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Ground state solution for differential equations with left and right fractional derivatives

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In this work, we study the existence of positive solutions for a class of fractional differential equation given by

$$\begin{aligned} {}_tD^{\alpha}_{\infty}\,{}_{-\infty}D^{\alpha}_{t}u(t) + u(t) &= f(t,u(t)),\\ u &\in H^{\alpha}(\mathbb{R}), \end{aligned} \tag{1}$$

where $\alpha \in (1/2, 1), t \in \mathbb{R}, u \in \mathbb{R}, f \in C(\mathbb{R}, \mathbb{R})$. Using the mountain pass theorem and comparison argument, we prove that (1) at least has one nontrivial solution.

Keywords: Liouville–Weyl fractional derivative; fractional Sobolev space; critical point theory; comparison argument; ground state

1. Introduction

The aim of this article is to study the fractional differential equation with left and right fractional derivative

$$\begin{aligned} {}_tD^{\alpha}_{\infty}\,{}_{-\infty}D^{\alpha}_{t}u(t) + u(t) &= f(t,u(t)),\\ u &\in H^{\alpha}(\mathbb{R}), \end{aligned} \tag{2}$$

where $\alpha \in (1/2, 1), t \in \mathbb{R}, u \in \mathbb{R}, f \in C(\mathbb{R}, \mathbb{R})$.

The study of fractional calculus (differentiation and integration of arbitrary order) has emerged as an important and popular field of research. It is mainly due to the extensive application of fractional differential equations in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, and so on [1–6]. An important characteristic of fractional-order differential operator that distinguishes it from the integer-order differential operator is its nonlocal behavior, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well as its past states. In other words, differential equations of arbitrary order describe memory and hereditary properties of various materials and processes. This is one of the features that has contributed to the popularity of the subject and has motivated the researchers to focus on fractional order models, which are more realistic and practical than the classical integer-order models.

Very recently, also equations including both left and right fractional derivatives were investigated [7–20]. Equations of this type are known in literature as the fractional Euler-Lagrange equation and are obtained by modifying the principle of least action and applying the rule of fractional integration by parts. Such differential equations mixing both types of derivatives found interesting applications in fractional variational principles, fractional control theory, fractional Lagrangian and Hamiltonian dynamics, as well as in the construction industry [21–28].

Although investigations concerning ordinary and partial fractional differential equations yield many interesting and important results for equations with operators including fractional derivatives of one type [2–4, 29], still the fractional differential equations with mixed derivatives need further study. This form of fractional operator makes it difficult to find an analytical solution of the considered

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equation. Some analytical results can be found in papers [7, 10, 11, 14], where a fixed point theorem was used. This solution has a complex form, that is, contains a series of alternately left and right fractional integrals. Using the Mellin transform, Klimek [12] obtained an analytical solution which was represented by a series of special functions. In both cases, the analytical results are very difficult for practical calculations.

On the other hand, it should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space [30, 31]. In [15] and [17], the authors showed that the critical point theory is an effective approach to tackle the existence of solutions for fractional boundary value problem (FBVP) with mixed derivatives. We note that it is not easy to use the critical point theory to study FBVP, because it is often very difficult to establish a suitable space and variational functional for the FBVP.

Inspired by this previous works, very recently in [18], the author considered the fractional Hamiltonian systems

$$ {}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t u(t)) + L(t)u(t) = \nabla W(t, u(t)), \tag{3}$$

where $\alpha \in (1/2, 1)$, $L(t)$ is a positive definite $n \times n$ matrix, W is assumed to be superquadratic at infinity and subquadratic at zero in u. It is worth noting that under the assumption $L(t) \to \infty$ as $|t| \to \infty$, the Palais–Smale condition holds, and the existence of nontrivial solution of (3) follows from the mountain pass theorem. In [19], the authors considered the potential $W(t, u) = a(t)V(u)$ and assumed that L is uniformly bounded from below and

$$\lim_{|t|\to+\infty} a(t) = 0.$$

It was shown that (3) possesses at least one nontrivial solution using the mountain pass theorem.

Our goal is to study the existence of ground states of (2). Before continuing, we make precise definitions of the notion of solutions for the equation

$$ {}_tD^{\alpha}_{\infty}\,{}_{-\infty}D^{\alpha}_t u(t) + u(t) = f(t, u(t)). \tag{4}$$

Definition 1.1
Given $f \in L^2(\mathbb{R})$, we say that $u \in H^{\alpha}(\mathbb{R})$ is a weak solution of (4) if

$$\int_{\mathbb{R}} \left[{}_{-\infty}D^{\alpha}_t u(t)\,{}_{-\infty}D^{\alpha}_t v(t) + u(t)v(t)\right] dt = \int_{\mathbb{R}} f(t, u(t))v(t)dt \ \text{ for all } \ v \in H^{\alpha}(\mathbb{R}).$$

Here, $H^{\alpha}(\mathbb{R})$ denotes the fractional Sobolev space (see Section 2).

Now, we state our main assumptions. In order to find solutions of (2), we will assume the following general hypotheses.

(f_0) $f(t, \xi) \geq 0$ if $\xi \geq 0$ and $f(t, \xi) = 0$ if $\xi \leq 0$, for all $t \in \mathbb{R}$.
(f_1) There exists $\theta > 2$ such that

$$0 < \theta F(t, \xi) \leq \xi f(t, \xi), \quad \forall (t, \xi),\ \xi \neq 0,$$

where $F(t, \xi) = \int_0^{\xi} f(t, \sigma)d\sigma$.
(f_2) $f(t, \xi) = o(|\xi|)$ uniformly in t.
(f_3) $\lim_{|\xi|\to\infty} \frac{f(t,\xi)}{|\xi|^{p_0}} = 0$ for some $p_0 + 1 > \theta$, uniformly in $t \in \mathbb{R}$.
(f_4) $\frac{f(t,\sigma\xi)\xi}{\sigma}$ is a increasing function for every $\sigma > 0, t, \xi \in \mathbb{R}$.
(f_5) There exist continuous functions $\bar{f}$ and a, defined in $\mathbb{R}$, such that $\bar{f}$ satisfies $(f_0) - (f_4)$ and

$$0 \leq f(t, \xi) - \bar{f}(\xi) \leq a(t)(|\xi| + |\xi|^{p_0}) \quad \text{for all } \ t, \xi \in \mathbb{R},$$
$$\lim_{|t|\to\infty} a(t) = 0,$$

and

$$m(\{t \in \mathbb{R} : f(t, \xi) > \bar{f}(\xi)\}) > 0,$$

where m denotes the Lebesgue measure.

At this point, we state our existence theorem for the autonomous equation, that is, when the nonlinearity f does not depend on t. This theorem will serve as a basis for the proof of the main existence theorem for the case where f depends on t.

Theorem 1.1
Assume that $\frac{1}{2} < \alpha < 1$ and that $\bar{f}$ satisfies $(f_0) - (f_4)$, then

$$ {}_tD^{\alpha}_{\infty}\,{}_{-\infty}D^{\alpha}_t u(t) + u(t) = \bar{f}(u) \ \text{ in } \ \mathbb{R},$$

has a nontrivial weak solution.

The simplest case of a function $\bar{f}$ satisfying the hypotheses $(f_0) - (f_4)$ is $f(s) = s_+^p$, where p is as in (f_3) and $s_+ = \max\{s, 0\}$. Naturally, the class of functions satisfying these hypotheses is much ampler than this homogeneous case.

In the t-dependent case, we have to consider the behavior of the nonlinearity for large values of t in order to obtain proper compactness conditions. In the simplest model case, we may consider the t-dependent nonlinearity $f(t,s) = b(t)s_+^p$, where $b(t) \geq 1$. If this inequality is strict somewhere and $\lim_{|t|\to\infty} b(t) = 1$, then we will prove that a solution of (2) exists. However, we could consider a more general class of t-dependent nonlinearities.

Now, we state our main existence theorem.

Theorem 1.2
Assume that $\frac{1}{2} < \alpha < 1$ and f satisfies $(f_0) - (f_5)$. Then Eq. (2) possesses at least one weak nontrivial solution.

We prove the existence of weak solution of (2) by applying the mountain pass theorem [32] to the functional I defined on $H^\alpha(\mathbb{R})$ as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left|{}_{-\infty}D_t^\alpha u(t)\right|^2 + u(t)^2 \right] dt - \int_{\mathbb{R}} F(t, u(t))dt. \tag{5}$$

However, the direct application of the mountain pass theorem is not sufficient, because the Palais-Smale sequence might lose compactness in the whole space $\mathbb{R}$. To overcome this difficulty, we use a comparison argument devised in [33], based on the energy functional

$$\bar{I}(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left|{}_{-\infty}D_t^\alpha u(t)\right|^2 + u(t)^2 \right] dt - \int_{\mathbb{R}} \bar{F}(u(t))dt. \tag{6}$$

The rest of the paper is organized as follows: In Section 2, we describe the Liouville–Weyl fractional calculus and we introduce the fractional space that we use in our work and some proposition are proven, which will aid in our analysis. In Section 3, we will prove Theorems 1.1 and 1.2.

2. Preliminary results

2.1. *Liouville–Weyl fractional calculus*

The Liouville–Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$${}_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-\xi)^{\alpha-1}u(\xi)d\xi, \tag{7}$$

$${}_xI_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (\xi-x)^{\alpha-1}u(\xi)d\xi. \tag{8}$$

The Liouville–Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville–Weyl fractional integrals

$${}_{-\infty}D_x^\alpha u(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}u(x), \tag{9}$$

$${}_xD_\infty^\alpha u(x) = -\frac{d}{dx}\,{}_xI_\infty^{1-\alpha}u(x). \tag{10}$$

The definitions (9) and (10) may be written in an alternative form:

$${}_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}}d\xi, \tag{11}$$

$${}_xD_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}}d\xi. \tag{12}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w}u(x)dx.$$

Let $u(x)$ be defined on $(-\infty, \infty)$. Then, the Fourier transform of the Liouville–Weyl integral and differential operator satisfies

$$\widehat{{}_{-\infty}I_x^\alpha u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w), \quad \widehat{{}_xI_\infty^\alpha u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w). \tag{13}$$

$$\widehat{{}_{-\infty}D_x^\alpha u(x)}(w) = (iw)^{\alpha}\widehat{u}(w), \quad \widehat{{}_xD_\infty^\alpha u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w). \tag{14}$$

2.2. Fractional derivative space

In this section, we introduce some fractional derivative space (for more details, see [34]).
Let $\alpha > 0$. Define the semi-norm

$$|u|_{I^{\alpha}_{-\infty}} = \|_{-\infty}D^{\alpha}_{x}u\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{I^{\alpha}_{-\infty}} = \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{I^{\alpha}_{-\infty}}\right)^{1/2}, \tag{15}$$

and let

$$I^{\alpha}_{-\infty}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|.\|_{I^{\alpha}_{-\infty}}}.$$

Now, we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, let the semi-norm

$$|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2(\mathbb{R})}, \tag{16}$$

and norm

$$\|u\|_{\alpha} = \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{\alpha}\right)^{1/2},$$

and let

$$H^{\alpha}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|.\|_{\alpha}}.$$

We note a function $u \in L^2(\mathbb{R})$ belongs to $I^{\alpha}_{-\infty}(\mathbb{R})$ if and only if

$$|w|^{\alpha}\widehat{u} \in L^2(\mathbb{R}), \tag{17}$$

especially

$$|u|_{I^{\alpha}_{-\infty}} = \left\||w|^{\alpha}\widehat{u}\right\|_{L^2(\mathbb{R})}. \tag{18}$$

Therefore, $I^{\alpha}_{-\infty}(\mathbb{R})$ and $H^{\alpha}(\mathbb{R})$ are equivalent with equivalent semi-norm and norm. Analogous to $I^{\alpha}_{-\infty}(\mathbb{R})$, we introduce $I^{\alpha}_{\infty}(\mathbb{R})$. Let the semi-norm

$$|u|_{I^{\alpha}_{\infty}} = \left\|_{x}D^{\alpha}_{\infty}u\right\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{I^{\alpha}_{\infty}} = \left(\|u\|^2_{L^2(\mathbb{R})} + |u|^2_{I^{\alpha}_{\infty}}\right)^{1/2}, \tag{19}$$

and let

$$I^{\alpha}_{\infty}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|.\|_{I^{\alpha}_{\infty}}}.$$

Moreover $I^{\alpha}_{-\infty}(\mathbb{R})$ and $I^{\alpha}_{\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm [34].

We recall the Sobolev lemma.

Theorem 2.1 ([18])
If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$, and there is a constant $C = C_{\alpha}$ such that

$$\|u\|_{\infty} \leq C\|u\|_{\alpha}. \tag{20}$$

Remark 1
If $u \in H^{\alpha}(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty]$, because

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|^{q-2}_{\infty}\|u\|^2_{L^2(\mathbb{R})}.$$

The following lemma is a version of the concentration compactness principle.

Lemma 2.1
Let $r > 0$ and $q \geq 2$. Let $(u_n) \in H^{\alpha}(\mathbb{R})$ be bounded. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_n(t)|^q dt \to 0, \tag{21}$$

then $u_n \to 0$ in $L^p(\mathbb{R})$ for any $p > 2$.

Proof
Let $q < s < \beta$ and $u \in H^{\alpha}(\mathbb{R})$. If $I_y = [y - r, y + r]$, the Hölder inequality implies that,

$$\begin{aligned}\|u\|_{L^s(I_y)} &= \left(\int_{I_y} |u(t)|^s dt\right)^{1/s} = \left(\int_{I_y} |u(t)|^{s(1-\lambda)}|u(t)|^{s\lambda} dt\right)^{1/s}\\ &\leq \left(\int_{I_y} |u(t)|^q dt\right)^{\frac{1-\lambda}{q}} \left(\int_{I_y} |u(t)|^{\beta} dt\right)^{\frac{\lambda}{\beta}}\\ &\leq \left(\int_{I_y} |u(t)|^q dt\right)^{\frac{1-\lambda}{q}} \|u\|_{\infty}^{\lambda} |I_y|^{\lambda/\beta},\end{aligned}$$

where $0 < \lambda < 1$ and $\frac{s(1-\lambda)}{q} + \frac{s\lambda}{\beta} = 1$. Now, by Theorem 2.1,

$$\|u\|_{\infty} \leq C\left(\int_{I_y} \left[\left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 + u(t)^2\right] dt\right)^{1/2}.$$

Thus,

$$\|u\|_{L^s(I_y)}^s \leq (2r)^{\frac{s\lambda}{\beta}} \left(\int_{I_y} |u(t)|^q dt\right)^{\frac{(1-\lambda)s}{q}} C^{\lambda s}\left(\int_{I_y} \left[\left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 + u(t)^2\right] dt\right)^{\frac{\lambda s}{2}}.$$

Choosing $\lambda s = 2$, i.e $s = 2 + q\left(1 - \frac{2}{\beta}\right)$, which gives, as $t > s$ is arbitrary, $2 < s < 2 + q$, we obtain

$$\|u\|_{L^s(I_y)}^s \leq C'\left(\int_{I_y} |u(t)|^q dt\right)^{\frac{(1-\lambda)s}{q}} \left(\int_{I_y} \left[\left|{}_{-\infty}D_t^{\alpha} u(t)\right|^2 + u(t)^2\right] dt\right).$$

Consequently,

$$\begin{aligned}\int_{\mathbb{R}} |u(t)|^s dt &= \sum_{k\in\mathbb{Z}} \int_{2rk}^{2r(k+1)} |u(t)|^s dt\\ &\leq C' \sum_{k\in\mathbb{Z}} \left\{\left(\int_{2rk}^{2r(k+1)} |u(t)|^q dt\right)^{\frac{(1-\lambda)s}{q}} \left(\int_{2rk}^{2r(k+1)} [|{}_{-\infty}D^{\alpha} u(t)|^2 + u(t)^2] dt\right)\right\}\\ &\leq \sup_{y\in\mathbb{R}} \left(\int_{y-r}^{y+r} |u(t)|^q dt\right)^{\frac{(1-\lambda)s}{q}} \|u\|_{\alpha}.\end{aligned}$$

Applying this inequality to each u_n, we see that $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < q + 2$. As $u_n \in L^r(\mathbb{R})$ for each $r > 2$, it follows by interpolation that $u_n \to 0$ in $L^p(\mathbb{R})$ for each $p > 2$. □

Now we introduce more notations and some necessary definitions. Let $\mathfrak{B}$ be a real Banach space, $I \in C^1(\mathfrak{B}, \mathbb{R})$, which means that I is a continuously Fréchet-differentiable functional defined on $\mathfrak{B}$. Recall that $I \in C^1(\mathfrak{B}, \mathbb{R})$ is said to satisfy the Palais–Smale condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \in \mathfrak{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathfrak{B}$.

Moreover, let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r denotes its boundary. For the reader convenience, we recall the Mountain Pass Theorems [31].

Theorem 2.2
Let $\mathfrak{B}$ be a real Banach space and $I \in C^1(\mathfrak{B}, \mathbb{R})$ satisfying (PS) condition. Suppose that $I(0) = 0$ and

(i) There are constants $\rho, \beta > 0$ such that $I|_{\partial B_{\rho}} \geq \beta$, and
(ii) There is and $e \in \mathfrak{B} \setminus \overline{B_{\rho}}$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \beta$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1], \mathfrak{B}) : \ \gamma(0) = 0, \ \gamma(1) = e\}.$$

3. Ground state

In this section, we consider the fractional differential equation with mixed derivatives, given by

$$\begin{aligned} {}_tD^{\alpha}_{\infty}{}_{-\infty}D^{\alpha}_t u(t) + u(t) &= f(t,u(t)), \quad t \in \mathbb{R}, \\ u &\in H^{\alpha}(\mathbb{R}), \end{aligned} \tag{22}$$

where $\frac{1}{2} < \alpha < 1$ and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ satisfies $(f_0) - (f_5)$. Our goal is to prove the existence of a ground state of Eq. (22), that is, a non-negative solution with lowest energy.

We prove the existence of weak solution of (22) finding a critical point of the functional I defined on $H^{\alpha}(\mathbb{R})$ as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left|{}_{-\infty}D^{\alpha}_t u(t)\right|^2 + u^2(t)\right] dt - \int_{\mathbb{R}} F(t,u(t))dt. \tag{23}$$

Using the properties of the Nemistky operators and the embeddings given in Remark 1, we can prove that the functional I is of class C^1, and we have

$$I'(u)v = \int_{\mathbb{R}} \left[{}_{-\infty}D^{\alpha}_t u(t){}_{-\infty}D^{\alpha}_t v(t)dt + u(t)v(t)\right] dt - \int_{\mathbb{R}} f(t,u(t))v(t)dt. \tag{24}$$

We define the Nehari manifold associated to the functional I as

$$\Lambda = \{u \in H^{\alpha}(\mathbb{R})\backslash\{0\} : \ \ I'(u)u = 0\},$$

and we observe that all non trivial solutions of (22) belong to Λ. Next, from (f_2) and (f_3), it is standard to prove that, for any $\epsilon > 0$, there exists C_ϵ such that

$$|f(t,\xi)| \le \epsilon|\xi| + C_\epsilon|\xi|^{p_0}, \quad \forall t \in \mathbb{R}, \tag{25}$$

and consequently

$$|F(t,\xi)| \le \frac{\epsilon}{2}|\xi|^2 + \frac{C_\epsilon}{p_0+1}|\xi|^{p_0+1}, \quad \forall t \in \mathbb{R}. \tag{26}$$

We start our analysis with.

Lemma 3.1
Assume the hypotheses (f_0)–(f_4) hold. For any $u \in H^{\alpha}(\mathbb{R}) \setminus \{0\}$, there is a unique $\sigma_u = \sigma(u) > 0$ such that $\sigma_u u \in \Lambda$, and we have

$$I(\sigma_u u) = \max_{\sigma\ge 0} I(\sigma u).$$

Proof
Let $u \in H^{\alpha}(\mathbb{R}) \setminus \{0\}$ and consider the function $\psi : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$\psi(\sigma) = I(\sigma u) = \frac{\sigma^2}{2}\|u\|_{\alpha}^2 - \int_{\mathbb{R}} F(t,\sigma u)dt.$$

Then, by (26), we have

$$\int_{\mathbb{R}} F(t,u)dt \le \frac{C\epsilon}{2}\|u\|_{\alpha}^2 + \frac{CC_\epsilon}{p_0+1}\|u\|_{\alpha}^{p_0+1}.$$

This implies that $\psi(\sigma) > 0$, for σ small. On the other hand, by (f_1), there exists $A > 0$ such that $F(t,\xi) \ge A|\xi|^{\theta}$, $\forall|\xi| > 1$. So

$$I(\sigma u) \le \frac{\sigma^2}{2}\|u\|_{\alpha}^2 - A\sigma^{\theta}\int_{\mathbb{R}} |u(t)|^{\theta}dt, \tag{27}$$

and because $\theta > 2$, we see that $\psi(\sigma) < 0$ for σ large. By (f_0), $\psi(0) = 0$, therefore there is $\sigma_u = \sigma(u) > 0$ such that

$$\psi(\sigma_u) = \max_{\sigma\ge 0}\psi(\sigma) = \max_{\sigma\ge 0} I(\sigma u) = I(\sigma_u u).$$

We see that $\psi'(\sigma) = 0$ is equivalent to

$$\|u\|_{\alpha}^2 = \int_{\mathbb{R}} \frac{f(t,\sigma u)u}{\sigma}dt,$$

from where, using (f_4), we prove that there is a unique $\sigma_u > 0$ such that $\sigma_u u \in \Lambda$. ☐

Now, we define two critical values as follows

$$c^* = \inf_{u\in\Lambda} I(u) \quad \text{and} \quad c = \inf_{\gamma\in\Gamma}\sup_{\sigma\in[0,1]} I(\gamma(\sigma)), \tag{28}$$

where Γ is given by

$$\Gamma = \{\gamma \in C([0,1], H^{\alpha}(\mathbb{R})/ \ \ \gamma(0) = 0, \ I(\gamma(1)) < 0\}.$$

Under our assumptions, certainly Γ is not empty and $c > 0$. The following lemma is crucial, and it uses (f_1).

Lemma 3.2

$$c^* = \inf_{u\in H^{\alpha}(\mathbb{R})\setminus\{0\}} \sup_{\sigma\geq 0} I(\sigma u) = c. \tag{29}$$

Proof

We notice that I is bounded below on Λ, because by (f_1), $I(u) > 0, \forall \ u \in \Lambda$, so that c^* is well defined. By Lemma 3.1 for any $u \in H^{\alpha}(\mathbb{R}) \setminus \{0\}$, there is a unique $\sigma_u = \sigma(u) > 0$ such that $\sigma_u u \in \Lambda$, then

$$c^* \leq \inf_{u\in H^{\alpha}(\mathbb{R})\setminus\{0\}} \max_{\sigma\geq 0} I(\sigma u).$$

On the other hand, for any $u \in \Lambda$, we have

$$I(u) = \max_{\sigma\geq 0} I(\sigma u) \geq \inf_{u\in H^{\alpha}(\mathbb{R})\setminus\{0\}} \max_{\sigma\geq 0} I(\sigma u),$$

so

$$c^* = \inf_{\Lambda} I(u) \geq \inf_{u\in H^{\alpha}(\mathbb{R})\setminus\{0\}} \max_{\sigma\geq 0} I(\sigma u),$$

therefore the first equality in (29) holds. Next, we prove the other equality, that is, $c^* = c$. We claim that for every $\gamma \in \Gamma$, there exists $\sigma_0 \in [0,1]$, such that $\gamma(\sigma_0) \in \Lambda$.

To prove the claim, we first see that, by (25) and the Remark 1, we have

$$\int_{\mathbb{R}} f(t,u)u dx \leq \epsilon C\|u\|_{\alpha}^2 + C_{\epsilon} C\|u\|_{\alpha}^{p_0+1}. \tag{30}$$

Hence, for $\gamma \in \Gamma$, we have

$$\begin{aligned} I'(\gamma(\sigma))\gamma(\sigma) &= \|\gamma(\sigma)\|_{\alpha}^2 - \int_{\mathbb{R}} f(t,\gamma(\sigma))\gamma(\sigma)dt \\ &\geq \left(1 - \epsilon C - C_{\epsilon} C\|\gamma(\sigma)\|_{\alpha}^{p_0-1}\right)\|\gamma(\sigma)\|_{\alpha}^2. \end{aligned}$$

If we take $r = \left(\frac{1-\epsilon C}{C_{\epsilon}C}\right)^{\frac{1}{p_0-1}}$, then we see that

$$I'(\gamma(\sigma))\gamma(\sigma) > 0 \quad \forall \ \sigma \in [0,1], \ \text{ such that, } \ \|\gamma(\sigma)\|_{\alpha} < r.$$

On the other hand, using (f_1) and since $I(\gamma(1)) < 0$, we have

$$\|\gamma(1)\|_{\alpha}^2 < \int_{\mathbb{R}} 2F(t,\gamma(1))dt < \int_{\mathbb{R}} \theta F(t,\gamma(1))dt \leq \int_{\mathbb{R}} f(t,\gamma(1))\gamma(1)dt,$$

that implies $I'(\gamma(1))\gamma(1) < 0$. Thus, by the Intermediate Value Theorem, there exists $\sigma_0 \in (\sigma_*, 1)$ such that $I'(\gamma(\sigma_0))\gamma(\sigma_0) = 0$ and so $\gamma(\sigma_0) \in \Lambda$, completing the proof of the claim. From this result, $\max_{\sigma\in[0,1]} I(\gamma(\sigma)) \geq I(\gamma(\sigma_0)) \geq \inf_{\Lambda} I$ and then

$$c \geq c^*. \tag{31}$$

In order to prove the other inequality, we see that from (27), there exists σ_u^* large enough such that $I\left(\sigma_u^* u\right) < 0$. Now, we define the curve $\gamma_u : [0,1] \to H^{\alpha}(\mathbb{R})$ as $\gamma_u(\sigma) = \sigma\left(\sigma_u^* u\right)$. Then $\gamma_u(0) = 0, \ I(\gamma(1)) = I\left(\sigma_u^* u\right) < 0$ and γ_u is continuous, so that $\gamma_u \in \Gamma$. Now, by definition of γ_u,

$$\max_{\sigma\geq 0} I(\sigma u) \geq \max_{\xi\in[0,1]} I(\gamma_u(\xi)), \quad \forall \ H^{\alpha}(\mathbb{R}) \setminus \{0\},$$

then $c^* \geq c$, completing the proof. □

Remark 2

Since $c = \inf_{\Lambda} I$ and any critical point of I lies on Λ, if c is a critical value of I, then it is the smallest positive critical value of I.

Lemma 3.3

Suppose $\{u_n\} \in H^{\alpha}(\mathbb{R})$ and there exists $b > 0$ such that

$$I(u_n) \leq b \ \text{ and } \ I'(u_n) \to 0. \tag{32}$$

Then either

(i) $u_n \to 0$ in $H^\alpha(\mathbb{R})$, or
(ii) there is a sequence $(y_n) \in \mathbb{R}$, and $r, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{y_n - r}^{y_n + r} |u_n(x)|^2 dx > \beta.$$

Proof
By (32), it is standard to check for k large enough

$$b + \|u_n\|_\alpha \geq I(u_n) - \frac{1}{\theta} I'(u_n)u_n \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\alpha^2, \tag{33}$$

and then $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R})$.

Suppose (ii) is not satisfied, then for any $r > 0$, (21) holds. Consequently by Lemma 2.1

$$\|u_n\|_{L^{p_0+1}} \to 0. \tag{34}$$

Then, noticing that

$$I'(u_n)u_n = \|u_n\|_\alpha^2 - \int_{\mathbb{R}} f(t, u_n)u_n dt, \tag{35}$$

by (25) and Remark 1, we have

$$\int_{\mathbb{R}} f(t, u_n)u_n dt \leq \epsilon C \|u_n\|_\alpha^2 + C_\epsilon \|u_n\|_{L^{p_0+1}}^{p_0+1},$$

where $p_0 + 1 > \theta$. So

$$I'(u_n)u_n \geq (1 - \epsilon C)\|u_n\|_\alpha^2 - C_\epsilon \|u_n\|_{L^{p_0+1}}^{p_0+1}. \tag{36}$$

Choosing an appropriate ϵ and using (32) and (34), we find that $u_n \to 0$ in $H^\alpha(\mathbb{R})$, that is, (i) holds. □

We define $\overline{\Lambda}, \overline{\Gamma}$, and $\bar{c}$, replacing f by $\bar{f}$. The following theorem gives the existence of a solution for the limit problem.

Theorem 3.1
$\bar{I}$ has at least one critical point with critical value $\bar{c}$.

Proof
By the Ekeland variational principle [30], there is a sequence u_n such that

$$\bar{I}(u_n) \to \bar{c} \quad \text{and} \quad \bar{I}'(u_n) \to 0. \tag{37}$$

By (37) and (f_1), given $\epsilon > 0$, for n large enough,

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\alpha^2 &\leq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\alpha^2 + \int_{\mathbb{R}} \left[\frac{1}{\theta}\bar{f}(u_n)u_n - \bar{F}(u_n)\right] dt \\ &= \bar{I}(u_n) - \frac{1}{\theta}\bar{I}'(u_n)u_n \leq \|u_n\|_\alpha + \bar{c} + \epsilon,\end{aligned}$$

so that (u_n) is a bounded sequence. Because $H^\alpha(\mathbb{R})$ is a reflexive space, there is a subsequence $(u_n) \in H^\alpha(\mathbb{R})$ converging weakly to u in $H^\alpha(\mathbb{R})$ and strongly in $L^p_{loc}(\mathbb{R})$ for $p \in [2, \infty)$. Thus, for such a subsequence and any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\lim_{n\to\infty} \bar{I}'(u_n)\varphi = \bar{I}'(u)\varphi = 0.$$

If we show that $u \neq 0$, then $\bar{I}'(u) = 0$, and then $\bar{I}(u) \geq \bar{c}$. On the other hand, using (f_1) again, we see that, for every $r > 0$,

$$\begin{aligned}\bar{I}(u_n) - \frac{1}{2}\bar{I}'(u_n)u_n &= \int_{\mathbb{R}} \left(\frac{1}{2}\bar{f}(u_n)u_n - \bar{F}(u_n)\right) dt \\ &\geq \int_{-r}^{r} \left(\frac{1}{2}\bar{f}(u_n)u_n - \bar{F}(u_n)dt\right. .\end{aligned} \tag{38}$$

Because $u_n \to u$ in $L^p_{loc}(\mathbb{R})$, for any $p \in [2, \infty)$, up to a subsequence,

$$u_n(t) \to u(t) \ \text{ a.e. on } \ (-r, r),$$

and there are $h \in L^2(-r,r)$ and $g \in L^{p_0+1}(-r,r)$, such that

$$|u_n(t)| \leq h(t) \quad \text{and} \quad |u_n(t)| \leq g(t) \quad \text{a.e on } (-r,r).$$

Moreover, by (25) and (26), we get

$$|\bar{f}(u_n(t))u_n(t)| \leq \epsilon h(t)^2 + C_\epsilon g(t)^{p_0+1} \in L^1(-r,r),$$
$$|\bar{F}(u_n(t))| \leq \frac{\epsilon}{2}h(t)^2 + \frac{C_\epsilon}{p_0+1}g(t)^{p_0+1} \in L^1(-r,r).$$

By Lebesgue's dominated convergence theorem

$$\int_{-r}^{r} \bar{f}(u_n)u_n dt \to \int_{-r}^{r} \bar{f}(u)u dt \quad \text{and} \quad \int_{-r}^{r} \bar{F}(u_n)dt \to \int_{-r}^{r} \bar{F}(u)dt.$$

Therefore, because r is arbitrary

$$\bar{c} \geq \int_{\mathbb{R}} \left(\frac{1}{2}\bar{f}(u)u - \bar{F}(u) \right) dt.$$

Now, because $\bar{I}'(u)u = 0$ and

$$\bar{I}(u) = \bar{I}(u) - \frac{1}{2}\bar{I}'(u)u = \int_{\mathbb{R}} \left(\frac{1}{2}\bar{f}(u)u - \bar{F}(u) \right) dt,$$

it follows that $\bar{I}(u) \leq \bar{c}$.

In order to complete the proof, we just need to show that u is non-trivial. For this purpose, by Lemma 3.3, it is possible to find a sequence $y_n \in \mathbb{R}, r > 0$ and $\beta > 0$ such that

$$\int_{y_n-r}^{y_n+r} u_n^2(t)dt > \beta, \quad \forall n.$$

Now, we define $\overline{u}_n(t) = u_n(t+y_n)$. We note that $\overline{u}_n$ is bounded in $H^\alpha(\mathbb{R})$, and so, up to a subsequence, weakly converges in $H^\alpha(\mathbb{R})$ to some $u \in H^\alpha(\mathbb{R})$ and strongly in $L^p(-r,r)$. But

$$\int_{-r}^{r} |u(t)|^p dt = \lim_{n\to\infty} \int_{-r}^{r} |\overline{u}_n(t)|^2 dt = \lim_{n\to\infty} \int_{y_n-r}^{y_n+r} |u_n(t)|^p dt > \beta,$$

that implies $u \neq 0$. □

Now, we prove our main result.

Theorem 3.2
I has at least one critical point with critical value $c < \bar{c}$.

Proof
By definition of c in (29), for every sequence $\{\epsilon_n\}$, there exists a sequence of $\{u_n\}$ in $H^\alpha(\mathbb{R})$ such that $\|u_n\|_\alpha = 1$,

$$c \leq \max_{\sigma\geq 0} I(\sigma u_n) \leq c + \epsilon_n \quad \text{and} \quad \max_{\sigma\geq 0} I(\sigma u_n) \to c. \tag{39}$$

As in the proof of Lemma 3.2, associated with each u_n, there is a function $\gamma_n \in \Gamma$ such that

$$\max_{\xi\in[0,1]} I(\gamma_n(\xi)) \leq \max_{\sigma\geq 0} I(\sigma u_n) \leq c + \epsilon_n. \tag{40}$$

Now, let $X = H^\alpha(\mathbb{R}), K = [0,1], K_0 = \{0,1\}, M = \Gamma, \varphi = \gamma_n$ and

$$c_1 = \max_{\gamma_n(K_0)} I = 0 < c,$$

then we can use Theorem 4.3 of [30] to find a sequence $\{w_n\}$ in $H^\alpha(\mathbb{R})$ and $\{\xi_n\} \subset [0,1]$ such that $I(w_n) \in (c-\epsilon_n, c+\epsilon_n)$,

$$\|w_n - \gamma_n(\xi_n)\|_\alpha \leq \epsilon_n^{1/2} \quad \text{and} \quad \|I'(w_n)\|_{(H^\alpha)'} \leq \epsilon_n^{1/2}. \tag{41}$$

Now, because

$$I(w_n) \to c \text{ in } \mathbb{R} \text{ and } I'(w_k) \to 0 \text{ in } (H^\alpha(\mathbb{R}))', \tag{42}$$

as in the proof of the Theorem 3.1, we show that $\{w_n\}$ is bounded in $H^\alpha(\mathbb{R})$. Moreover, up to a subsequence

$$w_n \rightharpoonup w \text{ in } H^\alpha(\mathbb{R}) \text{ and } w_n \to w \text{ in } L^p_{loc}(\mathbb{R}), \ 2 \leq p < \infty, \tag{43}$$

where w is weak solution of (22). By Lemma 3.3, there is a sequence $\{y_n\} \subset \mathbb{R}$, $\beta > 0$ and $r > 0$ such that

$$\liminf_{n\to\infty} \int_{y_n-r}^{y_n+r} w_n^2 dt \geq \beta. \tag{44}$$

If $\{y_n\}$ contains a bounded subsequence, then (44) guarantees that $w \not\equiv 0$, and the results follows. If $\{y_k\}$ is an unbounded sequence, we may assume that, for given $R > 0$,

$$\lim_{n\to\infty} \int_{-R}^{R} |u_n|^2 dt = 0, \tag{45}$$

because the contrary implies that $u \neq 0$ following the same argument as in the preceding text. In order to complete the proof, we first obtain that

$$c < \bar{c}. \tag{46}$$

To see this, we use the characterization of c and $\bar{c}$ as in Lemma 3.2. Let $\bar{u}$ be a non-trivial critical point of $\bar{I}$ given by Theorem 3.1 and let

$$\mathcal{A} = \{t \in \mathbb{R} : \ f(t,\xi) > \bar{f}(\xi) \ \text{ for all } \ \xi > 0\}.$$

Then, by (f_5) and the fact that $\bar{u}$ is non-zero, there exists $y \in \mathbb{R}$ such that the function u_y, defined as $u_y(t) = u(t+y)$, satisfies

$$m(.)\{t \in \mathbb{R} : \ |u_y(t)| > 0\} \cap \mathcal{A}| > 0,$$

where m denotes the Lebesgue measure. But then

$$\bar{c} = \bar{I}(u_y) \geq \bar{I}(\sigma u_y) > I(\sigma u_y) \quad \text{for all } \ \sigma > 0.$$

Choosing $\sigma = \sigma^* > 0$ such that $I(\sigma^* u_y) = \sup_{\sigma>0} I(\sigma u_y)$, we find $\sigma^* u_y \in \Lambda$ and we conclude that

$$\bar{c} > I(\sigma^* u_y) \geq \inf_{u\in\Lambda} I(u) = c,$$

proving (46). Now, we see that, for $\sigma \geq 0$, from (f_5), we have

$$\begin{aligned} I(\sigma u_n) &= \bar{I}(\sigma u_n) - \int_{\mathbb{R}} (F(t,u_n) - \bar{F}(u_n))dt \\ &\geq \bar{I}(\sigma u_n) - \int_{\mathbb{R}} Ca(t)(|\sigma u_n(t)|^2 + |\sigma u_n(t)|^{p_0+1})dt. \end{aligned}$$

Let $\epsilon > 0$, Then, by (f_5) again, there exists $R > 0$ such that

$$\int_{B(0,R)^c} Ca(t)(|\sigma u_n(t)|^2 + |\sigma u_n(t)|^{p_0+1})dt \leq \epsilon,$$

for σ bounded. Then, by (45),

$$\lim_{n\to\infty} \int_{B(0,R)} Ca(t)(|\sigma u_n(t)|^2 + |\sigma u_n(t)|^{p_0+1})dt = 0.$$

Choosing $\sigma = \sigma^*$ such that $\bar{I}(\sigma^* u_n) = \max_{\sigma\geq 0} \bar{I}(\sigma u_n)$, we see that $c \geq \bar{c} - \epsilon$. If $\epsilon > 0$ is chosen sufficiently small, this contradicts (46).

Having the existence of a critical point u of I in $H^\alpha(\mathbb{R})$, we just have to prove that $u \geq 0$ a.e. For this fact, we recall that

$$\int_{\mathbb{R}} |w|^{2\alpha} \widehat{u}\widehat{\varphi} dw = C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{1+2\alpha}} dxdy,$$

for all $\varphi \in H^\alpha(\mathbb{R})$ [35]. Testing with $u_- := \max\{-u, 0\}$, by the positive of $f(t,u(t))$, we obtain

$$\int_{\mathbb{R}} |w|^{2\alpha} \widehat{u}\widehat{\varphi} dw = \int_{\mathbb{R}} u_-^2 dx.$$

But this cannot occur for $u_- \not\equiv 0$, because

$$\begin{aligned} \int_{\mathbb{R}} |w|^{2\alpha} \widehat{u}\widehat{\varphi} dw &= C \int_{\{u<0\}} \int_{\{u>0\}} \frac{[u(x)-u(y)]u_-(x)}{|x-y|^{1+2\alpha}} dxdy \\ &+ C \int_{\{u>0\}} \int_{\{u<0\}} \frac{[u(x)-u(y)]u_-(y)}{|x-y|^{1+2\alpha}} dxdy \\ &+ C \int_{\{u<0\}} \int_{\{u<0\}} \frac{[u(x)-u(y)][u_-(x)-u_-(y)]}{|x-y|^{1+2\alpha}} dxdy. \end{aligned}$$

The last term can be written as

$$-C\int_{\{u<0\}}\int_{\{u<0\}}\frac{|u_-(x)-u_-(y)|^2}{|x-y|^{1+2\alpha}}dxdy,$$

which is strictly negative unless $u_- \equiv 0$ a.e. The other two terms are also negative, hence, $u_- \equiv 0$ and the conclusion follows. □

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