (46):

Since for all $s \in [\tau, \tau+\varsigma], u(t) \in S^*(t)$, where $t = T - e^{-s}$, by assumption, using (51) and (52), we see that when $s_0 \le \tau \le \sigma \le \tau \le \tau + \varsigma$, we have $\sigma \ge s/2$, hence

$$\|G(\sigma)\|_{L^\infty} \le \frac{C\eta_0}{\varepsilon_0}e^{-\frac{(p+1)\sigma}{2(p-1)}} \le \frac{C}{\varepsilon_0}e^{-\frac{(p+1)s}{4(p-1)}}, \tag{56}$$

$$\|xG(\sigma)\|_{L^\infty} \le C\eta_0 e^{-\frac{\sigma}{p-1}} \le Ce^{-\frac{s}{2(p-1)}}, \tag{57}$$

where G is defined by (25), remember that $\eta_0 \le 1$.

Using (iii) of Lemma 3.11, with $g = G(\sigma)$, we obtain

$$\|K(s,\sigma)\partial_x G\|_{L^\infty} \le Ce^{s-\sigma}\left\{\frac{e^{-\frac{(p+1)s}{4(p-1)}}}{\sqrt{1-e^{-(s-\sigma)}}} + \frac{(s-\sigma)}{s}(1+s-\sigma)\left(e^{-\frac{s}{2(p-1)}}(1+e^{\frac{s-\sigma}{2}}) + e^{\frac{s-\sigma}{2}}e^{-\frac{(p+1)s}{4(p-1)}}\right)\right\}.$$

Integrating in time, we get rid of the square rest term in the denominator, and see that

$$\left|\int_\tau^s K(s,\sigma)\partial_x G(x,\sigma)d\sigma\right| \le Ce^{s-\tau}(s-\tau)e^{-\frac{s}{2(p-1)}}\left(e^{-\frac{s}{4}} + \frac{(s-\tau)}{s}(1+s-\tau)(1+e^{(s-\tau)/2})\right).$$

Proceeding as for δ and using the fact that $0 \le s-\tau \le \varsigma$, we get

$$\sum_{m=0}^{2} |\tilde{\delta}_m(s)| + \left\| \frac{\tilde{\delta}_m(s)}{1+|y|^3} \right\|_{L^\infty} + \|\tilde{\delta}_e(s)\|_{L^\infty} \le \frac{C(s-\tau)}{s^3},$$

for s_0 large enough, which gives the desired estimates on $\tilde{\delta}$ in Lemma 3.9. Since the estimate notes for α, β and γ defined in (46) follow exactly as in Lemma 3.13 page 167 in [24], this concludes the proof of Lemma 3.9. ■

Conclusion of Part 1 and choice of parameters: Proceeding exactly as in [24, page 157], we derive estimate (44) from Proposition 3.8. This is possible for any $K_0 > 0, \varepsilon_0 > 0, A \ge A_4$, for some $A_4(K_0, \varepsilon_0) \ge 1$ and $s_0 \ge s_{0,6}$ for some $s_{0,6}(K_0, \varepsilon_0, A)$.

Since Proposition 3.8 follows directly from Lemma 3.9, this ends the proof of Proposition 3.8, as we mentioned right before the statement of Lemma 3.9 in the same way as in Proposition 3.11 page 161 in [24]. ■

Part 2: Estimates in $\mathcal{R}_2$.

The aim of this part is to show that

$$\text{if } \frac{\varepsilon_0}{2} \le |\theta| \le \pi, \quad \text{then } |u(\theta, t_*)| \le \frac{\eta_0}{2}, \tag{58}$$

provided the parameters satisfy some conditions. We proceed in 3 steps:

- In Step 1, we derive better bounds on the solution $u(\theta, t)$ in the intermediate region

$$K_0\sqrt{(T-t)\,|\log(T-t)|} \le |\theta| \le \frac{\varepsilon_0}{2}. \tag{59}$$

- In Step 2, we introduce a parabolic estimate on the solution in the region $\mathcal{R}_2$.
- Finally, in Step 3, we combine the previous steps to show (58).

Step 1: *Improved estimates in the intermediate region.*

Here, we refine the estimates on the solution in the region (59). In fact, we have from item (iii) of Lemma 3.10

$$\forall t \in [0, t_*],\ \forall \theta \in \mathbb{R}, \quad |u(t)| \le C(T-t)^{-\frac{1}{p-1}}, \tag{60}$$

valid in particular in the region (59). This bound is not satisfactory, since it goes to infinity as $t \to T$. In order to refine it, given a small θ, we use this bound when $t = t_0(\theta)$ defined by

$$|\theta| = K_0\sqrt{(T-t_0(\theta))|\log(T-t_0(\theta))|}, \tag{61}$$

to see that the solution is in fact flat at that time. Then, advancing the PDE (6), we see that the solution remains flat for later times. More precisely, we claim the following:

Lemma 3.12 (*Flatness of the Solution in the Intermediate Region in* (59)). *There exists $\zeta_0 > 0$ such that for all $K_0 > 0, \varepsilon_0 > 0, A \ge 1$, there exists $s_{0,9}(K_0, \varepsilon_0, A)$, such that if $s_0 \ge s_{0,9}$ and $0 < \eta_0 \le 1$, then,*

$$\forall t_0(\theta) \le t \le t_*, \quad \left| \frac{u(\theta,t)}{u^*(\theta)} - \frac{U_{K_0}(\theta)}{U_{K_0}(1)} \right| \le \frac{C}{|\log\theta|^{\zeta_0}},$$

where u^* is defined in (8) and

$$U_{K_0}(\tau) = \kappa\left((1-\tau) + \frac{(p-1)K_0^2}{4p}\right)^{-1/(p-1)}. \tag{62}$$

In particular, $|u(\theta,t)| \le 2|u^*(\theta)|$.

Proof. We argue as in Masmoudi and Zaag [17]. If $\theta_0 \neq 0$ is small enough, we introduce for all $(\xi,\tau) \in \mathbb{R}\times[-\frac{t_0(\theta_0)}{T-t_0(\theta_0)}, \tau_*)$, with $\tau_* = \frac{t_*-t_0(\theta_0)}{T-t_0(\theta_0)}$

$$U(\theta,\xi,\tau) = (T-t_0(\theta_0))^{1/(p-1)} u(\theta,t), \tag{63}$$

where

$$\theta = \theta_0 + \xi\sqrt{T-t_0(\theta_0)}, \quad t = t_0(\theta_0) + \tau(T-t_0(\theta_0)), \tag{64}$$

and $t_0(\theta_0)$ is uniquely defined by

$$|\theta_0| = K_0\sqrt{(T-t_0(\theta_0))|\log(T-t_0(\theta_0))|}. \tag{65}$$

From the invariance of problem (6) under dilation, $U(\theta_0,\xi,\tau)$ is also a solution of (6) on its domain. Since $u(t) \in S^*(t)$, from (35), using the definition of $S^*(t)$, Lemma 3.10 together with (61) and (63), we have

$$\begin{aligned}\sup_{|\xi|<2|\log(T-t_0(\theta_0))|^{1/4}} |U(\theta_0,\xi,0) - f(K_0)| &\le \frac{K_0}{2p\mathcal{S}_0} + \|q(\mathcal{S}_0)\|_{L^\infty}\\ &\le \frac{C}{\mathcal{S}_0} + \frac{CA^2}{\mathcal{S}_0^{1/2}}\\ &\le \frac{C}{\mathcal{S}_0^{1/4}},\end{aligned}$$

with $\mathcal{S}_0 = \mathcal{S}_0(\theta_0) = -\log(T-t_0(\theta_0))$ and f is defined by (5). Provided that $s_0(=-\log T)$ is large enough.

Using the continuity with respect to initial data for problem (6), associated to a space-localization in the ball $B(0,|\log(T-t_0(\theta_0))|^{1/4})$, we show as in Section 4 of [39] that

$$\sup_{|\xi|\le|\log(T-t_0(\theta_0))|^{1/4},\ 0\le\tau<\tau_*} |U(\theta_0,\xi,\tau) - U_{K_0}(\tau)| \le \frac{C}{\mathcal{S}_0^{\zeta_0}}, \tag{66}$$

$U_{K_0}(\tau)$ given by (62) is the solution of the PDE (6) with constant initial data $f(K_0)$. Since $U_{K_0}(\tau) \le U_{K_0}(1) = \kappa\left(\frac{(p-1)K_0^2}{4p}\right)^{-1/(p-1)}$ and we have from (65)

$$\log(T-t_0(\theta_0)) \sim 2\log\theta_0 \quad \text{and} \quad (T-t_0(\theta_0)) \sim \frac{\theta_0^2}{2K_0^2|\log(\theta_0)|} \quad \text{as } \theta_0 \to 0, \tag{67}$$

this yields $(T-t_0(\theta_0))^{1/(p-1)} \sim \frac{U_{K_0}(1)}{u^*(\theta_0)}$, by definition (8) of u^*. We obtain the desired conclusion from (63) and (66). This ends the proof of Lemma 3.12. ■

Step 2: A parabolic estimate in region $\mathcal{R}_2$

We recall from Definition 3.1 of $S^*(t)$ that

$$\forall\theta\in\mathbb{R} \text{ such that } \frac{\varepsilon_0}{2} \le |\theta| \le \pi, \quad |u(\theta,t)| \le \eta_0.$$

Here, we will obtain a parabolic estimate on the solution in $\mathcal{R}_2$. More precisely, we claim the following:

Proposition 3.13 (*A Parabolic Estimate in $\mathcal{R}_2$*). *For all $\varepsilon > 0, \varepsilon_0 > 0, \sigma_1 \geq 0, \exists T_4(\varepsilon, \varepsilon_0, \sigma_1) \geq 0$, such that for all $\bar{t} \leq T_4$, if u a periodic solution of*

$$\partial_t u = \partial_\theta^2 u + |u|^{p-1}u \quad \text{for all } \theta \in \mathbb{S},\ t \in [0, \bar{t}],$$

which satisfies:

(i) *for $|\theta| \in [\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{2}]$, $|u(\theta,t)| \leq \sigma_1$.*
(ii) *for $\frac{\varepsilon_0}{4} \leq |\theta| \leq \pi, u(\theta, 0) = 0$.*

Then, for all $t \in [0, \bar{t}]$, for all $\frac{\varepsilon_0}{2} \leq |\theta| \leq \pi$,

$$|u(\theta, t)| \leq \varepsilon.$$

Proof. Consider $\bar{u}$ defined in (11), which satisfies Eq. (12), recalled here, after a trivial chain rule to transform the $\partial_\theta u$ term:

$$\forall t \in [0, \bar{t}],\ \forall \theta \in \mathbb{R}, \quad \partial_t \bar{u} = \partial_\theta^2 \bar{u} + |u|^{p-1}\bar{u} - 2\partial_\theta(\bar{\chi}' u) + \bar{\chi}'' u.$$

Therefore, since $\bar{u}(\theta, 0) \equiv 0$, we write

$$\|\bar{u}(t)\|_{L^\infty} \leq \int_0^t \left| S(t-t') \left[|u|^{p-1} \mathbf{I}_{|\theta| \geq \frac{\varepsilon_0}{4}} \bar{u} - 2\partial_\theta \left(\bar{\chi}' u \mathbf{I}_{|\theta| \geq \frac{\varepsilon_0}{4}} \right) + \bar{\chi}'' u(t') \mathbf{I}_{|\theta| \geq \frac{\varepsilon_0}{4}} \right] \right| dt',$$

where $S(t)$ is the heat kernel.

Since $\bar{\chi}'$ and $\bar{\chi}''$ are supported by $\{\frac{\varepsilon_0}{4} \leq |\theta| \leq \frac{\varepsilon_0}{2}\}$ and satisfy $|\bar{\chi}'| \leq C/\varepsilon_0, |\bar{\chi}''| \leq C/\varepsilon_0^2$ and using parabolic regularity, we write

$$\begin{aligned} \|\bar{u}(t)\|_{L^\infty} &\leq \sigma_1^{p-1} \int_0^t \|\bar{u}(t')\| dt' + \frac{C\sigma_1}{\varepsilon_0} \int_0^t \frac{dt'}{\sqrt{t-t'}} + \frac{C\sigma_1}{\varepsilon_0^2} \int_0^t dt' \\ &\leq \sigma_1^{p-1} \int_0^t \|\bar{u}(t')\| dt' + \frac{C\sigma_1}{\varepsilon_0} \sqrt{\bar{t}} + \frac{C\sigma_1}{\varepsilon_0^2} \bar{t}. \end{aligned}$$

If $\bar{t} < 1$, by Gronwall estimate, this implies that

$$\|\bar{u}(t)\|_{L^\infty} \leq C e^{\sigma_1^{p-1}} \left(\frac{\sigma_1}{\varepsilon_0} \sqrt{\bar{t}} + \frac{\sigma_1}{\varepsilon_0^2} \bar{t} \right).$$

Taking $\bar{t}$ small enough, we can obtain

$$\forall t \in [0, \bar{t}], \quad \|\bar{u}(t)\|_{L^\infty} \leq \varepsilon.$$

Since $u = \bar{u}$ for all $\frac{\varepsilon_0}{2} \leq |\theta| \leq \pi$ by definition (11), this concludes the proof of Proposition 3.13. ∎

Step 3: Proof of the improvement in (58)

Here, we use Step 1 and Step 2 to prove (58), for a suitable choice of parameters.

Let us consider $K_0 > 0$, and $\delta_0(K_0) > 0$ defined in Lemma 3.12. Then, we consider $\varepsilon_0 \leq 2\delta_0, A \geq 1, 0 < \eta_0 \leq 1$, and

$$s_0 \geq s_{1,0} \equiv \max \left\{ s_{0,9}(K_0, \varepsilon_0, A), s_{0,5}(K_0, \varepsilon_0, A), -\log \left(T_4 \left(\frac{\eta_0}{2}, \varepsilon_0, 2 \left| u^* \left(\frac{\eta_0}{4} \right) \right| \right) \right) \right\},$$

where the different constants are defined in Lemmas 3.7 and 3.12 and Proposition 3.13.

Applying Lemma 3.12, we see that

$$\forall |\theta| \leq \delta_0,\ A \geq 1,\ \forall t \in [0, t_*], \quad |u(\theta, t)| \leq 2|u^*(\theta)|.$$

In particular,

$$\forall \frac{\varepsilon_0}{4} \le |\theta| \le \frac{\varepsilon_0}{2} \le \delta_0, \ \forall t \in [0, t_*], \quad |u(\theta,t)| \le 2|u^*(\frac{\varepsilon_0}{4})|.$$

Using item (iii) of Lemma 3.7, we see that $\forall \frac{\varepsilon_0}{4} \le |\theta| \le \pi, u(\theta, 0) = 0$.

Therefore Proposition 3.13 applies with $\varepsilon = \frac{\eta_0}{2}$ and $\sigma_1 = 2u^*(\frac{\varepsilon_0}{4})$ and we see that

$$\forall \frac{\varepsilon_0}{2} \le |\theta| \le \pi, \ \forall t \in [0, t_*], \quad |u(\theta,t)| \le \frac{\eta_0}{2} \tag{68}$$

and estimate (58) holds.

Conclusion of Part 2 and choice of parameters: From (68), we see that (58) holds for any $K_0 > 0, \varepsilon_0 \le 2\delta_0(K_0), A \ge 1, 0 < \eta_0 \le 1$ and $s_0 \ge s_{1,0}(K_0, \varepsilon_0, A, \eta_0)$.

Conclusion of the proof of Lemma 3.4: From the conclusion of Parts 1 and 2, if we take $K_0 > 0, \varepsilon_0 \le 2\delta_0(K_0), A \ge A_7(K_0, \varepsilon_0), 0 < \eta_0 \le 1$ and

$$s_0 \ge \max\{s_{0,6}(K_0, \varepsilon_0, A), s_{1,0}(K_0, \varepsilon_0, A, \eta_0)\},$$

then we see that (44) and (58) holds at $t = t_*$.

Recalling that $u(t_*) \in \partial S(t_*)$ by (36), we see from Definition 3.1 of S^* that only one of the components $q_0(s_*)$ or $q_1(s_*)$ may touch the boundary of $[-\frac{A}{s_*^2}, \frac{A}{s_*^2}]$. This concludes the proof of Lemma 3.4. ■

3.3.2. Transverse crossing on $V_{K_0,A}(s)$

We prove Lemma 3.5 here. The key estimate is to prove the following differential inequality on q_m for $m = 0, 1$:

$$\forall s \in [s_0, s_*], \quad \left| q_m'(s) - \left(1 - \frac{m}{2}\right) q_m(s) \right| \le \frac{C}{s^2}, \tag{69}$$

provided $s_0 \ge s_{0,3}(K_0, \varepsilon_0, A)$ and $0 < \eta_0 \le 1$, for some large enough $s_{0,3}$. Indeed if (39) holds, say $q_m(s_*) = \frac{\omega A}{s_*^2}$ for $m = 0, 1$ and $\omega = \pm 1$, then, we see that

$$\omega q_m'(s) \ge \left(1 - \frac{m}{2}\right) \frac{A}{s_*^2} - \frac{C}{s^2} \ge \left(1 - \frac{m}{2}\right) \frac{A}{2s_*^2},$$

assuming that A is large enough, which yields the conclusion of Lemma 3.5, assuming that (69) holds.

Let us briefly justify (69). Multiplying Eq. (22) by $h_m(y)\chi_1(y,s)$, defined in (27), we obtain the following estimate

$$(\partial_s q)_m = (\mathcal{L}q)_m + (Vq)_m + B_m + R_m + F_m.$$

From straightforward estimates, already used for the standard heat equation in $\mathbb{R}^N$ considered in [24] (see Lemma 3.8 page 158 there), we know that

$$|(\partial_s q)_m - q_m'| + \left| (\mathcal{L}q)_m - \left(1 - \frac{m}{2}\right) q_m \right| + |(Vq)_m + B_m + R_m| \le \frac{C}{s^2}.$$

It remains only to treat the new term F_m. In fact from (53), we see that

$$\|H(s)\|_{L^\infty} \le Ce^{-\frac{s}{2(p-1)}} \le \frac{C}{s^2}, \quad \text{for } s_0 \text{ large enough, provided that } 0 < \eta_0 \le 1.$$

Then integrating by parts and using (51), (52), (56) and (57), we get

$$\left|(\partial_y G)_m\right| \le \frac{C}{s^2} \quad \text{for } s_0 \text{ large enough,}$$

and we obtain the following

$$|F_m(s)| \le \frac{C}{s^2}.$$

This concludes the proof of (69) and Lemma 3.5 too. ■

4. Proof of Theorem 1

We prove Theorem 1 in this section. We will first derive (ii) from Section 3, then we will prove (i) and (iii).

Let us fix $K_0 > 0, \varepsilon_0 > 0, A > 0, 0 < \eta_0 \le 1$ and $T > 0$ so that Proposition 3.2 as well as all the statements of Section 3 apply hence, for some $d_0, d_1 \in \mathbb{R}^2$, Eq. (6) with initial data given by (31) has a solution $u(\theta, t)$ such that

$$T = t_*(d_0, d_1), \quad \forall t \in [0, T),\ u(t) \in S^*(K_0, \varepsilon_0, A, \eta_0, T, t). \tag{70}$$

(Note the fact that $t_*(d_0, d_1) = T$ follows from the conclusion of the topology argument given by (41).)

Applying item (ii) of Lemma 3.10, we see that

$$\forall y \in \mathbb{R},\ \forall s \ge -\log T, \quad |q(y,s)| \le \frac{CA^2}{\sqrt{s}}.$$

By definitions (15), (19) and (20), we see that

$$\forall s \ge -\log T,\ \forall |y| \le \varepsilon_0 e^{s/2}, \quad \left| W(y,s) - f\left(\frac{y}{\sqrt{s}}\right)\right| \le \frac{CA^2}{\sqrt{s}} + \frac{C}{s}.$$

By definition (13) of W, we see that

$$\forall t \in [0,T),\ \forall |\theta| \le \varepsilon_0, \quad \left|(T-t)^{1/(p-1)} u(\theta,t) - f\left(\frac{\theta}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| \le \frac{C(A)}{\sqrt{|\log(T-t)|}}.$$

Since u is 2π-periodic estimate (7) holds.

(i) If $\theta_0 = 2k\pi,\ k \in \mathbb{Z}$, then we see from (7) that $|u(0,t)| \sim \kappa (T-t)^{-1/(p-1)}$ as $t \to T$, where κ is defined in (21). Hence u blows up at time T at $\theta_0 = 2k\pi,\ k \in \mathbb{Z}$.

It remains to prove that any $\theta_0 \ne 2k\pi$ is not a blow-up point.

From periodicity, we may assume that $-\pi \le \theta_0 \le \pi$.

Since, we know from item (ii) in Definition 3.1, that if $\frac{\varepsilon_0}{2} \le |\theta| \le \pi$, and $0 \le t \le T, |u(\theta,t)| \le \eta_0$, it follows that θ_0 is not a blow-up point, provided

$$\frac{\varepsilon_0}{2} \le |\theta_0| \le \pi.$$

Now, if $0 < |\theta_0| \le \frac{\varepsilon_0}{2}$, the following result from Giga and Kohn [13] allows us to conclude.

Proposition 4.1 (*Giga and Kohn — No blow-up Under the ODE Threshold*)**.** *For all $C_0 > 0$, there is $\eta_0 > 0$ such that if $v(\xi, \tau)$ solves*

$$|v_t - \Delta v| \le C_0(1 + |v|^p)$$

and satisfies

$$|v(\xi,\tau)| \le \eta_0 (T-t)^{-1/(p-1)}$$

for all $(\xi,\tau) \in B(a,r) \times [T - r^2, T)$ for some $a \in \mathbb{R}$ and $r > 0$, then v does not blow up at (a, T).

Proof. See Theorem 2.1 page 850 in [13]. ■

Indeed, since $|\theta_0| \le \frac{\varepsilon_0}{2}$, it follows from (7) that

$$\sup_{|\theta - \theta_0| \le |\theta_0|/2} (T-t)^{\frac{1}{p-1}} |u(\theta,t)| \le \left| f\left(\frac{|\theta_0|/2}{\sqrt{(T-t)|\log(T-t)|}}\right)\right| + \frac{C}{\sqrt{|\log(T-t)|}} \to 0$$

as $t \to T$. Therefore, applying Proposition 4.1, we see that θ_0 is not a blow-up point of u. This concludes the proof of (i) of Theorem 1.

(iii) Arguing as Merle did in [21], we derive the existence of a blow-up profile $u(\theta, T) \in C^2(\mathbb{R}\backslash\{2k\pi, k \in \mathbb{Z}\})$ such that $u(\theta, t) \to u(\theta, T)$ as $t \to T$, uniformly on compact sets of $\mathbb{R} \setminus \{2k\pi, k \in \mathbb{Z}\}$. The profile $u(\theta, t)$ is not defined at the origin. In the following, we would like to find its equivalent as $\theta \to 2k\pi$, for any $k \in \mathbb{Z}$ and show that it is in fact singular at $\theta_0 = 2k\pi$.

From periodicity it is enough to take $\theta_0 = 0$. Since $t_*(d_0, d_1) = T$ from (70), applying Lemma 3.12 and making $t \to T$. We see that

$$\left|\frac{u(\theta, T)}{u^*(\theta)} - 1\right| \le \frac{C}{|\log\theta|^{\zeta_0}}.$$

Making $\theta \to 0$, we get the desired estimate in item (iii). This concludes the proof of Theorem 1. ■

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Appendix. Proof of (iii) of Lemma 3.11

This part relies mainly on the understanding of the behavior of the kernel $K(s, \sigma, y, x)$. This behavior follows from a perturbation method around $e^{(s-\sigma)\mathcal{L}}(y, x)$.

Since $\mathcal{L}$ is conjugated to the harmonic oscillator $e^{-x^2/8}\mathcal{L}e^{x^2/8} = \partial^2 - (x^2/16) + (1/4) + 1$, We use the definition of K as the semigroup generated by $\mathcal{L} + V$ defined in (23) and give a Feynman–Kac representation for K:

$$K(s, \sigma, y, x) = e^{(s-\sigma)\mathcal{L}}(y, x)E(y, x), \tag{71}$$

where

$$E(y, x) = \int d\mu_{yx}^{s-\sigma}(\omega)e^{\int_0^{(s-\sigma)} V(\omega(\tau), \sigma+\tau)}, \tag{72}$$

and $d\mu_{yx}^{s-\sigma}$ is the oscillator measure on the continuous paths $\omega : [0, s-\sigma] \to \mathbb{R}$ with $\omega(0) = x, \omega(s-\sigma) = y$, i.e., the Gaussian probability measure with covariance kernel

$$\Gamma(\tau, \tau') = \omega_0(\tau)\omega_0(\tau') + 2\left(e^{-\frac{1}{2}|\tau-\tau'|} - e^{-\frac{1}{2}|\tau+\tau'|} + e^{-\frac{1}{2}|2(s-\sigma)+\tau-\tau'|} - e^{-\frac{1}{2}|2(s-\sigma)-\tau-\tau'|}\right),$$

which yields $\int d\mu_{yx}^{s-\sigma}\omega(\tau) = \omega_0(\tau)$ with

$$\omega_0(\tau) = \left(\sinh\left(\frac{s-\sigma}{2}\right)\right)^{-1}\left(y\sinh\frac{\tau}{2} + x\sinh\frac{s-\sigma-\tau}{2}\right).$$

Consider $1 \le \tau \le s \le 2\tau$. From Bricmont and Kupiainen [2, Lemma 6 page 555] and Merle and Zaag [24, pages 183–184], we have the following estimates:

Claim A.1. *If $1 \le \tau \le s$ with $s \le 2\tau$, then*

$$0 \le E(y, x) \le C,$$
$$|\partial_x E(y, x)| \le \frac{C}{s}(s-\sigma)(1+s-\sigma)(|y| + |x|).$$

Consider $g \in L^\infty$ such that $xg \in L^\infty$. By (71), (72) and integration by parts, we obtain

$$\begin{aligned} K(s,\sigma)(\partial_x g)(y) &= \int_{\mathbb{R}} dx K(s,\sigma,y,x)\partial_x g(x,\sigma) = \int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}(y,x)E(y,x)\partial_x g(x) \\ &= -\int_{\mathbb{R}} dx \partial_x e^{(s-\sigma)\mathcal{L}}(y,x)E(y,x)g(x) - \int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}\partial_x E(y,x)g(x) \\ &= I_1 + I_2. \end{aligned} \tag{73}$$

By (49), we have

$$\begin{aligned} |\partial_x e^{(s-\sigma)\mathcal{L}}(y,x)| &= \left| \frac{e^{s-\sigma}}{\sqrt{4\pi(1-e^{-(s-\sigma)})}} \frac{-(x-ye^{-\frac{(s-\sigma)}{2}})}{2(1-e^{-(s-\sigma)})} \exp\left[-\frac{(x-ye^{-(s-\sigma)/2})^2}{4(1-e^{-(s-\sigma)})}\right]\right| \\ &= C\frac{|z|e^{-z^2}}{\sqrt{1-e^{-(s-\sigma)}}} \frac{e^{s-\sigma}}{\sqrt{1-e^{-(s-\sigma)}}} \end{aligned}$$

where $z = \frac{x-ye^{-(s-\sigma)/2}}{\sqrt{4(1-e^{-(s-\sigma)})}}$. Using Claim A.1, we see that

$$\begin{aligned} |I_1| &\le C\|g\|_{L^\infty} \int_{\mathbb{R}} dx \frac{|z|e^{-z^2}}{\sqrt{1-e^{-(s-\sigma)}}} \frac{e^{(s-\sigma)}}{\sqrt{1-e^{-(s-\sigma)}}} = C\|g\|_{L^\infty} \int_{\mathbb{R}} dz|z|e^{-z^2} \frac{e^{(s-\sigma)}}{\sqrt{1-e^{-(s-\sigma)}}} \\ &\le C\|g\|_{L^\infty} \frac{e^{(s-\sigma)}}{\sqrt{1-e^{-(s-\sigma)}}}, \end{aligned} \tag{74}$$

and

$$\begin{aligned} |I_2| &\le \int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}(y,x)|g(x)| \frac{C(s-\sigma)}{s}(1+s-\sigma)(|y|+|x|), \\ &\le \frac{C(s-\sigma)}{s}(1+s-\sigma)\int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}(y,x)|g(x)| \left(|z|e^{(s-\sigma)/2}\sqrt{1-e^{-(s-\sigma)}} + |x|(1+e^{(s-\sigma)/2})\right), \\ &= \frac{C(s-\sigma)}{s}(1+s-\sigma)(J_1+J_2), \end{aligned} \tag{75}$$

with $z = \frac{x-ye^{-(s-\sigma)/2}}{\sqrt{4(1-e^{-(s-\sigma)})}}$. Moreover

$$\begin{aligned} J_1 &\le \|g\|_{L^\infty} e^{(s-\sigma)/2}\sqrt{1-e^{-(s-\sigma)}} \int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}(y,x)|z|, \\ &= \|g\|_{L^\infty} e^{3(s-\sigma)/2}\left(1-e^{-(s-\sigma)}\right) \int dz e^{-z^2}|z|, \\ &\le Ce^{3(s-\sigma)/2}\|g\|_{L^\infty}. \end{aligned}$$

Furthermore using item (i) of Lemma 3.11, we write

$$\begin{aligned} J_2 &= 2\int_{\mathbb{R}} dx e^{(s-\sigma)\mathcal{L}}(y,x)|xg(x)|(1+e^{(s-\sigma)/2}), \\ &\le C\|xg\|_{L^\infty} e^{s-\sigma}(1+e^{(s-\sigma)/2}) \end{aligned}$$

which gives the desired estimates thanks to item (i) of Lemma 3.11. ■

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