Journal of Lie Theory
Volume **25** (2015) 1045–1071

On the Construction of a Finite Siegel Space

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Communicated by B. Ørsted

Abstract. We construct a finite analogue of classical Siegel's Space. This is made by generalizing Poincaré half plane construction for a quadratic field extension $E \supset F$, considering in this case an involutive ring A, extension of the ring fixed points $A_0 = A^\Gamma$, (Γ an order two group of automorphisms of A), and the generalized special linear group $SL_*(2, A)$, which acts on a $*-$plane $\mathcal{P}_A$. Classical Lagrangians for finite dimensional spaces over a finite field are related with Lagrangians for $\mathcal{P}_A$. We show $SL_*(2, A)$ acts transitively on $\mathcal{P}_A$ when A is a $*-$ euclidean ring, and we study extensibly the case where $A = M_n(E)$. The structure of the orbits of the action of the symplectic group over F on Lagrangians over a finite dimensional space over E are studied.
Mathematics Subject Classification 2010: Primary 20G40; Secondary 11E16, 14M20, 17B10.
Key Words and Phrases: Finite Siegel half space, star-analogue.

1. Introduction

As a motivation for the construction below, we consider a second degree extension of fields $E \supset F$. We recall that finite Poincaré half plane, more precisely the double cover of finite Poincaré half plane, may be realized as the set of lines through the origin in the usual plane $E^2 = E \times E$, whose slope does not lie in $F \cup \{\infty\}$. Lines through the origin are however just the Lagrangians for the symplectic bilinear form *determinant* on E^2, and the constraint that the slope of a Lagrangian L does not lie in $F \cup \{\infty\}$ amounts to saying that the form h_E given by Galois twisting of the determinant, that is, given by

$$h_E(x, y) = \bar{x}_1 y_2 - \bar{x}_2 y_1$$

(equivalently we may consider the form $h_E(x, y) = x_1\bar{y}_2 - x_2\bar{y}_1$)
for $x = {}^t(x_1, x_2), y = {}^t(y_1, y_2)$ in E, is *non degenerate* when restricted to L. Indeed, if the constraint on L is fulfilled, we may take a representative vector of

*The authors were partially supported by FONDECYT 1120578, the third author was also partially supported by CONICET, SECYTUNC (Argentina), and the first author was also partially supported by Pontificia Universidad Católica de Valparaíso.

ISSN 0949–5932 / \$2.50

the form ${}^t(z,1) \in L$ $(z \in E)$, so that $L = \{\,{}^t(zx_2, x_2) | x_2 \in E\}$ and then h_E on L is given by

$$h_E(\begin{pmatrix} zx_2 \\ x_2 \end{pmatrix}, \begin{pmatrix} zy_2 \\ y_2 \end{pmatrix}) = \bar{x}_2(\bar{z} - z)y_2,$$

so h_E non degenerate means just $z \neq \bar{z}$.

When E is finite, under the action of $SL(2,F)$ in the set $\mathcal{L}_{E,2}$ of all Lagrangians (see example 2 below) we have the generic orbit consisting of all Lagrangians on which h_E is non degenerate and the residual orbit consisting of all Lagrangians on which h_E is degenerate, equivalently, h_E is null on the subspace. This holds if $z = \bar{z}$, i.e. $z \in F$ or $z = \infty$. One of the aims of this work is to extend this example to a more general setting.

Classical Siegel's half space is a clever generalization of Poincaré's half plane. In [10], the starting idea is to replace the real base field $\mathbb{R}$ by the full matrix ring $M(n, \mathbb{R})$. Then, Siegel's half space which consists of all symmetric complex $n \times n$ matrices whose imaginary part is positive definite, may be seen as a set of "slopes" of lines in $M_n(\mathbb{R}) \times M_n(\mathbb{R})$.

Our approach to obtain the finite analogue of Siegel's half space is to extend the universal (double cover of) Poincaré's half plane construction given in [12]) to the case where the field E is replaced by a ring A with involution denoted $*$. A ring with involution is also called involutive ring, as in [6, 7]. Instead of the groups $SL(2,E), SL(2,F)$ we have now their star-analogues [7], $SL_*(2,A), SL_*(2,A_0)$ (A a Galois extension of A_0). A natural $SL_*(2,A)-$ space is the $*-$plane $\mathcal{P}_A$ consisting of all points $x = {}^t(x_1, x_2) \in A^2 = A \times A$ whose coordinates x_1 and x_2 star-commute, i.e. $x_1^* x_2 = x_2^* x_1$. Notice *en passant* the analogy with Manin's $q-$plane, whose points have coordinates that anti-commute.

We introduce the $A-$valued canonical $*-$anti-hermitian form ω_A on A^2 given by

$$\omega_A(x,y) = x_1^* y_2 - x_2^* y_1 \qquad (a)$$

for all $x, y \in A^2$. We have then

$$\omega_A(y,x) = -\omega_A(x,y)^*$$

for all $x, y \in A^2$, and we see that the $*-$plane $\mathcal{P}_A$ consists of all isotropic vectors for ω_A. We also notice that if we write

$$x^* = (x_1^*, x_2^*)$$

for $x = {}^t(x_1, x_2) \in \mathcal{P}_A$, then we have $\omega_A(x,y) = x^* J y$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A way to stratify the $*-$plane $\mathcal{P}_A$ is to choose a suitable equivalent relation on the set of left ideals in A so that the family of subsets $\mathcal{P}_A(\mathcal{K})$ of $\mathcal{P}_A$ given by the condition $Ax + Ay = C(\mathcal{K})$ for each $\mathcal{K}$ left ideal in A, is a $SL_*(2,A)$ invariant partition of $\mathcal{P}_A$. In section 6 we explicit this point of view for the ring $A = M_n(E)$ endowed with the transpose map.

In this note we show $SL_*(2,A)$ acts transitively on $\mathcal{P}_A(A)$ for any $*-$euclidean ring A. For the particular case, $A = M_n(E)$, a consequence of a theorem of Witt give us that $SL_*(2,A)$ acts transitively on $\mathcal{P}_A(\mathcal{K})$, for an arbitrary ideal $\mathcal{K}$. One of our main results yields the orbits structure of the group $SL_*(2,A_0)$ ($A_0 = M_n(F)$) in $\mathcal{P}_A(A)$, this is presented in section 6. A consequence of our result is an analysis of the set of inner anti-involutions of $Sp(n,F)$.

2. Preliminaries and main result

2.1. General setup.

Let $(A,*)$ be an involutive ring. We will consider below the $SL_*(2,A)-$ space $\mathcal{P}_A = \{x : x = {}^t(x_1,x_2) \in A\times A, x_1^* x_2 = x_2^* x_1\}$. Define $Aut(A)$ to be the group of automorphisms or anti-automorphisms of A. Let $G = \{id_A, \tau\}$ be a subgroup of $Aut(A)$ of order 2, and let $A_0 = A^G = \{x \in A : \tau(x) = x\}$. We have

Lemma 1. *$A \supset A_0$ is a Galois extension, i.e., $A_0 = A^{Aut_{A_0}(A)}$.*

Proof. It is clear that $A^{Aut_{A_0}(A)} \supset A_0$. On the other hand, if $x \in A\backslash A_0$, then $\tau(x) \neq x$. So x cannot be an element of $A^{Aut_{A_0}(A)}$. ■

In what follows, the elements of A^2 will be considered as column vectors and A^2 as a left $A-$module.

Remark 1. *$\mathcal{P}_A$ is the additive subgroup of A^2 consisting of the ω_A-isotropic vectors.*

Definition 1. *For a matrix $M = (m_{ij})$ in $M_{n\times m}(A)$ we set $M^* = (m_{ji}^*)$, which lies in $M_{m\times n}(A)$.*

Set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. According to [8], we recall the group

Definition 2. $SL_*(2,A) = \{g \in M_2(A) : g^* J g = J\}$

Lemma 2. *$\mathcal{P}_A$ is stable by the natural (left) action of $SL_*(2,A)$ on A^2.*

Proof. Let g be an element of $SL_*(2,A)$ acting on A^2 and $x = {}^t(x_1,x_2)$ in $\mathcal{P}_A$. Then $\omega_A(gx,gx) = (gx)^* J gx = x^* g^* J g x = x^* J x = \omega_A(x,x) = 0$. ■

For each order two subgroup $G = \{id_A, \tau\}$ we define a $*$ -τ-antihermitian form on A^2 with values in A, which we denote by h_G, by

$$h_G(x,y) = \omega_A(\tau(x),y) \qquad x,y \in A^2. \tag{1}$$

We observe that when we pick $x = {}^t(x_1,x_2)$, $y = {}^t(y_1,y_2)$, $z \in A$, with $x_1 = zx_2$, $y_1 = zy_2$, we have

$$h_G(x,y) = \tau(x_2)^*(\tau(z)^* - z)y_2. \tag{2}$$

$$h_G(xp, yp) = \tau(p)^* h_G(x,y)\, p, \text{for}\, x, y \in A^2, p \in A. \tag{3}$$

We would like to point out that each level set $\{x \in \mathcal{P}_A : h_G(x,x) = c\}$ is $SL_*(2, A_0)-$invariant.

2.2. $*-$Euclidean Rings.

Definition 3. *A unitary ring with involution $*$ is called a $*-$ euclidean ring if given $a, c \in A$ such $a^*c = c^*a$ and $Aa + Ac = A$, there is a finite sequence $s_0, s_1, ..., s_{n-1} \in A^{sym} = \{s \in A : s^* = s\}$ and $r_1, r_2, ..., r_n \in A$, with $r_n \in A^\times$ such that*

$$\begin{aligned} a &= s_0 c + r_1 \\ c &= s_1 r_1 + r_2 \\ . &= . \\ . &= . \\ . &= . \\ r_{n-2} &= s_{n-1} r_{n-1} + r_n. \end{aligned} \tag{1}$$

Examples of such rings are, among others, the integers endowed with the identity as involution and $End_E(V)$, endowed with $*$ map the associated adjoint map coming from a non-degenerate symmetric bilinear form on the finite dimensional space V ([11] page 154, lemme 2).

Lemma 3. *Let A be a ring with involution. Let $a, c \in A$ be such that $a^*c = c^*a$ and let $q, r \in A$, with q a symmetric element, be such that $c = qa + r$. Then $a^*r = r^*a$.*

Proof. Since $r = c - qa$ we have $a^*r = a^*c - a^*qa$ and $r^*a = (c^* - a^*q)a$, from which the result follows. ∎

Lemma 4. *Let $Ac + Aa = A$, assume q is symmetric so that $c = qa + r$. Then $Aa + Ar = A$.*

Proof. There exist $x, y \in A$ such that $1 = xa + yc$, so $1 = xa + y(qa + r)$, which implies $1 = (x + yq)a + yr$. Thus, the lemma follows. ∎

Lemma 5. *If $Aa + Ac = A$, $a^*c = c^*a$, $c = qa + r$, $a = q_1 r + r_1$ with q, q_1 symmetric, and r_1 invertible, then there exist $b, d \in A$ such that*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \textit{ belongs to } SL_*(2, A).$$

Proof. We have $r_1 = a - q_1 r = a - q_1(c - qa) = (1 + q_1 q)a - q_1 c$, then $1 = r_1^{-1}(1 + q_1 q)a - r_1^{-1} q_1 c$.

We set $b = (r_1^{-1}q_1)^*$ and $d = (r_1^{-1}(1+q_1q))^*$, then $1 = d^*a - b^*c$. From lemma 3 we obtain $r^*r_1 = r_1^*r$, which yields $(r_1^*)^{-1}r^* = rr_1^{-1}$. Next, we show

i) $ab^* = ba^*$

ii) $cd^* = dc^*$

iii) $b^*d = d^*b$

Then, in accordance with [8], this shows g belongs to $SL_*(2,A)$. So, we verify i), ii) and iii).
i) $ab^* = ar_1^{-1}q_1 = (q_1r + r_1)r_1^{-1}q_1 = (q_1rr_1^{-1} + 1)q_1$
$ba^* = q_1(r_1^*)^{-1}a^* = q_1(r_1^*)^{-1}(r^*q_1 + r_1^*) = q_1(r_1^*)^{-1}r^*q_1 + q_1$,
hence $ba^* = q_1rr_1^{-1}q_1 + q_1$ and we have verified i).
ii) Since $c = qa + r = q(q_1r + r_1) + r$ we have $c^* = (r_1^* + r^*q_1)q + r^*$.

hence, $cd^* = (qq_1r + qr_1 + r)(r_1^{-1}(1+q_1q))$ and
$dc^* = (1+qq_1)(r_1^*)^{-1}(r_1^*q + r^*q_1q + r^*) = (1+qq_1)(q + rr_1^{-1}q_1q + rr_1^{-1})$
after we compute the multiplications we obtain ii).
iii)We have $b^*d = r_1^{-1}q_1(1+qq_1)(r_1^*)^{-1}$ and $d^*b = r_1^{-1}(1+q_1q)q_1(r_1^*)^{-1}$. ■

Proposition 1. *Let A be a $*-$ euclidean ring. If $Aa + Ac = A$ and $a^*c = c^*a$, then there exist b, d in A such that*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \textit{belongs to}\, SL_*(2,A)$$

Proof. Given that A is a $*-$euclidean ring, there is a finite sequence of symmetric elements $q_0, q_1, ..., q_{n-1}$ and $r_1, r_2, ..., r_n \in A$, with $r_n \in A^\times$ such that

$$\begin{aligned} a &= q_0c + r_1 \\ c &= q_1r_1 + r_2 \\ . &= . \\ . &= . \\ . &= . \\ r_{n-2} &= q_{n-1}r_{n-1} + r_n. \end{aligned}$$

Applying lemmas 3 and 4 to the sequence, yields $Ar_{n-2} + Ar_{n-3} = A$, $r_{n-3}^*r_{n-2} = r_{n-2}^*r_{n-3}$.
On the other hand, $r_{n-3} = q_{n-2}r_{n-2} + r_{n-1}$ and $r_{n-2} = q_{n-1}r_{n-1} + r_n$ where q_{n-2}, q_{n-1} are symmetric and r_n invertible.
We can apply then lemma 5 to get b_{n-2}, d_{n-2} such that

$$g_{n-2} := \begin{pmatrix} r_{n-2} & b_{n-2} \\ r_{n-3} & d_{n-2} \end{pmatrix} \text{ belongs to}\, SL_*(2,A).$$

We multiply on the left this last equality by the element

$\begin{pmatrix} 1 & q_{n-3} \\ 0 & 1 \end{pmatrix}$ of $SL_*(2, A)$, to get

$$\begin{pmatrix} 1 & q_{n-3} \\ 0 & 1 \end{pmatrix} g_{n-2} = \begin{pmatrix} r_{n-4} & -d_{n-3} \\ r_{n-3} & -b_{n-3} \end{pmatrix} \text{for some } b_{n-3}, d_{n-3} \in A.$$

By proposition 3 of [8]

$$g_{n-3} := \begin{pmatrix} r_{n-3} & b_{n-3} \\ r_{n-4} & d_{n-3} \end{pmatrix} \text{ belongs to } SL_*(2, A).$$

This process, applied $n-2$ times, give us $b, d \in A$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $SL_*(2, A)$.

From which the proposition follows. ■

Definition 4. *Given* ${}^t(a, c) \in \mathcal{P}_A$ *, with* $Aa + Ac = A$*. The generic vector line* $\mathbf{L}_{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)}$ *in* $\mathcal{P}_A$ *is the subset of* A^2 *consisting of all right multiples* ${}^t(a, c)r$ *where* r *runs over the set of invertible elements of* A*. We write* $\mathbb{P}^1_\times(\mathcal{P}_A)$ *for the set of generic vector lines.*

Corollary 1. *For an Euclidean ring* A*, the group* $SL_*(2, A)$ *acts transitively on the set* $\mathbb{P}^1_\times(\mathcal{P}_A)$*.*

Proof. We observe first that ${}^t(1, 0)$ defines a generic line. Owing to the proposition, given a generator of a generic line $\mathbf{L}_{{}^t(a,c)}$, there exists $b, d \in A$ so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_*(2, A)$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, the corollary follows. ■

Remark 2. *We have that* $\mathbf{L}_{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)} = \mathbf{L}_{\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right)}$ *if and only if* $a' = ar, c' = cr$ *for an invertible element* r *in* A*.*

2.3. The full matrix ring case.

We now specialize to the case where the involutive ring $(A, *)$ is the full matrix ring $M_n(E)$ over a finite field E, endowed with the transpose mapping. We assume that E is a quadratic extension of a subfield F with Galois group $\Gamma := \{Id, \tau\}$. Then τ extends to an automorphism τ of $A = M_n(E)$, and A is a Galois extension of $A_0 = M_n(F)$, with Galois group $\Gamma := \{Id, \tau\}$. Thus, we have two special linear groups, defined respectively, over the rings A and A_0, the group $SL_*(2, A)$ and the special linear group obtained restricting the coefficients from A to the fixed subring A_0. From now on, we write $\tau(x) =: \bar{x}$. Henceforth, we consider the symplectic vector space E^{2n}, of column vectors, endowed with the canonical symplectic form ω, that in terms of the canonical basis $e_1, \cdots, e_{2n}$ for E^{2n} is given by $\omega(e_j, e_{n+j}) = -\omega(e_{n+j}, e_j) = 1, j = 1, \ldots, n$ and $\omega(e_k, e_s) = 0$ for $|k - s| \neq n$.

It follows from [6] that the elements of $SL_*(2,A) = Sp(n,E)$, are described as the $2n\times 2n$ matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\ A,B,C,D \in M_n(E): \qquad {}^tAD - {}^tCB = 1,\ {}^tAC = {}^tCA,\ {}^tBD = {}^tDB. \quad \text{(SE)}$$

We have an analogous description for $SL_*(2,A_0) = Sp(n,F)$.
The set of classical Lagrangian subspaces for ω in E^{2n} is denoted by $\mathcal{L}_{E,2n}$.

A generic vector line $\mathbf{L}_{(a,b)} \subset \mathcal{P}_{M_n(E)}$ may be readily identified with classical Lagrangians. Indeed, in [6] a Lagrangian subspace L in E^{2n} is described as $L = L_{(a,b)} = L_{\left(\begin{smallmatrix} a \\ b\end{smallmatrix}\right)} = \langle Pa + Qb\rangle \quad (a,b\in A, Aa + Ab = A, a^*b = b^*a)$ where the (row) vectors P and Q are given by $P = (e_1,\cdots,e_n), Q = (e_{n+1},\cdots,e_{2n})$ and $\langle u\rangle$ stands for the vector subspace of E^{2n} spanned by the components $u_1,...,u_n$ of any $u\in M = (E^{2n})^n$. We have

$$L_{(a,b)} := L_{\left(\begin{smallmatrix} a \\ b\end{smallmatrix}\right)} := \left\{ \begin{pmatrix} ax \\ bx \end{pmatrix}, x\in E^n \right\}.$$

(2.3.1) We recall $L_{(a,b)} = L_{(c,d)}$ if and only if there exists $p\in GL_n(E)$ so that $a = cp, b = dp$. This is so, owing to the simple fact that two liner transformations s,t from one vector space into other, have the same image if and only if there exists an invertible linear operator p on the initial vector space such that $s = tp$. Therefore, remark 2, allow us to conclude: the generic vector line $\mathbf{L}_{(a,c)}$ can be identify with the Lagrangian $L_{(a,c)}$, Thus, the classical Lagrangian set $\mathcal{L}_{E,2n}$ is in bijective correspondence with the space $\mathbb{P}^1_\times(\mathcal{P}_{M_n(E)})$ of non commutative vector lines through the origin in $\mathcal{P}_{M_n(E)}$.

The group $SL_*(2, M_n(E)) = Sp(n,E)$, acts in both spaces $\mathbb{P}^1_\times(\mathcal{P}_{M_n(E)})$, $\mathcal{L}_{E,2n}$, obviously the map $\mathbf{L}_{(a,b)} \mapsto L_{(a,b)}$ is equivariant. In Corollary 1, we have shown that the group $Sp(n,E)$ acts transitively on the set of generic vector lines $\mathbb{P}^1_\times(\mathcal{P}_{M_n(E)})$, hence it acts transitively in the space $\mathcal{L}_{E,2n}$. This last observation is also a consequence of a theorem of Witt.

It is known that for a finite field E and a hermitian form (W,h) on a finite dimensional vector space W over E, there always exists an ordered basis $w_1,\ldots$ of W and a nonnegative integer r so that $h(w_k,w_s) = \delta_{ks}$ (δ_{ij} is as usual Kronecker delta) for $k,s\le r$ and $h(w_k,w_s) = 0$ for $k > r$ or $s > r$. In this situation we define the type of the form (W,h) to be r.

We denote the set of $n\times n$ symmetric matrices with coefficients in E by $Sym(E^n)$. The isotropy subgroup for the subspace L_+ spanned by $e_1,\ldots,e_n$ is the semidirect product of the subgroups

$$K := \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, A\in GL_n(E) \right\} \tag{4}$$

$$P^+ = \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, B\in Sym(E^n) \right\} \tag{5}$$

On the other hand, the isotropy subgroup for the subspace L_- spanned by the vectors $e_{n+1}, \ldots, e_{2n}$ is the semidirect product of K times the subgroup

$$P^- = \left\{ \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}, B \in Sym(E^n) \right\} \tag{6}$$

Let $\mathcal{L} : Sym(E^n) \to \mathcal{L}_{E,2n}$ be the Siegel map defined by the formula

$$\mathcal{L}(Z) = \left\{ \begin{pmatrix} Zx \\ x \end{pmatrix}, x \in E^n \right\} = L_{\left(\begin{smallmatrix} Z \\ I_n \end{smallmatrix}\right)} \tag{7}$$

The Siegel Lagrangian $\mathcal{L}(Z) = L_{Z,I_n}$ in the notation of [6] . $L_- = L_{0,I_n}$.

Remark 3. *Whenever $F = \mathbb{R}$, we have that $\mathcal{L}(Z)$ is equal to the action on the subspace L_- of the exponential of the Lie algebra element* $(0, Z, 0, 0) \in \mathfrak{sp}(n, \mathbb{C})$.

We define, the *bar*$-$anti-hermitian form h_E on E^{2n} by the equality

$$h_E(v, w) = \omega(\bar{v}, w) = {}^t\bar{x}s - {}^t\bar{y}r, \;\; v, w \in E^{2n}, v = \begin{pmatrix} x \\ y \end{pmatrix}, w = \begin{pmatrix} r \\ s \end{pmatrix}. \tag{8}$$

Then, h_E is a non-degenerate, *bar*$-$anti-hermitian form, i.e, $\overline{h_E(x, y)} = -h_E(y, \bar{x})$, we have $Sp(n, F) = U(E^{2n}, h_E) \cap Sp(n, E)$.

Proposition 2. *We have the decomposition into $Sp(n, F)-$invariant subsets*

$$\mathcal{L}_{E,2n} = \bigcup_{0 \le r \le n} \mathcal{H}_r,$$

where $\mathcal{H}_r$ stands for the set of all $W \in \mathcal{L}_{E,2n}$ such that the type (rank) of h_E restricted to $W \times W$ is r.

In order to carry out the proofs of the results, it is going to be useful to consider the hermitian form

$$h_0 : E^{2n} \times E^{2n} \to E \tag{9}$$

defined so that the canonical basis is an orthogonal basis for h_0, $h_0(e_j, e_j) = -1$ for $1 \le j \le n$ and $h_0(e_j, e_j) = 1$ for $n + 1 \le j \le 2n$.
We define the group

$$Sp_0(n, F) := U(E^{2n}, h_0) \cap Sp(n, E). \tag{10}$$

Later on, for a finite field F in proposition 3, we recall a generalized Cayley transform, C in $Sp(n, E)$, studied by [11], i.e., we show there exists an element which conjugates $Sp(n, F)$ into $Sp_0(n, F)$. That is, $C^{-1}Sp_0(n, F)C = Sp(n, F)$, a well known result for $F = \mathbb{R}$. Actually, we verify in (conf),

$$h_0(Cv, Cw) = c_n h_E(v, w). \tag{11}$$

Among the objectives of this note are, for a finite field F, to determine the orbits of both groups $Sp(n,F), Sp_0(n,F)$ in $\mathcal{L}_{E,2n}$ and the intersection of each orbit with the image of the Siegel map. When $F=\mathbb{R}, E=\mathbb{C}$ this problem has been considered and solved by [14], [5] and references therein. In [3] a description for $Sp(n,F)\backslash Sp(n,E)/P_0$, is given, here P_0 is a minimal parabolic subgroup for $Sp(n,E)$. See also [9]. In [4], the computation of the compression semigroup of each of the orbits is treated for the case $F=\mathbb{R}$.

Let $\mathcal{O}_r$ the set of Lagrangian subspaces $W\in\mathcal{L}_{E,2n}$ so that the form h_0 restricted to W is of type r. Obviously $Sp_0(n,F)$ leaves invariant the subset $\mathcal{O}_r$ and $\mathcal{L}_{E,2n}=\mathcal{O}_n\cup\mathcal{O}_{n-1}\cup\cdots\cup\mathcal{O}_0$.
One of the main results of this work is:

Theorem 1. *Assume F is a finite field, then*

- *The orbits of $Sp_0(n,F)$ in $\mathcal{L}_{E,2n}$ are exactly the sets $\mathcal{O}_j, j=0,\cdots,n$.*
- *The orbits of $Sp(n,F)$ in $\mathcal{L}_{E,2n}$ are exactly the sets $\mathcal{H}_j, j=0,\cdots,n$.*
- *Any orbit of either $Sp(n,F)$ or $Sp_0(n,F)$ intersects the image of the Siegel map.*
- *Except for $n=1$ and the orbit $\mathcal{O}_0$, no orbit of $Sp_0(n,F)$ is contained in the image of the Siegel map.*
- *$\mathcal{H}_n$ is the unique orbit of $Sp(n,F)$ contained in the image of the Siegel map.*
- *$C\mathcal{H}_j=\mathcal{O}_j$.*

Next, we reformulate the classical statements in theorem 1, in the language of the form h_Γ defined in (1). To begin with, we point out the equality

$$ {}^t\overline{x}\, h_\Gamma\left(\left(\begin{smallmatrix} r \\ s\end{smallmatrix}\right),\left(\begin{smallmatrix} t \\ u\end{smallmatrix}\right)\right) y = h_E\left(\left(\begin{smallmatrix} rx \\ sx\end{smallmatrix}\right),\left(\begin{smallmatrix} ty \\ uy\end{smallmatrix}\right)\right), \text{for } x,y\in E^n, r,s,t,u\in M_n(E). $$

Hence, for $x\in\mathcal{P}_{M_n(E)}$ the rank of the matrix $h_\Gamma(x,x)$ is equal to the rank of the form h_E restricted to the lagrangian subspace L_x. Thus, $\mathcal{H}_r$ is equal to the image, under the map $\mathbf{L}_x\mapsto L_x$, of the set $\tilde{\mathcal{H}}_r$ defined by $\mathbf{L}\in\mathbb{P}^1_\times(\mathcal{P}_{M_n(E)})$ so that rank of $h_\Gamma(x,x)$ is equal to r for some representative x of $\mathbf{L}$. Since the map $\mathbf{L}_x\mapsto L_x$ is equivariant, the second affirmation in theorem 1 may be stated as: The group $SL_*(2,A^\Gamma)$ acts transitively in $\tilde{\mathcal{H}}_r$.

For the orbits $\mathcal{O}_r$ we have a somewhat similar way to restate the fact that $Sp_0(n,F)$ acts transitively. For this, we consider the form H_0 on A^2 defined by $H_0(v,w)=-{}^t\overline{x}r+{}^t\overline{y}s=h_\Gamma(v,\iota(w)),\ v=\begin{pmatrix} x\\ y\end{pmatrix}, w=\begin{pmatrix} r\\ s\end{pmatrix}$. Here $\iota(a,b)=(b,a)$. Since $h_0(x,y)=h_E(x,\iota y)$, it readily follows that

$$ {}^t\overline{x}\, H_0\left(\left(\begin{smallmatrix} r \\ s\end{smallmatrix}\right),\left(\begin{smallmatrix} t \\ u\end{smallmatrix}\right)\right) y = h_0\left(\left(\begin{smallmatrix} rx \\ sx\end{smallmatrix}\right),\left(\begin{smallmatrix} ty \\ uy\end{smallmatrix}\right)\right), \text{for } x,y\in E^n, r,s,t,u\in M_n(E). $$

It follows from (11) and the two previous equalities that the Cayley transform C is a conformal map between $(A^2\times A^2, h_\Gamma)$ and $(A^2\times A^2, H_0)$. Thus, the

subgroup $Sp_0(n,F)$ of $Sp(n,E)$ is the subgroup $SL_*(2,H_0)$ of $SL_*(2,M_n(E))$ of elements leaving invariant the form H_0. Now, we consider the set $\tilde{\mathcal{O}}_r$ defined by $\mathbf{L}\in\mathbb{P}^1_\times(\mathcal{P}_{M_n(E)})$ so that rank of $H_0(x,x)$ is equal to r for some representative x of $\mathbf{L}$. Since the map $\mathbf{L}_x\mapsto L_x$ is $SL_*(2,H_0)=Sp_0(n,F)-$equivariant, the first affirmation in theorem 1 may be stated as: The group $SL_*(2,H_0)$ acts transitively on $\tilde{\mathcal{O}}_r$.
In the setting $\mathcal{P}_A$, the image of the Siegel map turns out to be the set $\mathcal{S}_A$ of vector lines $\mathbf{L}_{(a,b)}$ so that $b\in A^\times=GL_n(E)$. Some of the remaining statements are restated as: For $n>1$, the orbit $\tilde{\mathcal{H}}_n$ is contained in $\mathcal{S}_A$ and the orbits $\tilde{\mathcal{H}}_r, r=0,\dots n-1$ intersects non trivially $\mathcal{S}_A$ as well its complement.

3. Proofs

In order to write down the proof of theorem 1 and some of its consequences, we need to set up some notation and recall some known facts.
Following Siegel, we write sometimes (A,B,C,D) for the $2n\times 2n$ matrix

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\ A,B,C,D\in M_n(E).$$

${}^tA$ denotes the transpose of the matrix A. Vectors v in E^k are column vectors, so that we write ${}^tv$ for the row vector corresponding to v.
In particular, we will use

$$E^{2n}\ni v=\begin{pmatrix} x\\ y\end{pmatrix}, x,y\in E^n, E^{2n}\ni w=\begin{pmatrix} r\\ s\end{pmatrix}, r,s\in E^n.$$

Let I_n denote the $n\times n$ identity matrix and 0 denotes the zero matrix. We set

$$J:=\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}.$$

Hence, $\omega(v,w)={}^txs-{}^tyr={}^tvJw$. Let $G_n(E^{2n})$ denote the Grassmannian of the $n-$dimensional subspaces of E^{2n}. Hence, any of the groups $Sp(n,E), Sp(n,F)$, $Sp_0(n,F)$ acts on $G_n(E^{2n})$ by $TW=T(W)$.

A $n-$dimensional linear subspace W of (E^{2n},ω) is a Lagrangian subspace if and only if for every $v,w\in W$, ${}^tvJw=0$, if and only if ${}^txs-{}^tyr=0$ for every $v,w\in W$. For $R,S\in E^{n\times n}$ the subspace $L_{(R,S)}$ is Lagrangian, if and only if ${}^tRS-{}^tSR=0$ and the matrix $\left(\begin{smallmatrix} R\\ S\end{smallmatrix}\right)$ has rank n. Actually, any Lagrangian subspace may be written as in the previous example (see also [8]). Particular examples of Lagrangian subspaces are $L_+,L_-=\mathcal{L}(0),\mathcal{L}(Z)$, $(Z\in Sym(E^n))$. Needles to say, the image of $\mathcal{L}$ is equal to the orbit L_- under the subgroup P^+, hence, Bruhat's decomposition yields that the image of $\mathcal{L}$ is "open and dense" in $\mathcal{L}_{E,2n}$. Let $p:E^{2n}\to E^n$ denotes projection onto the second component. That is, $p\left(\begin{smallmatrix} x\\ y\end{smallmatrix}\right)=y$. It follows that:
(3.1) A subspace $W\in G_n(E^{2n})$ belongs to the image of $\mathcal{L}$ if and only if W is Lagrangian and $p(W)$ is equal to E^n. Since $p(L_{(a,b)})=Im(b)=b(E^n)$, we have $L_{(a,b)}$ belongs to image of the Siegel set if and only if b is an invertible matrix. We

obtain,
(3.2) For $(A,B,C,D) \in Sp(n,E), Z \in Sym(E^n)$, the subspace $(A,B,C,D)\mathcal{L}(Z) = L_{(AZ+B),CZ+D)}$ belongs to the image of $\mathcal{L}$ if and only if $(CZ+D)$ is an invertible matrix. Therefore,
(3.3) Let G be either $Sp(n,F)$ or $Sp_0(n,F)$ and fix $Z \in Sym(E^n)$. Then the orbit $G\mathcal{L}(Z)$ is contained in the image of $\mathcal{L}$ if and only if for every $(A,B,C,D) \in G$ the matrix $(CZ+D)$ is invertible. In this case, $(A,B,C,D)\mathcal{L}(Z) = \mathcal{L}((AZ+B)(CZ+D)^{-1})$. We give a necessary and sufficient condition on $g \in Sp(n,E)$ so that $g.\mathcal{L}(Z)$ belongs to the image of the Siegel map in proposition 6.

Example 1. Orbits of $Sp_0(1,F)$ in the space of Lagrangians $\mathcal{L}_{E,2}$. We assume F is a finite field. Let $N(e) = e\bar{e}$ be the norm of the extension E/F. The hypothesis on F implies N is a surjective map onto F. After a computation, we obtain that $Sp_0(1,F)$ is the set of matrices

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in E, \alpha\bar{\alpha} - \beta\bar{\beta} = 1 \right\}.$$

In this case $\mathcal{L}_{E,2} = G_1(E^2)$. Since $h_0((\begin{smallmatrix} z \\ 1 \end{smallmatrix}), (\begin{smallmatrix} w \\ 1 \end{smallmatrix})) = 1 - \bar{z}w$, it readily follows:

$$\mathcal{O}_1 = \{L_{(z,1)}, z \in E, N(z) \neq 1\} \cup \{L_+\},$$
$$\mathcal{O}_0 = \{L_{(z,1)}, z \in E, N(z) = 1\}.$$

For z so that $N(z) \neq 1$ we have $(1 - z\bar{z})^{-1} = t\bar{t}, t \in E$. For the matrix

$$A := \begin{pmatrix} \bar{t} & zt \\ \bar{z}\bar{t} & t \end{pmatrix}$$

we have $AL_{(0,1)} = L_{(zt,t)} = L_{(z,1)}$. Obviously $A \in Sp_0(1,F)$. We are left to transform $L_{(0,1)}$ into $L_{(1,0)}$. For this, we fix $z \neq 0$ such that $N(z^{-1}) \neq 1$. Then, by means of A the line $L_{(1,0)}$ is transformed into the line $L_{(1,\bar{z})}$, which is equal to the line $L_{(\bar{z}^{-1},1)}$. Thus, $Sp_0(1,F)$ acts transitively on $\mathcal{O}_1$.
We now show $Sp_0(1,F)$ acts transitively in $\mathcal{O}_0$. We fix $L_{(a,1)}$ so that $a\bar{a} = 1$. Let $L_{(b,1)}$ in $\mathcal{O}_0$. Then $N(a) = N(b)$, owing to theorem 90 of Hilbert we have $\frac{a}{b} = d\bar{d}^{-1}$. Since the characteristic of F is different from two, the pair of vectors $(\begin{smallmatrix} a \\ 1 \end{smallmatrix}), (\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix})$, as well as $(\begin{smallmatrix} b \\ 1 \end{smallmatrix}), (\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix})$ determine two ordered basis for E^2. Let T be the linear operator defined by $T((\begin{smallmatrix} a \\ 1 \end{smallmatrix})) = d(\begin{smallmatrix} b \\ 1 \end{smallmatrix})$ and $T((\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix})) = d^{-1}(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix})$. A short computation gives $h_0(T(\begin{smallmatrix} a \\ 1 \end{smallmatrix}), T(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix})) = h_0(d(\begin{smallmatrix} b \\ 1 \end{smallmatrix}), d^{-1}(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}))$ and that $\omega(T(\begin{smallmatrix} a \\ 1 \end{smallmatrix}), T(\begin{smallmatrix} 1 \\ -\bar{a} \end{smallmatrix})) = \omega(d(\begin{smallmatrix} b \\ 1 \end{smallmatrix}), d^{-1}(\begin{smallmatrix} 1 \\ -\bar{b} \end{smallmatrix}))$, hence T lies in $U(E^2, h_0) \cap Sp(1,E) = Sp_0(1,F)$. Whence, $\mathcal{O}_0$ is an orbit of $Sp_0(1,F)$.

Remark 4. *The orbit $\mathcal{O}_0$ is contained in the image of the Siegel map, whereas the orbit $\mathcal{O}_1$ does contain a point in the complement to the image of the Siegel map. This observation shows that for a finite field F and $n = 1$ our conclusions are in concordance with the results obtained by other authors for the case of $F = \mathbb{R}$. More precisely in the real case, $\mathcal{O}_1$ splits in the union of two orbits, one orbit is the set of lines where h_0 is positive definite and the other is the set of lines where h_0 is negative definite. In this case the orbit corresponding to the set of lines where h_0 is positive definite is contained in the image of the Siegel map, whereas the orbit corresponding to the set of lines where h_0 is negative definite is not contained in* *the image of the Siegel map. The orbit corresponding to the set of lines where h_0 vanishes is contained in the image of the Siegel map.*

Lemma 6. *Z be an element of $Sym(E^n)$. Then $\mathcal{L}(Z)$ belongs to $\mathcal{H}_r$ if and only if the anti-hermitian form on E^n defined by $Z - \bar{Z}$ has rank r.*

Proof. In fact, the form h_E on $\mathcal{L}(Z)$ is given by

$$h_E(\left(\begin{smallmatrix} Zx \\ x \end{smallmatrix}\right), \left(\begin{smallmatrix} Zy \\ y \end{smallmatrix}\right)) = {}^t\bar{x}(\bar{Z} - Z)y \quad x, y \in E^n,$$

from which the lemma follows. ∎

Example 1. *For a finite field F, the orbits of $Sp(1, F)$ in $\mathcal{L}_{E,2}$ are $\mathcal{H}_r, r = 0, 1$. In fact,*

$$\mathcal{H}_1 = \{L_{(z,1)} : z - \bar{z} \neq 0\} \text{ and } \mathcal{H}_0 = \{L_{(z,1)} : z \in F\} \cup \{L_+ = L_{(1,0)}\}.$$

Since $J \in Sp(n, F)$ we have that $L_{(1,0)}$ is in the orbit of $L_{(0,1)}$. Since the matrix $(1, s, 0, 1) \in Sp(1, F), s \in F$ and $(1, s, 0, 1)\,{}^t(0, 1) = {}^t(s, 1)$ we have that $Sp(1, F)$ acts transitively in $\mathcal{H}_0$.
Next, we show that $Sp(1, F)$ acts transitively in $\mathcal{H}_1$. Let $L_{(z,1)}, L_{(w,1)}$ so that $z - \bar{z} \neq 0, w - \bar{w} \neq 0$, Since F is a finite field, there exists $t_0 \in E$ so that $z - \bar{z} = t_0\bar{t}_0(w - \bar{w})$. We define

$$A := \frac{1}{z - \bar{z}} \begin{pmatrix} t_0 w - \bar{t}_0\bar{w} & z\bar{t}_0\bar{w} - \bar{z}t_0 w \\ t_0 - \bar{t}_0 & z\bar{t}_0 - \bar{z}t_0 \end{pmatrix}$$

The coefficients of A belong to F and

$$A\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = \frac{z}{z - \bar{z}} \begin{pmatrix} t_0 w - \bar{t}_0\bar{w} \\ t_0 - \bar{t}_0 \end{pmatrix} + \frac{1}{z - \bar{z}} \begin{pmatrix} z\bar{t}_0\bar{w} - \bar{z}t_0 w \\ z\bar{t}_0 - \bar{z}t_0 \end{pmatrix} = t_0 \begin{pmatrix} w \\ 1 \end{pmatrix}.$$

$$detA = \frac{(z - \bar{z})(w - \bar{w})t_0\bar{t}_0}{(z - \bar{z})^2} = 1.$$

We note that $\mathcal{H}_1$ is contained in the image of the Siegel map, whereas $\mathcal{H}_0$ is not.

3.1. Orbits of $Sp(n, F), Sp_0(n, F)$.

Lemma 7. *$Sp_0(n, F)$ acts transitively on $\mathcal{O}_n$.*

Proof. For a matrix A, we writte $A^\star = {}^t\bar{A}$. We have that $L_- = \mathcal{L}(0)$ is an element of $\mathcal{O}_n$. First, we will prove that given $\mathcal{L}(Z) \in \mathcal{O}_n$, there is an element of $Sp_0(n, F)$ which carries $\mathcal{L}(Z)$ onto L_-.
The matrix of the form h_0 restricted to $\mathcal{L}(Z)$ is $I_n - \bar{Z}Z$. Since $\mathcal{L}(Z) \in \mathcal{O}_n$, there exists an invertible matrix A so that $A(I_n - Z\bar{Z})\,{}^t\bar{A} = I_n$. Set $B := -AZ$. Then, since

$$A\,{}^t(-AZ) = -AZ\,{}^tA, \text{ and } A\,{}^t\bar{A} - (-AZ)(-\,{}^t(\bar{A}\bar{Z}) = A(I_n - Z\bar{Z})\,{}^t\bar{A} = I_n,$$

the matrix $(A, B, \bar{B}, \bar{A})$ belongs to $Sp_0(n, F)$ (it satisfies (SOR)).

On the other hand,

$$(A, B, C, D)\mathcal{L}(Z) = \left\{ \begin{pmatrix} (AZ+(-AZ))x \\ (\bar{B}Z+\bar{A})x \end{pmatrix}, \ x \in E^n \right\} = \left\{ \begin{pmatrix} 0 \\ \bar{A}(I_n-\bar{Z}Z)x \end{pmatrix}, \ x \in E^n \right\}.$$

Since the matrix $\bar{A}(I_n - \bar{Z}Z)$ is invertible, $\mathcal{L}(Z)$ belongs to the orbit of L_-.
Next, we will show that if $W := L_{(R,S)} = \{\left(\begin{smallmatrix} Rx \\ Sx \end{smallmatrix}\right) : x \in E^n\} \in \mathcal{O}_n$, then there exists an element g in $Sp_0(n, F)$ so that $gW \in Image(\mathcal{L})$.
In fact, we will show there exists $g \in Sp_0(n, F)$ so that $gW = \{(Cx, Dx) : x \in E^n\}$ with C invertible, then, by means of a matrix $(0, dI_n, \bar{d}I_n, 0)$ we transform gW into an element of the image of the Siegel map. Since W is in $\mathcal{O}_n$, there exists an invertible matrix A such that

$$A(-R^\star R + S^\star S)A^\star = I_n.$$

Let us consider $g = (-AR^\star, AS^\star, \bar{A}\bar{S}^\star, -\bar{A}\bar{R}^\star)$. Then

$$gW = \{((A^\star)^{-1}x, (\bar{A}\bar{S}^\star R - \bar{A}\bar{R}^\star S)x)x \in E^n\}.$$

Since

$$-AR^\star(-AR^\star)^\star - AS^\star(AS^\star)^\star = A(-R^\star R + S^\star S)A^\star = I_n$$

because W is a Lagrangian subspace, ${}^tRS = {}^tSR$, hence, we have $-AR^\star\,{}^t(AS^\star) = -AR^\star\bar{S}\,{}^tA = -A\,{}^t\bar{S}\bar{R}\,{}^tA = AS^\star\,{}^t(AR^\star)$, and so the matrix g belongs to $Sp_0(n, F)$. This concludes the proof that $\mathcal{O}_n$ is the orbit of L_- under the group $Sp_0(n, F)$. We sketch a different proof in Note 1. ∎

Proposition 3. *There exists an element C_n in $Sp(n, E)$ so that C_n^{-1} conjugates $Sp_0(n, F)$ onto $Sp(n, F)$.*

Proof. We follow the proof in [11]. We choose $v, b \in E \backslash F$ so that $N(v) = -1$, and $b + \bar{b} = 0$. Then, $b(v^2 - 1)$ is a square in E. We define

$$D_n := \frac{1}{\sqrt{b(v^2-1)}} \begin{pmatrix} vI_n & bI_n \\ I_n & vbI_n \end{pmatrix}$$

Then, $D_n \in Sp(n, E)$ and a computation gives ${}^t\bar{D}_n D_n = \frac{\sqrt{v^2}}{v}(0, I_n, I_n, 0)$. We fix a square root $i \in E$ of -1. Let $C_n := diag(iI_n, I_n)D_n diag(-iI_n, I_n)$. Then ${}^t\bar{C}_n diag(-I_n, I_n)C_n = i\frac{\sqrt{v^2}}{v}J$. Since conjugation by the matrix $diag(iI_n, I_n)$ leaves invariant $Sp(n, E)$ and any automorphism of $Sp(n, E)$ is inner [2], we obtain $C_n \in Sp(n, E)$. Moreover, for a suitable element $c_n \in E$ we have,

$$h_0(C_n v, C_n w) = c_n h_E(v, w). \qquad (conf)$$

Hence, conjugation by C_n carries $Sp(n, F)$ onto $Sp_0(n, F)$.
When -1 is not an square in F, a Cayley transform is given by the matrix

$$C_n := \frac{1}{\sqrt{-2}} \begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix}$$

From here the result ∎

Corollary 2. *The group $Sp(n,F)$ acts transitively on $\mathcal{H}_n$.*

Proof. Since the groups $Sp(n,F)$ and $Sp_0(n,F)$ are conjugated by the Cayley transform and the Cayley transform is a conformal map for the pair of bilinear forms h_0, h_E the corollary follows. We sketch a different proof for this corollary in Note 1. ■

For a subset W of E^{2n}, we define $\overline{W} = \{\bar{w} : w \in W\}$. For the linear subspace W, we denote by r_W the rank of the form h_E restricted to W.

Lemma 8. *For a Lagrangian subspace W of E^{2n} we have:*

$$dim(W + \overline{W}) = n + r_W$$

$$dim(W \cap \overline{W}) = n - r_W$$

Furthermore, $W \cap \overline{W} = (W + \overline{W})^{\perp_\omega} = (W + \bar{W})^{\perp_{h_E}}$.

Proof. We use the identities

$$Z^{\perp_\omega} \cap U^{\perp_\omega} = (Z+U)^{\perp_\omega}, (Z \cap U)^{\perp_\omega} = Z^{\perp_\omega} + U^{\perp_\omega}.$$

Since $W, \overline{W}$ are Lagrangian subspaces we have

$$W \cap \overline{W} = W^{\perp_\omega} \cap \overline{W}^{\perp_\omega} = (W + \overline{W})^{\perp_\omega}.$$

Fix $y = \bar{z} \in W \cap \overline{W}, z \in W, \text{and}\, x \in W$, then $h_E(x,y) = \omega(x,\bar{y}) = \omega(x,z) = 0$. We have then, $y \in W^{\perp_{h_E}}$. Next, for $y \in W^{\perp_{h_E}}$, we have $\omega(\bar{x}, y) = 0$ for every $x \in W$. The hypothesis W is Lagrangian forces $\bar{y} \in W$, hence $y = \bar{\bar{y}} \in W \cap \overline{W}$. ■

Proposition 4. *For a finite field F and $k = 0, \ldots, n$, the group $Sp(n,F)$ acts transitively on $\mathcal{H}_k$.*

Proof. We make the following induction hypothesis: for every $m < n$ and for every $k \leq m$ the group $Sp(m,F)$ acts transitively on the $\mathcal{H}_k$ determinated by the corresponding form h_E on (E^{2m}, ω).
Since we have already shown that $Sp(1,F)$ acts transitively on $\mathcal{H}_k, k = 0, 1$, the first step of the induction process follows.
We recall also that for n and $k = n$ we have shown that $Sp(n,F)$ acts transitively on $\mathcal{H}_n$. We are left to consider $r < n$.
We fix $W, Y \in \mathcal{H}_r$ with $r = r_W < n$, we must find $g \in Sp(n,F)$ so that $gW = Y$. Since each of the subspaces $W \cap \overline{W}, W + \cap\overline{W}$ are invariant under the Galois automorphism, it follows that the subspaces are the complexification of, respectively, $F^{2n} \cap W \cap \overline{W}, F^{2n} \cap (W + \cap\overline{W})$. We notice that the quotient space $(W + \cap\overline{W})/(W \cap \overline{W})$ is of dimension $n + r - (n - r) = 2r < 2n$. Now, by above we have that the push forward to $(W + \overline{W})/(W \cap \overline{W})$ of the form ω is a non degenerate form, and the same holds for h_E.

Thus, the inductive hypothesis gives a linear transform

$$T : F^{2n} \cap (W + \overline{W})/(F^{2n} \cap W \cap \overline{W}) \to F^{2n} \cap (Y + \overline{Y})/(F^{2n} \cap Y \cap \overline{Y})$$

such that $T^{\star}\omega = \omega$, and the complex extension transforms $W/(W \cap \overline{W})$ onto $Y/(Y \cap \overline{Y})$. We lift T to a linear transform

$$T : F^{2n} \cap (W + \overline{W}) \to F^{2n} \cap (Y + \overline{Y})$$

so that $T^{\star}\omega = \omega$ and the complex extension transforms W onto Y. Now we apply the theorem of Witt to T to get an element g of $Sp(n, F)$ which carries W into Y. This completes the induction process and we have the result ■

Corollary 3. *$Sp_0(n, F)$ acts transitively in $\mathcal{O}_k, k = 1, \ldots, n$.*

(3.1.1) Corollary 3 together with Proposition 4 show the first and second statement in Theorem 1.

Note 1. A different proof for Corollary 2 runs as follows, let W, U be elements of $\mathcal{H}_n$. Since E is a finite field, there exists a linear isometry $t : (W, h_E) \to (U, h_E)$. Owing to lemma 9, we have the orthogonal decompositions for the form h_E, $E^{2n} = W \oplus \overline{W} = U \oplus \overline{U}$. Thus, we may and will extend t to a linear operator T of E^{2n} by the formulae $T(w_1, \bar{w}_2) = (t(w_1), \overline{t(w_2)})$. Thus, $T \in U(E^{2n}, h_E)$ and since T commutes with the bar linear operator we have $T \in Sp(n, F)$. A similar proof can be carried out for Lemma 7, replacing $\overline{U}$ for $\iota(\overline{U})$ and h_E for h_0.

3.2. Relative position between orbits and image Siegel map. Let $g \in Sp_0(n, F)$, then $g^{-1} = diag(-I_n, I_n)\ {}^t\bar{g}\, diag(-I_n, I_n)$. Therefore, the elements of $Sp_0(n, F)$ are the matrices

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} A, B \in M_n(E),\ {}^t\bar{A}B = {}^tB\bar{A},\ {}^tA\bar{A} - {}^t\bar{B}B = I \qquad (SOI)$$

Since $Sp_0(n, F)$ is invariant under the map $g \mapsto {}^tg$, we get the characterization of $Sp_0(n, F)$ obtained by [10], namely,
$(R, S, T, V) \in Sp_0(n, F)$ if and only if

$$T = \bar{S},\ V = \bar{R},\ R\,{}^tS = S\,{}^tR,\quad R\,{}^t\bar{R} - S\ {}^t\bar{S} = I_n. \qquad (SOR)$$

A simple computation shows:

$$Sp_0(n, F) \cap KP^{+} = Sp_0(n, F) \cap KP^{-} = \{diag(A, \bar{A}), A \in U(n, E)\}.$$

(3.2.1) Next, assuming that F is a finite field, we show: any set $\mathcal{O}_r$ intersects nontrivially the image of the Siegel map, and for $r > 0$, $\mathcal{O}_r$ contains a point in the complement of the image of the Siegel map.

We observe that the form h_0 restricted to $\mathcal{L}(diag(d_1, \ldots, d_n))$ is the form

$$(1 - d_1\bar{d}_1)\bar{x}_1 y_1 + \cdots + (1 - d_n\bar{d}_n)\bar{x}_n y_n.$$

Thus, for $r = 0, 1, \dots, n$, $\mathcal{L}(diag(0, \dots, 0, 1, \dots, 1))$ (r zeros) belongs to $\mathcal{O}_r$.

We fix $0 < r \leq n$ and $d \in E \backslash F$ such that $d\bar{d} = 1$. Let W_r denote the subspace spanned by the vectors $e_1, \dots, e_r, de_{r+1} + e_{n+r+1}, \dots, de_n + e_{2n}$. Then, W_r is $n-$dimensional and isotropic for ω. The matrix of the form h_0 restricted to W_r, on the defining basis for W_r, is $diag(-1, \dots, -1, 0, \dots, 0)$, (here -1 appears r-times) yields W_r belongs to $\mathcal{O}_r$. Moreover, for $0 < r$ and p defined in (3.1), the dimension of $p(W_r)$ is $n - r < n$. Therefore, W_r does not belong to the image of the Siegel map and we have verified (3.2.1).

(3.2.2) We show for $n > 1$ that $\mathcal{O}_0$ contains points in the complement to the image of the Siegel map.

To begin with, we consider $n = 3$. We fix $d, c \in E$ so that $0 = 1 + c\bar{c} + d\bar{d}$ and $c\bar{d} \in F$. We set

$$A := \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & -d \\ \bar{c} & \bar{d} & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 0 \end{pmatrix}$$

Then,

$${}^tAB = \begin{pmatrix} 1 + c\bar{c} & \bar{c}d & 0 \\ c\bar{d} & 1 + d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}^tBA = \begin{pmatrix} 1 + c\bar{c} & c\bar{d} & 0 \\ \bar{c}d & 1 + d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Given that $\bar{c}d \in F$, ${}^tAB = {}^tBA$ and hence the subspace $L_{(A,B)}$ is Lagrangian.

$${}^t\bar{A}A = \begin{pmatrix} 1 + c\bar{c} & c\bar{d} & 0 \\ \bar{c}d & 1 + d\bar{d} & 0 \\ 0 & 0 & c\bar{c} + d\bar{d} + 1 \end{pmatrix}, \quad {}^t\bar{B}B = \begin{pmatrix} 1 + c\bar{c} & \bar{c}d & 0 \\ c\bar{d} & 1 + d\bar{d} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

hence, the hypothesis gives that both matrices are equal, and therefore h_0 restricted to $L_{(A,B)}$ is the zero form. Also, $detA = 1 + d\bar{d} + c\bar{c} = detB = 0$. Whence, $L_{(A,B)}$ is an element of $\mathcal{O}_0$ which is not in the image of the Siegel map.

In order to produce an element of $\mathcal{O}_0$ in the complement of the image of the Siegel map for odd n with $n > 3$, we write $n = 3 + n - 3$. Then the subspace $L_{(A,B)} \oplus E(e_4 + e_{n+4}) \oplus \cdots \oplus E(e_n + e_{2n})$ satisfies the requirement.

Next, we consider n even and we construct an element in $\mathcal{O}_0$ in the complement of image of the Siegel map. We fix $b \in E$ such that $b\bar{b} = -1$. For arbitrary $c \in E$. We set

$$A := \begin{pmatrix} -bc & -b \\ c & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & 0 \\ b & 0 \end{pmatrix}$$

Then ${}^tAB = {}^tBA = (0, 0, 0, 0)$, and, $L_{(A,B)} := \{(Ax, Bx), x \in E^2\}$ is a Lagrangian subspace. Given that ${}^t\bar{A}A = {}^t\bar{B}B = (0, 0, 0, 0)$, we see that, h_0 restricted to $L_{(A,B)}$ is the null form, that is, $L_{(A,B)} \in \mathcal{O}_0$. Furthermore, neither A nor B is invertible, hence, $L_{(A,B)}$ is not in the image of the Siegel map. For $n = 2k$, it readily follows that the subspace $L_{(A,B)} \oplus \cdots \oplus L_{(A,B)}$ ($k-$times) belongs to $\mathcal{O}_0$ and it does not belong to the image of the Siegel map. Thus, we have concluded (3.2.2).

(3.2.3) Obviously, the subspace $\mathcal{L}(I_n)$ is an element of $\mathcal{O}_0$. Hence, (3.2.1), (3.2.2) and example 1 imply the third and fourth item in theorem 1 for the orbits $\mathcal{O}_r$.

(3.2.4) For any permutation matrix $T \in GL_n(E)$, the matrix $({}^tT^{-1}, 0, 0, T)$ belongs to $Sp(n,F)_0 \cap Sp(n,F)$.

Lemma 9. *$\mathcal{H}_n$ is contained in the image of the Siegel map.*

Proof. Let $W \in \mathcal{H}_n$. We write $W = L_{(R,S)} = \{\left(\begin{smallmatrix} Rx \\ Sx \end{smallmatrix}\right), x \in E^n\}$ with ${}^tRS - {}^tSR = 0$, $\left(\begin{smallmatrix} R \\ S \end{smallmatrix}\right)$ of rank n. Since $W \in \mathcal{H}_n$ the matrix ${}^t\bar{R}S - {}^t\bar{S}R$ is invertible. The matrix S has rank r, with $0 \le r \le n$. We show $r = n$. For this, after a change of representative (R,S) for W, we choose two $n \times n$ permutation matrices P, Q in $GL_n(F)$ such that

$$PSQ = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where B_1 is an invertible $r \times r$ matrix and each of B_2, B_4 is the null matrix. We may write $W = \{\left(\begin{smallmatrix} RQx \\ SQx \end{smallmatrix}\right), x \in E^n\}$ and

$$\begin{pmatrix} ({}^tP)^{-1} & 0 \\ 0 & P \end{pmatrix} W = \left\{ \left(\begin{smallmatrix} ({}^tP)^{-1}RQx \\ (B_1, B_2, B_3, B_4)x \end{smallmatrix}\right), x \in E^n \right\}.$$

We set $A := ({}^tP)^{-1}RQ$. Thus, $W = \{\left(\begin{smallmatrix} Ax \\ (B_1, 0, B_3, 0)x \end{smallmatrix}\right), x \in E^n\}$.
Write $A := (A_1, A_2, A_3, A_4)$, with $A_1 \in M_r(F), A_4 \in M_{n-r}(F)$. The hypothesis W is a Lagrangian implies ${}^tA(B_1, 0, B_3, 0) = {}^t(B_1, 0, B_3, 0)A$, from which ${}^tA_1 + {}^tA_3B_3 = A_1 + {}^tB_3A_3, A_2 + {}^tB_3A_4 = 0$. The hypothesis the rank of $(A, (B_1, 0, B_3, 0))$ is n implies A_4 is invertible.
Hence, replacing x by $(diag(I_r), 0, 0, A_r^{-1})x$ gives $W = \{\left(\begin{smallmatrix} Cx \\ Dx \end{smallmatrix}\right), x \in E^n\}$, with $C = (A_1, 0, A_3, I_{n-r})$ and $D = (diag(I_r), 0, B_3, 0)$. The matrix of h_E in this new coordinates is

$${}^t\bar{C}D - {}^t\bar{D}C = \begin{pmatrix} \bullet & 0 \\ \bullet & 0 \end{pmatrix}.$$

The hypothesis h_E restricted W has rank n forces $n - r = 0$. ∎

(3.2.4) For $0 \le k < n$, the set $\mathcal{H}_k$ contain points in the complement to the image of the Siegel map. Indeed, for $k = 0$ we have $L_+ \in \mathcal{H}_0$. For k positive, we fix $c \in E : c - \bar{c} \neq 0$ and set W_k to be the linear subspace of E^{2n} spanned by $e_1 + ce_n, e_2 + ce_{n+1}, \dots, e_k + ce_{n+k}, e_{k+1}, \dots, e_n$. Then, E_k is lagrangian, and the form h_E restricted to W_k is $(\bar{c} - c)(\bar{x}_1 y_1 + \dots, \bar{x}_k y_k)$. Hence, $W_k \in \mathcal{H}_k$. The projection $p(W_k)$ has dimension $k < n$, hence, W_k is not in the image of the Siegel map.

(3.2.5) Next, we determine the intersection of each orbit $\mathcal{H}_k$ and the image of the Siegel map $\mathcal{L}$. For this, we write $K(F) = Sp(n,F) \cap K \equiv U(n,F), P^+(F) = P^+ \cap Sp(n,F)$. Hence, $K(F)P^+(F)$ is a parabolic subgroup for $Sp(n,F)$ and obviously it acts on $\mathcal{H}_k \cap Im(\mathcal{L})$. We now show,

Proposition 5. *For $0 \le k < n$, we have $\mathcal{H}_k \cap Im(\mathcal{L})$ is the union of two orbits for $K(F)P^+(F)$.*

(As a consequence we show that convenient representatives of the two orbits of $K(F)P^+(F)$ in $\mathcal{H}_k \cap Im(\mathcal{L})$ are conjugated by the symplectic group of the

$F-$plane spanned by e_k, e_{n+k+1}. For $k = n$ we already have shown that the intersection is equal to $\mathcal{H}_n$, later on we show $\mathcal{H}_n$ an orbit of $Sp(n, F)$).

Proof. Once for all, we fix $\alpha \in E \backslash F$ so that $\alpha^2 \in F, \bar{\alpha} = -\alpha$. Hence, $\{1, \alpha\}$ is a basis for the vector space E over F. We write, for a matrix $E^{n \times n} \ni Z = Z_1 + \alpha Z_2$ with $Z_1, Z_2 \in F^{n \times n}$. Let $\mathcal{L}(Z)$ be an element of $\mathcal{H}_k$. Then, $(I, -Z_1, 0, I)Z = (\alpha Z_2)$. Thus, Z_2 is a symmetric matrix of rank k. Owing to the theory of quadratic forms over F^n there exists $A \in Gl(n, F)$ and $c \in F^\star$ so that $AZ_2{}^tA = diag(1, \ldots, 1, c, 0, \ldots, 0)$, (0 repeats $n - k$ times). We recall we may arrange matters so that either $c = 1$ or $c = \alpha^2$. Therefore, any element of $\mathcal{H}_k \cap Im(\mathcal{L})$ is conjugated by an element of $K(F)P^+(F)$ to one of

$$\mathcal{L}(\alpha diag(1, \ldots, 1, 0, \ldots, 0)), \ \mathcal{L}(\alpha diag(1, \ldots, 1, \alpha^2, 0, \ldots, 0))$$

and we have shown the proposition. Carrying out the same computation as in Example 2, for the symplectic group for the form ω restricted to the $F-$plane spanned by e_k, e_{n+k+1} we obtain an element s_n of $Sp(n, F)$ which transform the first subspace into the second. The element s_n is not in the subgroup $K(F)P^+(F)$. ∎

(3.2.6) Lemma 9, Proposition 5, (3.2.5) imply the third, fourth and fifth statement in theorem 1 about the orbits $\mathcal{H}_r$.

Note 2. For different reasons we would like to know, when for $g = (A, B, C, D) \in Sp(n, F)$ or $g \in Sp_0(n, F)$ and $Z \in Sym(E^n)$ the matrix $CZ + D$ is invertible. According to (3.2.1) up to (3.2.5), lemma 9 and theorem 1, an answer is: i) For any element $Z \in Sym(E^n)$ such that $Z - \bar{Z}$ is invertible and for any $(A, B, C, D) \in Sp(n, F)$, the matrix $CZ + D$ is invertible. ii) For a symmetric matrix Z such that $Z - \bar{Z}$ is not invertible, there exists $(A, B, C, D), (M, N, R, S) \in Sp(n, F)$ such that $CZ + D$ is invertible and $RZ + S$ is not invertible. iii) For $n > 1$ and any symmetric matrix Z there exists $(A, B, C, D), (M, N, R, S) \in Sp_0(n, F)$ so that $CZ + D$ is invertible and $RZ + S$ is not invertible. From (3.2), we may recall, $(A, B, C, D)\mathcal{L}(Z) \in Im(\mathcal{L})$ if and only if $(CZ + D)$ is an invertible map. We obtain a somewhat more precise statement.

Proposition 6. *We fix $g \in Sp(n, F)$ and $Z \in Sym(E^n)$. Then $g\mathcal{L}(Z) \in Im(\mathcal{L})$ if and only if g belongs to*

$$K(F)P^+(F)\mathcal{E}_{Sp(n,F)}(\mathcal{L}(Z)) \cup K(F)P^+(F)s_n\mathcal{E}_{Sp(n,F)}(\mathcal{L}(Z))$$

$\mathcal{E}_{Sp(n,F)}(\mathcal{L}(Z)$ as defined in section 4.

Proof. Let g so that $g\mathcal{L}(Z) \in Im(\mathcal{L})$. We fix r so that $Z \in \mathcal{H}_r$. Owing to Proposition 5 there exists $q \in K(F)P^+(F)$ so that $g\mathcal{L}(Z) = q\mathcal{L}(Z)$ or $g\mathcal{L}(Z) = qs_n\mathcal{L}(Z)$, then, the proposition follows. ∎

4. Isotropy subgroups

The purpose of this section is to explicitly compute the structure of $\mathcal{O}_k, \mathcal{H}_k, k = 0, \dots, n$ as homogeneous spaces. For the real case, this has been accomplished by [13] [14] and references therein.

An element of $\mathcal{O}_{n-k}$ is constructed as follows: we define V_k to be the subspace spanned by the vectors $e_1 + e_{n+1}, \dots, e_k + e_{n+k}, e_{k+1}, \dots, e_n$. Then, $V_0 = L_+$. A simple computation shows that the form h_E restricted to $V_k \times V_k$ is the null form, whereas the type of the form h_0 restricted to $V_k \times V_k$ is $n-k$,. Obviously V_k is a lagrangian subspace. Henceforth, for $x \in Sp(n, E)$, $Ad(x)$ denotes the inner automorphism defined by x. Let t_k be the partial Cayley transform

$$t_k := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

where, $D_1 = D_4 = diag(\frac{\sqrt{2}}{2} I_k, I_{n-k}), D_2 = diag(-\frac{\sqrt{2}}{2} I_k, 0), D_3 = -D_2$. Then, t_k is an element of $Sp(n, E)$ and $t_k L_+ = t_k V_0 = V_k$. A computation gives

$$t_k^{-1} = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$$

where, $L_1 = L_4 = diag(\frac{\sqrt{2}}{2} I_k, I_{n-k}), L_2 = diag(\frac{\sqrt{2}}{2} I_k, 0), L_3 = -L_2$.

Let $\mathcal{E}_{Sp_0(n,F)}(V_k)$ denote the set stabilizer of V_k in $Sp_0(n, F)$. The equality $\mathcal{E}_{Sp(n,E)}(V_0) = KP^+$ implies

$$\mathcal{E}_{Sp_0(n,F)}(V_k) = Ad(t_k)\mathcal{E}_{Sp(n,E)}(V_0) \cap Sp_0(n, F) = Ad(t_k)(KP^+) \cap Sp_0(n, F).$$

The stabilizer of V_0 in $Sp(n, F)$ is

$$KP_+ \cap Sp_0(n, F) = K \cap Sp_0(n, F) = \{diag(T, \bar{T}) : T \in U(n, E)\}$$

Thus, the stabilizer of V_0 in $Sp_0(n, F)$ is isomorphic to $U(n, E)$.

The main result of this section is

Theorem 2. *The stabilizer group $\mathcal{E}_{Sp_0(n,F)}(V_k)$ is isomorphic to the semidirect product of the group $O(k, F) \times U(n-k, E)$ times the unipotent subgroup $Ad(t_k)(P^+) \cap Sp_0(n, F)$.*

The proof of the result requires some computations, which we carry out. First, we verify that the subgroup of $Sp_0(n, F)$, $diag(S, T, S, \bar{T}), S$ in $O(k, F), T$ in $U(n-k, E)$ is contained in $\mathcal{E}_{Sp_0(n,F)}(V_k)$. For this, we write for $v \in V_k, v = \begin{pmatrix} x \\ y \\ x \\ 0 \end{pmatrix}$ with $x \in E^k, y \in E^{n-k}$. Hence,

$$diag(S, T, S, \bar{T})v = \begin{pmatrix} Sx \\ Ty \\ Sx \\ 0 \end{pmatrix} \in V_k.$$

Is clear that the unipotent subgroup is contained in $\mathcal{E}_{Sp_0(n,F)}(V_k)$.
For a matrix $T \in E^{n\times n}$ we write

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, T_1 \in E^{k\times k}, T_2 \in E^{k\times n-k}, T_3 \in E^{n-k\times k}, T_4 \in E^{n-k\times n-k}$$

And for $(A,B,0,D) \in KP^+$ we have $Ad(t_k)(A,B,0,D) =$

$$\begin{pmatrix} \frac{1}{2}(A_1 - B_1 + D_1) & \frac{\sqrt{2}}{2}A_2 & \frac{1}{2}(A_1 + B_1 - D_1) & \frac{\sqrt{2}}{2}(B_2 - D_2) \\ \frac{\sqrt{2}}{2}(A_3 - B_3) & A_4 & \frac{\sqrt{2}}{2}(A_3 + B_3) & B_4 \\ \frac{1}{2}(A_1 - B_1 - D_1) & \frac{\sqrt{2}}{2}A_2 & \frac{1}{2}(A_1 + B_1 + D_1) & \frac{\sqrt{2}}{2}(B_2 + D_2) \\ -\frac{\sqrt{2}}{2}D_3 & 0 & \frac{\sqrt{2}}{2}D_3 & D_4 \end{pmatrix}.$$

Next, we show that $Ad(t_k)K \cap Sp_0(n,F)$ is equal to the subgroup $\{diag(S,T,S,{}^tT^{-1}) : S \in O(k,F), T \in U(n,E)\}$. In fact, the computation for $Ad(t_k)X$ gives for $S \in O(k,F), T \in U(n,E)$ that

$$Ad(t_k)(diag(S,T,S,{}^tT^{-1})) = diag(S,T,S,{}^tT^{-1}).$$

Now for $diag(A,D) = diag(A,{}^tA^{-1}) \in K$, such that

$$Ad(t_k)(diag(A,D)) \in Sp_0(n,F),$$

(1.2) and the formula for $Ad(t_k)X$ imply the equalities

$$\overline{(A_1 + D_1)} = A_1 + D_1, \quad \bar{A}_2 = D_2 \quad \bar{A}_3 = D_3, \quad \bar{A}_4 = D_4$$

and

$$\overline{A_1 - D_1} = A_1 - D_1, \quad A_2 = -\bar{D}_2, \quad A_3 = -\bar{D}_3$$

So

$$D_2 = A_2 = 0, \quad D_3 = A_3 = 0, \quad , \bar{A}_1 = A_1, \bar{D}_1 = D_1.$$

Hence, $A_1 \in O(n,F)$. Finally, the equality $D = {}^tA^{-1}$ yields, $A_1 = D_1$, which shows the claim.
Now $Ad(t_k)P^+ \cap Sp_0(n,F) = \{Ad(t_k)(I_n,B,0,I_n) : {}^tB = B \text{ and } \bar{B}_1 = -B_1, B_3 = {}^tB_2 = 0, B_4 = 0\}$. In fact, the formula for $Ad(t_k)X$ leads us to

$$Ad(t_k)(I,B,0,I) = \begin{pmatrix} \frac{1}{2}(2I - B_1) & 0 & \frac{1}{2}B_1 & \frac{\sqrt{2}}{2}B_2 \\ -\frac{\sqrt{2}}{2}B_3 & I & \frac{\sqrt{2}}{2}B_3 & B_4 \\ -\frac{1}{2}B_1 & 0 & \frac{1}{2}(2I + B_1) & \frac{\sqrt{2}}{2}B_2 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

From (1.2) we get $\bar{B}_1 = -B_1, \quad B_2 = 0, \quad B_4 = 0$, and the equality follows.
(E) We will show at this point the equality

$$\mathcal{E}_{Sp_0(n,F)}(V_k) = (Ad(t_k)K \cap Sp_0(n,F))(Ad(t_k)P^+ \cap Sp_0(n,F))$$

Let $X \in KP^+$ so that $Ad(t_k)X \in Sp_0(n,F)$. Condition (1.2) gives us the following equalities,

$$\bar{A}_1 - \bar{B}_1 + \bar{D}_1 = A_1 + B_1 + D_1, \quad B_2 + D_2 = \bar{A}_2, \quad \bar{A}_3 - \bar{B}_3 = D_3, \quad \bar{A}_4 = D_4$$

$$\bar{A}_1 + \bar{B}_1 - \bar{D}_1 = A_1 - B_1 - D_1, \quad \bar{B}_2 - \bar{D}_2 = A_2, \quad \bar{A}_3 + \bar{B}_3 = -D_3, \quad B_4 = 0.$$

From the second equality on each line, we deduce $D_2 = 0$. Thus, $B_2 = \bar{A}_2$. From the third equality in both lines we obtain $\bar{A}_3 = 0$. Hence $A_3 = 0$ and $B_3 = -\bar{D}_3$. Next ${}^tAD - {}^tB0 = I$ give us $D = {}^tA^{-1}$. Explicitly $D = ({}^tA_1^{-1}, 0, -{}^t(A_1^{-1}A_2A_4^{-1}), {}^tA_4^{-1})$. Since $(A, B, 0, D) \in Sp(n, E)$ and so ${}^tBD = {}^tDB$. The computation of the last equality lead us to

$$\begin{pmatrix} A_1^{-1}B_1 - {}^t\bar{Y}\bar{Y} & A_1^{-1}\bar{A}_2 \\ A_4^{-1}\bar{Y} & 0 \end{pmatrix} = \begin{pmatrix} {}^tB_1\,{}^tA_1^{-1} - {}^t\bar{Y}Y & {}^t\bar{Y}\,{}^tA_4^{-1} \\ -\,{}^t\bar{A}_2^{-1}\,{}^tA_1^{-1} & 0 \end{pmatrix}$$

where $Y := {}^t(A_1^{-1}A_2A_4^{-1})$
Now, the equality of the (2,1)-coefficients gives

$$A_4^{-1}\,{}^t\bar{A}_4^{-1}\,{}^t\bar{A}_2\bar{A}_1^{-1} = -\,{}^t\bar{A}_2\,{}^tA_1^{-1}$$

which, after we transpose both members of the last equality, we obtain

$$\bar{A}_1^{-1}\bar{A}_2\bar{A}_4^{-1}\,{}^tA_4^{-1} = -A_1^{-1}\bar{A}_2.$$

From, equality of the (1,2)-coefficients implies

$$\bar{A}_1^{-1}\bar{A}_2\bar{A}_4^{-1}\,{}^tA_4^{-1} = A_1^{-1}\bar{A}_2.$$

Thus, $A_2 = 0$ and we have that

$$(A, B, 0, D) = (diag(A_1, A_4), diag(B_1, 0), 0, diag({}^tA_1^{-1}, {}^tA_4^{-1})).$$

The hypothesis $Ad(t_k)(A, B, 0, D) \in Sp_0(n, F)$ let us conclude that
$A_1 \in O(k, F)$, $A_4 \in U(n-k, E)$. Therefore, (E) is shown, and the theorem follows.

5. Anti-involutions in $Sp(n, F)$

In this section we analyze the structure on the set of anti-involutions in the group $Sp(n, F)$. We will show that this set is a homogeneous space for $Sp(n, F)$.
The denote by $\mathcal{C}(n, F)$ the set of anti-involutions ,i.e.,

$$\mathcal{C}(n, F) = \{T \in Sp(n, F) : T^2 = -1\}.$$

Proposition 7. *$\mathcal{C}(n, F)$ is equivariant isomorphic to $\mathcal{H}_n$ when -1 is not a square in F, whereas is isomorphic to*
$Sp(n, F)/(Sp(n, F) \cap K)$ when -1 is a square in F.

It is clear that $\mathcal{C}(n, F)$ is invariant under conjugation.
Since $J = (0, I_n, -I_n, 0)$ is an element of $Sp(n, F)$ we have that JT is an element of $Sp(n, F)$. The poof of the proposition will follow from the next three lemmas

Lemma 10. *i) Let T be an involution, then JT is a symmetric matrix. That is, ${}^t(JT) = JT$*

ii) For $T \in Sp(n, F)$, such that JT is symmetric, we have that T is an involution.

Proof. Recall ${}^tJ = -J$, ${}^tTJT = J, T^2 = -1$ Hence, ${}^t(JT) = -{}^tTJ = -JT^{-1} = JT$. For the second statement, we have ${}^t(JT) = JT$ hence $J = -{}^tT^{-1}JT = {}^tTJT$, thus $T^2 = -I$. ■

According to lemma 6, to each involution T in $Sp(n,F)$ we naturally associate a symmetric non-degenerate bilinear form b_T on F^{2n}. The matrix of the form b_T in the canonical basis is JT.
Now, from the classification of symmetric non-degenerate bilinear forms on F^{2n} we have that b_T is either equivalent to the Euclidean form $x_1^2 + \cdots + x_{2n}^2$ or to the non-Euclidean form $x_1^2 + \cdots + x_{2n-1}^2 + cx_{2n}^2$ where $c \in F$ is not a square.
Since $det(JT) = 1$, we obtain

Remark 5. *The form b_T is always equivalent to the Euclidean form. The group $Sp(n,F)$ acts on $Sp(n,F) \cap Sym(F^{2n})$ by the formula*

$$(g, S) \to {}^t(g^{-1})Sg^{-1}.$$

It readily follows that the map $\mathcal{C}(n,F) \ni T \to JT \in Sp(n,F) \cap Sym(F^{2n})$ intertwines the respective actions of $Sp(n,F)$. Hence, for $g \in Sp(n,F)$ the forms b_T and $b_{gTg^{-1}}$ are equivalent.

To continue, we split up the analysis of $\mathcal{C}(n,F)$ into the two possible cases, namely, -1 is either a square in F or -1 is not a square in F. To begin with, we assume -1 is not an square in F. Let us fix a square root $i \in E$ of -1.

For an anti involution $T \in Sp(n,F)$ we have that T is a semisimple linear map with possible eigenvalues $i, -i$ because the minimal polynomial of T divides x^2+1. Let $V_i(T)$ (resp $V_{-i}(T)$) the corresponding possible eigenspace in E^{2n}. Hence, $E^{2n} = V_i(T) \oplus V_{-i}(T)$, and we have

Proposition 8. i) *Both subspaces $V_i(T), V_{-i}(T)$ are nonzero.*

ii) $\overline{V_i(T)} = V_{-i}(T)$.

iii) $F^{2n} \cap V_i(T) = F^{2n} \cap V_{-i}(T) = \{0\}$.

iv) *The map $F^{2n} \ni v \to v - iTv \in V_i(T)$ is linear bijection over F.*

v) *$V_i(T)$ (resp $V_{-i}(T)$) is a lagrangian subspace.*

vi) $h_E(v - iTv, w - iTw) = 2\omega(v,w) + 2ib_T(v,w)$, *for* $v, w \in F^{2n}$.

vii) *The decomposition $E^{2n} = V_i(T) \oplus V_{-i}(T)$ is orthogonal with respect to h_E.*

viii) *h_E restricted to $V_i(T)$ is non degenerate.*

Proof. The result follows from the hypothesis $T \in U(h_E, E^{2n}) \cap Sp(n,E)$ and $i \notin F$. In particular, viii) follows from vii) and that h_E is non degenerate. For $x, y \in V_i(T), \omega(x,y) = \omega(Tx,Ty) = ii\omega(x,y) = -\omega(x,y)$. ■

Let $v_j - iTv_j, j = 1, \ldots, n$ denote an orthonormal basis of $V_i(T)$ for the restriction of $\frac{1}{2i}h_E$. Then, $v_1, \ldots, v_n$ span a lagrangian subspace of F^{2n} and $v_1, \ldots, v_n, Tv_1, \ldots, Tv_n$ is a basis for F^{2n}.
In fact, from vi) we obtain $w(v_k, v_s) = 0, b_T(v_k, v_s) = \delta_{k,s}$. The last statement follows from $T^2 = -1$ applied to $\sum_{1\leq j\leq n} c_j v_j + d_j T v_j = 0$ for $c_j, d_j \in F$ and a short computation.

Lemma 11. *Assume -1 is not a square in F. Then, the action of $Sp(n, F)$ in $\mathcal{C}(n, F)$ is transitive.*

Proof. Proposition 6 gives rise to a map from $\mathcal{C}(n, F)$ to $\mathcal{L}_{E,2n}$ by the rule

$$\mathcal{C}(n, F) \ni T \longrightarrow V_i(T)$$

From viii) we have the image of the map is contained in $\mathcal{H}_n$. For $g \in Sp(n, F)$ we have the equality $gV_i(T) = V_i(gTg^{-1})$, which shows that the map is equivariant. The maps is obviously injective. Since $\mathcal{H}_n$ is an orbit of $Sp(n, F)$ (Theorem 1) we have that the map is a bijection and hence the result ■

Next, we assume $-1 = i^2$ with $i \in F$. Then, due to the semisimplicity of T we have the decomposition $F^{2n} = (F^{2n} \cap V_i(T)) \oplus (F^{2n} \cap V_{-i}(T))$.
From the equalities $\omega(x, y) = -\omega(x, y)$ for $x, y \in V_i(T)$, we have that the subspaces $F^{2n} \cap V_i(T), F^{2n} \cap V_{-i}(T)$ are isotropic, Corollary 3 pag 81 in [1] gives us that both subspaces are lagrangian. Therefore, the anti hermitian form h_E restricted to $F^{2n} \cap V_i(T)$ is the null form, which forces to $V_i(T)$ to be an element of $\mathcal{H}_0$.

Lemma 12. *Assume -1 is a square in F. Then, $\mathcal{C}(n, F)$ is a homogeneous space equivalent to $Sp(n, F)/(Sp(n, F) \cap K)$.*

Remark 6. *The map $\mathcal{C}(n, F) \ni T \longrightarrow V_i(T) \in \mathcal{H}_0$ is equivariant for $Sp(n, F)$ and in this case is no longer injective (c.f. example 3-a) , due to theorem 1, $\mathcal{H}_0$ is a homogeneous space for $Sp(n, F)$, hence, the map is surjective.*

Proof. We now show lemma 11. We set

$$H := \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix}.$$

Then, $H \in \mathcal{C}(n, F)$. Let T be an anti involution in $Sp(n, F)$ we will show that T is conjugated in $Sp(n, F)$ to the matrix H. For this, we define $D := J^{-1}TJ$, which is another anti involution in $Sp(n, F)$. The minimal polynomial of $J^{-1}TJ$ divides the polynomial $x^2 + 1 = (x - i)(x + i)$. Hence, $D := J^{-1}TJ$ is diagonalizable over F.
Let $W_{\pm i}$ the associated eigenspaces. Thus, $F^{2n} = W_i \oplus W_{-i}$.
Since for every $v, w \in F^{2n}, \omega(Dv, Dw) = \omega(v, w)$, we have that $W_{\pm i}$ are isotropic subspaces for ω. The hypothesis that ω is non degenerate forces, $W_{\pm i}$ to be lagrangian subspaces. Thus, there exists $P \in Sp(n, F)$ so that

$Pe_1, \ldots, Pe_n$ is a basis for W_i, $Pe_{n+1}, \ldots, Pe_{2n}$ is a basis for W_{-i}

We have

$$DPe_j = iPe_j = P(ie_j) = PH(e_j), j = 1, \ldots, n,$$
$$DPe_j = -iPe_j = P(-ie_j) = PH(e_j), j = n+1, \ldots, 2n.$$

Hence, $DP = PH$. That is, $PH = DP = J^{-1}TJP$. Therefore, $H = P^{-1}J^{-1}TJP = (JP)^{-1}T(JP)$. The matrices in $Gl(2n, F)$ which commute with H are the matrices $diag(A, B), A, B, \in Gl_n(F)$. Thus, the isotropy at H is $Sp(n, F) \cap K$. ■

Remark 7. *A particular element of $Sp(n, F)$ which conjugates H onto J is the Cayley transform*

$$C(e_j) = \frac{1}{-2i}(e_j + ie_{n+j}), j = 1, \ldots, n, \quad C(e_{n+j}) = e_j - ie_{n+j}, j = 1, \ldots, n.$$

Example 2. *We assume $-1 = i^2, i \in F$. A simple calculation yields $\mathcal{C}(1, F) =$*

$$\left\{ \begin{pmatrix} \pm i & x \\ 0 & - \pm i \end{pmatrix}, \begin{pmatrix} \pm i & 0 \\ y & - \pm i \end{pmatrix}, x \in F, y \in F^\times \right\}$$

union the set

$$\left\{ \begin{pmatrix} a & -\frac{1+a^2}{c} \\ c & -a \end{pmatrix}, c \in F^\times, a \in F \backslash \{\pm i\} \right\}$$

Hence, the cardinal of the set of involutions is $2(q + q - 1) + (q - 2)(q - 1) = q(q + 1)$. The isotropy at $diag(i, -i)$ is the subgroup $diag(a, -a), a \in F^\times$. Hence $card(Sl(2, F_q))/card(F^\times) = q(q - 1)(q + 1)/(q - 1) = card(\mathcal{C}(1, F))$. Also,

$$V_i(\begin{pmatrix} -i & 0 \\ x & i \end{pmatrix}) = F\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad V_i(\begin{pmatrix} i & x \\ 0 & -i \end{pmatrix}) = F\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$V_i(\begin{pmatrix} a & -\frac{1+a^2}{c} \\ c & -a \end{pmatrix}) = F\begin{pmatrix} \frac{1+a^2}{c} \\ a - i \end{pmatrix},$$

$$V_i(\begin{pmatrix} -i & x \\ 0 & i \end{pmatrix}) = F\begin{pmatrix} x \\ 2i \end{pmatrix}, V_i(\begin{pmatrix} i & 0 \\ x & -i \end{pmatrix}) = F\begin{pmatrix} 2i \\ x \end{pmatrix}.$$

5.1. The case $T^2 = a, a$ square. Let F be a field of odd characteristic, and let ω be a non degenerate alternating form in $V = F^{2n}$. We fix $a \in F$ and define

$$S_a := \{T \in Sp(n, F) : T^2 = aId\}$$

For $a = 1$ the identity matrix belongs to S_a.
For $a = -1$ the matrix J belongs to $S_{-1} = \mathcal{C}(n, F)$.

Proposition 9. *For $a \notin \{1, -1\}$ and $a = b^2, b \in F$ the set S_a is empty.*

Proof. Let $T \in S_a$, then the eigenvalues of T belong to the set $\pm b$. Let W_b, W_{-b} be the eigenspaces of V.
The equality $\frac{1}{2}(bI - T) + \frac{1}{2}(bI + T) = bI$ implies that $V = W_b \oplus W_{-b}$.

For $x, y \in W_b$, we have $\omega(x,y) = 0$ ($\omega(x,y) = \omega(Tx,Ty) = b^2\omega(x,y)$). Similarly, for $x, y \in W_{-b}$ we have $\omega(x,y) = 0$. Therefore, both subspaces are isotropic.
We now verify for $x, \in W_b, y \in W_{-b}$ that $\omega(x,y) = 0$. In fact, $\omega(x,y) = \omega(Tx,Ty) = b(-b)\omega(x,y) = -a\omega(x,y)$. Since $a \neq -1$, we get $\omega(x,y) = 0$.
Then, assuming S_a is not empty, unless $a \in \{1,-1\}$ we have ω equal to the null form, and the result follows.

Another proof follows along the following lines :
For a symplectic matrix, if λ is an eigenvalue, then $1/\lambda$ is also an eigenvalue.
So if $b, -b$ are the unique eigenvalues, and $b \notin \{\pm 1, \pm i\}$ we must have $-b = 1/b$ from which $b^2 = -1$ so $a = -1$. ∎

5.2. The case $a = 1$. Let W be any subspace of V such that ω restricted to W is non degenerate, so $V = W \oplus W^{\perp}$.
Define T_W to be the linear operator equal to the identity in W and equal to $-I$ in $W^{\perp}$.
It readily follows that $T_W \in Sp(n,F)$ and T_W is an involution.

Proposition 10. *Any involution T in $Sp(n,F)$ is equal to a T_W for a convenient W.*

Proof. In fact, the eigenvalues of T belongs to the set ± 1 Let W_1, W_{-1} be the eigenspaces of V the equality $\frac{1}{2}(I-T)+\frac{1}{2}(I+T) = I$ implies that $V = W_1 \oplus W_{-1}$.

For $x, \in W_1, y \in W_{-1}$ we have $\omega(x,y) = 0$. In fact, $\omega(x,y) = \omega(Tx,Ty) = 1(-1)\omega(x,y) = -\omega(x,y)$.

It follows: ω restricted to any of the subspaces in non degenerate. Hence, $T = T_{W_1}$. ∎

Corollary 4. *The orbits of $Sp(n,F)$ in S_1 are parameterized by the set* $\{0, 1, \ldots, 2n\}$.
Indeed, for each k the set of involutions T such that its $1-$eigenspace is of dimension k, is an orbit for $Sp(n;F)$.

6. Ideals

We fix $A = M_n(E)$. To begin with for ${}^t(x,y) \in A \times A$ we analyze the left ideal $Ax + Ay$. We recall that any left ideal J in $M_n(E)$ is principal and generated by an idempotent matrix e of certain rank r_J. The number r_J is an invariant that determine the ideal J owing to that whenever e, d are idempotents so that $M_n(E)e = M_n(E)d$, then there exists an invertible matrix p so that $e = p^{-1}dp$. The last statement holds because the equality of ideals and the inequality $rank\,(AB) \leq min\{rank\,A, rank\,B\}$ forces $rank\,e = rank\,d$, hence, the existence of the matrix p. Therefore, for two left ideals I, J in $M_n(E)$ there exists an invertible matrix p so that $I = Jp$ if and only if $r_I = r_J$. Next, we observe that the rank if the ideal $Ax + Ay$ is equal to the rank of the $2n \times n$ matrix ${}^t(x,y)$. Indeed, if r denotes the rank of the matrix ${}^t(x,y)$, then there exists invertible matrices p, q so that ${}^t(x,y) = q diag(I_{r\times r}, 0)p$. Hence $diag(I_{r\times r}, 0)p$

belongs to the ideal $Ax + Ay$, which yields that $Ax + Ay$ contains an idempotent of rank r. Thus $r \leq r_{Ax+Ay}$. The reverse inequality follows from the inequality $rank(AB) \leq min\{rankA, rankB\}$ and that $diag(I_{r\times r}, 0)p$ actually spans the ideal $Ax + Ay$. On the set of left ideals in A we define the equivalence relation I is equivalent to J if and only if there exists $p \in A^{\times}$ so that $I = Jp$. Let $C(J)$ denote the equivalence class for an ideal J. The preceding statements let as conclude: For a fixed left ideal J, the set of pairs ${}^t(x, y)$ so that $C(Ax + Ay) = C(J)$ is equal to the set of pairs ${}^t(x, y)$ so that the rank of the matrix ${}^t(x, y)$ is r_J. Let $\tilde{\mathcal{P}}_A$ denote the set of pairs ${}^t(a, b)$ so that ${}^tab = {}^tba$. That is, $\tilde{\mathcal{P}}_A$ is the set of pairs so that the subspace $L_{(a,b)}$ of E^{2n} is isotropic for ω. Obviously, $\mathcal{P}_A \subset \tilde{\mathcal{P}}_A$. The group $SL_*(2, A) = Sp(n, E)$ acts on the left on $\tilde{\mathcal{P}}_A$. Moreover, $Gl_n(E) = A^{\times}$ acts on the right. Because of a theorem of Witt and the previous observations, we have that the orbits of $Sp(n, E)$ in $\mathbb{P}^1_{\times}(\tilde{\mathcal{P}}_A)$ are exactly the $n+1$ subsets, $\mathbb{T}^r$, $r = 0, 1, \ldots, n$, which has as a representative a vector line $\mathbb{L}_{{}^t(x,y)}$ so that r_{Ax+Ay} is equal to r. Thus, $\mathbb{T}^n = \mathbb{P}^1_{\times}(\mathcal{P}_{M_n(E)})$. The preceding statements let as conclude: The set of orbits of $SL_*(2, M_n(E))$ in $\mathbb{P}^1_{\times}(\tilde{\mathcal{P}}_A)$ is in a bijective correspondence with the set of equivalence classes for the equivalence relation in the set of left ideals in A. We would like to generalize the last statement to other involutive rings. We also point out, that theorem 1 gives that the set of orbits of $SL_*(2, H_0) = Sp(n, F)$ in $\mathbb{P}^1_{\times}(\mathcal{P}_{M_n(E)})$ is parameterized by the set of left ideals in $A_0 = M_n(F)$ module the analogous equivalence relation.

Acknowledgements. Part of the work was done during the meeting "Representation theory days in Patagonia" organized by University of Talca. The authors are grateful to the organizers of the workshop, Stephen Griffeth, Steen Ryom-Hansen, Jean F. van Diejen, Luc Lapointe for their kind invitation to attend to such a nice meeting. The authors want also to thank Pierre Cartier for illuminating discussions and suggestions related to this work.

References

[1] Bourbaki, N., «Éléments de Mathématique, Livre II, Algèbre», Chapter IX, Herman et Cie, 1959.

[2] Dieudonné, J., «Le Géométrie des groupes classiques», Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag , 1955.

[3] Helminck, A. G., *On the orbits of symmetric spaces under the action of parabolic subgroups,* in: Invariant theory (Denton, TX, 1986), 435–447, Contemp. Math.**88**, Amer. Math. Soc., Providence, RI, 1989.

[4] Hilgert, J., and K.-H. Neeb, *Compression semigroups of open orbits in complex manifolds*, Ark. Mat. **33** (1995), 293–322.

[5] Kaneyuki, S., *Pseudo-hermitian symmetric spaces and Siegel domains over nondegenerate cones,* Hokkaido Math. J. **20** (1991), 213–239.

[6] Pantoja, J., and J. Soto-Andrade, *Représentations de $SL_*(2, A)$ et $SL(n, q)$*, C.R. Acad. Sci. Paris, Série I, **323** (1996), 1109–1112.

[7] —, *A Bruhat decomposition of the group $SL_*(2,A)$*, J. Algebra **262** (2003), 401–412.

[8] —, *Bruhat presentations for $*-$ classical groups,* Comm. in Alg. **37** (2009), 149–167.

[9] Richardson, R., G. Röhrle, and R. Steinberg, *Parbolic subgroups with Abelian unipotent radical,* Invent Math. **110** (1992), 649–671.

[10] Siegel, C., *Symplectic geometry,* Amer. J. of Math. **65** (1943), 1–86.

[11] Soto Andrade, J., *Représentations de certain groupes symplectiques finis,* Bull. Soc. Math. France **55-56** (1978), 5–334.

[12] Soto-Andrade, J., and J. Vargas, *Twisted spherical functions on the Poincaré upper half plane,* J. Algebra **248** (2002), 724–246.

[13] Takeuchi, M., *On Orbits in a Compact Hermitian Symmetric Space,* Amer. J. of Math. **90** (1968), 657–680.

[14] Wolf, J., *Fine Structure of Hermitian Symmetric Spaces,* in: Boothby, and Weiss, Eds., "Symmetric Spaces," Short courses presented at Washington University, Marcel Dekker, Inc, New York, 1972, 271–357.

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Received August 22, 2014
and in final form February 15, 2015