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ORIGINAL ARTICLE

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# On the consequences of the constraint of incompressibility with regard to a new class of constitutive relations for elastic bodies: small displacement gradient approximation

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**Abstract** Recently, there has been an interest in the development of implicit constitutive relations between the stress and the deformation gradient, to describe the response of elastic bodies as such constitutive relations are capable of describing physically observed phenomena, in which classical models within the construct of Cauchy elasticity are unable to explain. In this paper, we study the consequences of the constraint of incompressibility in a subclass of such implicit constitutive relations.

## 1 Introduction

All materials are compressible; it is merely a matter of how high the applied pressure ought to be in order to discern the compressibility. Thus, for example, under the normal range of pressure (mean normal stresses) that water is subjected to, it is modeled as an incompressible material as the volume changes that occur are negligible. This idealization of water as an incompressible material ceases to be valid when the pressure range becomes large enough that volume changes need to be taken into account to describe the response of the material. Solid materials such as rubber are also idealized as incompressible materials under a certain range of pressures. The assumption that a body is incompressible leads to a constraint on the class of motions that it can undergo, namely those motions that are volume preserving (isochoric). In the development of constitutive relations for the stress in terms of the kinematics for such incompressible bodies that are constrained to describe only isochoric motions, it is traditional to follow the seminal ideas of Bernoulli and D'Alembert (see the discussion in [10]) and to require that such internal constraints do not perform any work. This procedure leads to the splitting of the stress into a part, referred to as the reaction to the constraint, that does no work, and another part that is constitutively determined. The reaction stress due to the constraint is referred to as the Lagrange multiplier, and this quantity needs to be determined as a part of the solution of the problem of interest. The above-mentioned procedure that appeals to the constraint reaction doing no work works only for certain class of bodies, and in general dissipative systems, such a procedure leads to incorrect conclusions as pointed out first by Gauss [9] within the context of rigid body dynamics, wherein he stated that the constraint should lead to a force that is the minimum to enforce the constraint. This result of Gauss was extended by Rajagopal and Srinivasa [14] to the case of continua.

Dedicated to our good friend David Steigmann for his insightful and seminal contributions to elasticity.

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When dealing with implicit constitutive relations, wherein one has an implicit relationship between the stress and appropriate kinematical variables, one does not need to introduce notions such a constraint reaction (Lagrange multiplier), and the constraint becomes a natural part of the constitutive relation (see [16,17]). Such is also the case when constitutive expressions are provided for the kinematics in terms of the stress, and it is to the investigation of such a class of constitutive expression that this paper is devoted to.

In this paper, we consider the subclass of a new class of elastic bodies introduced by Rajagopal [13], wherein an implicit relation exists between the stress and the deformation gradient. The special subclass that we consider is those wherein one had an expression for the Cauchy–Green tensor **B** in terms of the stress **T**. We consider a further subset that arises when one assumes that the deformations are small in the sense that the gradient of the displacement field is sufficiently small that higher-order terms of the gradient of the displacement can be neglected, leading to the constitutive expression (4) (see (15)). We observe that the constraint of incompressibility follows directly from the requirement that $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, leading to a constraint relationship between the terms that appear on the right-hand side of (15), and one does not introduce an unknown Lagrange multiplier that needs to be determined.

Implicit constitutive relations and constitutive relations wherein one expresses the strain in terms of the stress, namely the class of constitutive relations considered in this paper, are very useful in describing the response of biological matter such as collagen, proteins and DNA, and silks (see [8]). Since many biological tissues are also approximated as being incompressible, this study is particularly pertinent to the response of biological matter.

After considering the consequences of the constraint on the form of the constitutive expression (15), we consider some simple boundary value problems within the context of the theory. We first consider some homogeneous deformations, namely the uniform extension of a cylinder, simple shear of a slab, and the biaxial tension of a thin sheet. This is followed by a discussion of the consequences or lack thereof of the imposition of an additional state of spherical stress on such bodies. We end the paper with a discussion concerning the extensions of the results presented here.

## 2 Basic equations

### 2.1 Kinematics and the equation of equilibrium

Let $\mathscr{B}$ denote the abstract body and $\kappa_r(\mathscr{B})$ denote its reference configuration. A particle $X$ belonging to $\mathscr{B}$ occupies the position $\mathbf{X} = \boldsymbol{\kappa}_r(\mathscr{B})$ belonging to the reference configuration. In the current deformed configuration $\kappa_t(\mathscr{B})$, each particle $X$ occupies the position $\mathbf{x} = \boldsymbol{\kappa}_t(X, t)$, where $t$ is the time. We assume there exists a one-to-one mapping $\boldsymbol{\chi}$ such that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The deformation gradient and the linearized strain tensor are defined, respectively as:

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right), \tag{1}$$

where $\nabla$ is the gradient operator with respect to $\mathbf{X}$, $J = \det \mathbf{F} > 0$, and $\mathbf{u} = \mathbf{x} - \mathbf{X}$ is the displacement field.

In this study, we consider only quasi-static deformations; therefore, the Cauchy stress tensor must satisfy the equilibrium Eq.

$$\mathrm{div}\, \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \tag{2}$$

where div is the divergence operator, $\rho$ is the density of the body in the current configuration, and $\mathbf{b}$ is the body force.

The above definitions and equations suffice for this study, and more details about kinematics and the balance of linear momentum can be found, for example, in [7,25].

### 2.2 Some new classes of constitutive relations

As mentioned in the introduction, Rajagopal and his co-workers have proposed several forms of constitutive relations for elastic bodies, which cannot be classified as either Cauchy elastic bodies or Green elastic bodies

[13,16–20]. If $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, and $\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathrm{T}}$, some of these forms are[1]:

$$\mathfrak{A}(\mathbf{S}, \mathbf{E})\dot{\mathbf{S}} + \mathfrak{B}(\mathbf{S}, \mathbf{E})\dot{\mathbf{E}} = \mathbf{0}, \quad \mathfrak{F}(\mathbf{S}, \mathbf{E}) = \mathbf{0}, \quad \mathfrak{H}(\mathbf{T}, \mathbf{B}) = \mathbf{0}, \tag{3}$$

where in general it is not possible to express either the second Piola–Kirchhoff stress tensor in terms of $\mathbf{E}$ or viceversa or the Cauchy stress tensor $\mathbf{T}$ in terms of $\mathbf{B}$. Two special cases of (3) are $\mathbf{T} = \mathfrak{G}(\mathbf{B})$ and $\mathbf{B} = \mathfrak{K}(\mathbf{T})$ where the first relation is the classical constitutive equation for an isotropic homogeneous Cauchy elastic body, while the second is a new class of constitutive relation that has been studied recently.

When $|\nabla\mathbf{u}| \sim O(\delta)$, $\delta \ll 1$ we have $\mathbf{B} \approx \mathbf{I} + 2\boldsymbol{\varepsilon}$, and from (3), it is possible to prove (under certain conditions that are not discussed here for brevity) that the correct relation between the stresses and the linearized strain tensor $\boldsymbol{\varepsilon}$ should be of the form [1,2,11,12,21]

$$\boldsymbol{\varepsilon} = \mathfrak{f}(\mathbf{T}). \tag{4}$$

In the present work, we shall consider the constitutive expression (4).

## 2.3 Boundary value problems

Expressing the constitutive relation in the form of (4), especially when the relationship is not invertible, leads to difficulties with regard to the solution of boundary value problems as one cannot substitute the inverted expression for the stress in terms of the linearized strain in the equations of equilibrium (in general into the balance of linear momentum) to obtain an equation governing the displacement. One has to solve the equilibrium equations (in general the balance of linear momentum) and the constitutive relation simultaneously. Also, in large deformation theories, one has to also append the balance of mass, which is ignored when considering small strains as the density is viewed as being the same as the reference density. For the problem under consideration, the unknowns would be the stress and the displacement, making up nine scalar unknowns with the equations of equilibrium and (4) making up nine equations. That is, in view of $(1)_5$ and (2), we need to solve simultaneously (see, for example, [3,4,22]):

$$\frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right) = \mathfrak{f}(\mathbf{T}), \tag{5}$$

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \mathbf{0}, \tag{6}$$

$$\mathbf{T}\mathbf{n} = \breve{\mathbf{t}} \quad \text{on} \quad \partial\kappa_t^t(\mathscr{B}), \quad \mathbf{u} = \breve{\mathbf{u}} \quad \text{on} \quad \partial\kappa_t^u(\mathscr{B}) \tag{7}$$

where the boundary of the body in the current configuration is $\partial\kappa_t(\mathscr{B})$ and $\partial\kappa_t(\mathscr{B}) = \partial\kappa_t^u(\mathscr{B}) \cup \partial\kappa_t^t(\mathscr{B})$, $\partial\kappa_t^u(\mathscr{B}) \cap \partial\kappa_t^t(\mathscr{B}) = \emptyset$, and $\breve{\mathbf{t}}$ and $\breve{\mathbf{u}}$ are the known external traction and a known displacement field.

## 2.4 Constraints on the deformation in the classical theory of elasticity

In the classical theory of elasticity, constraints on the deformation are defined as restrictions, which the field $\boldsymbol{\chi}$ must satisfy for any deformation for a given family of bodies. In terms of the deformation gradient, such restrictions are usually written as [15,26]

$$\lambda(\mathbf{F}) = 0, \tag{8}$$

where $\lambda$ is a scalar function. The function $\lambda$ must be Galilean invariant; therefore, the restriction (8) is written as

$$\lambda(\mathbf{C}) = 0. \tag{9}$$

The classical approach to the treatment of constraints, which traces its roots to the contributions of D'Alembert, Bernoulli, and Lagrange, is based on assuming that the stress tensor can be divided in two parts

$$\mathbf{T} = \mathfrak{F}(\mathbf{F}) + \mathbf{T}_{\mathrm{N}}, \tag{10}$$

[1] In (3) $\mathfrak{A}(\mathbf{S}, \mathbf{E})$ and $\mathfrak{B}(\mathbf{S}, \mathbf{E})$ are fourth order tensor functions, where, for example, the expression $\mathfrak{A}(\mathbf{S}, \mathbf{E})\dot{\mathbf{S}}$ becomes $\mathfrak{A}_{\alpha\beta\gamma\delta}(\mathbf{S}, \mathbf{E})\dot{S}_{\gamma\delta}$ in index notation (Cartesian coordinates).

where on part $\mathfrak{F}(\mathbf{F})$ depends on the particular material being considered (in the family for which the constraint holds), and $\mathbf{T}_\mathrm{N}$ would be a part of the stress that do not do work with any deformation compatible with the constraint (9) (see §30 of [26]). It follows from the above assumption that the stress tensor is given by

$$\mathbf{T}_\mathrm{N} = q\mathbf{F}\frac{\partial \lambda}{\partial \mathbf{C}}\mathbf{F}^\mathrm{T}, \tag{11}$$

where $q$ is referred to as the Lagrange multiplier.

In the case of incompressible bodies, $\lambda$ takes the form:

$$\lambda(\mathbf{C}) = \det \mathbf{C} - 1. \tag{12}$$

If $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, then the above equation becomes

$$\mathrm{tr}\,(\boldsymbol{\varepsilon}) = 0. \tag{13}$$

## 3 Incompressibility

Let us now study the consequences of the constraint of incompressibility in the case of the new class of models (4). First, let us consider the special case wherein the function $\mathfrak{f}(\mathbf{T})$ is derivable from a potential, that is, $\mathfrak{f}(\mathbf{T}) = \frac{\partial W}{\partial \mathbf{T}}$, where $W = W(\mathbf{T})$ is a scalar function [1]. Furthermore in this section, let us assume that $f$ is an isotropic function,[2] in which case $W$ depends on the three invariants

$$I_1 = \mathrm{tr}\mathbf{T}, \quad I_2 = \frac{1}{2}\mathrm{tr}\,(\mathbf{T}^2), \quad I_3 = \frac{1}{3}\mathrm{tr}\,(\mathbf{T}^3); \tag{14}$$

therefore, using the chain rule for the derivative, we have (see [1])

$$\boldsymbol{\varepsilon} = W_1\mathbf{I} + W_2\mathbf{T} + W_3\mathbf{T}^2, \tag{15}$$

where $W_i = \frac{\partial W}{\partial I_i}$, $i = 1, 2, 3$.

Replacing (15) in (13), we obtain the first-order linear partial differential Eq. (see [1])

$$3W_1 + W_2 I_1 + 2W_3 I_2 = 0, \tag{16}$$

whose solution is

$$W = \bar{W}(\bar{I}_1, \bar{I}_2), \tag{17}$$

where

$$\bar{I}_1 = I_2 - \frac{I_1^2}{6}, \quad \bar{I}_2 = I_3 + \frac{2}{27}I_1^3 - \frac{2}{3}I_1 I_2. \tag{18}$$

Using (17) in (15) we have the general expression for $\mathfrak{f}(\mathbf{T})$ in the case of an isotropic body that satisfies (13)

$$\boldsymbol{\varepsilon} = \left(\mathbf{T} - \frac{I_1}{3}\mathbf{I}\right)\frac{\partial \bar{W}}{\partial \bar{I}_1} + \left[2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2\right]\frac{\partial \bar{W}}{\partial \bar{I}_2}. \tag{19}$$

[2] Material symmetry classifications for implicit constitutive relations can be found in [23].

### 3.1 The effect of a spherical state of stress

Let us consider the response of a model described by the constitutive relation (19) to a spherical state of stress. Let us assume that the stress tensor is of the form $\mathbf{T} = -p\mathbf{I}$, where $p = p(\mathbf{x})$ is a scalar field. It follows from (14) that the invariants are given by

$$I_1 = -3p, \quad I_2 = \frac{3}{2}p^2, \quad I_3 = -p^3. \tag{20}$$

Therefore, from (18), (19), we have $\bar{I}_1 = \bar{I}_2 = 0$ and $\mathbf{T} - \frac{I_1}{3}\mathbf{I} = \mathbf{0}$, $2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2 = \mathbf{0}$, i.e., from (19) we have that $\boldsymbol{\varepsilon} = \mathbf{0}$, which means that such stress does not generate any strain in the case of an elastic body described by (19).

Next, let us work with a more general case, where the stress tensor is of the form

$$\mathbf{T} = \mathbf{T}_{\mathrm{o}} - p\mathbf{I}, \tag{21}$$

where

$$\mathbf{T}_{\mathrm{o}} = \mathbf{T} - \frac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}, \quad p = -\frac{1}{3}(\mathrm{tr}\,\mathbf{T})\mathbf{I}. \tag{22}$$

Let us define

$$I_1^{\mathrm{o}} = \mathrm{tr}\,\mathbf{T}_{\mathrm{o}}, \quad I_2^{\mathrm{o}} = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}_{\mathrm{o}}^2\right), \quad I_3^{\mathrm{o}} = \frac{1}{3}\mathrm{tr}\left(\mathbf{T}_{\mathrm{o}}^3\right), \tag{23}$$

from $(22)_1$ we have that $I_1^{\mathrm{o}} = 0$. It follows from (21), (23) and (14), on using (23), that

$$I_1 = -3p, \quad I_2 = \frac{1}{2}(2I_2^{\mathrm{o}} + 3p^2), \quad I_3 = \frac{1}{3}(3I_3^{\mathrm{o}} - 6pI_2^{\mathrm{o}} - 3p^3). \tag{24}$$

Using these expressions in (18), it is possible to prove that $\bar{I}_1 = \bar{I}_1^{\mathrm{o}}$, $\bar{I}_2 = \bar{I}_2^{\mathrm{o}}$, i.e., the invariants $\bar{I}_1$, $\bar{I}_2$ only depend on $\mathbf{T}_{\mathrm{o}}$; as a result $\bar{W}_i(\mathbf{T}) = \bar{W}_i(\mathbf{T}^{\mathrm{o}})$, $i = 1, 2, 3$. In addition to this, on using (21), we also have

$$\mathbf{T} - \frac{I_1}{3}\mathbf{I} = \mathbf{T}_{\mathrm{o}} - \frac{I_1^{\mathrm{o}}}{3}\mathbf{I} = \mathbf{T}_{\mathrm{o}},$$
$$2\left(\frac{I_1^2}{9} - \frac{I_2}{3}\right)\mathbf{I} - \frac{2I_1}{3}\mathbf{T} + \mathbf{T}^2 = 2\left(\frac{I_1^{\mathrm{o}2}}{9} - \frac{I_2^{\mathrm{o}}}{3}\right)\mathbf{I} - \frac{2I_1^{\mathrm{o}}}{3}\mathbf{T}_{\mathrm{o}} + \mathbf{T}_{\mathrm{o}}^2 = -\frac{2I_2^{\mathrm{o}}}{3}\mathbf{I} + \mathbf{T}_{\mathrm{o}}^2,$$

and consequently, for a stress field given by (21), from (19) we obtain

$$\boldsymbol{\varepsilon}(\mathbf{T}) = \boldsymbol{\varepsilon}(\mathbf{T}_{\mathrm{o}}), \tag{25}$$

i.e., we confirm again that the spherical part of the stress given by $-p\mathbf{I}$ does not cause any strain (see [24] for a discussion of the distinction between the Lagrange multiplier that enforces a constraint and the mean value of the stress).

Thus, in the case of an incompressible body the boundary value problem (5), (6) can be rewritten in the form

$$-\mathrm{grad}\, p + \mathrm{div}\,\mathbf{T}_{\mathrm{o}} = \mathbf{0}, \quad \nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} = 2\mathfrak{f}(\mathbf{T}_{\mathrm{o}}). \tag{26}$$

We discuss about (26) in more detail in Sect. 5.

### 3.2 Dimensionless expressions

In order to discuss the relationship between (19) and the linearized case, Eq. (19) is rewritten in a dimensionless form. In order to do so, let us define the dimensionless stress tensor $\hat{\mathbf{T}}$ as[3]

$$\hat{\mathbf{T}} = \frac{1}{\sigma_0}\mathbf{T}, \tag{27}$$

where $\sigma_0$ is a characteristic value for the stress. Using (27) it is possible to define invariants in terms of this dimensionless stress tensor as (see (14)):

$$\hat{I}_1 = \operatorname{tr}\hat{\mathbf{T}} = \frac{I_1}{\sigma_0}, \quad \hat{I}_2 = \frac{1}{2}\operatorname{tr}\hat{\mathbf{T}}^2 = \frac{I_2}{\sigma_0^2}, \quad \hat{I}_3 = \frac{1}{3}\operatorname{tr}\hat{\mathbf{T}}^3 = \frac{I_3}{\sigma_0^3}. \tag{28}$$

Let us define the dimensionless function $\hat{W}$ as

$$\hat{W} = \frac{W}{\sigma_0}. \tag{29}$$

On considering (28), (29) and using the chain rule for the derivatives, we have $\frac{\partial W}{\partial I_1} = \frac{\partial W}{\partial \hat{I}_1}\frac{1}{\sigma_0}$, $\frac{\partial W}{\partial I_2} = \frac{\partial W}{\partial \hat{I}_2}\frac{1}{\sigma_0^2}$, $\frac{\partial W}{\partial I_3} = \frac{\partial W}{\partial \hat{I}_3}\frac{1}{\sigma_0^3}$, therefore, from (15) it is possible to show that

$$\boldsymbol{\varepsilon} = \hat{W}_1\mathbf{I} + \hat{W}_2\hat{\mathbf{T}} + \hat{W}_3\hat{\mathbf{T}}^2. \tag{30}$$

Appealing to the above expression for the strain tensor and following the same methodology presented in Sect. 3, for an incompressible body, the scalar function $\hat{W}$ should be written as

$$\hat{W} = \tilde{W}(\tilde{I}_1, \tilde{I}_2), \tag{31}$$

where

$$\tilde{I}_1 = \hat{I}_2 - \frac{\hat{I}_1^2}{6} = \frac{\bar{I}_1}{\sigma_o^2}, \quad \tilde{I}_2 = \hat{I}_3 + \frac{2}{27}\hat{I}_1^3 - \frac{2}{3}\hat{I}_1\hat{I}_2 = \frac{\bar{I}_2}{\sigma_o^3}, \tag{32}$$

and from (30) we have the equivalent dimensionless form of (19)

$$\boldsymbol{\varepsilon} = \left(\hat{\mathbf{T}} - \frac{\hat{I}_1}{3}\mathbf{I}\right)\frac{\partial \tilde{W}}{\partial \tilde{I}_1} + \left[2\left(\frac{\hat{I}_1^2}{9} - \frac{\hat{I}_2}{3}\right)\mathbf{I} - \frac{2\hat{I}_1}{3}\hat{\mathbf{T}} + \hat{\mathbf{T}}^2\right]\frac{\partial \tilde{W}}{\partial \tilde{I}_2}. \tag{33}$$

### 3.3 Linearized constitutive equations

The constitutive equation for incompressible isotropic bodies leading to the classical linearized theory of elasticity can be derived from (33), assuming that the stress tensor is very small in comparison with a characteristic value $\sigma_0$. Let us suppose that $|\hat{\mathbf{T}}| \sim O(\delta)$, where $\delta \ll 1$, then it is easy to show that $\hat{I}_1 \sim O(\delta)$, $\hat{I}_2 \sim O(\delta^2)$. From (33) discarding terms of order $O(\delta^n)$, $n \geq 2$, we obtain

$$\boldsymbol{\varepsilon} \approx \left(\hat{\mathbf{T}} - \frac{\hat{I}_1}{3}\mathbf{I}\right)\frac{\partial \tilde{W}}{\partial \tilde{I}_1}, \tag{34}$$

where the function $\frac{\partial \tilde{W}}{\partial \tilde{I}_1}$ can be approximated as a Taylor series around $\hat{\mathbf{T}} = \mathbf{0}$ as

$$\frac{\partial \tilde{W}}{\partial \tilde{I}_1} \approx \left.\frac{\partial \tilde{W}}{\partial \tilde{I}_1}\right|_{\hat{\mathbf{T}}=\mathbf{0}} + \left.\frac{\partial^2 \tilde{W}}{\partial \tilde{I}_1^2}\right|_{\hat{\mathbf{T}}=\mathbf{0}}\tilde{I}_1 + \left.\frac{\partial^2 \tilde{W}}{\partial \tilde{I}_1 \partial \tilde{I}_2}\right|_{\hat{\mathbf{T}}=\mathbf{0}}\tilde{I}_2. \tag{35}$$

[3] Dimensionless quantities will be denoted with a hat, i.e., as $(\hat{\ })$.

Again, neglecting terms of order $O(\delta^n)$, $n \geq 2$ from (34), (35), we obtain

$$\boldsymbol{\varepsilon} \approx \aleph \left( \hat{\mathbf{T}} - \frac{\hat{I}_1}{3} \mathbf{I} \right), \tag{36}$$

where $\aleph = \left. \frac{\partial \tilde{W}}{\partial \tilde{I}_1} \right|_{\hat{\mathbf{T}}=\mathbf{0}}$.

The constitutive equation for an isotropic linearized elastic solid (written in terms of the dimensionless stress (27)) is

$$\boldsymbol{\varepsilon} = \frac{(1+\nu)}{\hat{E}} \hat{\mathbf{T}} - \frac{\nu}{\hat{E}} \hat{I}_1 \mathbf{I}, \tag{37}$$

where $\hat{E} = \frac{E}{\sigma_0}$, where $E$ and $\nu$ are the Young modulus and the Poisson ratio, respectively. For the case of an incompressible body, we have $\nu = 1/2$ so

$$\boldsymbol{\varepsilon} = \frac{3}{2\hat{E}} \left( \hat{\mathbf{T}} - \frac{\hat{I}_1}{3} \mathbf{I} \right), \tag{38}$$

and on comparing (38) with (36) we obtain $\aleph = \frac{3}{2\hat{E}}$.

## 4 Study of some boundary value problems

In this section, we first obtain some simple solutions for boundary value problems where the linearized strain is related nonlinearly with respect to the stress, where the distributions of stresses and strains are homogeneous, and then we explore some consequences of considering the superposition of a spherical state of stress in the resolution of some boundary value problems, where the stress and strain distributions are in general nonhomogeneous.

### 4.1 The uniform extension of a cylinder

Let us consider the cylinder $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq L$, which we assume is under the influence of a stress field of the form:

$$\mathbf{T} = \sigma_z \mathbf{e}_z \otimes \mathbf{e}_z, \tag{39}$$

where $\sigma_z$ is a constant. On using this expression for the stress in (14), we have

$$I_1 = \sigma_z, \quad I_2 = \frac{\sigma_z^2}{2}, \quad I_3 = \frac{\sigma_z^3}{3}, \tag{40}$$

and from (18), we obtain

$$\bar{I}_1 = \frac{\sigma_z^2}{3}, \quad \bar{I}_2 = \frac{2}{27} \sigma_z^3, \tag{41}$$

while from (19) we have for the nonzero components of the strain tensor

$$\varepsilon_{zz} = \frac{2}{3} \frac{\partial \bar{W}}{\partial \bar{I}_1} \sigma_z + \frac{2}{9} \frac{\partial \bar{W}}{\partial \bar{I}_2} \sigma_z^2, \quad \varepsilon_{rr} = \varepsilon_{\theta\theta} = -\frac{1}{3} \frac{\partial \bar{W}}{\partial \bar{I}_1} \sigma_z - \frac{1}{9} \frac{\partial \bar{W}}{\partial \bar{I}_2} \sigma_z^2. \tag{42}$$

Since $\sigma_z$ is constant, this is a very simple solution of the boundary value problem (6) (when there is no body force). It is easy to see that $\operatorname{tr} \boldsymbol{\varepsilon} = 0$ independently of the magnitude of $\sigma_z$ or of the particular expression for $\bar{W}$.

### 4.2 The simple shear of a slab

Another simple problem that is usually considered in the theory of elasticity corresponds to the shear of a slab. Let us consider the slab $-L_i/2 \leq x_i \leq L_i/2$, $i = 1, 2, 3$, and let us assume that the slab is under the effect of the following stress tensor field

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \tag{43}$$

where $\tau$ is a constant. From (14), (18), we obtain

$$\bar{I}_1 = \bar{I}_3 = 0, \quad \bar{I}_2 = \tau^2, \tag{44}$$

while from (19), we have for the nonzero components of the strain tensor

$$\varepsilon_{12} = \frac{\partial \bar{W}}{\partial \bar{I}_1}\tau, \quad \varepsilon_{11} = \varepsilon_{22} = \frac{1}{3}\frac{\partial \bar{W}}{\partial \bar{I}_2}\tau^2, \quad \varepsilon_{33} = -\frac{2}{3}\frac{\partial \bar{W}}{\partial \bar{I}_2}\tau^2, \tag{45}$$

which since $\tau$ is a constant is also a solution for the boundary value problem.

### 4.3 The biaxial tension of a thin sheet

Let us study now the behavior of the thin plate $-L_1/2 \leq x_1 \leq L_1/2$, $-L_2/2 \leq x_2 \leq L_2/2$, $0 \leq x_3 \leq h$, where $h \ll L_1$ and $h \ll L_2$ described by the constitutive Eq. (19) when subject to external stimuli. We assume that this plate is under a state of stress of the form:

$$\mathbf{T} = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2, \tag{46}$$

where $\sigma_1$ and $\sigma_2$ are constants. Using this in (14) we have

$$I_1 = \sigma_1 + \sigma_2, \quad I_2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right), \quad I_3 = \frac{1}{3}\left(\sigma_1^3 + \sigma_2^3\right), \tag{47}$$

and from (18) we obtain that

$$\bar{I}_1 = \frac{1}{3}\left(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2\right), \quad \bar{I}_2 = -\frac{7}{27}\left(\sigma_1^2 + \sigma_2^3\right) - \frac{4}{9}\left(\sigma_1^2\sigma_2 + \sigma_2\sigma_2^2\right), \tag{48}$$

therefore from (19) the nonzero components of the strain tensor are:

$$\varepsilon_{11} = \left(\frac{2\sigma_1 - \sigma_2}{3}\right)\frac{\partial \bar{W}}{\partial \bar{I}_1} + \frac{1}{9}\left(2\sigma_1^2 - 2\sigma_1\sigma_2 - \sigma_2^2\right)\frac{\partial \bar{W}}{\partial \bar{I}_2}, \tag{49}$$

$$\varepsilon_{22} = \left(\frac{2\sigma_2 - \sigma_1}{3}\right)\frac{\partial \bar{W}}{\partial \bar{I}_1} + \frac{1}{9}\left(2\sigma_2^2 - 2\sigma_1\sigma_2 - \sigma_1^2\right)\frac{\partial \bar{W}}{\partial \bar{I}_2}, \tag{50}$$

$$\varepsilon_{33} = -\left(\frac{\sigma_1 + \sigma_2}{3}\right)\frac{\partial \bar{W}}{\partial \bar{I}_1} + \frac{1}{9}\left(-\sigma_1^2 + 4\sigma_1\sigma_2 - \sigma_2^2\right)\frac{\partial \bar{W}}{\partial \bar{I}_2}. \tag{51}$$

## 5 Consequences of the application of an additional arbitrary spherical stress

In this section, we explore whether the constraint of incompressibility facilitates the possibility to establish some exact solutions of boundary value problems defined by (26) as we have an arbitrariness with regard to the spherical part of the stress that can be adjusted at will.

### 5.1 Deformations of a slab

Let us consider the body[4] $a \leq x \leq b$, $-L_2/2 < y < L_2/2$, $-L_3/2 < z < L_3/2$, and let us assume that this body is subject to a stress field which is decomposed into a part $\mathbf{T}_\mathrm{o}$ and a spherical part, that is, $\mathbf{T} = \mathbf{T}_\mathrm{o} - p\mathbf{I}$, where:

$$\mathbf{T}_\mathrm{o} = \mathbf{T}_\mathrm{o}(x) = \sum_{i=1}^{3} \sigma_i(x)\mathbf{e}_i \otimes \mathbf{e}_i + \tau_{12}(x)(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + \tau_{13}(x)(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) + \tau_{23}(x)(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2). \tag{52}$$

$$p = p(x, y, z) = \zeta(x) + c_1 y + c_2 z, \tag{53}$$

where $c_1, c_2$ are constants and

$$\sigma_1(x) + \sigma_2(x) + \sigma_3(x) = 0. \tag{54}$$

Under the previous assumptions the equilibrium Eq. $(26)_1$ (without body forces) becomes

$$\frac{\mathrm{d}\zeta}{\mathrm{d}x} = \frac{\mathrm{d}\sigma_1}{\mathrm{d}x}, \quad c_1 = \frac{\mathrm{d}\tau_{12}}{\mathrm{d}x}, \quad c_2 = \frac{\mathrm{d}\tau_{13}}{\mathrm{d}x}, \tag{55}$$

and hence we obtain the solutions

$$\zeta(x) = \sigma_1(x) + p_0, \quad \tau_{12} = c_1 x + d_1, \quad \tau_{13} = c_2 x + d_2, \tag{56}$$

where $p_0$, $d_1$ and $d_2$ are constants.

Let us assume that the stress tensor field (52), (53) produces the following displacement field

$$u_i(x, y, z) = \upsilon_i(x) + g_i y + h_i z, \quad i = 1, 2, 3, \tag{57}$$

where $g_i, h_i, i = 1, 2, 3$ are constants. For these expressions for the components of the displacement field, from $(1)_5$ and $(26)_2$, we obtain the relations

$$\frac{\mathrm{d}\upsilon_1}{\mathrm{d}x} = \mathfrak{f}_{11}(\mathbf{T}_\mathrm{o}(x)), \quad g_2 = \mathfrak{f}_{22}(\mathbf{T}_\mathrm{o}(x)), \quad g_3 = \mathfrak{f}_{33}(\mathbf{T}_\mathrm{o}(x)), \tag{58}$$

$$g_1 + \frac{\mathrm{d}\upsilon_2}{\mathrm{d}x} = 2\mathfrak{f}_{12}(\mathbf{T}_\mathrm{o}(x)), \quad h_1 + \frac{\mathrm{d}\upsilon_3}{\mathrm{d}x} = 2\mathfrak{f}_{13}(\mathbf{T}_\mathrm{o}(x)), \quad h_2 + g_3 = 2\mathfrak{f}_{23}(\mathbf{T}_\mathrm{o}(x)). \tag{59}$$

From $(58)_1$, $(59)_{1,2}$, (57) it is possible to obtain the solutions for[5] $u_i(x)$, $i = 1, 2, 3$ as

$$u_1 = \int_a^x \mathfrak{f}_{11}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi + g_1 y + h_1 z, \quad u_2 = 2\int_a^x \mathfrak{f}_{12}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi - g_1 x + g_2 y + h_2 z, \tag{60}$$

$$u_3 = 2\int_a^x \mathfrak{f}_{13}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi - h_1 x + g_3 y + h_3 z. \tag{61}$$

It is necessary now to consider some boundary conditions for the body, let us assume that at $x = a$ we are applying an external traction $\breve{\mathbf{t}}$, while at $x = b$ the displacement field vector is assumed to be known and is equal to $\breve{\mathbf{u}}$. Considering the expressions for the stress tensor (53) and (56) from $\mathbf{Tn} = \breve{\mathbf{t}}$ at the boundary $x = a$ we obtain

$$p_0 = \breve{t}_1, \quad -c_1 a - d_1 = \breve{t}_2, \quad -c_2 a - d_2 = \breve{t}_3, \tag{62}$$

while from (57), (60), (61) and $\mathbf{u} = \breve{\mathbf{u}}$ at the boundary $x = b$ we have

$$\int_a^b \mathfrak{f}_{11}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi + g_1 y + h_1 z = \breve{u}_1, \quad 2\int_a^b \mathfrak{f}_{12}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi - g_1 b + g_2 y + h_2 z = \breve{u}_2, \tag{63}$$

$$2\int_a^b \mathfrak{f}_{13}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi - h_1 b + g_3 y + h_3 z = \breve{u}_3. \tag{64}$$

A solution of this boundary value problem may be found in the following manner:

[4] We use the notation $x, y, z$ for $x_1, x_2, x_3$. An interesting case can be the semi-infinite body $L_2 \gg |a|$, $L_2 \gg |b|$, $L_3 \gg |a|$, $L_3 \gg |b|$.

[5] We do not add a constant when integrating $(58)_1$ and $(59)_{1,2}$.

- Let us assume that $\breve{t}_i$, $i = 1, 2, 3$ are given data, then from (62)$_1$ we can find $p_0$ and, for example, $d_1, d_2$.
- Let us assume that $c_1$, $c_2$, $g_2$ and $g_3$ are arbitrary.
- As a consequence of the above assumptions, from (56)$_1$ we can find an expression for $\zeta(x)$.
- The algebraic Eqs. (58)$_{2,3}$, (59)$_3$ and (54) can be used to find, for example, $\sigma_1(x)$, $\sigma_2(x)$, $\sigma_3(x)$ and $\tau_{23}(x)$.
- Considering now the expression for the stress tensor, from (63) and (64), we can obtain the components $\breve{u}_i$, $i = 1, 2, 3$ of the external displacement field, for which (52), (53), (57) are solutions to the boundary value problem.

### 5.2 Deformations of a cylindrical tube

Let us assume that the infinitely long tube $a \leq r \leq b$ is under the following state of stress:

$$\mathbf{T}_\mathrm{o} = \mathbf{T}_\mathrm{o}(r) = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z + \tau_{r\theta}(r)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \tau_{rz}(r)(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) + \tau_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_\theta), \quad (65)$$

$$p = p(r, z) = \zeta(r) + cz, \quad (66)$$

where $c$ is a constant and

$$\sigma_r(r) + \sigma_\theta(r) + \sigma_z(r) = 0. \quad (67)$$

In this case, the equilibrium equations (26)$_1$ are of the form

$$\frac{\mathrm{d}\zeta}{\mathrm{d}r} = \frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{1}{r}(\sigma_r - \sigma_\theta), \quad 0 = \frac{\mathrm{d}\tau_{r\theta}}{\mathrm{d}r} + \frac{2}{r}\tau_{r\theta}, \quad c = \frac{1}{r}\tau_{rz}. \quad (68)$$

Let us assume that the stress field associated with (65), (66) generates the displacement field

$$u_r = \upsilon_r(r) + h_r z, \quad u_\theta = \upsilon_\theta(r), \quad u_z = \upsilon_z(r) + h_z z, \quad (69)$$

where $h_r$, $h_z$ are constants. Using these expressions for the displacement field in (1) and considering the constitutive relation (26)$_2$, we have

$$\frac{\mathrm{d}\upsilon_r}{\mathrm{d}r} = \mathfrak{f}_{rr}(\mathbf{T}_\mathrm{o}(r)), \quad \frac{\upsilon_r}{r} = \mathfrak{f}_{\theta\theta}(\mathbf{T}_\mathrm{o}(r)), \quad h_z = \mathfrak{f}_{zz}(\mathbf{T}_\mathrm{o}(r)), \quad (70)$$

$$\frac{\mathrm{d}\upsilon_\theta}{\mathrm{d}r} - \frac{\upsilon_\theta}{r} = 2\mathfrak{f}_{r\theta}(\mathbf{T}_\mathrm{o}(r)), \quad h_r + \frac{\mathrm{d}\upsilon_z}{\mathrm{d}r} = 2\mathfrak{f}_{rz}(\mathbf{T}_\mathrm{o}(r)), \quad 0 = \mathfrak{f}_{\theta z}(\mathbf{T}_\mathrm{o}(r)). \quad (71)$$

From (68)$_1$, we can obtain $\zeta(r)$ as

$$\zeta(r) = \sigma_r(r) + \int_a^r \frac{(\sigma_r(\xi) - \sigma_\theta(\xi))}{\xi}\,\mathrm{d}\xi + p_0, \quad (72)$$

where $p_0$ is a constant. From (68)$_{2,3}$, we have

$$\tau_{r\theta} = \frac{d}{r^2}, \quad \tau_{rz} = cr, \quad (73)$$

where $d$ is a constant.

From (70)$_2$, (71)$_1$ and (71)$_2$, we obtain

$$\upsilon_r(r) = r\mathfrak{f}_{\theta\theta}(\mathbf{T}_\mathrm{o}(r)), \quad \upsilon_\theta(r) = 2r\int_a^r \frac{\mathfrak{f}_{r\theta}(\mathbf{T}_\mathrm{o}(\xi))}{\xi}\,\mathrm{d}\xi, \quad \upsilon_z(r) = 2\int_a^r \mathfrak{f}_{rz}(\mathbf{T}_\mathrm{o}(\xi))\,\mathrm{d}\xi - h_r r + h_z z. \quad (74)$$

Finally from (70)$_{1,2}$ we obtain the differential equation

$$\mathfrak{f}_{rr}(\mathbf{T}_\mathrm{o}(r)) = \frac{\mathrm{d}}{\mathrm{d}r}[r\mathfrak{f}_{\theta\theta}(\mathbf{T}_\mathrm{o}(r))], \quad (75)$$

which needs to be satisfied to have a unique solution for $\upsilon_r(r)$. It is important to recognize that this is not necessarily the unique solution to the problems as we are using an inverse method and assuming a special form for the solution, other forms of the solution might be possible.

Solving the (in general nonlinear) ordinary differential equation (75) and the algebraic equations (70)$_3$, (71)$_3$ and (67), we could find $\sigma_r(r)$, $\sigma_\theta(r)$, $\sigma_z(r)$ and $\tau_{\theta z}(r)$. Regarding the values for the different constants in the above expressions for the stress tensor and the displacement field, they can be found considering some specific boundary conditions at $r = a$ and $r = b$, but we do not study that here.

## 6 Conclusions

In the present work, we have studied how the constraint of incompressibility can be dealt within the context of a new class of constitutive relations, wherein the linearized strain tensor is a function of the Cauchy stress tensor. From the results established, in particular in Sect. 3, it is possible to see the simple and straightforward nature in which such constraints are forced for these new classes of constitutive relations, where we do not need to assume a priori the existence of a Lagrangian multiplier. After having developed constitutive relations that satisfy the constraint that the body is incompressible and hence can only undergo isochoric motions, we study several boundary value problems such as simple shear, and uniaxial and biaxial extension for a particular class of the new constitutive relations.

Although in this communication we have studied only the case of small gradient of the displacement field $|\nabla \mathbf{u}| \sim O(\delta)$, $\delta \ll 1$, the extension of this theory to the case of large deformation gradients is straightforward. In such a case, the counterpart of (16) is a first-order nonlinear partial differential equation, which is being analyzed by the authors [5]. Other constraints can also be studied in the same manner as presented here for these new classes of constitutive relation, such as the case of a nonlinear elastic body that is inextensible in a given direction. This problem is also being studied by the present authors [6].

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