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Weighted pseudo almost periodic functions, convolutions and abstract integral equations

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ABSTRACT

This paper deals with a systematic study of the convolution operator $\mathcal{K}f = f * k$ defined on weighted pseudo almost periodic functions space $PAP(\mathbb{X}, \rho)$ and with $k \in L^1(\mathbb{R})$. Upon making several different assumptions on k, f and ρ, we get five main results. The first two main results establish sufficient conditions on k and ρ such that the weighted ergodic space $PAP_0(\mathbb{X}, \rho)$ is invariant under the operator $\mathcal{K}$. The third result specifies a sufficient condition on all functions (k, f and ρ) such that the $\mathcal{K}f \in PAP_0(\mathbb{X}, \rho)$. The fourth result is a sufficient condition on the weight function ρ such that $PAP_0(\mathbb{X}, \rho)$ is invariant under $\mathcal{K}$. The hypothesis of the convolution invariance results allows to establish a fifth result related to the translation invariance of $PAP_0(\mathbb{X}, \rho)$. As a consequence of the fifth result, we obtain a new sufficient condition such that the unique decomposition of a weighted pseudo almost periodic function on its periodic and ergodic components is valid and also for the completeness of $PAP(\mathbb{X}, \rho)$ with the supremum norm. In addition, the results on convolution are applied to general abstract integral and differential equations.

1. Introduction

In [17], Diagana introduced weighted pseudo almost periodic functions theory as an extension of the pseudo almost periodic functions theory introduced by Zhang [41] (see also [8,37–39,42–44]) as a natural generalization of the almost periodicity notion started by H. Bohr [6,7] (see also [2,12–16,26–29,32]) and continued by several researchers like V.V. Stepanov, S. Bochner, J. Von Neumann and S.L. Sobolev [11,18].

We now briefly describe some generalities, terminology and notation. Overall, the central and original idea of Diagana [17] (see also [17–24]) was the enlargement of the ergodic component space with the help of

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a so-called *weighted measure* $d\mu(t) = \rho(t)dt$, with $\rho : \mathbb{R} \to \mathbb{R}^+$ a *locally integrable* function commonly called *weight*. More precisely, a continuous function f defined from $\mathbb{R}$ to the Banach space $\mathbb{X}$ is called a weighted pseudo-almost periodic function if it can be written as follows: $f = g + \phi$, with g an almost periodic function and ϕ a weighted ergodic function in the sense that $\phi : \mathbb{R} \to \mathbb{X}$ is a bounded continuous function such that

$$\lim_{r\to+\infty} \frac{1}{\int_{-r}^{r} \rho(t)dt} \int_{-r}^{r} \|\phi(t)\|_{\mathbb{X}}\, \rho(t)dt = 0, \text{ with } \lim_{r\to+\infty} \int_{-r}^{r} \rho(t)dt = \infty. \tag{1}$$

The space of all weighted almost periodic functions and the weighted ergodic functions from $\mathbb{R}$ to $\mathbb{X}$ are denoted by $PAP(\mathbb{X}, \rho)$ and $PAP_0(\mathbb{X}, \rho)$, respectively. It is clear that $PAP(\mathbb{X}, \rho)$ is more general, richer and have become more important than the standard space of pseudo-almost periodic functions $PAP(\mathbb{X})$. In particular, we note two facts. First, if we consider that ρ is a constant function, then the first limit given in (1) defines the mean of ϕ, i.e. the ergodic functions space, $PAP_0(\mathbb{X})$, is a particular case of $PAP_0(\mathbb{X}, \rho)$. Second, we note that

$$\phi_1(x) = \frac{1}{1+|x|}, \quad \phi_2(x) = \frac{1}{1+x^2} \quad \text{and} \quad \phi_3(x) = \exp(-x^2)$$

are ergodic functions and may not necessarily be weighted ergodic functions. Thus, the Diagana's definition of weighted pseudo almost-periodic functions, allows to distinguish between ϕ_1, ϕ_2 and ϕ_3 according to the size of the disturbance measured by ρ.

The initial motivation of this paper was the fact that the general theory of weighted pseudo-almost periodic functions is far to be closed, since several questions remain still open. For instance, the existence of the weighted mean for a general almost periodic function [25], the unique decomposition of a weighted pseudo almost periodic function in its periodic and ergodic components and the characterization of the cases when the set of weighted pseudo almost periodic functions is a Banach space with the supremum norm [45], the ergodicity of the weighted mean and the convolution invariance of the weighted ergodic space [1,33]. For a recent list of some of these problems we refer the reader to [45] and for the partial solutions we refer to [1,25,33].

In this paper, we are interested on the convolution invariance of $PAP(\mathbb{X}, \rho)$. Indeed, we recall that it is well known that $PAP(\mathbb{X})$ is a convolution invariant space, in the sense that if $f \in PAP(\mathbb{X})$, then $f * k \in PAP(\mathbb{X})$ for $k \in L^1(\mathbb{R})$. Now, for $f \in PAP(\mathbb{X}, \rho)$, we note that $f * k = g * k + \phi * k$ and $g * k$ is almost periodic function but $\phi * k$ is not necessarily in $PAP_0(\mathbb{X}, \rho)$. Then, the study of the convolution invariance of the spaces $PAP(\mathbb{X}, \rho)$ and $PAP_0(\mathbb{X}, \rho)$ are equivalent. Thus, we focus in the following task: is the space $PAP_0(\mathbb{X}, \rho)$ convolution invariant?

We found four sufficient conditions which imply positive answers to the question of convolution invariance of $PAP_0(\mathbb{X}, \rho)$. Moreover, we establish results with the consequences of the convolution invariance of $PAP_0(\mathbb{X}, \rho)$ in the translation invariance, the unique decomposition of $PAP(\mathbb{X}, \rho)$, and the completeness of $PAP(\mathbb{X}, \rho)$ with the supremum norm.

We now state more precisely the sufficient conditions. Let us denote by $\rho \in \mathbb{U}_\infty$ the set of bounded weights such that the second limit in (1) is valid. First, by assuming that $\rho \in \mathbb{U}_\infty$ satisfies the condition

$$\sup_{r\in\mathbb{R}_+} \sup_{t\in\Omega_{r,s}} \frac{\rho(t+s)}{\rho(t)} < \infty, \qquad \Omega_{r,s} = \Big\{ t \in \mathbb{R} \;:\; |t| < |s| + r \Big\},\ r \in \mathbb{R}_+, \tag{2}$$

for each $s \in \mathbb{R}$ and (k, f) arbitrary selected in $L^1(\mathbb{R}) \times PAP_0(\mathbb{X}, \rho)$, we get that $f * k \in PAP_0(\mathbb{X}, \rho)$. Second, by considering that ρ and k satisfy the conditions

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{1}{\rho(s)} \int_s^r |k(t-s)|\rho(t)dt < \infty, \tag{3}$$

$$\sup_{|s|\le r, r\in\mathbb{R}} \frac{1}{\rho(s)} \int_{-r}^s |k(t-s)|\rho(t)dt < \infty, \tag{4}$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} ds \int_{-r}^{r} |k(t-s)|\rho(t)dt = 0,$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{r}^{+\infty} ds \int_{-r}^{r} |k(t-s)|\rho(t)dt = 0$$

we prove that for f arbitrary selected in $PAP_0(\mathbb{X},\rho)$, we get that $f * k \in PAP_0(\mathbb{X},\rho)$. Third, considering (3)–(4) together with one of the following conditions

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \left[\left(\int_{-\infty}^{t-r} + \int_{t+r}^{+\infty} \right) |k(s)|ds \right] \rho(t)dt = 0,$$

$$\sup_{r\in\mathbb{R}^+} \int_{-r}^{r} \left(\int_{-\infty}^{t-r} + \int_{t+r}^{\infty} \right) |k(s)|ds\rho(t)dt < \infty,$$

we introduce one additional affirmative answer to the question of convolution invariance of $PAP_0(\mathbb{X},\rho)$. Fourth, by assuming that k, f and ρ are such that

$$\exists \phi : \mathbb{R} \to \mathbb{R}_+ \quad : \quad k\phi \in L^1(\mathbb{R}) \quad \text{and} \quad \sup_{|u+s|\le r, r\in\mathbb{R}^+, s\in\mathbb{R}} \frac{\rho(u+s)}{\rho(u)\phi(s)} < \infty,$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \left(\int_{-r-|s|}^{-r} + \int_{r}^{r+|s|} \right) \|f(\tau)\|_{\mathbb{X}}\, \rho(\tau)d\tau = 0 \quad \text{for all } s \in \mathbb{R},$$

we prove that $f * k \in PAP_0(\mathbb{X},\rho)$. Note that the last result is not a result on the convolution invariance of $PAP(\mathbb{X},\rho)$, since it requires a restriction on f.

In this paper, we also get three further results of the application of convolution results. In the first application, we get a result on the uniqueness of a weighted pseudo almost periodic solution of an integral equation, see Theorem 4.1. Then, we introduce a second result for the uniqueness of pseudo almost periodic mild solution of an evolution equation, see Theorem 4.2. Finally, in Theorem 4.3, we give a sufficient condition for a unique weighted pseudo almost periodic of the heat equation where the source function is given by $\gamma H(t)\sin(u(t,x))$, where γ is a positive parameter and H is the function defined as follows $H(t) = \cos(t) + \cos(\sqrt{2}t) + \phi(t)$ with ϕ such that $\phi(t)e^t$ is bounded.

The paper is organized as follows. In section 2 we introduce the notation and recall some concepts and previous results. In section 3, we state and prove the main results. To close the paper, in section 4 we introduce some applications in integral equations, abstract differential equations and partial differential equations.

2. Preliminaries

In this section we present the concept of a weighted pseudo almost periodic function and related concepts like the weighted mean of a function, the weighted ergodic function. Moreover, we also recall some useful results.

Hereinafter, the notation $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ will be used for the general Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ with norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, respectively. Furthermore, we will use the notations $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $BC(\mathbb{R},\mathbb{X})$ for the jointly continuous function form $\mathbb{R}\times\mathbb{Y}$ to $\mathbb{X}$ and the Banach space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{X}$ dotted with the sup norm, respectively.

2.1. Weight notion and related notation

We say that a function ρ is a weight if it has the following properties: (i) ρ is defined from $\mathbb{R}$ to $[0,\infty)$, (ii) ρ is locally integrable over $\mathbb{R}$, and (iii) ρ is strictly positive almost everywhere on $\mathbb{R}$. The set of such functions is denoted by $\mathbb{U}$. Now, in relation to the weight notion, we introduce the function μ defined as follows

$$\mu(r,\rho) := \int_{-r}^{r} \rho(x)dx \tag{5}$$

as well as the following sets notation

$$\mathbb{U}_\infty = \Big\{\rho\in\mathbb{U} \ :\ \lim_{r\to\infty}\mu(r,\rho)=\infty\Big\},$$
$$\mathbb{U}_B = \Big\{\rho\in\mathbb{U}_\infty \ :\ \rho \text{ is bounded with } \inf_{x\in\mathbb{R}}\rho(x)>0\Big\}.$$

Note that $\mathbb{U}$, $\mathbb{U}_\infty$ and $\mathbb{U}_B$ are the collections of all possible weight functions, the weights such that belong $L^1_{\text{loc}}(\mathbb{R}) - L^1(\mathbb{R})$, and the positive bounded weights, respectively. Clearly, the sets $\mathbb{U},\mathbb{U}_\infty$ and $\mathbb{U}_B$ are not empty and $\mathbb{U}_B\subset\mathbb{U}_\infty\subset\mathbb{U}$. Two examples of weight functions are given by $\rho_0,\rho_1:\mathbb{R}\to[0,\infty)$ defined as follows

$$\rho_0(x) = \frac{a+b|x|}{1+b|x|} \text{ with } a\ge 1,\ b>0 \quad\text{and}\quad \rho_1(x)=\exp(1-x).$$

Indeed, we have that $\rho_0\in\mathbb{U}$ since $\rho_0\in L^1_{\text{loc}}(\mathbb{R})$ and $\rho_0(x)>0$ for all $x\in\mathbb{R}$; $\rho_0\in\mathbb{U}_\infty$ since $\lim_{r\to\infty}\mu(r,\rho_0)=\infty$; and also $\rho_0\in\mathbb{U}_B$ since $1<\rho_0(x)\le a$ for $x\in\mathbb{R}$. Similarly we can prove that $\rho_1\in\mathbb{U}_\infty-\mathbb{U}_B$.

Definition 2.1. Let $\rho_1,\rho_2\in\mathbb{U}_\infty$. Then, we say that ρ_1 is equivalent to ρ_2 if $\rho_1/\rho_2\in\mathbb{U}_B$. The equivalence of ρ_1 and ρ_2 is denoted by $\rho_1\sim\rho_2$.

We note that $\rho_1\sim\rho_2$ if and only if there exist $\alpha_i>0$ for $i=1,2$ such that $\alpha_1\rho_2\le\rho_1\le\alpha_2\rho_2$ and it implies $\alpha_1\mu(r,\rho_2)\le\mu(r,\rho_1)\le\alpha_2\mu(r,\rho_2)$. Hence $\sim$ is a binary equivalence relation on $\mathbb{U}_\infty$. Thus the equivalence class of a given weight $\rho\in\mathbb{U}_\infty$ will then be denoted by $cl(\rho)$ and is naturally defined as follows

$$cl(\rho) = \Big\{\bar\omega\in\mathbb{U}_\infty \ :\ \bar\omega\sim\rho\Big\}.$$

It is then clear that $\mathbb{U}_\infty=\cup_{\rho\in\mathbb{U}_\infty}cl(\rho)$. Moreover, this notion of equivalence implies the identification of some weighted pseudo almost periodic spaces, see Theorem 2.3 below.

We also have the monotony property

$$\rho(t+s) \le C_s \rho(t) \text{ implies that } \mu(t+s,\rho) \le C_s \mu(t,\rho), \tag{6}$$

for some positive constant C_s and for each $s \ge 0$, since

$$\mu(r+s,\rho) = \int_{-r-s}^{r+s} \rho(t)dt = \int_{-r}^{r} \rho(t+s)dt \le C_s \int_{-r}^{r} \rho(t)dt = C_s \mu(r,\rho).$$

2.2. Weighted pseudo almost periodic functions definition

In order to introduce the weighted pseudo almost periodic functions, we firstly need to define the "weighted ergodic" space $PAP_0(\mathbb{X},\rho)$. Then, the weighted pseudo almost periodic functions appear as perturbations of almost periodic functions by elements of $PAP_0(\mathbb{X},\rho)$ (see Definition 2.2). Indeed, firstly we introduce some notation and then we precise the definition of $PAP_0(\mathbb{X},\rho)$. Let us recall that the weighted mean of g is denoted by $\mathcal{M}(g)$ and is defined by the following limit

$$\mathcal{M}(g) := \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} g(s)\ \rho(s)ds,$$

when this limit exists (see (5) for μ notation). It is well known that in general the weighted mean does not exist for any pseudo almost periodic function or equivalently, there exists $g \in PAP(\mathbb{X})$ such that $\mathcal{M}(g)$ may not exist, see for instance J. Liang et al. [34]. Now, let us assume that $\rho \in \mathbb{U}_\infty$, then we define the weighted ergodic space associated to ρ as follows

$$PAP_0(\mathbb{X},\rho) := \Big\{ f \in BC(\mathbb{R},\mathbb{X}) \ :\ \mathcal{M}(\|f\|_{\mathbb{X}}) = 0 \Big\}. \tag{7}$$

Sometimes the space $PAP_0(\mathbb{X},\rho)$ is referenced as the $\rho - PAP_0(\mathbb{X})$ space. We note that by considering $\rho = 1$, we recover the so-called ergodic space of Zhang, that is, $PAP_0(\mathbb{X},1)$ is the ergodic space of Zhang which is briefly denoted by $PAP_0(\mathbb{X})$. Moreover, we observe that for several $\rho \in \mathbb{U}_\infty$, the spaces $PAP_0(\mathbb{X},\rho)$ are richer than $PAP_0(\mathbb{X})$ and, naturally, gives rise to an enlarged space of pseudo almost periodic functions. Furthermore, analogously to (7), we define $PAP_0(\mathbb{Y},\mathbb{X},\rho)$ as the collection of jointly continuous function $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ such that $F(\cdot,y)$ is bounded for each $y \in \mathbb{Y}$ and $\mathcal{M}(\|F(\cdot,y)\|) = 0$ uniformly in $y \in \mathbb{Y}$. Now, we are ready to recall the definition of weighted pseudo almost periodic functions.

Definition 2.2. Let $\rho \in \mathbb{U}_\infty$. Then, we have that

(a) A function $f \in BC(\mathbb{R},\mathbb{X})$ is called "ρ-pseudo almost periodic" or briefly "weighted pseudo almost periodic" if it can be expressed as follows $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X},\rho)$. The collection of such kind of functions will be denoted by $PAP(\mathbb{X},\rho)$.
(b) A function $F \in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is called "ρ-pseudo almost periodic" or briefly "weighted pseudo almost periodic" in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if it can be expressed as $F = G + \Phi$, where $G \in AP(\mathbb{Y},\mathbb{X})$ and $\Phi \in PAP_0(\mathbb{Y},\mathbb{X},\rho)$. The collection of such functions will be denoted by $PAP(\mathbb{Y},\mathbb{X},\rho)$.

Remark 2.1. The definition of mean carries implicitly a measure μ absolutely continuous with respect to the Lebesgue measure and its Radom–Nikodym derivative is ρ, since $d\mu(t) = \rho(t)dt$. Then, the results of the paper can be extended to more general and recent concepts for weighted pseudo-almost periodic functions in the context of measure theory [4,5].

Example 2.1. Here we introduce some examples of weighted pseudo almost periodic functions.

(a) Let $\rho_0(x) = 1 + x^2$ for each $x \in \mathbb{R}$. We note that $\rho_0 \in \mathbb{U}_B$, since $\mu(r, \rho_0) = 2(r + r^3/3)$. Now, we define the function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \sin(x) + \sin(\sqrt{2}x) + \phi(x) \text{ with } \phi \text{ such that } \phi\rho_0 \text{ is bounded.}$$

Clearly, f belongs to $PAP(\mathbb{R}, \rho_0)$. Namely, $\sin(x) + \sin(\sqrt{2}x)$ is its almost periodic component, while $\phi \in PAP_0(\mathbb{R}, \rho_0)$ since

$$\mathcal{M}(\phi) = \lim_{r\to\infty} \frac{1}{2(r + r^3/3)} \int_{-r}^{r} |\phi(x)|(1 + x^2)dx = 0.$$

Moreover, $\phi \in PAP(\mathbb{R}, \hat{\rho})$ if $\phi\rho_0 \le \hat{\rho}$, with $\mu(r, \hat{\rho})[\mu(r, \rho_0)]^{-1} \to 0$ as $r \to \infty$. This assertion is valid, for example, with $\hat{\rho}(x) = 1 + |x|$.

(b) Let $\rho_1(x) = |x|^d$, $d \in \mathbb{N}$ for each $x \in \mathbb{R}$. Clearly $\rho_1 \in \mathbb{U}_\infty - \mathbb{U}_B$ since $\mu(r, \rho_1) = 2r^{d+1}/(d + 1)$. Now, we define the function $f : \mathbb{R} \to \mathbb{R}$ as follows

$$f(x) = \sin(x) + \sin(\sqrt{2}x) + \phi(x) \text{ with } \phi\rho_1 \in L^\infty(\mathbb{R}).$$

We note that f belongs to $PAP(\mathbb{R}, \rho_1)$. Indeed, $\sin(x) + \sin(\sqrt{2}x) \in AP(\mathbb{R})$ and $\phi \in PAP_0(\mathbb{R}, \rho_1)$ since

$$\mathcal{M}(\phi) = \lim_{r\to\infty} \frac{d + 1}{2r^{d+1}} \int_{-r}^{r} |\phi(x)||x|^d dx = 0.$$

(c) In this example, we present two weights which are equivalent. Let $\rho_1(x) = 2 + \sinh(|x|)$ and $\rho_2(x) = 1 + \cosh(|x|)$ for each $x \in \mathbb{R}$. It can be easily seen that $\rho_1, \rho_2 \in \mathbb{U}_\infty$ and that

$$\frac{\rho_1(x)}{\rho_2(x)} = \frac{2 + \sinh(|x|)}{1 + \cosh(|x|)} = \frac{4 + e^{|x|} - e^{-|x|}}{2 + e^{|x|} + e^{-|x|}} \in \mathbb{U}_B.$$

Then $\rho_1 \sim \rho_2$. □

2.3. Some results for weighted pseudo almost periodic functions

In this section we list four useful results. First, we recall a result of J. Liang et al. [34] about the non-uniqueness of the decomposition given in Definition 2.2 when $\rho \in \mathbb{U}_\infty$, see Lemma 2.1. Second, we remember that $PAP(\mathbb{X}, \rho)$ has a natural structure of Banach space with the sup norm when $\rho \in \mathbb{U}_B$ (see [17,35,46]). However, it is also known that, the completeness of $PAP(\mathbb{X}, \rho)$ in the sup norm topology and when $\rho \in \mathbb{U}_\infty - \mathbb{U}_B$ is not a trivial problem, see Theorem 2.2. Third, we have a result related to the equivalence relation, see Theorem 2.3. Fourth, we recall a composition theorem of weighted pseudo almost periodic functions, which will be important in the study of weighted pseudo almost periodic solution of differential equations, see Theorem 2.5. Moreover, we present two immediate consequences of Theorem 2.5, see Corollaries 2.6 and 2.7.

Lemma 2.1. *Fix $\rho \in \mathbb{U}_\infty$. The decomposition of a ρ-pseudo almost periodic function $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$ is not unique.*

We recall that the proof of this Lemma was given by Liang and collaborators in [34] through the construction of some examples. Moreover, by the application of Lemma 2.1, we observe that when $\rho \in \mathbb{U}_\infty$ the space $PAP(\mathbb{X}, \rho)$ cannot be always decomposed as a direct sum of $AP(\mathbb{X})$ and $PAP_0(\mathbb{X}, \rho)$. The immediate consequence of this fact is that we do not know precisely the subset of $\mathbb{U}_\infty - \mathbb{U}_B$ such that $PAP(\mathbb{X}, \rho)$ is a Banach space with the induced norm $\|f\|_\infty = \|g\|_\infty + \|\phi\|_\infty$ although $AP(\mathbb{X})$ and $PAP_0(\mathbb{X}, \rho)$ are closed linear subspaces of $BC(\mathbb{R}, \mathbb{X})$. Then, in order to overcome this troublesome situation Zhang and collaborators in [45] introduce the following norm

$$\|f\|_\rho = \inf_{i\in I} \left(\sup_{t\in\mathbb{R}} \|g_i(t)\|_{\mathbb{X}} + \sup_{t\in\mathbb{R}} \|\phi_i(t)\|_{\mathbb{X}} \right) \tag{8}$$

and prove the completeness of $\big(PAP(\mathbb{X}, \rho), \|\cdot\|_\rho\big)$. Here $\{g_i + \phi_i,\ i \in I\}$ denotes all possible decomposition of $f \in PAP(\mathbb{X}, \rho)$. To be more precise the result of completeness of $PAP(\mathbb{X}, \rho)$ is given by the following theorem (see also Corollary 3.9)

Theorem 2.2. *Fix $\rho \in \mathbb{U}_\infty$. $PAP(\mathbb{X}, \rho)$ is a Banach space with the norm $||\cdot||_\rho$ defined on (8).*

In order to characterize the Zhang's space $PAP(\mathbb{X})$ in terms of the new space $PAP(\mathbb{X}, \rho)$, Diagana [17] (see also [17–19,21,35]) introduce and analyze the equivalence relation given on Definition 2.2-(c). Moreover, he gives a generalization of the well known composition results for the pseudo almost periodic functions.

Theorem 2.3. *Let $\rho \in \mathbb{U}_\infty$. If $\rho_1, \rho_2 \in cl(\rho)$, then*

(a) *$PAP(\mathbb{X}, \rho_1 + \rho_2) = PAP(\mathbb{X}, \rho_1) = PAP(\mathbb{X}, \rho_2)$, and*
(b) *$PAP(\mathbb{X}, \rho_1/\rho_2) = PAP(\mathbb{X}, cl(1)) = PAP(\mathbb{X})$.*

Corollary 2.4. *Let $\rho \in \mathbb{U}_B$. Then, $PAP(\mathbb{X}, \rho) = AP(\mathbb{X}) \oplus PAP_0(\mathbb{X}, \rho)$ and $(PAP(\mathbb{X}, \rho), ||\cdot||_\infty)$ is a Banach space equivalent to the spaces $(PAP(\mathbb{X}, \rho), ||\cdot||_\rho)$ and $(PAP(\mathbb{X}, 1), ||\cdot||_1)$.*

Theorem 2.5. *(See [17–19,21].) Let $\rho \in \mathbb{U}_\infty$ and let $f \in PAP(\mathbb{Y}, \mathbb{X}, \rho)$ satisfying the Lipschitz condition*

$$\|f(t, u) - f(t, v)\|_{\mathbb{X}} \le L\|u - v\|_{\mathbb{Y}} \text{ for all } u, v \in \mathbb{Y}, t \in \mathbb{R}. \tag{9}$$

If $g \in PAP(\mathbb{Y}, cl(\rho))$, then $f(\cdot, g(\cdot)) \in PAP(\mathbb{X}, cl(\rho))$.

Corollary 2.6. *Let $\rho_1, \rho_2 \in \mathbb{U}_\infty$ with $\rho_2 \in cl(\rho_1)$. Let $f \in PAP(\mathbb{Y}, \mathbb{X}, cl(\rho_1))$ satisfying the Lipschitz's condition (9). If $g \in PAP(\mathbb{Y}, \rho_2)$, then $f(\cdot, g(\cdot)) \in PAP(\mathbb{X}, cl(\rho_1))$.*

Corollary 2.7. *Let $\rho \in \mathbb{U}_B$. Let $f \in PAP(\mathbb{Y}, \mathbb{X}, cl(\rho))$ satisfying the Lipschitz's condition (9). If $g \in PAP(\mathbb{Y}, \rho)$, then $f(\cdot, g(\cdot)) \in PAP(\mathbb{X})$.*

Here, we introduce two comments related with Theorem 2.3. Firstly, we note that the Theorem 2.3 enables us to identify the Zhang's space $PAP(\mathbb{X})$ with a weighted pseudo almost periodic class $PAP(\mathbb{X}, \rho)$. Indeed, if $\rho \in \mathbb{U}_B$, then $PAP(\mathbb{X}, \rho) = PAP(\mathbb{X}, cl(1)) = PAP(\mathbb{X})$. Secondly, by considering ρ_1 and ρ_2 as given on Example 2.1-(c) an application of Theorem 2.3 implies that $PAP(\mathbb{X}, \rho_1) = PAP(\mathbb{X}, \rho_2) = PAP(\mathbb{X}, \rho_1 + \rho_2)$ and $PAP(\mathbb{X}, cl(1)) = PAP(\mathbb{R}, \rho_1/\rho_2)$.

3. The convolution and some consequences

In this section we focus on the following question: When the convolution operator $\mathcal{K}$, defined as follows

$$(\mathcal{K}f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)ds \quad \text{for a given } k \in L^1(\mathbb{R}), \tag{10}$$

maps $PAP_0(\mathbb{X}, \rho)$ into itself? In the best of our knowledge, there are some isolated results for this problem, but there is not yet a systematic study, see for instance [3,9,10,31,40]. Here, we recall that an effective way to construct weighted pseudo almost periodic functions is through the convolution operator. Indeed Diagana [17–19,21] proved the following result

Proposition 3.1. *Fix $\rho \in \mathbb{U}_B$. Let $f \in PAP_0(\mathbb{X}, \rho)$ and $k \in L^1(\mathbb{R})$. Then $\mathcal{K}f = f * k$, the convolution of f and k on $\mathbb{R}$, belongs to $PAP_0(\mathbb{X}, \rho)$.*

However, the restriction $\rho \in \mathbb{U}_B$ reduces $PAP_0(\mathbb{X}, \rho)$ to $PAP_0(\mathbb{X}, 1)$ the ergodic space of Zhang (see Theorem 2.3). In order to overcome this problem, in this section we assume that $\rho \in \mathbb{U}_\infty$. To be more precise we assume that

$$\text{Fix } \rho \in \mathbb{U}_\infty \text{ and } k \in L^1(\mathbb{R}). \tag{11}$$

Now, under the general hypothesis (11), we obtain four main results related to the general question which proves that $f \in PAP_0(\mathbb{X}, \rho)$ implies that $\mathcal{K}f \in PAP_0(\mathbb{X}, \rho)$. In a broad sense, we get the results by three types of additional conditions:

(a) The first two results (see Theorems 3.2 and 3.3) are obtained by requiring explicit conditions between k and ρ.
(b) The third result (see Theorem 3.4) is deduced by imposing conditions on k, ρ and f.
(c) The fourth result (see Theorem 3.5) is a result where we assume conditions for the weight ρ.

Then, to have that the convolution $(\mathcal{K}f)$ is $PAP_0(\mathbb{X}, \rho)$, first we use Theorem 3.5. If this does not work we check Theorems 3.2 and 3.3, where k helps to ρ. Finally, we prove with Theorem 3.4, where f helps to k and ρ, see Liang et al. [34].

Moreover, we obtain a fifth result (see Theorem 3.7) related to the translation invariance property of the space $PAP_0(\mathbb{X}, \rho)$.

3.1. Convolution invariance of $PAP_0(\mathbb{X}, \rho)$

Theorem 3.2. *Consider that the condition (11) is satisfied and $\mathcal{K}$ is defined by (10). Assume that ρ and k satisfy the following requirements:*

$$\left.\begin{aligned} &\sup_{|s| \le r, r \in \mathbb{R}_+} \frac{1}{\rho(s)} \int_s^r |k(t-s)|\rho(t)dt < \infty, \\ &\sup_{|s| \le r, r \in \mathbb{R}_+} \frac{1}{\rho(s)} \int_{-r}^s |k(t-s)|\rho(t)dt < \infty, \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} &\lim_{r \to \infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} ds \int_{-r}^{r} |k(t-s)|\rho(t)dt = 0, \\ &\lim_{r \to \infty} \frac{1}{\mu(r,\rho)} \int_{r}^{+\infty} ds \int_{-r}^{r} |k(t-s)|\rho(t)dt = 0. \end{aligned}\right\} \tag{13}$$

Then $f \in PAP_0(\mathbb{X}, \rho)$ implies $\mathcal{K}f \in PAP_0(\mathbb{X}, \rho)$.

Proof. By the properties of convolution we have that $f \in BC(\mathbb{X}, \mathbb{R})$ implies that $f * k \in BC(\mathbb{R}, \mathbb{X})$ for $k \in L^1(\mathbb{R})$. Then, in order to get that $f * k \in PAP_0(\mathbb{X}, \rho)$ we must to prove that $\mathcal{M}(\|f * k\|_{\mathbb{X}}) = 0$. Indeed, we proceed in two stages. Firstly, we assume that $k(t - s) = 0$ for $t > s$. Then, by applying the Fubini Theorem, we deduce that

$$
\begin{aligned}
|\mathcal{M}(\|f * k\|_{\mathbb{X}})| &= \left| \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \big\|(f * k)(t)\big\|_{\mathbb{X}} dt \right| \\
&\le \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \int_{-\infty}^{t} \|f(s)\|_{\mathbb{X}} |k(t-s)| dt ds \\
&\le \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \rho(t) \left(\int_{-\infty}^{-r} \|f(s)\|_{\mathbb{X}} |k(t-s)| ds \right. \\
&\qquad \left. + \int_{-r}^{t} \|f(s)\|_{\mathbb{X}} |k(t-s)| ds \right) dt \\
&= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \|f(s)\|_{\mathbb{X}} \int_{-r}^{r} |k(t-s)| \rho(t) dt ds \\
&\quad + \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(s)\|_{\mathbb{X}} \int_{s}^{r} |k(t-s)| \rho(t) dt ds \\
&= \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-\infty}^{-r} \|f(s)\|_{\mathbb{X}} \int_{-r}^{r} |k(t-s)| \rho(t) dt ds \\
&\quad + \lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(s)\|_{\mathbb{X}} \rho(s) \left[\frac{1}{\rho(s)} \int_{s}^{r} |k(t-s)| \rho(t) dt \right] ds.
\end{aligned} \tag{14}
$$

Now, we note that these two limits tend to zero. Indeed, we deduce that the first limit vanishes by using the fact that $f \in BC(\mathbb{R}, \mathbb{X})$ and the first limit of the hypothesis (13). Meanwhile, the deduction that the second limit vanishes is proved by application of the first part of the condition (12) and the fact that $\mathcal{M}(\|f\|_{\mathbb{X}}) = 0$. This concludes the first stage, since we have that $\mathcal{M}(\|f * k\|_{\mathbb{X}}) = 0$ for $k \in L^1(\mathbb{R})$ such that $k(t - s) = 0$ for $t > s$. Now, in the case of a general k we deduce the result similarly using the fact that $\int_{-\infty}^{\infty} = \int_{-\infty}^{t} + \int_{t}^{+\infty}$ and the second parts of the hypothesis (12)–(13) to estimate the terms corresponding to $\int_{t}^{+\infty}$. □

Theorem 3.3. *The result in Theorem 3.2 is true if (13) is replaced by*

$$
\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \left[\left(\int_{-\infty}^{t-r} + \int_{t+r}^{+\infty} \right) |k(s)| ds \right] \rho(t) dt = 0. \tag{15}
$$

In particular, Theorem 3.2 is valid, if (13) is replaced by

$$
\sup_{r\in\mathbb{R}^+} \int_{-r}^{r} \left(\int_{-\infty}^{t-r} + \int_{t+r}^{\infty} \right) |k(s)| ds \rho(t) dt < \infty. \tag{16}
$$

Proof. We follow the proof by applying similar arguments to the proof of Theorem 3.2. Mainly, we note that the hypothesis (13) and (15) are equivalent. This fact can be proved by using the identities

$$\int_{-r}^{r} \rho(t) \int_{-\infty}^{-r} |k(t-s)|dsdt = \int_{-r}^{r} \rho(t) \left(\int_{t+r}^{\infty} |k(s)|ds \right) dt \tag{17}$$

$$\int_{-r}^{r} \rho(t) \int_{r}^{\infty} |k(t-s)|dsdt = \int_{-r}^{r} \rho(t) \left(\int_{-\infty}^{t-r} |k(s)|ds \right) dt. \tag{18}$$

More precisely, if in (14) we use (17)–(18) instead of the Fubini Theorem, we can follow the proof of the theorem by using the facts that $f \in PAP_0(\mathbb{X}, \rho)$ and the pair (ρ, k) satisfies the hypothesis (12) and (15). □

Example 3.1. Let us consider $h_\alpha(t) = e^{-\alpha t}$ and $\rho_\sigma(t) = e^{\sigma t}$ with $\alpha_0 := \alpha - \sigma > 0$. We note that

$$\begin{aligned}\frac{1}{\rho(s)} \int_{s}^{r} k_\alpha(t-s)\rho_\sigma(t)dt &= \frac{1}{e^{\sigma s}} \int_{s}^{r} e^{-\alpha(t-s)} e^{\sigma t} dt = e^{\alpha_0 s} \int_{s}^{r} e^{-\alpha_0 t} dt \\ &= e^{\alpha_0 s} \left[\frac{-1}{\alpha_0} (e^{-\alpha_0 r} - e^{-\alpha_0 s}) \right] < 1,\end{aligned}$$

which implies that (k_α, ρ_σ) satisfies the conditions in Theorems 3.2 and 3.3. □

Here, we recall that Diagana [17,19] considers the condition

$$\sup_{r>0} \left\{ \int_{-r}^{r} e^{-\alpha(r+t)} \rho(t)dt \right\} < \infty,$$

which corresponds to the particular case $k(t) = e^{-\alpha t}$ of (16) in Theorem 3.3, but he does not consider other condition and does not the convolution operator either. Moreover, we have two observations. First, the convolution $e^{-\alpha \cdot} * f$ is not necessarily in $PAP_0(\mathbb{X}, \rho)$ as is shown by $\rho(t) = |t|^\beta e^{\alpha t}$, see examples 2.1, 2.2 in Liang et al. [34]. Second, let $\varepsilon > 0$, our results allow consider $(e^{-(2+\varepsilon)t}, |t|e^{2t})$-convolutions.

Theorem 3.4. *Consider that the condition (11) is satisfied. Assume that ρ, k and $f \in PAP(\mathbb{X}, \rho)$ such that*

$$\exists \phi : \mathbb{R} \to \mathbb{R}_+ \quad : \quad k\phi \in L^1(\mathbb{R}) \quad \textit{and} \quad \sup_{|u+s| \le r, r \in \mathbb{R}^+, s \in \mathbb{R}} \frac{\rho(u+s)}{\rho(u)\phi(s)} < \infty, \tag{19}$$

$$\lim_{r \to \infty} \frac{1}{\mu(r, \rho)} \left(\int_{-r-|s|}^{-r} + \int_{r}^{r+|s|} \right) \|f(\tau)\|_{\mathbb{X}} \, \rho(\tau)d\tau = 0 \quad \textit{for all } s \in \mathbb{R}. \tag{20}$$

Then the (k, ρ)-convolution of f is in $PAP_0(\mathbb{X}, \rho)$.

Proof. Clearly $k * f \in BC(\mathbb{X}, \mathbb{R})$ since $f \in PAP(\mathbb{X}, \rho)$ and $k \in L^1(\mathbb{R})$. Now, by Fubini Theorem, we get

$$\begin{aligned}\mathcal{M}(\|k * f\|_{\mathbb{X}}) &= \lim_{r \to \infty} \frac{1}{\mu(r, \rho)} \int_{-r}^{r} \|(k * f)(t)\|_{\mathbb{X}} \rho(t)dt \\ &= \lim_{r \to \infty} \frac{1}{\mu(r, \rho)} \int_{-\infty}^{\infty} |k(s)| \int_{-r}^{r} \|f(t-s)\|_{\mathbb{X}} \rho(t)dtds\end{aligned}$$

$$= \lim_{r\to\infty} \int_{-\infty}^{\infty} |k(s)| \left[\frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(t-s)\|_{\mathbb{X}} \rho(t) dt \right] ds$$

$$= \lim_{r\to\infty} \int_{-\infty}^{\infty} f_{r,\rho}(s)|k(s)|ds, \tag{21}$$

where $f_{r,\rho}$ is defined by the term between the brackets $[\ldots]$. We note that $f_{r,\rho}$ has the following properties

(f$_1$) By definitions of $f_{r,\rho}$ and μ we can deduce that $|f_{r,\rho}(s)| \leq \|f\|_\infty$, which implies that $\|f_{r,\rho}k\|_{L^1(\mathbb{R})} \leq \|f\|_\infty \|k\|_{L^1(\mathbb{R})}$ for all $r \in \mathbb{R}_+$.

(f$_2$) By a change of variable and hypothesis (19) we have that there exists a positive constant C such that the following bound holds

$$\begin{aligned} f_{r,\rho}(s) &= \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(t-s)\|_{\mathbb{X}} \rho(t) dt \\ &= \frac{1}{\mu(r,\rho)} \int_{-r-s}^{r-s} \|f(u)\|_{\mathbb{X}} \rho(u+s) du \\ &\leq C \frac{1}{\mu(r,\rho)} \int_{-r-|s|}^{r+|s|} \|f(u)\|_{\mathbb{X}} \rho(u)\phi(s) du \\ &= C\phi(s) \left[\frac{1}{\mu(r,\rho)} \int_{-r-|s|}^{r+|s|} \|f(u)\|_{\mathbb{X}} \rho(u) du \right]. \end{aligned}$$

Now, noticing that $f_{r,\rho} \geq 0$, the application of condition (20) implies that $f_{r,\rho}(s) \to 0$ when $r \to \infty$ for all $s \in \mathbb{R}$. Then, naturally $f_{r,\rho}(s)|k(s)| \to 0$ when $r \to \infty$ for all $s \in \mathbb{R}$.

(f$_3$) By definition of the limit given on hypothesis (20) we have that for all $\epsilon > 0$ there exists $N > 0$ such that $r > N$ implies that

$$\frac{1}{\mu(r,\rho)} \int_{-r-|s|}^{r+|s|} \|f(u)\|_{\mathbb{X}} \rho(u) du \leq \epsilon.$$

Then, the bound deduced in (f$_2$) implies that $f_{r,\rho}(s) \leq C\,\epsilon\,\phi(s)$ for all $s \in \mathbb{R}$ and $r > N$. Thus, using additionally the fact that $k\phi \in L^1(\mathbb{R})$ by the assumption (19), we deduce that there exists $g(s) = C\,\epsilon\,\phi(s)|k(s)| \in L^1(\mathbb{R})$ such that $f_{r,\rho}(s)|k(s)| \leq g(s)$ for all $s \in \mathbb{R}$.

From the properties (f$_1$)–(f$_3$), the Lebesgue dominated convergence theorem and (21) we follow that $\mathcal{M}(\|k * f\|_{\mathbb{X}}) = 0$. Hence $k * f \in PAP_0(\mathbb{X},\rho)$. $\square$

Theorem 3.5. *Consider that the condition (11) is satisfied and $\mathcal{K}$ is defined by (10). Assume that ρ is such that for each $s \in \mathbb{R}$ the inequality*

$$\sup_{r\in\mathbb{R}_+} \sup_{t\in\Omega_{r,s}} \frac{\rho(t+s)}{\rho(t)} < \infty, \quad \Omega_{r,s} = \Big\{ t \in \mathbb{R} \,:\, |t| < |s| + r \Big\},\ r \in \mathbb{R}_+, \tag{22}$$

holds. Then $f \in PAP_0(\mathbb{X},\rho)$ implies $\mathcal{K}f \in PAP_0(\mathbb{X},\rho)$.

Proof. Denote by C_s the supremum given in (22). By the definition of $C_{|s|}$, $\rho(t+|s|) \le C_{|s|}\rho(t)$ for $|t| < |s|+r$. By (6), we deduce that the following inequality holds

$$\mu(r+|s|,\rho) \le C_{|s|}\ \mu(r,\rho), \quad \text{for each } (s,r) \in \mathbb{R}\times\mathbb{R}_+. \tag{23}$$

Now the proof continues using the ideas and the same notation as in the proof of Theorem 3.4. Indeed, by the estimate (23), the hypothesis (22), and the fact that $f \in PAP_0(\mathbb{X},\rho)$, we deduce that

$$\begin{aligned}
0 \le f_{r,\rho}(s) &= \frac{1}{\mu(r,\rho)} \int_{-r-s}^{r-s} \|f(t)\|_{\mathbb{X}}\rho(t+s)dt \\
&\le \frac{1}{\mu(r,\rho)} \int_{-r-|s|}^{r+|s|} \|f(t)\|_{\mathbb{X}}\rho(t+s)dt \\
&\le \frac{\mu(r+|s|,\rho)}{\mu(r,\rho)} \frac{1}{\mu(r+|s|,\rho)} \int_{-r-|s|}^{r+|s|} \|f(t)\|_{\mathbb{X}}\rho(t+s)dt \\
&\le C_{|s|} \frac{1}{\mu(r+|s|,\rho)} C_s \int_{-r-|s|}^{r+|s|} \|f(t)\|_{\mathbb{X}}\rho(t)dt.
\end{aligned}$$

Then

$$\begin{aligned}
0 \le \lim_{r\to\infty} f_{r,\rho}(s) &\le C_{|s|}C_s \lim_{r\to\infty} \frac{1}{\mu(r+|s|,\rho)} \int_{-r-|s|}^{r+|s|} \|f(t)\|_{\mathbb{X}}\rho(t)dt \\
&= C_{|s|}C_s\mathcal{M}(\|f\|_{\mathbb{X}}) = 0,
\end{aligned}$$

i.e. $f_{r,\rho}(s) \to 0$ when $r \to \infty$. Thus the property (f_2) is again valid. Meanwhile, to prove (f_1) and (f_3) we proceed similarly. Then, the proof finishes as in Theorem 3.4, i.e. by application of Lebesgue dominated convergence theorem. □

Example 3.2. In this example, we introduce some possible functions where the hypotheses given on (19), (20) or (22) are valid. First, conditions (19) or (22) are satisfied with $\phi = \rho$ if ρ is sub-multiplicative, i.e. there exists $c_s > 0$ such that $\rho(s+t) \le c_s\rho(s)\rho(t)$ for all $t, s \in \mathbb{R}$. Now, condition (20) follows if f and ρ satisfy at least one of the following requirements

$$f\rho \in L^p(\mathbb{R}) \text{ for some } p \text{ such that } 1 \le p \le \infty, \text{ or}$$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \left(\int_{-r-|s|}^{-r} \rho(t)dt + \int_{r}^{r+|s|} \rho(t)dt \right) = 0.$$

Moreover, the condition (20) in Theorem 3.4 is related with the condition $f \in PAP_0(\mathbb{X},\rho)$ "enlarged", see Liang et al. [34]. Meanwhile, the condition (22) is easy to verify when ρ is sub-multiplicative or for instance, when ρ is even, $\|\rho\|_{L^1(0,t)} \to \infty$ when $t \to \infty$ and the limit $\lim_{t\to\infty} \rho(t+|s|)[\rho(t)]^{-1}$ exists. □

Here, we comment two important facts. First, note that the bound in Theorem 3.5 is not uniform in $s \in \mathbb{R}$. Then the limit of the mean is not uniform. However, these conditions are satisfied by a wide range of weight functions. For instance if ρ verifies the sub-multiplicativity condition (see Example 3.2). A function of this type is given by $\rho(t) = (1 + |t|^{\beta})e^{\alpha t}$ for $\alpha \geq 0$ and $\beta \geq 0$. Second, more particular conditions of type (22) have been obtained previously by Agarwal et al. [1].

Example 3.3. Consider the weight defined by $\rho(x) = (1 + |x|^{\beta})e^{\alpha x}$ for each $x \in \mathbb{R}$. Clearly, $\rho \in \mathbb{U}_{\infty} - \mathbb{U}_B$. Set $k(x) = e^{-x^2}$ and $f(x) = \sin(x) + \sin(\sqrt{2}x) + e^{-|x|}$. It is easily seen that f belongs to $PAP(\mathbb{R}, \rho)$ and satisfies (22). Then,

$$\begin{aligned}(\mathcal{K}f)(u) &= (f * k)(u) \\ &= \int_{-\infty}^{\infty} \left(\sin(u-r) + \sin(\sqrt{2}(u-r)) + e^{-|u-r|}\right) e^{-r^2} dr\end{aligned}$$

is in $PAP(\mathbb{R}, \rho)$, by application of Theorem 3.5. $\square$

3.2. Some consequences of convolution invariance of $PAP_0(\mathbb{X}, \rho)$

In this subsection we introduce some consequences of the fourth Theorems 3.2–3.5. Indeed, let us start by recalling that when $f \in AP(\mathbb{X})$ the standard mean satisfies the translation invariance property: $\mathcal{M}(f) = \mathcal{M}(f_\xi)$ for any $\xi \in \mathbb{R}$ with $\rho = 1$. Here f_ξ denotes a ξ-translation of f, i.e. $f_\xi(t) = f(\xi + t)$ for all $t \in \mathbb{R}$. Therefore, the following question naturally appears: is the translation invariance property valid for the weighted mean when $f \in BC(\mathbb{R}, \mathbb{X})$ and $\rho \in \mathbb{U}$ are arbitrary selected? The answer to this question was focused by Ji and Zhang in [33]. In particular, they prove the following result

Theorem 3.6. *Consider the notation*

$$\begin{aligned}\mathbb{U}^0_{\infty} &= \left\{\rho \in \mathbb{U}_{\infty} \ : \ \lim_{r\to\infty} \frac{\mu(r+\tau, \rho)}{\mu(r, \rho)} = 1, \ \forall \tau \in \mathbb{R}\right\}, \\ a(\lambda, f) &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} f(t)e^{-i\lambda t} dt, \\ \sigma_b(f) &= \{\lambda \in \mathbb{R} \ : \ a(\lambda, f) \neq 0\}.\end{aligned}$$

Suppose that $\rho \in \mathbb{U}^0_{\infty}$ and $f : \mathbb{R} \to \mathbb{X}$ an almost periodic function such that

$$\lim_{r\to\infty} \left| \frac{1}{\mu(r, \rho)} \int_{-r}^{r} \rho(s)e^{-i\lambda s} ds \right| = 0$$

for all $\lambda \in \sigma_b(f)\backslash\{0\}$. Then a translation invariance of the following type

$$\begin{aligned}\lim_{r\to\infty} \frac{1}{\mu(r, \rho)} \int_{-r}^{r} (f\rho)_\xi(s) ds &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} f_\xi(s) ds \\ &= \lim_{r\to\infty} \frac{1}{2r} \int_{-r+\xi}^{r+\xi} f(s) ds, \quad \forall \xi \in \mathbb{R},\end{aligned}$$

is satisfied.

Moreover, Ji and Zhang assuming that $PAP_0(\mathbb{X},\rho)$ is translation invariant for $\rho \in \mathbb{U}_\infty$ prove that $PAP_0(\mathbb{X},\rho)$ is convolution invariant.

On the other hand, Ji and Zhang [33] note that the problem of existence and the ergodicity property of the weighted mean for almost periodic functions are involved in a systemic answer to the translation invariance property of $PAP_0(\mathbb{X},\rho)$. In the case of the existence of the weighted mean for almost periodic functions, by applying the Theorem 2.3 of [25], the authors of [33] prove two results (see Theorems 3.3 and 3.5 of [33]). Meanwhile, concerning the ergodicity, they introduce a result (see Theorems 3.11 of [33]).

In this subsection, we prove that $PAP_0(\mathbb{X},\rho)$ is convolution invariant under (22). More precisely, we have the following result.

Theorem 3.7. *Consider that the condition (11) is satisfied. Then, the following assertions are valid*

(a) *If the space $PAP_0(\mathbb{X},\rho)$ is translation invariant, then $PAP_0(\mathbb{X},\rho)$ is convolution invariant.*
(b) *If (22) holds, then the space $PAP_0(\mathbb{X},\rho)$ is translation invariant.*

Proof. The proof of item (a) follows by application of Theorem 2.4 of [33]. Now, we prove that $f \in PAP_0(\mathbb{X},\rho)$ implies that $f_s(\cdot) \in PAP_0(\mathbb{X},\rho)$. Indeed, let us consider that $f \in PAP_0(\mathbb{X},\rho)$, then by condition (22), we have that for any $s \in \mathbb{R}$

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} \|f(t-s)\|_{\mathbb{X}}\rho(t)dt = 0,$$

and the translation $f_s(\cdot) \in PAP_0(\mathbb{X},\rho)$. $\square$

Corollary 3.8. *Consider that the condition (11) is satisfied. Then, the space $PAP_0(\mathbb{X},\rho)$ is translation invariant, whether (22) is valid or even if we impose that one of the following two pairs of conditions hold: (12) and (13) or (12) and (15).*

Using Theorem 3.2 of Liang et al. [34] we have

Corollary 3.9. *If (12) and (13), or (12) and (15), or (22) hold, then the decomposition of $PAP(\mathbb{X},\rho)$ is unique. Furthermore, under the same hypothesis, $PAP(\mathbb{X},\rho)$ is a Banach space with the sup norm.*

4. Applications to abstract integral and differential equations

Consider the integral equation [37]

$$u(t) = \int_{-\infty}^{t} R(t,s)f(s,u(s))ds, \tag{24}$$

where f and R satisfy the following hypothesis

(H1) The function f belongs to $PAP(\mathbb{Y},\mathbb{X},\rho)$ (see Definition 2.2-(b)) and satisfying the Lipschitz condition (9).
(H2) The kernel R satisfies the inequality $\|R(t,s)\| \le Mk(t-s)$ for all $t \ge s$, some $k \in L_1([0,\infty))$ and some positive constant M. Moreover, $R(t,s)$ is k-bi-almost periodic [37], that is, for every $\varepsilon > 0$ there are $\ell_\varepsilon > 0$ and $c > 0$ such that each interval of length ℓ_ε contains τ for which the inequality

$$\|R(t+\tau, s+\tau) - R(t,s)\| \leq \varepsilon\, c\, k(t-s), \quad t \geq s,$$

is satisfied.

We note that, for an arbitrary almost periodic function $a > 0$, the kernel defined by the following relation $R(t,s) = \exp\left(-\int_s^t (\alpha + a(r))dr\right)$ satisfies the condition (H2) with $k(t) = \exp(-\alpha t)$ for each $t \geq 0$.

Theorem 4.1. *Consider $\rho \in \mathbb{U}_\infty$ satisfying (22) or the conditions in Theorems 3.2 and 3.3. Assume that (H1) and (H2) hold. Then, if $LM \|k\|_1 < 1$, the integral equation (24) has a unique $cl(\rho)$-pseudo almost periodic solution.*

Proof. Let us consider the operator

$$(\mathcal{N}u)(t) := \int_{-\infty}^{t} R(t,s) f(s, u(s)) ds.$$

Then the integral equation (24) can be rewritten equivalently as the operator equation of second kind $\mathcal{N}u = u$. Then the proof of the theorem follows by application of fixed point arguments. Indeed, we firstly obtain some estimates and then we specify the application of fixed point argument.

Let us consider $u \in PAP(\mathbb{X}, \rho)$. Then, by Definition 2.2, we can write u as follows $u = y + z$ with $y \in AP(\mathbb{X})$ and $z \in PAP_0(\mathbb{X}, \rho)$. Now, using the hypothesis (H1) and the composition result given in Theorem 2.5, it follows that $f(\cdot, u(\cdot)) \in PAP(\mathbb{X}, \rho)$ and naturally, by Definition 2.2, f can be expressed as follows

$$\begin{aligned}
f(\cdot, u(\cdot)) &= G(\cdot, u(\cdot)) + \Phi(\cdot, u(\cdot)), \\
G(\cdot, u(\cdot)) &= g(\cdot, y(\cdot)), \\
\Phi(\cdot, u(\cdot)) &= f(\cdot, u(\cdot)) - f(\cdot, y(\cdot)) + \phi(\cdot, y(\cdot)),
\end{aligned}$$

where $f = g + \phi$ with $g \in AP(\mathbb{Y}, \mathbb{X})$ and $\phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \rho)$. Then, we can split $\mathcal{N}$ in two operators $\mathcal{N}_1$ and $\mathcal{N}_2$, since we can rewrite $(\mathcal{N}u)$ as follows $(\mathcal{N}u) = (\mathcal{N}_1 u) + (\mathcal{N}_2 u)$, where for each $t \in \mathbb{R}$ the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ are defined by

$$(\mathcal{N}_1 u)(t) := \int_{-\infty}^{t} R(t,s) G(s, u(s)) ds,$$

$$(\mathcal{N}_2 u)(t) := \int_{-\infty}^{t} R(t,s) \Phi(s, u(s)) ds.$$

Moreover, we note $G(\cdot, u(\cdot)) \in AP(\mathbb{X})$ and $\Phi(\cdot, u(\cdot)) \in PAP_0(\mathbb{X}, \rho)$. Then, by the definition of the space $AP(\mathbb{X})$, the fact $G(\cdot, u(\cdot)) \in AP(\mathbb{X})$ implies that for each $\varepsilon > 0$, there exists $\ell(\varepsilon) > 0$ such that every interval of length $\ell(\varepsilon)$ contains a number τ such that

$$\left\| G\big(t+\tau, u(t+\tau)\big) - G\big(t, u(t)\big) \right\| < \frac{\varepsilon}{M \|k\|_1} \quad \text{for all } t \in \mathbb{R}.$$

Now, using the assumption (H2) it follows that $||(\mathcal{N}_1 u)(t+\tau) - (\mathcal{N}_1 u)(t)|| < \varepsilon$ for all $t \in \mathbb{R}$. Hence $\mathcal{N}_1 u \in AP(X)$. The next step consists of showing that

$$\lim_{r\to\infty} \frac{1}{\mu(r,\rho)} \int_{-r}^{r} ||(\mathcal{N}_2 u)(t)||\rho(t)dt = 0. \tag{25}$$

Indeed, from hypothesis (H2) we deduce that

$$\|(\mathcal{N}_2 u)(t)\| \leq \int_{-\infty}^{t} Mh(t-s)\|\Phi(s,u(s))\|ds.$$

Then, the fact that $\Phi(\cdot, u(\cdot)) \in \rho\text{-}PAP_0(\mathbb{X})$ implies that $\mathcal{N}_2 u \in \rho\text{-}PAP_0(\mathbb{X})$, since by Theorem 3.5 any (h,ρ)-convolution of a $PAP_0(\mathbb{X},\rho)$ function is $PAP_0(\mathbb{X},\rho)$. Then $\mathcal{N}_2 u \in PAP_0(\mathbb{X},\rho)$ since $\Phi(\cdot, u(\cdot)) \in \rho\text{-}PAP_0(\mathbb{X})$. Thus (25) is satisfied.

To complete the proof, we apply the fixed-point principle of Banach to the nonlinear operator $\mathcal{N}$. Based on the above, it is clear that $\mathcal{N}$ maps $PAP(\mathbb{X},\rho)$ into itself. Moreover, for all $u, v \in PAP(\mathbb{X},\rho)$, it is easy to see that

$$||(\mathcal{N}u) - (\mathcal{N}v)||_\infty \leq LM \, \|k\|_1 \, ||u-v||_\infty,$$

and hence $\mathcal{N}$ has a unique fixed-point, which obviously is the unique ρ-pseudo almost periodic solution to (26). □

As a simple application we obtain a ρ-pseudo almost periodic solution to the abstract differential equation

$$u'(t) = Au(t) + f(t, u(t)), \ t \in \mathbb{R}, \tag{26}$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ and $f \in PAP(\mathbb{X}, \mathbb{X}, cl(\rho))$ [30,36]. Moreover, we assume that

(H3) There exist constants $M > 0$ and $\alpha > 0$ such that $||T(t)|| \leq Me^{-\alpha t}$ for each $t \geq 0$.
(H4) (e,ρ)-conditions. The weight ρ satisfies at least one of the following conditions:

ρ is such that (22) holds, or

$$\sup_{|s|\leq r, r\in\mathbb{R}} \frac{e^{\alpha s}}{\rho(s)} \int_{s}^{r} e^{-\alpha t}\rho(t)dt < \infty, \quad \lim_{r\to\infty} \frac{e^{-\alpha r}}{\mu(r,\rho)} \int_{-r}^{r} e^{-\alpha t}\rho(t)dt = 0.$$

Thus, we have the following result related to the pseudo almost periodic mild solutions of (26).

Theorem 4.2. *Fix $\rho \in \mathbb{U}_\infty$. Suppose that assumptions (H1), (H3)–(H4) hold. Then (26) has a unique $cl(\rho)$-pseudo almost periodic mild solution whenever $L < \alpha/M$.*

To illustrate Theorem 4.2 we consider the existence and uniqueness of weighted pseudo almost periodic solutions to the heat equation given by the system

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \gamma H(t)\sin(u(t,x)), \qquad (x,t) \in [0,\pi]\times\mathbb{R}, \tag{27}$$

$$u(t,0) = u(t,\pi) = 0, \qquad t \in \mathbb{R}, \tag{28}$$

where γ is a positive parameter and H is a function defined as follows $H(t) = \cos(t) + \cos(\sqrt{2}t) + \phi(t)$ for each $t \in \mathbb{R}$ and with ϕ such that $\phi(t)e^t$ is bounded. Now, in order to analyze (27)–(28), we suppose that $\mathbb{X} = \left(L^2[0,\pi], ||\cdot||_2\right)$ and define the notation

$$D(A) = \left\{u \in L^2[0,\pi] : u'' \in L^2[0,\pi], u(0) = u(\pi) = 0\right\}$$
$$Au = \Delta u = u''(\cdot), \qquad \forall u(\cdot) \in D(A).$$

It is well-known that A is the infinitesimal generator of an analytic semigroup $T(t)$ on $L^2[0,\pi]$ with $M = \alpha = 1$: $||T(t)|| \leq e^{-t}$ for $t \geq 0$. More precisely

$$T(t)\varphi = \sum_{n=1}^{\infty} e^{-n^2 t} < \varphi, \psi_n > \psi_n,$$

for $\varphi \in L^2[0,\pi]$ and $\psi_n(t) = \sqrt{2/\pi}\sin(nt)$ with $n \in \mathbb{N}$. Now, set $\rho(t) = e^t$ and $f(t, u(t)) = \gamma H(t)\sin(u(t,\cdot))$ for $t \in \mathbb{R}$. Clearly f satisfies the Lipschitz condition (H2) since

$$\|f(t, u(t,\cdot)) - f(t, v(t,\cdot))\|_2 \leq \gamma\|H\|\|u(t,\cdot) - v(t,\cdot)\|_2,$$

for all $u(t,\cdot), v(t,\cdot) \in L^2[0,\pi]$ and $t \in \mathbb{R}$. Moreover, it is straightforward to check that $H \in PAP(\mathbb{R}, cl(e^t))$ with $\phi(t)$ as its weighted ergodic component and $\cos(t) + \cos(\sqrt{2}t)$ as its almost periodic component. Consequently, f is e^t-pseudo almost periodic in $t \in \mathbb{R}$ uniformly in the second variable. Note that the hypothesis (H4) is also satisfied. Hence an application of Theorem 4.2 implies the following result.

Theorem 4.3. *The heat equation with Dirichlet condition in (27)–(28) has a unique weighted* e^t*-pseudo almost periodic solution, whenever* γ *is small enough.*

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