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Boundedness and convergence on fractional order systems

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ABSTRACT

We establish conditions to guarantee boundedness and convergence of signals described by non integer order equations using Caputo derivatives. The case of linear time-varying unforced equations is first studied, and later, results for linear time-varying forced equations and time-varying unforced non linear equations are presented and discussed.

1. Introduction

Though there is no unique nor equivalent definition of non integer derivative, systems defined by Caputo fractional order derivative are widely used because it makes use of initial conditions similar as in the integer order case and also because of the non local behavior, which seems to be the distinctive character that one could expect for non integer dynamics.

Since classical definition of dynamical system (with a specific evolution function, manifold or monoid background and so on) does not completely hold for fractional systems (whether Caputo or other type of derivative is used in its rule of evolution), we will simply call fractional system (of equations) to the object of our study instead of dynamical fractional system.

Like in the integer order case, one of the main topic of research in fractional systems is the study of its asymptotic properties such as convergence and boundedness. In the simplest systems, the linear time invariant systems, those properties can be directly analyzed by using the analytic solution. The reader is referred for example to [1]. The next simplest fractional systems, the linear forced systems and linear time varying systems, which are the main object of study of our work, have received comparatively less attention in the specialized literature. We mention [2] for the latter (scalar case) and [1] for the former (BIBO stability for time invariant systems). Again, in both cases, properties are deduced by appealing to schematic solutions of such equations.

For most complex systems, however, a generic analytic or schematic solution is not possible or not available in the literature and therefore specific tools must be employed or developed instead. Among those tools, we will stand out the Lyapunov functions and the comparison principle [3].

The paper is organized in the following way: Section 2 gives some basic notions and properties of fractional order operators. Section 3 studies fractional linear unforced time variant systems, whereas in Section 4 fractional forced linear systems are analyzed. Next, in Section 5 the study of fractional nonlinear unforced systems is presented. Finally, Section 6 offers general conclusions and future work.

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2. Preliminaries

Some useful definitions and properties (taken mainly from [4] except where indicated) are presented in this section.

Definition 1 (*Fractional Integral [4, page 69]*). The fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ on the half axis $\mathbb{R}^+$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau$ is the Gamma function.

We denote $I_T^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} f(\tau) d\tau$ with $t > T$ and $I_{[T_1,T_2]}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{T_1}^{T_2} (t-\tau)^{\alpha-1} f(\tau) d\tau$ with $T_2 > T_1$.

In the following, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = [\alpha]$ otherwise, where $[\alpha]$ denotes the integer part of the real number α.

Definition 2 (*Caputo Derivative [5, Definition 3.1]*). The Caputo derivative of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ on the half axis $\mathbb{R}^+$ is defined as

$${}^C D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t) \tag{2}$$

whenever f belongs to $\mathcal{L}^1(a,b)$, the Lebesgue space of functions for which $|f|$ is Lebesgue integrable on the interval (a,b).

It must be noted that Caputo derivative requires that $f^{(n)}$ be differentiable a.e. and if f has well defined Caputo derivative then $f^{(n)}$ must be differentiable a.e.

To simplify the notation, we will denote ${}^C D^\alpha f(t)$ as $D^\alpha f(t)$ or $f^{(\alpha)}$ since we will be using only the Caputo fractional derivative throughout the paper.

An analogue to the fundamental theorem of integer calculus is stated in the next two properties for Caputo fractional derivative.

Property 1 (*[4, Lemma 2.22]*). *If f belongs to $C^n[a,b]$, the space of continuous functions on $[a,b]$ that have continuous first n derivatives (or f belongs to $AC^n[a,b]$, the space of absolutely continuous functions on $[a,b]$ that have continuous first n derivatives), and $\alpha > 0$, then for all $t \in [a,b]$*

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{3}$$

Property 2 (*[4, Lemma 2.21]*). *If f belongs to $\mathcal{L}^\infty(a,b)$, the Lebesgue space of bounded functions on the interval (a,b) (or f belongs to $C[a,b]$), and $\alpha > 0$ with $\alpha \notin \mathbb{N}$, then for all $t \in (a,b)$*

$$D^\alpha I^\alpha f(t) = f(t). \tag{4}$$

The next properties will be regularly cited along the proofs of the next sections. It is assumed that $0 < \alpha \le 1$.

Property 3 (*Caputo Derivative Property [6, Lemma 1]*). *Let $x(t) \in \mathbb{R}^n$ be a differentiable vector function, then for all $t \ge 0$ it holds that*

$$D^\alpha x^T x \le 2x^T D^\alpha x. \tag{5}$$

For the proof, the reader is referred to [6]. The proofs of the following two properties can be found at [7].

Property 4 (*Decaying Property*). *If $f(t) \in \mathbb{R}$ is a bounded function that vanishes for all $t > T$ then $I^\alpha f \to 0$ and $D^\alpha f \to 0$ as $t \to \infty$. Moreover, $I^\alpha f$ will be a uniformly continuous function and if $D^\alpha f$ is continuous, $D^\alpha f$ will be a uniformly continuous function.*

Property 5 (*Limit of Integrals Property*). *Let $f(t) \in \mathbb{R}$ be a bounded function, if $I^\alpha f \to L$ as $t \to \infty$ then $I_T^\alpha f \to L$ as $t \to \infty$.*

Finally, we recall the following lemma from [3]

Lemma 1 (*Comparison Principle [3, Lemma 10]*). *If $x(0) = y(0)$ and $D^\beta x \ge D^\beta y$ for $0 < \beta < 1$, then $x(t) \ge y(t)$ for all $t \ge 0$.*

3. Linear time-varying systems

In this section we analyze the case of systems described by fractional order linear time-varying (non-autonomous) differential equations, deriving conditions for boundedness and convergence on system signals.

Let $A(t) \in \mathbb{R}^{n\times n}$ be a matrix and $0 < \alpha \leq 1$. Let us consider the system defined by

$$x^{(\alpha)} = A(t)x, \tag{6}$$

with $x(0) = x_0 \in \mathbb{R}^n$.

When A is a constant matrix, the condition for asymptotic convergence of x to zero becomes $arg(spec(A)) > \alpha\frac{\pi}{2}$ [1]. According to the scalar solution in the same reference, we have $\partial x/\partial a \leq sgn(x(0))$ by using the monotony of $E_\alpha(-at^\alpha)$ [8], whereby the greater is $|a|$ the faster x converges to zero. For the vector case, one can deduce the same role about the eigenvalues. Thus, the convergence speed is given by the eigenvalues of A.

We say that A is α-stable if $arg(spec(A)) \geq \alpha\frac{\pi}{2}$. Hence, if $\alpha \leq \beta \leq 1$ and A is β-stable then A is α-stable. This suggests that, in the case A is time-varying, the results of the integer case could also be valid in non-integer case. A positive example is given by the following theorem.

Theorem 1. *Let $A(t)$ a real matrix such that for all $t \geq 0$, x is differentiable for the system (6) and $A(t) \leq -\varepsilon I$ (i.e. for all vector $y \in \mathbb{R}^n$ and for all $t \geq 0$, it holds that $y^TA(t)y \leq -\varepsilon y^Ty$) then x asymptotically converges to zero. In particular, the convergence is $O(t^{-\alpha})$.*

Proof. By Property 3 and hypothesis, $D^\alpha x^Tx \leq 2x^TD^\alpha x \leq 2x^TA(t)x$. Then $D^\alpha x^Tx \leq -2\varepsilon x^Tx$. By calling $V(t) = x^Tx(t)$, we note that $V(\cdot)$ satisfies the following inequality, $D^\alpha V \leq -2\varepsilon V$. By applying Lemma 1 and since the equation $D^\alpha V = -2\varepsilon V$ has asymptotic behavior $O(t^{-\alpha})$ as shown in [1], the claim follows since $x^Tx \to 0$ implies $x \to 0$. □

Remark 1. In the above proof, no restriction is imposed on initial condition x_0 except that is finite. Therefore convergence is global as long as condition $A(t) \leq -\varepsilon I$ globally holds. A local result is obtained by stating that $x^TA(t)x \leq -\varepsilon x^Tx$ holds for a region defined by $\|x\| < c$. This observation is implicit in the results that follow.

Remark 2. Note that in the case of a symmetric matrix if $A(t) \geq \varepsilon I$ then $\lambda_m = \lambda_m(t) > \varepsilon$ where λ_m is the smallest eigenvalue of $A(t)$. This is because if x is any eigenvector of $(A - \varepsilon I)$ with eigenvalue λ then $A(t)x = (\lambda + \varepsilon)x$ and since $A(t) - \varepsilon I \geq 0$ then $\lambda \geq 0$. Therefore $\lambda_m = \lambda_m(t) > \varepsilon$. In particular A is positive-definite.

Remark 3. Note that the system with $A_1 = -\varepsilon I$ has speed of convergence ruled by ε. On the other hand, by the comparison principle, the rate of convergence of solutions for $A_2(t) \leq -\varepsilon I$ is faster or equal to that of $A_1 = -\varepsilon I$. Then it can be inferred that $k > 1$ in kA is an acceleration factor.

Hereinafter we will look for weaker conditions than those of the previous theorem. We will distinguish the scalar and the vector cases to simplify the analysis.

3.1. Scalar case

Let us consider the following equation

$$x^{(\alpha)} = a(t)x, \tag{7}$$

with $x(0) = x_0$ and $a(t) \in \mathbb{R}$ is a continuous bounded function such that for all $t \geq 0$, $a(t) \leq 0$.

Property 6. *The solution to the Eq. (7) exists and it is unique.*

Proof. Calling $f(t, x) = a(t)x$ then f is Lipschitz respect to its second variable since $a(\cdot)$ is bounded (the bound of $a(\cdot)$ being the Lipschitz parameter). On the other hand, f is continuous since it is (bi)linear. Then applying Theorem 6.5 of [5] one obtains existence and uniqueness of the solution. □

Property 7. *$x = 0$ is an equilibrium point of (7) and it is unique.*

Proof. Since the Caputo derivative of a constant function is zero, $x \equiv 0$ is a solution of Eq. (7) with initial condition $x(0) = 0$. By uniqueness of the solution, if $x(0) = 0$ then for all $t > 0$, $x(t) = 0$, which is the definition of equilibrium point. For the uniqueness part, note that for all other initial condition the system will have non null dynamic since $a(\cdot)$ is not identically null and the Caputo derivative is zero only for the constant function. □

The next property is an obvious result by the linearity of Eq. (7), so its proof is omitted.

Property 8. *The solutions of Eq. (7) are linear with respect to the initial conditions.*

Remark 4. By this Property, the asymptotic properties for Eq. (7) have a global character. For instance, for fixed function $a(\cdot)$, if for initial condition $x(0)$ the system variable $x(t)$ converges asymptotically to zero then it also converges asymptotically for the initial condition $rx(0)$ with $r \in \mathbb{R}$ different from zero. On the contrary, if there exists $x(0)$ such that x does not converge to zero, then there will be no convergence for none initial condition.

Property 9. *$x(t) = x(0) + I^{\alpha}[ax](t)$ is a schematic solution of Eq. (7).*

Proof. It follows directly from Lemma 6.2 of [5]. As alternative demonstration, given the uniqueness of the solution of (7), it is sufficient α derive the ansatz $x(t)$. □

Remark 5. Such a solution is in fact an integral equation of second type with discontinuous kernel.

Property 10. *The equilibrium point of Eq. (7) is uniformly Lyapunov stable at $t = 0$.*

Proof. Without loss of generality, by Property 8, let us suppose that $x(0) > 0$. By Corollary 6.16 in [5], for all $t > 0$ it holds that $x(t) > 0$. On the other hand, since $a(t) \leq 0$, it holds

$$0 < x(t) \leq x(0). \tag{8}$$

Therefore, if $\|x(0)\| \leq \delta = \epsilon$ then $\|x(t)\| \leq \epsilon$ for all $t > 0$, which is the definition of Lyapunov stability for initial time $t = 0$ given in [9]. □

Note that if a is not equal to zero, then for all $t > 0$, $0 < x(t) < x(0)$.

For the next property we will need the following proposition.

Proposition 1. *Let f be a bounded function i.e. $|f(t)| < f_M$, $\forall t \geq t_0$. Then $I^{\alpha}[f](t)$ is a Hölder-α continuous function (i.e. there exists constant C such that $|f(x) - f(y)| \leq C|x - y|^{\alpha}$ for any x, y). In particular, if $D^{\alpha} f$ is a bounded function and $f \in C^1(\mathbb{R})$ then f is a Hölder-α continuous function.*

Proof. Without loss of generality, let us assume that $t_1 \geq t_2$.

$$|I^{\alpha}[f](t_1) - I^{\alpha}[f](t_2)|\Gamma(\alpha) = \left| \int_0^{t_2} [(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}] f(\tau) d\tau + \int_{t_2}^{t_1} (t_1 - \tau)^{\alpha-1} f(\tau) d\tau \right|.$$

$$|I^{\alpha}[f](t_1) - I^{\alpha}[f](t_2)|\Gamma(\alpha) \leq \int_0^{t_2} [|(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}|] |f(\tau)| d\tau + \int_{t_2}^{t_1} (t_1 - \tau)^{\alpha-1} |f(\tau)| d\tau.$$

Using the fact that f is a bounded function we have

$$|I^{\alpha}[f](t_1) - I^{\alpha}[f](t_2)|\Gamma(\alpha) \leq f_M \left[\int_0^{t_2} |(t_1 - \tau)^{\alpha-1} - (t_2 - \tau)^{\alpha-1}| d\tau + \int_{t_2}^{t_1} (t_1 - \tau)^{\alpha-1} d\tau \right].$$

Resolving the integrals

$$|I^{\alpha}[f](t_1) - I^{\alpha}[f](t_2)|\Gamma(\alpha) \leq \frac{f_M}{\alpha} [t_2^{\alpha} - t_1^{\alpha} + (t_1 - t_2)^{\alpha} + (t_1 - t_2)^{\alpha}].$$

Then

$$|I^{\alpha}[f](t_1) - I^{\alpha}[f](t_2)| \leq \frac{2 f_M}{\alpha \Gamma(\alpha)} (t_1 - t_2)^{\alpha}.$$

Thus f is a Hölder-α continuous function. Now, if $D^{\alpha} f$ is bounded then $I^{\alpha} D^{\alpha} f = f(t) - f(0)$ is a Hölder-α continuous function. Then $(f(t) - f(0)) + f(0) = f(t)$ is a Hölder-α continuous function since adding a constant to the function leaves invariant the difference $|f(t_2) - f(t_1)|$ which defines the Hölder-α continuity. □

Remark 6. There is an analogy with the integer case, namely, if Df is a bounded function (and therefore a Lipschitz continuous function) then f is uniformly continuous and if $D(If) = f$ is a bounded function then If is uniformly continuous.

Corollary 1. *If Df is a bounded function, then $D^{\alpha} f$ is a Hölder-$(1 - \alpha)$ continuous function.*

Proof. It follows by noting that $D^{\alpha} f = I^{1-\alpha}[Df]$ and using the last proposition. □

Property 11. *The solutions of (7) are Hölder-α continuous functions.*

Proof. Using Proposition 1 and the fact that $x(\cdot)$, $a(\cdot)$ are bounded functions ($x(\cdot)$ is bounded by Property 10), it follows that x is Hölder-α continuous. □

In order to apply Property 3, we assure differentiability with the following property.

Property 12. *The solutions of (7) are a.e. differentiable functions. Moreover, if* $a(\cdot) \in C^1[0, T]$*, then* $x \in C^1(0, T]$.

Proof. Since $x(\cdot)$, $a(\cdot)$ are bounded functions, by Property 9, x is well defined for all $t > 0$. Thereby, $D^\alpha x = ax$ is well defined as well. Since $I^{1-\alpha}[Dx] = D^\alpha x$, Dx must be a.e. differentiable. By assuming that a is a bounded function and that belongs to C^1, hypotheses (A1–A4) of Theorem 1 of [10] hold and the claim follows. □

Hereafter, we assume that $a \in C^1(\mathbb{R}_+)$.

Proposition 2. *For Eq. (7) there exist constants* $C_1, C_2 > 0$*, such that for all* $t > 0$ *we have* $I^\alpha[-ax](t) < C_1$ *and* $I^\alpha[-ax^2](t) < C_2$.

Proof. The first follows directly from Property 10 together with the schematic solution in Property 9 for Eq. (7). For the second one, defining $2V = x^2$ and using Property 3, it follows that $D^\alpha V \leq ax^2 \leq 0$. Integrating, one gets $V(0) - V(t) \geq I^\alpha[-ax^2](t)$, and therefore $V(0) \geq V(0) - V(t) \geq I^\alpha[-ax^2](t)$. □

When $\alpha = 1$ and $a(\cdot)$ is a uniformly continuous function that does not converge to zero, $x(\cdot)$ converges to zero by using $I^1[-ax](t) < C_1$ and corollary of Barbalat Lemma [11]. Nevertheless, for $\alpha < 1$ if $I^\alpha f < C$ we can only affirm that $\liminf_{t\to\infty} f = 0$ (see [7]).

The following theorem gives a sufficient condition for asymptotic convergence of Eq. (7).

Theorem 2. *Let us consider Eq. (7) with* $x(0) \in \mathbb{R}$ *any initial condition and* $a \in C^1(\mathbb{R}_+)$ *a bounded function. If* $a(\cdot)$ *satisfies that* $\lim_{t\to\infty} I^\alpha[-a] = \infty$ *then* $\lim_{t\to\infty} x = 0$.

Proof. Without loss of generality, let us suppose that $x(0) > 0$. Defining $2V = x^2$, by applying Property 3, one gets $D^\alpha V \leq ax^2$. Integrating the previous expression we get $I^\alpha D^\alpha V \leq I^\alpha[ax^2](t)$, therefore we have $x^2(t) \leq x^2(0) + I^\alpha[2ax^2](t)$. Let τ be the integration variable. Rewriting $I^\alpha[2ax^2](t)$ as $I^\alpha[(x^2(t) + x^2(\tau) - x^2(t))2a(\tau)](t) = x^2(t)I^\alpha[2a(\tau)](t) + I^\alpha[(x^2(\tau) - x^2(t))2a(\tau)](t)$, we obtain

$$x^2(t) \leq \frac{x^2(0) + B}{1 + I^\alpha[2a](t)}, \tag{9}$$

where $B = B(t) = I^\alpha[(x^2(\tau) - x^2(t))[2a](\tau)](t)$. Since B is the fractional integral of a bounded function (let say, bounded by constant C), B is a continuous function. Therefore for B to diverge from above it must take infinite time since $B = B(t)$ is lesser than Ct^α, which diverges only at infinite.

We prove next that if $I^\alpha[-a] \to \infty$ then $\liminf_{t\to\infty} x = 0$. In fact, reasoning by contradiction, let us assume that there is an instant of time T such that $x > \epsilon$ for any $t > T$. It follows that $I_T^\alpha[-ax] \to \infty$ since if $I^\alpha[-a] \to \infty$ then $I_T^\alpha[-a] \to \infty$ because the term ${}_0I_T^\alpha[-ax]$ is bounded. Therefore $I_T^\alpha[-ax] > \epsilon_T I^\alpha[-a] \to \infty$. But $I_T^\alpha[-ax] \to \infty$ contradicts the precedent proposition.

Let us define the following sequence $t_i \equiv \min\{t \mid x(t) \leq 1/i\}$ for any $i \neq 0, i \in \mathbb{N}$. Since $\liminf_{t\to\infty} x = 0$ and x is a continuous function, every t_i exists and is well defined because this minimum is always reached since it is equivalent to find the first time that $x(t) = 1/i$.

The sequence $(t_i)_i$ is divergent, because x is a continuous function and reaches its minimum in closed interval but x has no global minimum but global ínfimum (by Property 10, $x(\cdot)$ is never zero), which is zero. By the definition of t_i, one has that $x^2(t_i) \to 0$ and $x^2(t_i) \leq x^2(t)\ \forall t < t_i$, therefore $B(t_i) < 0$. On the other hand, the separation between t_k and t_{k+1} is finite for any k. In fact, if it were infinite $I^\alpha ax$ will be unbounded because $x(t) > 1/(k+1)$ for an infinite interval, therefore by Proposition 2, $\infty > I_{t_k}^\alpha[-a]x^2 > (1/(k+1))I_{t_k}^\alpha[-a] \to \infty$ which is a contradiction.

Since $B(t)$ is a continuous function, the intervals are finite and $B(t_k) < 0$, $B(t_{k+1}) < 0$, it follows that $B(t)$ cannot diverge to $+\infty$ between t_k and t_{k+1}. Therefore $B(t)$ is bounded from above. Taking limit of (9) when $t \to +\infty$, one concludes by algebra of limits that $x \to 0$ when $t \to \infty$. □

Remark 7. When $\alpha = 1$ one can use the general solution ($x = \exp(-\int_0^t a d\tau)$) to obtain the same conclusion as in Theorem 2. The precedent proof gives an alternative way to demonstrate it. In effect, since $Dx \leq 0$, x cannot increase and B is always non positive and in particular bounded from above. Therefore $x^2(t) \leq \frac{x^2(0)}{1+I2a(t)}$ converges to zero when $t \to \infty$.

Remark 8. If $I^\alpha a$ is bounded from below, since x is a bounded function, B will be bounded and inequality (9) can be used to bound x.

In the following theorem, we prove that condition in Theorem 2 is not only sufficient but also necessary.

Theorem 3. *If* $x \to 0$ *then* $I^\alpha[-a] \to \infty$ *as* $t \to \infty$.

Proof. Since $x \to 0$, and using that $x = x(0) + I^\alpha[ax]$, we have that $I^\alpha[-ax] \to x(0)$. By using property of limits of integrals (Property 5), $I_T^\alpha[ax] \to x(0)$ for any finite $T > 0$. On the other hand, since $x \to 0$ there exists T_ϵ such that for all $t > T_\epsilon$ it holds that $|x| < \epsilon$, then it follows that $\left|I_T^\alpha(ax)\right| \leq \varepsilon \left|I_T^\alpha(a)\right|$. Since $\epsilon > 0$ is an arbitrarily small number and the integral converges to $x(0)$, it is necessary that $\left|I_T^\alpha(a)\right| \to \infty$. Therefore, since $a(\cdot)$ is a bounded function and T is finite, we conclude that $|I^\alpha(a)| \to \infty$. □

Although Theorem 2 gives a condition that guarantees convergence, it does not tell us about the rate of convergence. The following proposition allows us to have a criterion in order to estimate rates of convergence.

Proposition 3. *Let us consider $x_1(t)$ and $x_2(t)$ the solutions of two systems of the form (7) defined by $a_1(t)$ and $a_2(t)$, respectively. If $(a_1) \leq (a_2)$ for all t then $|x_1| \leq |x_2|$ for all t when $x_1(0) = x_2(0) = x(0)$. In particular, if $[-a(t)] \geq C > 0$ for all t, with C a constant, then $|x(t)| \leq E_\alpha(-Ct^\alpha)$.*

Proof. Without loss of generality, we will take $x(0) > 0$. By defining $\epsilon = \epsilon(t)$ a function such that $x_2 = x_1 + \epsilon$, we have

$$D^\alpha \epsilon = x_1(a_2 - a_1) + \epsilon a_2,$$

with $\epsilon(0) = 0$. Since $a_2 - a_1 \geq 0$ we have

$$D^\alpha \epsilon \geq \epsilon a_2.$$

The equation $D^\alpha \eta = \eta a_2$ with $\eta(0) = 0$ has a unique solution $\eta = 0$ for all $t > 0$. Therefore, by applying the comparison principle (Lemma 1), $\epsilon(t) \geq 0$ and hence $x_2(t) \geq x_1(t)$ for all $t \geq 0$.

The particular case follows from comparing $a_2 = -C$ with $a_1 = a$. □

Remark 9. Taking pulse functions, the parameters θ of the pulse (length of the cycle, period and height) allow to classify the functions $a(\cdot)$ by imposing that $(-a) > pulse(\theta)$ for all $t > 0$ and determine the rates of convergence as a function of θ.

Remark 10. By Proposition 3, one can relax the differentiability condition on $a(\cdot)$ in Theorem 2 by imposing instead that $a(t) \leq b(t)$ for all $t > 0$ where b is a differentiable function.

Next, we give some examples of functions $a(\cdot)$ satisfying that $\lim_{t\to\infty} I^\alpha[-a] = \infty$.

Example 1. $a(t) = C < 0$ where C is a constant, since $I^\alpha[-C] = -Ct^\alpha \to \infty$.

Example 2. $a(t) = -\sin^2(t)$. In effect, since $I^\alpha[1] \to +\infty$, it follows that $I^\alpha[\sin^2(t)+\cos^2(t)] = I^\alpha[\sin^2(t)]+I^\alpha[\cos^2(t)] \to \infty$. Therefore at least one of both integrals diverge. Let us suppose that only one diverges, then $I^\alpha[\cos 2t] = I^\alpha[\cos^2 t] - I^\alpha[\sin^2 t] \to \pm\infty$. But this contradicts the known result that the fractional integral of $\cos(t)$ is bounded [12].

Example 3. Let p be a periodic pulse of values 0 and 1. If p has 100% cycle, its fractional integral diverges since $p \equiv 1$. Note that we can write $p = p_1 + p_2$, where p_1 is a pulse of 50% cycle and p_2 is a pulse of 50% cycle such that when p_1 is 0 p_2 is 1 and viceversa. Therefore $I^\alpha p = I^\alpha p_1 + I^\alpha p_2$. Since $I^\alpha p$ diverges, either $I^\alpha p_1$ or $I^\alpha p_2$ or both must diverge. Intuitively, both must diverge. In effect, let T be half of the period of p_1 that starts in 1, then we can express $p_1(t) = p_2(t - T)$ and therefore $I^\alpha p_1 = I_T^\alpha p_2 - I_{[0,T]}^\alpha p_2 + I_{[0,T]}^\alpha 1$, where the last terms are bounded. (Alternatively, by observing that $I^\alpha p_1(t + (2n+1)T) > I^\alpha p_2(t+(2n+1)T)$, because p_1 will have more pulses in 1 up to time $t+(2n+1)T$ than p_2, but $I^\alpha p_1(t+2nT) < I^\alpha p_2(t+2nT)$ since p_2 has the same amount of pulses that p_1 at the time $(t + 2nT)$ but those of the latter happened before whereby they decay faster than the former.) Therefore since at least one of the integrals diverges and both of them are continuous, both of them diverge. Then any pulse of 50% cycle has fractional integral divergent.

Further, if p_3 is a pulse of cycle > 50% we have that $I^\alpha p_3 \geq I^\alpha p_1$, and then its fractional integral diverges too.

Since any pulse of 50% cycle can be written as a sum of two pulses of 25% cycles, we conclude that integrals of pulses of 25% cycle diverge.

Recursively, any pulse of finite cycle has fractional integral that diverges.

Example 4. Let f be a positive function such that it greater than ϵ periodically in a finite interval, then it has a divergent fractional integral. In effect, there exists a periodic pulse function p with values 0 and ϵ which by the previous example has divergent fractional integral so that $I^\alpha f \geq I^\alpha p$.

Example 5. Let p be a pulse function with values 0, 1 of cycle I/T with $T > I > 0$ which starts with $p(0) = 1$. Let $\bar{p}$ an aperiodic pulse with values 1 at intervals of large I such that such an interval happens at least one time in the interval $[nT, (n+1)T)$, and 0 else. Evaluating at $t = T$, we have $I^\alpha\bar{p}(T) \geq I^\alpha p(T)$ since the cycle of $\bar{p}$ happens after from that of p and the integrals decay when the intervals of cycle is over (decay property). Recursively, as any pulse which compose $\bar{p}$ starts just or after from that of p and supposing that $I^\alpha\bar{p} \geq I^\alpha p$ holds for $t = (n-1)T$, by separating the integral up to and after of $t = (n-1)T$, we have $I^\alpha\bar{p}(nT) \geq I^\alpha p(nT)$. Taking limits when n goes to infinity and since the integral of periodic pulse diverges, we conclude that $\limsup_{n\to\infty} I^\alpha\bar{p} = \infty$. On the other hand, as T is a finite number, the integral of the pulse can decay finitely in each interval $[nT, (n+1)T)$ (since it is continuous function as the pulse is bounded) and together with the fact that $\limsup_{n\to\infty} I^\alpha\bar{p} = \infty$, it follows that $\lim_{t\to\infty} I^\alpha\bar{p} = \infty$.

We end this sub-section with other results of convergence for Eq. (7).

Proposition 4. *Let us consider Eq. (7). If* $I^\alpha[-a] \to 0$ *then* $x \to x(0)$.

Proof. Without loss of generality, let us suppose that $x(0) = x_0 > 0$. By Eq. (8) we have $0 \leq x \leq x_0$. Since $-a > 0$ we have $0 \leq [-xa] \leq [-x_0 a]$. Thereby $0 \leq I^\alpha[-xa] \leq x_0 I^\alpha[-a]$. Using that $x = x_0 + I^\alpha[xa]$ and taking limit when t goes to ∞, the conclusion follows. □

Remark 11. This property can be read as that initial conditions are attractive when $a(\cdot)$ has fractional integral that converges to zero.

Remark 12. Note that although $D^\alpha x \leq 0$, for $x(0) > 0$ there exists necessarily an instant of time (and by continuity, intervals of time) where $x(\cdot)$ is increasing. The above proposition is vacuously true for $\alpha = 1$ since never happens that $I^1[-a] \to 0$ because the integral is monotonically increasing for positive argument. Moreover, $Dx \leq 0$ implies for $x(0) > 0$ that $x(\cdot)$ can never increase.

Remark 13. Since x converges to $x(0)$, if a is a uniformly continuous function then a necessarily converges to zero. For a proof of this statement refer to [13].

Remark 14. When $\alpha = 1$ we have that x always converges, since it is bounded and monotone. When $\alpha = 0$ (consider as the right hand limit), under smoothness assumption, we have $\lim_{\alpha\to 0^+} I^\alpha f = f$ and we can write the solution as $x(t) = x(0) - ax(t)$. Then, (a) $x(t) \leq x(0)$ and $x(t) > 0$ (without loss of generality $x(0) > 0$) (b) if a converges to zero, x converges to $x(0)$, in accord with Remark 13 (c) x converges iff a converges. Therefore the claim that x always converges, would not hold for every α (d) if a diverges, then x converges, in accord with Theorem 2 where $\lim I^\alpha a = -\infty$ with $\alpha \to 0^+$.

We will prove that x converges only if $I^\alpha|a|$ converges. Let us suppose that x converges to L. Then $I^\alpha ax$ converges to $x(0)-L$. As x converges, there exists T such that for all $t > T$ we have $|x(t) - L| \leq \varepsilon$, thereby $(L-\varepsilon)I_T^\alpha|a| \leq I_T^\alpha|a|x \leq (L+\varepsilon)I_T^\alpha w^2$. By property of limits of integrals, $I_T^\alpha|a|x$ converges also to $L - x(0)$. In particular, $I_T^\alpha|a| \leq \frac{1}{L}I_T^\alpha|a|x + \varepsilon_1$, where $\varepsilon_1 = \varepsilon_1(\varepsilon)$ which converges to zero when ε converges to zero. Similarly, using the other inequality, $I_T^\alpha|a| \geq \frac{1}{L}I_T^\alpha|a|x - \varepsilon_1$. On the other hand, for all $t > T$ sufficiently larger, there exists ε_2 such that $I^\alpha|a| \geq I_T^\alpha|a| + \varepsilon_2$, since $I_{[0,T]}^\alpha|a|$ converges to zero. Consequently, there exists ε_3 such that $I^\alpha|a| \geq \frac{1}{L}I_T^\alpha|a|x + \varepsilon_3$ (Similar for the other inequality). Then $I^\alpha|a|$ converges to $\frac{L-x(0)}{L}$.

We arrive to the following classification of linear systems according to the integral of its function $a(\cdot)$: If $I^\alpha|a| \to \infty$ then $x(\cdot)$ converges to zero. If $I^\alpha|a| \leq C$ implies $x(\cdot)$ converges to a non zero number or it does not converge (since $I^\alpha|a| \to \infty$ is condition necessary for convergence to zero). If $I^\alpha|a| \to 0$ implies that $x(\cdot)$ converges to $x(0)$.

3.2. Vector case

We assume that the components of matrix A are of class $C^1(\mathbb{R}_+)$ and the matrix A is bounded (in matrix norm), whereby the solutions of (6) become of class $C^1(\mathbb{R}_+)$ by similar arguments of [10] generalized to the vector case. For the boundedness of the trajectories of the system (6), we define the function $2V = x^T x$ since, by applying Caputo derivative property, its derivative satisfies $V^{(\alpha)} \leq x^T Ax \leq 0$ thereby, by comparison principle, $V(t) \leq V(0)$. For the asymptotic convergence, Theorem 1 provides a useful tool but not quite general since in many practical cases the matrix $A(t)$ could not be negative-definite for each instant. For these cases, we state the following theorem.

Theorem 4. *Let $f(t)$ be a scalar non negative differentiable function such that $I^\alpha[f] \to \infty$. Let $x(0) \in \mathbb{R}^n$ be any initial condition. If $A(t) \leq -f(t)I$ holds for all $t > 0$ and the components of matrix A are of class $C^1(\mathbb{R}_+)$ then $x \to 0$ (where I is the identity matrix).*

Proof. By using Property 3 and the hypothesis, we can write

$$(x^T x)^{(\alpha)}/2 \leq x^T A(t)x \leq -f(t)x^T x.$$

Noting that for system $(x^T x)^{(\alpha)} = -2f(t)x^T x$, $x^T x$ converges asymptotically to zero by Theorem 2, since it is equivalent to $V^{(\alpha)} = -2f(t)V$ with $V(0) \geq 0$ where $V(t) = x^T x$. The claim follows by applying the comparison principle. □

Remark 15. Theorem 1 can be seen as a direct corollary by using $f(t) = \epsilon$ for all $t > 0$ and Example 1. Therefore, if $f(t) > \epsilon$ for all $t > 0$ we get $t^{-\alpha}$ stability.

Remark 16. Since it is possible for $f(t)$ to take the zero value for some instants, this theorem does not restrict $A(t)$ to be positive-definite as Theorem 1 does. Further, if $A(t)$ is a symmetric matrix, a sufficient condition for asymptotic convergence is obtained by defining $\lambda_M = \lambda_M(A(t))$ as the largest eigenvalue of A and imposing that $-\lambda_M(t)$ has divergent fractional integral (since $A \leq \lambda_M I$). In the same way, the condition of theorem holds only if $-\lambda_m(I^\alpha[A])$ has divergent fractional integral, where $\lambda_m(A)$ is the smallest eigenvalue of A. When A is not symmetric one must use $Re(\lambda)$ instead of λ.

Remark 17. Let u be any unitary vector of dimension n. $A(t) \leq -f(t)I$ implies, by definition, $u^T A(t) u \leq -f(t)$. Since that if $f(t)$ converges fast to zero its fractional integral does not diverge (for instance, if $f(t)$ is $O(e^{-t})$), A must have full range at least in a divergent sequence of instants in order to satisfy the condition of last theorem.

The condition could not be necessary as shown in [14] for $\alpha = 1 \wedge A = -ww^T$, where it is enough that the integral diverges in a finite set of directions.

Example 6. If $A(t)$ continuous differentiable, periodic and uniformly continuous matrix function such that $A(t) \leq 0$ for all $t > 0$ and there exists one instant where $A < 0$, then $A(t)$ holds hypothesis of Theorem 4 (it follows by using Example 3).

As in the scalar case, the rates of convergence can be relatively estimated by using the following comparison proposition.

Proposition 5. *If $f_1(t) \geq f_2(t)$ for all t where f_i are continuous differentiable functions and $A_i(t) \leq -f_i(t)I$, then $x_1^T x_1(t) \leq x_2^T x_2(t)$ when $x_1(0) = x_2(0)$.*

Proof. Using Property 3, hypothesis and defining $2V_i = x_i^T x_i$, we have $V_1^{(\alpha)} \leq -f_1(t)V_1 \leq -f_2(t)V_1$ and $V_2^{(\alpha)} \leq -f_2(t)V_2$. Using Lemma 1 and that $V_1(0) - V_2(0) = 0$, we have $V_1(t) \leq V_2(t)$ and the claim follows. □

4. Forced linear systems

In this section we analyze the case of systems described by fractional order linear time-varying forced differential equations, deriving conditions for boundedness and convergence on system signals. Before going deeper in the subject of this section, we will establish the following lemma.

Lemma 2. *Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a function belonging to $\mathcal{L}^1$. If $u : \mathbb{R}^+ \to \mathbb{R}$ is a bounded function (whose bound is u_M) such that u converges to zero as $t \to \infty$, then $y = h * u$ also converges to zero, where $*$ denotes the convolution operator.*

Proof. Let us consider any $\epsilon > 0$. Since h is in $\mathcal{L}^1$ there exists $T_1 > 0$ such that $\int_{T_1}^{\infty} h < \epsilon/(2u_M)$. Since u converges to zero, there exists $T_2 > 0$ such that $u(t) < \epsilon/(2l)$ where l is a real number such that $l \geq \int_0^{\infty} |h|$. By taking $T = \max\{T_1, T_2\}$ we can write

$$\int_0^{\infty} |h(\tau)|\,|u(t-\tau)|d\tau = \int_0^{T} |h(\tau)|\,|u(t-\tau)|d\tau + \int_T^{\infty} |h(\tau)|\,|u(t-\tau)|d\tau,$$

and by the choice of each T_i, have for all $t > T_2 + T$ that

$$\int_0^{\infty} |h(\tau)|\,|u(t-\tau)|d\tau < \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore

$$0 \leq \left| \int_0^{\infty} h(\tau)u(t-\tau)d\tau \right| \leq \int_0^{\infty} |h(\tau)|\,|u(t-\tau)|d\tau < \epsilon.$$

Thereby y converges to zero. □

Using Lemma 2 we can prove the following theorem.

Theorem 5. *Let $H(s)$ be a linear time-invariant filter defined by a rational transfer function with polynomials of integer order or commensurate fractional order relative to $\alpha < 1$, which is asymptotically stable. If the input to the filter u is a bounded function that converges to zero then the output of the filter y also converges to zero.*

Proof. For $\alpha = 1$, the filter impulse response h is in $\mathcal{L}^1$ since it is a sum of decaying exponentials (eventually multiplied by polynomials functions of t in the case of poles with multiplicity different from unity). For $\alpha < 1$, by using results of [1], the impulse response h belongs to $\mathcal{L}^1$ since the linear system is BIBO stable. Thereby taking $u \equiv 1$ the output y is just the integral of the impulse response, which by BIBO stability turns out to be bounded. (Moreover, for the scalar case it is monotonically and therefore for the general case it is also a sum of $\mathcal{L}^1$ terms). The claim follows by applying Lemma 2. Since the filter is asymptotically stable, the terms associated to initial conditions also decay to zero. □

Remark 18. The previous theorem allows us to get rid of terms in $\mathcal{L}^1$ from inputs in forced time invariant linear systems, when studying asymptotic convergence to zero of outputs.

Corollary 2. *Let us consider the system defined by*

$$e^{(\alpha)} = -\lambda e + \phi^T w,$$

with $e(t) \in \mathbb{R}$; $\phi(t), w(t) \in \mathbb{R}^n$; *the components of* $w(\cdot)$ *are of class* $C^1(\mathbb{R}_+)$; $\lambda > 0$ *and* $\phi(t)$ *satisfies that*

$$\phi^{(\alpha)} = -ew.$$

If w *converges to zero then* e *converges to zero.*

Proof. In order to apply preceding theorem, we must show that ϕ is a bounded function. In effect, choosing the function $2V = \phi^2 + e^2$ we have $D^\alpha V \le -\lambda e^2 \le 0$ (the trajectories are differentiable by using Theorem 1 of [10]). By α-integrating the last inequality, $V(t) \le V(0)$, whereby the trajectories of the system are bounded and therefore ϕ is bounded. □

The next theorem studies the above issue for linear time-variant systems. However, it holds only for integer order.

Theorem 6. *Let us consider the following system*

$$\dot{x} = A(t)x + f(t), \tag{10}$$

where $A(t) \in \mathbb{R}^{n\times n}$ *is a matrix function bounded (in norm) and* $f(t) \in \mathbb{R}^n$. *Let us suppose that when* $f(t) \equiv 0$ *the resulting system is asymptotically stable. Then if* $f(t)$ *is any bounded function belonging to* $\mathcal{L}^2$ *the trajectories of the system asymptotically converge to zero for any bounded initial condition.*

Proof. Let us consider the following function $V = x^T P(t)x$ with $P(t) \in \mathbb{R}^{n\times n}$. Since the unforced system is asymptotically stable, then there exists a matrix $P(t)$ satisfying the following conditions

$$\begin{aligned} &0 < c_1 I \le P(t) \le c_2 I, \\ &-\dot{P} = PA + A^T P + Q, \\ &Q \ge c_3 I. \end{aligned}$$

The existence of such $P(t)$ and $Q(t)$ follows from Theorem 3.10 in [15] since $A(t)$ is asymptotically stable by hypothesis. Then using Cauchy–Schwarz inequality we can write

$$\dot{V} \le -c_3\|x\|^2 + c_2\|x\|\,\|f\|. \tag{11}$$

Using that $2ab \le a^2 + b^2$ with $a = \epsilon\|x\|$ and $b = c_2\|f\|/\epsilon$ (with $\epsilon^2 = 2(1-\rho)c_3$ and $\rho < 1$), we have that $\dot{V} \le -c_3\rho\|x\|^2 + c_2^2\|f\|^2/(2\epsilon^2) \le c_2^2\|f\|^2/(2\epsilon^2)$. By integrating, we obtain $V(t) \le V(0) + c_2^2\|f\|_2/(2\epsilon^2)$. Therefore $V(t)$ is bounded and so it is $x(t)$ and therefore $\|x(t)\| < C$.

Integrating (11) and using that $V(t)$ is bounded, it follows that $\|x\|_2 \le (V(0) + c_2^2 C\|f\|_2/(2\epsilon^2))/c_3$. Therefore x belongs to $\mathcal{L}^2$. Since f and x are bounded functions, x is Lipschitz continuous by Eq. (10). Applying Barbalat Lemma [11], we conclude that $\|x\| \to 0$ as $t \to \infty$. □

Remark 19. Theorem 6 allows us to ignore terms in $\mathcal{L}^2$, in the analysis of asymptotic convergence of systems where an Eq. (11) can be demonstrated. For example, in the context of adaptive observers instead of studying equation $\dot{\phi} = ww^T\phi + c^T\exp(Ft)\phi_0$ [11], it is enough to study $\dot{\phi} = ww^T\phi$ when F is a constant asymptotically stable matrix.

Corollary 3. *Let us consider the system* (10). *If* $f \in \mathcal{L}^\infty$ *then* $x \in \mathcal{L}^\infty$. *In particular, the system is BIBO stable for an output defined as* $y = c^T(t)x$ *with* $c \in \mathcal{L}^\infty$.

Proof. Using Eq. (11), it follows that there exists a constant $C_1 > 0$ sufficiently large such that for $\|x\| > C$ we have $\dot{V} \le 0$. Therefore we can apply Theorem 4 in [16] to conclude that $x(t)$ a is bounded function. The rest of the claim follows by using Cauchy–Schwarz inequality to get $|y| \le \|c\|\,\|x\| \le C_2$. □

Corollary 4. *Let* $D^\alpha x = A(t)x + f(t)$ *be a system where* $f \in \mathcal{L}^2_\alpha := \{f : \mathbb{R}_+ \to \mathbb{R}^n | (\forall t > 0) I^\alpha[\|f\|^2](t) < \infty\}$, f *continuously differentiable, the components of matrix* A *are of class* $C^1(\mathbb{R}_+)$ *and* $A(t) \le -\epsilon I$ *for all* $t > 0$. *Then the trajectories* $x(t)$ *of the system are bounded functions.*

Proof. By Property 3 we have $D^\alpha x^T x \le 2x^T D^\alpha x = 2x^T A(t)x + 2x^T f \le -2\epsilon\|x\|^2 + 2\|x\|\,\|f\|$. By a similar argument to that of the proof of Theorem 6, it follows that $D^\alpha x^T x \le 2\|f\|$. Thereby, after integrating and using hypothesis upon f, we conclude that x is a bounded function. □

5. Non linear systems

In this section we analyze the case of systems described by fractional order nonlinear differential equations, deriving conditions for boundedness and convergence on system signals. We begin by generalizing Theorem 4 of Section 3 to systems described by

$$D^{\alpha}x = f(t,x), \tag{12}$$

with $x(t), f(t,x(t)) \in \mathbb{R}^n$, for the initial condition $x(0) \in \mathbb{R}^n$. It will be assumed that $f(t,0) = 0$ for all $t \geq 0$.

Theorem 7. *Let us consider system (12). If there exist a scalar function $V = V(x)$, a scalar class-K function γ and, for any x (or locally around a ball of the origin $x = 0$), a bounded continuous differentiable function $g(\cdot, x)$ such that*

(i) $V \geq \gamma(x)$;
(ii) $D^{\alpha}V \leq -|g(x,t)|^2\gamma(x)$;
(iii) $I^{\alpha}[\|g(x,t)\|^2] \to \infty$ *for any fixed* x *as* $t \to \infty$.

Then $\|x(t)\| \to 0$ *as* $t \to \infty$.

Proof. From (i) and (ii) it follows that $D^{\alpha}V \leq -\|g(x,t)\|^2 V$. Using the comparison principle (Lemma 1) and (iii) we conclude that V converges to zero, and then, using (i), the trajectories of (12) are globally asymptotically convergent to zero (or locally around a ball of the origin $x = 0$). □

Similarly, we can prove the following claim.

Proposition 6. *Let us consider system (12). If we can find a positive-definite function $V = V(x) \in \mathbb{R}$ such that $D^{\alpha}V = g(V)$ locally (globally) holds with $g(\cdot)$ a concave function such that $g(x) = 0$ implies $x = 0$ and $g'(0) < 0$, then we conclude that trajectories of (12) are locally (globally) asymptotically convergent to zero.*

Proof. From the concavity of g we have $D^{\alpha}V \leq g'(0)V$. By using the comparison principle (Lemma 1) and since $D^{\alpha}y = g'(0)y$ holds that y converges to zero and V is not negative, we conclude that V converges to zero and since $g(0) = 0$, the trajectories of (12) are locally (globally) asymptotically convergent to zero. □

It would be desirable to have a convergence condition of the type $\int_t^{t+T} \dot{V} \leq -\gamma(\|x\|)$ since it would allow to generalize the notion of persistent excitation of adaptive theory in a natural way. In fact, we can prove it with a weaker hypothesis where γ is a positive constant, namely

$$\int_t^{t+T} \dot{V} \leq -\gamma V(t+T), \tag{13}$$

so we can write $V(t+T) \leq \frac{1}{1+\gamma}V(t)$. Therefore we have convergence to zero of V since $\frac{1}{1+\gamma} < 1$.

By the same reasons, with the condition $\int_t^{t+T} \dot{V} \leq -\gamma \|V(t)\|$ we have convergence to zero of V when $0 < \gamma < 1$ is a constant.

A fractional (non local) version can be obtained by the following condition

$$I^{\alpha}D^{\alpha}V(t) \leq -g(t)V(t), \tag{14}$$

where $g(t) \to \infty$ as $t \to \infty$. From this condition it follows that V converges to zero.

The following theorem generalizes to fractional order claims for the integer orders of theorems in [16].

Theorem 8. *Let $V(t,x)$ be an associated scalar continuous function for the system with continuous trajectories $x(t) \in \mathbb{R}^n$ such that $V(t,x) > \gamma(x)$ with γ radially unbounded (i.e. a function $\gamma : \mathbb{R}^n \to \mathbb{R}$ such that $\|x\| \to \infty \Rightarrow \gamma(x) \to \infty$) non negative function (for example, $\|x\|^2 = \gamma(x)$), for all $t > 0$. Let us assume that $D^{\alpha}V(\cdot, x)$ is differentiable and $0 < \alpha \leq 1$.*

(a) *If there exists $T > 0$ such that $D^{\alpha}V \leq 0$ for any $t > T$ then the trajectories of the system are bounded.*
(b) *In the case that $V = V(x)$, let Ω be a compact neighborhood around the origin $x = 0$ such that $D^{\alpha}V \leq 0$ for all $x \in \Omega^c$. Then the trajectories of the system are bounded.*

Proof. (a) Let us denote $D^{\alpha}V(t) := g(t)$. Since T is finite and $D^{\alpha}V$ is continuous, $g(t)$ turns out to be bounded on $[0,T]$. Further, since it is differentiable, we can integrate to obtain $V(t) = V(0) + I^{\alpha}_{[0,T]}g(t) + I^{\alpha}_{T}g(t) < C + I^{\alpha}_{T}g < C$, since $I^{\alpha}_{[0,T]}g(t)$ converges to zero (Property 4), there exists T_1 such that for all $t > T_1$ we bound $I^{\alpha}_{[0,T]}f(t) < C_1$ and by continuity, on $[0, T_1]$ its value is bounded by C_2; choosing $C = \max(C_i)$ we get the bound. Therefore as $\gamma(x) < V < V(0) + C$ we have that x is bounded since γ is radially unbounded then by counter reciprocal if γ is bounded, x is bounded.

(b) If x is always in Ω it is bounded. If x is always in Ω^c it is bounded since we have $D^{\alpha}V \leq 0$ then $V \leq V(0)$, by comparison principle. If x is always on Ω after a finite time, it will be bounded since it is continuous. If x is always on Ω^c after a finite time, we conclude boundedness by using part (a). The remaining case is if x is in Ω and in Ω^c alternately and endlessly. Observing that Ω is compact, V turns out to be bounded on $\partial\Omega$, the border of Ω, (say) by C_{Ω}. Also, when x crosses to the set Ω^c, since it is well definite and x is a continuous function, x crosses necessarily by $\partial\Omega$. Let T be any instant when this cross to Ω^c occurs, then

$$V(x(T)) = V(0) + I^\alpha D^\alpha V(x(T)) < C_\Omega.$$

Since C_Ω is a constant independent of the instant T, we have $I^\alpha_{[0,T]} D^\alpha V(T) < C_\Omega + V(0)$ with $C_\Omega + V(0)$ independent of the instant T and $I^\alpha_{[0,T]} D^\alpha V(t)$ converges to zero (Property 4) as $t \to \infty$. Therefore, for all $T' > t > T$ where T' is the instant when x returns to Ω

$$V(t) = V(0) + I^\alpha_{[0,T]} D^\alpha V(t) + I^\alpha_T D^\alpha V(t) < C + I^\alpha_T D^\alpha V(t) < C,$$

where C is chosen in the same way as in part (a) since $I^\alpha_{[0,T]} D^\alpha V(t)$ is continuous because $D^\alpha V$ is bounded on $[0, T]$. Thereby, V is bounded when x belongs to Ω^c, and it is also bounded when x belongs to Ω. Since T is an arbitrary crossing time, we conclude boundedness of $V(x)$ and, by a similar reasoning as in the part (a), we conclude boundedness of x. □

Remark 20. For $\alpha = 1$ the proof can be simplified by noting that in Ω^c, V cannot increase since $DV \leq 0$ (a fact not necessarily true for fractional derivative of any kind) and therefore $x(t)$ is bounded.

Remark 21. The part (b) can be generalized for $V = V(x, t)$ if Ω is a compact time invariant set.

Hypothesis of Theorem 8 can be seen as conditions on the (fractional) integral of (integer) derivative of (Lyapunov) function in order to get a bounded system or formally asymptotically stable system, since, by definition, Caputo derivative is the (fractional) integral of (integer) derivative. In this direction, we include the following results.

Let $V(\cdot, x)$ be a differentiable function for any x such that $V(t, x) \geq \gamma(x)$ with $\gamma(\cdot)$ a radially unbounded function, defined on all the trajectories $x(t)$ for $t \geq 0$ of a system $\mathcal{S}$.

The first result allows to conclude boundedness even if $D^\alpha V$ could take positive values at some instants of time (the case $D^\alpha V(t) \leq 0$ for all $t \geq 0$ being a particular case).

Proposition 7. *If there exists α in $(0, 1]$ such that for any $t \geq 0$ it holds that $I^\alpha D^\alpha V \leq 0$ for any trajectory $x(t)$ then the system $\mathcal{S}$ has bounded trajectories.*

Proof. $I^\alpha D^\alpha V = V(t) - V(0) \leq 0$, then $\gamma(x(t)) \leq V(t, x(t)) \leq V(x(0))$ for every $t > 0$. Since $\gamma(\cdot)$ is radially unbounded, if it is bounded, its argument is bounded. □

The second result requires a weaker hypothesis for a weaker conclusion.

Proposition 8. *Let $V = V(t)$ be a uniformly continuous function. If there exist α in $(0, 1]$, a divergent sequence (t_i) such that $|t_i - t_{i+1}| \leq T$ where T is a sufficiently small number (depending on the grade of boundedness $T(\epsilon)$) and it holds that $I^\alpha D^\alpha V(t_i) \leq 0$ for any trajectory $x(t)$, then the system $\mathcal{S}$ has bounded trajectories.*

Proof. $I^\alpha D^\alpha V(t_i) = V(t_i) - V(0) \leq 0$ whereby $V(t_i) \leq V(0)$. By uniform continuity hypothesis, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|t - t_i| < T < \delta$ then $|V(t) - V(t_i)| \leq \epsilon$. Thereby, $V(t) \leq \epsilon + V(0)$ for all t since $t_i \to \infty$, and therefore $\|x(t)\|$ is bounded for all $t \geq 0$. □

The third result is restricted to the integer case.

Proposition 9. *If there exists $T > 0$ such that for all $t > T > 0$, $\int_t^{t+T} \dot{V} \leq 0$ then the system $\mathcal{S}$ has bounded trajectories.*

Proof. Since V is a continuous function of time and $[0, T]$ is a compact set, V reaches its maximum, say, $V_M < \infty$ whereby for any $t \in [0, T]$, it holds that $V(t) \leq V_M$. Therefore for any $0 < t < T$ and since $\int_t^{t+T} \dot{V} \leq 0$ we have $V(t + T) \leq V(t) \leq V_M$. Taking now $0 < t < 2T$ one can put the same argument as before and recursively conclude that since for any $t > 0$, $\gamma(x) \leq V(t) \leq V_M$, any trajectory will be bounded. □

6. Conclusions

Sufficient conditions for asymptotic convergence of fractional linear systems have been presented in this paper for both, scalar and vector case. In the vector case, some additional work has yet to be done in order to clarify how far from necessity are these conditions and how practical turn out to be for evaluate if any specific matrix satisfies them. Besides, the rate of convergence may still be better characterized.

Some rather simple but useful results of convergence and boundedness were presented for fractional forced linear systems. However, the case of strictly fractional forced non autonomous linear systems could not be fully encompassed in the proposed results for convergence and must be treated in a different way; in the integer case, we make use of Barbalat Lemma and the existence of Lyapunov functions for linear stable systems.

Finally, we obtained boundedness conditions and, by abstraction of the method used to get results for linear systems, asymptotic convergence conditions for fractional nonlinear systems were studied. Additionally, a couple of ways that could be explored in order to have more general results were stated.

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