POMS PRODUCTION AND OPERATIONS MANAGEMENT

PRODUCTION AND OPERATIONS MANAGEMENT POMS

Vol. 25, No. 1, January 2016, pp. 114–127
ISSN 1059-1478 | EISSN 1937-5956 | 16 | 2501 | 0114

DOI 10.1111/poms.12402

Product Assortment and Price Competition under Multinomial Logit Demand*

Omar Besbes
Graduate School of Business, Columbia University, New York City, New York 10027, USA, ob2105@columbia.edu

Denis Sauré
Industrial Engineering Department, University of Chile, Republica 701, Santiago, Chile, dsaure@dii.uchile.cl

The role of assortment planning and pricing in shaping sales and profits of retailers is well documented and studied in monopolistic settings. However, such a role remains relatively unexplored in competitive environments. In this study, we study equilibrium behavior of competing retailers in two settings: (i) when prices are exogenously fixed, and retailers compete in assortments only; and (ii) when retailers compete jointly in assortment and prices. For this, we model consumer choice using a multinomial Logit, and assume that each retailer selects products from a predefined set, and faces a display constraint. We show that when the sets of products available to retailers do not overlap, there always exists one equilibrium that Pareto-dominates all others, and that such an outcome can be reached through an iterative process of best responses. A direct corollary of our results is that competition leads a firm to offer a broader set of products compared to when it is operating as a monopolist, and to broader offerings in the market compared to a centralized planner. When some products are available to all retailers, that is, assortments might overlap, we show that display constraints drive equilibrium existence properties.

Key words: assortment planning; competition; choice models; multinomial Logit; pricing
History: Received: May 2014; Accepted: May 2015 by Bill Cooper, after 1 revision.

1. Introduction

Assortment planning decisions are fundamental drivers of consumers' purchase decisions and ultimately of a retailer's profitability. Retailers face significant challenges to understand the mapping from assortment decisions to consumer behavior as this mapping should synthesize complex aspects of purchase decisions such as, for example, substitution behavior, consumers' collection and aggregation of information, consumer heterogeneity, and the effect of competition. A key input in most assortment models is a consumer choice model. In this regard, and despite its documented deficiencies, the multinomial Logit model (MNL) of consumer choice has been widely used in the economics, operations and marketing literatures (see Ben-Akiva and Lerman 1985, Guadagni and Little 1983, Train 2002), and also in practice. Thus, it is important to study its properties and its implications on decision making of firms in competitive settings, which predominate in practice. However, the theory on competitive outcomes in assortment and/or prices appears underdeveloped. For example, recently assortment and pricing decisions under the MNL have been analyzed in Misra (2008), but only best responses, specific to the *joint* selection of assortment and prices, were studied. In particular, no results on general equilibrium properties, such as existence or uniqueness, or the structure of the equilibrium set are known.

The present study aims to develop a general framework that enables to analyze equilibrium outcomes from competition in assortment-only or in joint assortment and pricing decisions. In particular, the goals of the present study are two-fold: our main objective is, given the widespread use of the MNL model, to advance the theory pertaining to equilibrium outcomes under this model. In addition, we also aim to derive insights on assortment and pricing actions in competitive settings using the MNL model, complementing those in the existing literature.

To this end, we analyze a model of assortment and price competition in a duopolistic setting. On the consumers' side, we assume that customers select from the set of products offered by both retailers according to an MNL model. On the firms' side, each retailer

*An early version of this paper was circulated under the title: "Product Assortment and Price Competition with Informed Consumers."

has access to a set of products from which to select an assortment and is constrained by limited display capacity. The set of products from which retailers may select products are general and may overlap. In particular, we differentiate between *common* products, that is, those that are available to select in an assortment for both competitors, and *exclusive* products, those that are unavailable to a firm's competitors. We first analyze competition when product prices are exogenous, a setting we refer to as assortment-only competition. We then analyze competition when prices are endogenous, that is, are decided by the retailers. In this case, we consider a general formulation when firms face arbitrary minimum margins, and analyze the setting in which firms simultaneously select assortments and prices. Unless otherwise stated, we analyze the interactions between retailers as a game and focus on pure-strategy Nash equilibria.

Regarding our first goal above, our results may be summarized as follows:

1. We establish that, when retailers have access only to exclusive products, an equilibrium in pure strategies always exists for both assortment-only and joint assortment and price competition.
2. We prove that, in those settings, when multiple equilibria exist, an equilibrium will always Pareto-dominate all others. In other words, all retailers would prefer to settle at the same equilibrium. Moreover, we show that this equilibrium arises naturally starting from a monopolistic setup, when retailers periodically react through best responses to the competitors' last observed offerings.
3. We establish that when retailers have access to exclusive and common products, an equilibrium in pure strategies is guaranteed to exist as long as there are no shelf space constraints. When shelf space constraints bind, it is possible that the number of non-Pareto-dominated equilibria grow exponentially with the retailers' display capacity. In addition, while an equilibrium in mixed strategies always exists, it is now possible that an equilibrium in pure strategies fails to exist.

Many of these results extend to the case of an oligopoly with an arbitrary number of firms and we comment on how to do so throughout the study.

Regarding our second goal, our results allow to measure the impact of competition on offerings in the market. For example, when retailers compete only through exclusive products, the introduction of a product by a retailer leads the competitor to also broaden her assortment. Also, we show that the competitor might be better off reducing the breadth of the assortment offered, so as to induce the retailer to reduce her assortment as well, which leads both retailers to increase their profit (this is the case when the original equilibrium is not Pareto-dominant). In terms of comparisons with monopolists, our analysis shows that a retailer will offer a broader set of products than if she was operating as a monopolist (with the same capacity), and that the set of retailers will jointly offer a broader set of products than if a central planner (facing the same display constraints) coordinated the assortment decisions. In this regard, our results complement those from studies in assortment selection and pricing in different competitive settings (e.g., Cachon et al. 2008, Coughlan and Shaffer 2009, Dukes et al. 2009). Our analysis also indicates that when retailers have access to both common and exclusive products, the interactions between retailers may take a fundamentally different form. The presence of common products introduces an interplay between retailers' decisions at the product level: a product that otherwise should not be included in an assortment might be so if the competitor includes it in her assortment, and vice versa. We show that this interplay is softened when display constraints are absent since there is no longer an opportunity cost associated with offering a given product. In particular, we show that display constraints are the driver of the possibility of equilibrium non-existence in pure strategies.

Assortment optimization is in general a complex combinatorial problem. Thus, characterizing and analyzing properties of the outcome of competition among retailers may appear to be a daunting task *a priori*. This is probably why most formulations in the existing literature abstract away from this combinatorial structure. The main contribution of the present study is to establish that under an MNL model, the problems of assortment-only and assortment and price competition are actually amenable to analysis, despite the combinatorial nature of the problems solved by the retailers. From a methodological viewpoint, the analysis builds on the idea of computing best responses via a problem transformation. Such an approach has been previously used in various settings when faced with a combinatorial optimization problem with a rational objective function. It was, for example, used by Dantzig et al. (1967) for finding the minimal cost-to-time ratio cycle in a network, by Megiddo (1979) for computational complexity results on the optimization of rational objective functions, and more recently by Rusmevichientong et al. (2010) in the closely related context of monopolistic assortment optimization with Logit demand. The current paper leverages this transformation in a novel fashion, and shows that it can serve as one of the building blocks for a unified framework to analyze a *competitive* setting. In addition, the resulting framework is shown to be fairly flexible, enabling one to, for example, incorporate endogenous prices.

1.1. Literature Review

Misra (2008) studies joint assortment and price competition of retailers offering exclusive products with MNL demand and in the presence of display constraints, and conducts an empirical study to analyze the impact of competition on assortment size and prices. The analytical results obtained focus only on best response analysis and do not provide equilibrium existence or uniqueness results, which may be obtained through our framework. Furthermore, the present study also develops theory for the case of assortment-only competition. We also refer the reader to Draganska and Jain (2006) and Draganska et al. (2009) for empirical investigations of assortment and pricing strategies in oligopolistic markets.

Additional dimensions of competition as well as alternative consumer choice models have also been analyzed (see Anderson and de Palma 2006, Cachon and Kök 2007, Hopp and Xu 2008, Kök and Xu 2011, Symeonidis 2009). The present study is the first, to the best of our knowledge, to study assortment-only competition and to provide a framework that applies to both the latter and joint assortment and pricing competition.

The possibility of offering overlapping assortments has been considered before in the literature. The challenges introduced by common products are highlighted in Cachon et al. (2008) when modeling competition with consumers that sequentially search for products (see also Iyer and Kuksov (2012) for an analysis of the role of search cost in competitive environments), and by Dukes et al. (2009) in a competitive setting dominated by a retailer. Similarly, Coughlan and Shaffer (2009) highlights the interaction between common and exclusive products in the context of price match guarantees when retailers compete in price and assortment.

The interplay between product introduction and price competition has been studied in Thomadsen (2012) to highlight that a rival may benefit from a firm introducing an additional product. Price-only competition under choice models has been studied and is still an active area of research. Anderson et al. (1992) study oligopoly pricing for single-product firms under Logit demand and study pricing and assortment depth for multi-product firms in a duopoly with a nested Logit demand, restricting attention to symmetric equilibria. When firms offer a single product and customers' choice is described by an attraction model, Bernstein and Federgruen (2004) establish existence and uniqueness of an equilibrium for profit-maximizing firms and Gallego et al. (2006) generalize this result for different cost structures. Gallego and Wang (2014) study price competition under the nested Logit model. For the Logit model, Konovalov and Sándor (2009) provide guarantees for the existence and uniqueness of an equilibrium for affine cost functions when firms may have multiple products. Allon et al. (2013) provide conditions that ensure existence and uniqueness of an equilibrium under MNL demand with latent classes.

While the studies above focus on assortment competition, there is a large body of work that focuses on monopolistic assortment optimization, and approaches to compute optimal strategies given the combinatorial nature of the problem. The problem of assortment planning has often been studied in conjunction with inventory decisions, starting with the work of van Ryzin and Mahajan (1999), who consider Logit demand and assume that customers do not look for a substitute if their choice is stocked out. They identify a tractable set of candidates that contains the optimal assortment. Maddah and Bish (2007) study a similar model, where in addition, the retailer could select prices; see also Aydin and Ryan (2000) for a study in the absence of inventory considerations. More recently, dynamic multi-period assortment optimization has been analyzed; see, for example, Caro et al. (2014). The case of customers looking for substitutes if their choice is stocked out, known as stock-out-based substitution, was studied in Smith and Agrawal (2000), Mahajan and van Ryzin (2001) and more recently Goyal et al. (2009). We also refer the reader to Rooderkerk et al. (2013) and references therein for a recent study of attribute-based assortment optimization.

In the present work, we do not consider inventory decisions and assume that products that are included in a retailer's assortment are always available when requested; hence stock-out-based substitution does not arise. In particular, we focus on the case in which the retailers face display constraints. Such a setting with Logit demand and fixed prices in a monopolistic context has been studied in Chen and Hausman (2000), where the authors analyze mathematical properties of the problem, and in Rusmevichientong et al. (2010), where the authors provide an efficient algorithm for finding an optimal assortment. Fisher and Vaidyanathan (2009) also study assortment optimization under display constraints and highlight how such constraints arise in practice. When demand is generated by a mixture of Logit, Miranda-Bront et al. (2009) show that when the number of classes is sufficiently high, the assortment optimization problem is NP-Hard (see also Rusmevichientong et al. 2014). A review of the literature on monopolistic assortment optimization and of industry practices can be found in Kök et al. (2008). At a higher level, the move from a single-agent optimization to competition relates to the work of Immorlica et al. (2011).

1.2. The Remainder of the Paper

Section 2 formulates the model of competition. Sections 3 and 4 present our analysis of the assortment only and joint assortment and price competition settings, respectively. Section 5 presents extensions and concluding remarks. Proofs are relegated to Appendix S1.

2. Model of Assortment and Price Competition

We next describe the setting in which retailers compete and the demand model considered, and then present two competitive settings: one where retailers compete on assortments in which prices are predetermined and one in which retailers compete on both assortments and prices.

2.1. Setting

We consider duopolistic retailers that compete in product assortment and pricing decisions. We index retailers by 1 and 2, and whenever we use n to denote a retailer's index, we use m to denote her/his competitor's index (e.g., if $n = 1$, then $m = 2$).

We assume that retailer n has access to a subset $\mathcal{S}_n$ of products, from which she or he must select her or his product assortment. In addition, we assume that, due to display space constraints, retailer n can offer at most $C_n \geq 1$ products. Such display constraints have been used and motivated for various settings in previous studies (see, e.g., Fisher and Vaidyanathan 2009, Misra 2008, Rusmevichientong et al. 2010). Without loss of generality, we assume that $C_n \leq |\mathcal{S}_n|$, where $|A|$ denotes the cardinality of a set A. We let $\mathcal{S}$ denote the set of all products, that is, $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2$, and denote its elements by $\{1, \ldots, S\}$. For each product $i \in \mathcal{S}$ and $n = 1,2$, we let $c_{n,i} \geq 0$ denote the marginal cost to retailer n resulting from acquiring a unit of the product, which is assumed constant.

We say that product i is *exclusive* to retailer n if it belongs to $\mathcal{S}_n$ but not to $\mathcal{S}_m$; we denote the set of exclusive products for retailer n by $\mathcal{S}_n \backslash \mathcal{S}_m$, where $A \backslash B := A \cap B^c$ stands for the set difference between sets A and B, and the complement of a set is taken relative to $\mathcal{S}$. Similarly, we say that product i is *common* if it is available to both retailers, that is, if i belongs to $\mathcal{S}_1 \cap \mathcal{S}_2$. An example of exclusive products would be private labels and of common products would be national brands.

For $n \in \{1,2\}$, we define $\mathcal{A}_n$ to be the set of feasible assortment selections for retailer n, that is,

$$\mathcal{A}_n := \{A \subseteq \mathcal{S}_n : A \leq C_n\}.$$

We let A_n denote the assortment selection and $p_n := (p_{n,1}, \ldots, p_{n,S})$ the vector of prices offered by retailer n. Note that p_n specifies a price for all products in $\mathcal{S}$, for notational convenience: it should be clear that only prices that correspond to the assortment selection of the retailer matter. In addition, we will omit the dependence of various quantities on the price decisions when possible, making them explicit only when deemed necessary.

2.2. Demand Model and Retailers' Objective

We assume that customers have perfect information about product assortments and prices offered by both retailers. (Here, we ignore search costs as, for example, in Thomadsen (2012); see, e.g., Kuksov and Villas-Boas (2010) for a study that accounts for such costs.) We assume that customer t assigns a utility $U_{n,i}(t)$ to buying product i from retailer n, and utility $U_{n,0}(t)$ to not purchasing any product, where

$$U_{n,i}(t) := \mu_{n,i} - \alpha_{n,i} p_{n,i} + \xi_i^t, \quad n = 1,2$$
$$U_{n,0}(t) := \mu_0 + \xi_0^t.$$

In the above, $\mu_{n,i}$ represents the adjusted mean utility associated with buying product i from retailer n. Similarly, $\alpha_{n,i} > 0$ is a parameter of price sensitivity. To obtain MNL demand, we assume that $\{\xi_i^t : i \in \mathcal{S} \cup \{0\}\}$ are i.i.d. random variables following a standard Gumbel distribution. Note that these random variables, which represent idiosyncratic shocks to utility, are independent of the retailer n and hence consumers identify common products as such. (Considering idiosyncratic shocks of the form $\xi_{i,n}^t$ would lead to MNL demand with only exclusive products, a special case of the analysis. We discuss a setting with nested Logit demand in section 5).

Without loss of generality, we set $\mu_0 := 0$. Customers are utility maximizers; customer t computes the best option from each retailer, $i_n \in \operatorname{argmax}\{U_{n,i}(t) : i \in A_n \cup \{0\}\}$, for $n \in \{1,2\}$, and then selects option i that belongs to $\operatorname{argmax}\{U_{1,i_1}(t), U_{2,i_2}(t)\}$. Note that the assumption above implies that utility maximization may be attained simultaneously at a common product offered by both retailers *with positive probability* (e.g., when $\mu_{1,i} - \alpha_{1,i} p_{1,i} = \mu_{2,i} - \alpha_{2,i} p_{2,i}$). In such a case, we assume that customers select any of the retailers, with equal probability.

REMARK 1. Note that in our basic model, if customers do not prefer any retailer, a retailer offering a lower price for a common product will capture the whole market for that product (we discuss extensions in section 5). Thus, we have assumed that customers have perfect information and are rational. An alternative interpretation of the model in this special case is that products are sold through an intermediary and the demand is fulfilled by the cheaper supplier.

For n = 1,2 define the *attraction factor* of product $i \in \mathcal{S}_n$ when offered by retailer n as follows

$$v_{n,i} := e^{\mu_{n,i} - \alpha_{n,i} p_{n,i}}.$$

The above setup leads to MNL demand where the customers' consideration set is obtained after eliminating options that are strictly dominated. These are product–retailer pairs such that the same product is offered by the competitor and provides a higher utility when bought from the competitor. In particular, one can show that for given assortment and price decisions $\{(A_n, p_n) : n = 1, 2\}$, the probability that a customer elects to purchase product $i \in A_n$ from retailer n, $q_{n,i}$, is given by

$$q_{n,i}(A_n, p_n, A_m, p_m) := \frac{v_{n,i}\Big(\mathbf{1}\{i \notin A_m\} + \delta_{n,i}\mathbf{1}\{i \in A_m\}\Big)}{1 + \sum_{i \in A_n \setminus A_m} v_{n,i} + \sum_{i \in A_n \cap A_m} (\delta_{n,i} v_{n,i} + \delta_{m,i} v_{m,i}) + \sum_{i \in A_m \setminus A_n} v_{m,i}},$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function and

$$\delta_{n,i} := \mathbf{1}\{v_{n,i} > v_{m,i}\} + \frac{1}{2}\mathbf{1}\{v_{n,i} = v_{m,i}\}$$

defines the split of product i's market share between the retailers (when offered by both). The expected profit per customer for retailer n, is then written as

$$\pi_n(A_n, p_n, A_m, p_m) = \sum_{i \in A_n} (p_{n,i} - c_{n,i}) q_{n,i}(A_n, p_n, A_m, p_m).$$

Each retailer's objective is to maximize her expected profit per customer, given the competitor's decision.

3. Assortment-Only Competition: Main Results

In this section, prices are assumed to be predetermined and not under the control of retailers. We further assume that all products have a positive profit margin (i.e., $p_{n,i} > c_{n,i}$). This accommodates settings where, for example, prices are set by the manufacturers/service providers and not the retailers. Here and throughout the study, we will abstract away from strategic interactions between retailers and manufacturers.

Given retailer m's assortment decision, retailer n selects an assortment so as to maximize her/his expected profit per customer subject to the display constraint on the number of products that can be offered. Mathematically, the problem that retailer n solves can be written as follows

$$\max_{A_n \in \mathcal{A}_n} \{\pi_n(A_n, A_m)\}, \tag{1}$$

where we have omitted the dependence of the expected profit on prices. In problem (1), the retailer attempts to find the best set of products to offer among a combinatorial number of possibilities in $\mathcal{A}_n$. We say that a feasible assortment A_n is a best response to A_m if A_n maximizes the profit per customer for retailer n, that is, if A_n solves problem (1). Given that there is a finite number of feasible assortments, there always exists at least one best response to each assortment $A_m \in \mathcal{A}_m$. We say that an assortment pair (A_1, A_2) is an equilibrium if A_n is a best response to A_m for n = 1,2. Formally, this corresponds to the concept of a *pure-strategy Nash equilibrium*.

3.1. Best Response Correspondence

Our analysis of the best response correspondence relies on a simple equivalent formulation of the profit maximization problem. This idea of focusing on such an equivalent formulation when faced with a combinatorial problem with an objective function in the form of a ratio has previously been used in various settings as mentioned in the introduction. The reader is in particular referred to Gallego et al. (2004) and Rusmevichientong et al. (2010) for applications to monopolistic assortment optimization with Logit demand. The treatment that follows develops the appropriate modifications for the *competitive* setting we study. Consider the following problem

$$\max \ \lambda \tag{2a}$$

$$\text{s.t.} \quad \max_{A \in \mathcal{A}_n} \Big\{ \sum_{i \in A \setminus A_m} (p_{n,i} - c_{n,i} - \lambda) v_{n,i} + \sum_{i \in A \cap A_m} \Big(\delta_{n,i}(p_{n,i} - c_{n,i}) - \lambda\Big)(\delta_{n,i} v_{n,i} + \delta_{m,i} v_{m,i}) - \lambda \sum_{i \in A_m \setminus A} v_{m,i} \Big\} \geq \lambda. \tag{2b}$$

Lemma 1. *Problems* (1) *and* (2) *are equivalent in the following sense: the optimal values for both problems are equal and an assortment is optimal for problem* (1) *if and only if it maximizes the left-hand side of* (2*b*) *when* $\lambda = \lambda^*$, *where* λ^* *corresponds to the optimal objective function of* (2).

Lemma 1 exploits the rational form of the profit function, by first finding any assortment that surpasses a given profit level, and then looking for the highest profit level attainable. Hence, in theory, one could solve for the best response to A_m by solving the maximization in Equation (2b) for all possible values of λ, and then selecting any assortment maximizing the left-hand side of Equation (2b) for λ^*. This way of envisioning solving Equation (2) will prove useful in the equilibrium analysis we conduct for this setting, as well as throughout the rest of the study. We now outline how to solve the maximization in Equation (2b). To that end, for $i \in \mathcal{S}_n$, define

$$\theta_{n,i}(\lambda) := \begin{cases} (p_{n,i} - c_{n,i} - \lambda)v_{n,i} & \text{if } i \notin A_m, \\ \delta_{n,i}((p_{n,i} - c_{n,i} - \lambda)v_{n,i} + \lambda v_{m,i}) & \text{if } i \in A_m. \end{cases} \tag{3}$$

Given this, formulation (2) can be rewritten as

$$\max\left\{\lambda \in \mathbb{R} : \max_{A \in \mathcal{A}_n}\left\{\sum_{i \in A} \theta_{n,i}(\lambda)\right\} \geq \lambda\Big(1 + E_m(A_m)\Big)\right\}, \tag{4}$$

where $E_m(A)$, referred to as the *attractiveness* of an assortment A, is defined as follows

$$E_m(A) := \sum_{i \in A} v_{m,i}, \quad m = 1, 2.$$

For given non-overlapping assortment offerings, A_1 and A_2, this quantity is related to an aggregate measure of the market share of retailer n, $E_n(A_n)/(1 + E_1(A_1) + E_2(A_2))$. One can solve the inner maximization in Equation (4) by ordering the products in $\mathcal{S}_n$ according to the corresponding values of $\theta_{n,i}(\lambda)$, from highest to lowest, and selecting the maximum number of products in the assortment (up to C_n) with positive values of $\theta_{n,i}(\lambda)$.

3.1.1. Product Ranking. As already highlighted in the monopolistic setting by Rusmevichientong et al. (2010), the $\theta_{n,i}(\lambda)$-ranking for an optimal value of λ need not to coincide with the ranking of the profit margins (which is always the case in the absence of capacity constraints: see van Ryzin and Mahajan 1999). In addition, note that in the competitive setting under analysis, the product ranking according to the $\theta_{n,i}$'s (and hence the selected assortment) will vary depending on the value of λ and on which products are included in the competitor's assortment A_m. This last observation implies that, for a fixed value of λ, a product that is not "appealing" (i.e., a product that is not included in a best response) if not offered by the competitor might become appealing when the latter offers it. This can be seen from the second case in Equation (3) where $\theta_{n,i}(\lambda)$ might increase by $\lambda v_{m,i}$ when product i is offered by retailer m. This gain can be interpreted as the value of profiting from product i without having to expand the consideration set of products.

3.2. The Case of Exclusive Products

This section studies the case of retailers having only *exclusive products*, that is, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. We begin by specializing the best response computation to this setting, and then study equilibrium behavior. In this setting, one has that for each product $i \in \mathcal{S}_n$

$$\theta_{n,i}(\lambda) = (p_{n,i} - c_{n,i} - \lambda)v_{n,i},$$

independent of A_m, thus the solution to the inner maximization in Equation (4) depends on A_m only through $E_m(A_m)$. Define $\lambda_n(e)$ as retailer n's expected profit per customer when retailer m offers assortment A_m with attractiveness e. That is

$$\lambda_n(e) := \max\left\{\lambda \in \mathbb{R} : \max_{A \in \mathcal{A}_n}\left\{\sum_{i \in A} \theta_{n,i}(\lambda)\right\} \geq \lambda(1 + e)\right\}. \tag{5}$$

Similarly, let $a_n(e)$ denote retailer n's best response correspondence to assortments with attractiveness e, that is,

$$a_n(e) := \operatorname*{argmax}_{A \in \mathcal{A}_n}\left\{\sum_{i \in A} \theta_{n,i}(\lambda_n(e))\right\}. \tag{6}$$

The next result establishes monotonicity properties of the best response correspondence in terms of attractiveness and profit level.

Proposition 1 (best response properties). *Suppose that* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

(i) Retailer n's best response profit is decreasing in the attractiveness of its competitor's assortment, that is, $\lambda_n(e)$ *is decreasing in e.*
(ii) The attractiveness of retailer n's best response assortments is non-decreasing in the attractiveness of the competitor's assortment, e, in the following sense: for any $e > e' \geq 0$, $E_n(a') \leq E_n(a)$ *for all* $a \in a_n(e)$ *and* $a' \in a_n(e')$.

Proposition 1(i) states that a retailer's (optimized) profits will decrease if the competitor increases the attractiveness/breadth of its offerings, which is in line with intuition. Proposition 1(ii) provides an important qualitative insight: if one retailer increases the attractiveness of the products it is offering, then so will the other one.

The conclusions of Proposition 1 are usually obtained in the context of supermodular games. However, it is worth noting that, in general, it is not clear whether one could obtain a supermodular representation of the assortment game with the exception of the case where margins are equal across products. (In the latter case, the game can be seen to be log-supermodular on the discrete lattice of possible attractiveness levels induced by all feasible assortments.) However, once properties outlined in Proposition 1 are at hand, one can establish existence and ordering of equilibria in a similar fashion as is usually performed for supermodular games (see e.g., Vives 2000), which we do next. The following result guarantees that an equilibrium exists.

THEOREM 1 (EQUILIBRIUM EXISTENCE). *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Then there always exists an equilibrium in assortment decisions.*

The proof of this result rests on the fact that assortments with higher attractiveness levels lead the competitor to also offer assortments with higher attractiveness. Since there is a finite number of possible attractiveness levels to offer, attractiveness of best responses will necessarily settle at a certain level, with the corresponding assortments forming an equilibrium.

Theorem 1 establishes the existence of an equilibrium, but leaves open the possibility of having multiple equilibria. While there might indeed exist multiple equilibria, we show next that if such a case occurs, both retailers will prefer the same equilibrium.

PROPOSITION 2 (BEST EQUILIBRIUM). *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and that multiple equilibria exist. Then, one equilibrium Pareto-dominates all equilibria, and such an equilibrium minimizes the attractiveness of the offerings of each retailer among all equilibria.*

In other words, when multiple equilibria exist, retailers would prefer to select one with the least breadth of offerings. The result is a direct consequence of the relationship between profit level and attractiveness of the offering, established in Proposition 1(i).

REMARK 2 (ARBITRARY NUMBER OF FIRMS). Proposition 1, Theorem 1, and Proposition 2 can be generalized to an arbitrary number of retailers and we briefly indicate how one might do so in the proofs of those results.

The next example illustrates the possibility of multiple equilibria

EXAMPLE 1 (MULTIPLE EQUILIBRIA). Let $\mathcal{S}_n := i_1^n, i_2^n, \ldots\}$ be such that $v_{n,i_j^n} > v_{n,i_{j+1}^n}$ for all $j < C_n$, n = 1,2, and suppose that for all i in $\mathcal{S}_n \backslash \{i_1^n\}$, $p_{n,i} - c_{n,i} = r_n$, where

$$r_1 := \frac{\left(p_{1,i_1^1} - c_{1,i_1^1}\right) v_{1,i_1^1}}{1 + v_{1,i_1^1} + \sum_{j \le C_2} v_{2,i_j^2}}, \quad r_2 := \frac{\left(p_{2,i_1^2} - c_{2,i_1^2}\right) v_{2,i_1^2}}{1 + v_{2,i_1^2} + v_{1,i_1^1}}.$$

In other words, for each retailer, all products except the one with the highest attraction factor have the same profit margin, and the latter margin is strictly lower than the former. The construction above is such that when retailer 1 selects the assortment $\{i_1^1\}$, retailer 2's best response is any assortment of the type $\{i_1^2, \ldots, i_j^2\}$ for $j \le C_2$. Similarly, when retailer 2 selects the assortment $\{i_j^2 : j \le C_2\}$, retailer 1's best response is any assortment of the type $\{i_1^1, \ldots, i_j^1\}$ for $j \le C_1$. This, in conjunction with Proposition 2(ii) implies that

$$(\{i_1^1\}, \{i_1^2, \ldots i_j^2\}), j \le C_2, \quad (\{i_1^1, \ldots i_j^1\}, \{i_1^2, \ldots i_{C_2}^2\}), j \le C_1,$$

are all equilibria.

Note that the number of equilibria in Example 2 is $C_1 + C_2 - 1$, and that in each of these equilibria, retailers offer different total attractiveness, thus they do not lead to the same outcomes. The following result shows that such a number is the highest possible in the setting.

PROPOSITION 3 (BOUND ON THE NUMBER OF EQUILIBRIA). *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ and let $\mathcal{E}_n$ denote all possible attraction levels offered in a best response by retailer n, that is,*

$$\mathcal{E}_n := \{E_n(a) \in \mathbb{R}_+ : a \in a_n(e), e \ge 0\}, \quad \text{for } n = 1, 2.$$

There are at most $|\mathcal{E}_1| + |\mathcal{E}_2| - 1$ equilibria in which retailers offer different attraction levels.

A priori, a trivial bound on the number of fundamentally different equilibria is the number of combinations of best response attractiveness levels, $|\mathcal{E}_1|\mathcal{E}_2|$. Proposition 3 provides a significantly sharper bound. The proof of Proposition 3 relies on the strong monotonicity property established in Proposition 1(ii), which enables one to eliminate a large set of equilibrium candidates. In Example 1, we had that $\mathcal{E}_n = C_n$, n = 1,2, and $C_1 + C_2 - 1$ equilibria. Thus, the bound in Proposition 3 is tight. The next result, which we state without proof, is a direct consequence of Proposition 3.

COROLLARY 1 (SUFFICIENT CONDITION FOR A UNIQUE EQUILIBRIUM OUTCOME). *Suppose that all products offered by a given retailer have the same margin, that is, $p_{n,i} - c_{n,i} = r_n$ for all $i \in \mathcal{S}_n$, $n = 1,2$, where r_1 and r_2 are given positive constants. Then, retailers offer the same attraction level in all equilibria.*

3.2.1. Tatônement Stability. Proposition 1 has also important implications for stability. Consider the following discrete-time interaction dynamics: suppose that initially one of the retailers operates as a monopolist until a competitor enters the market, and that retailers adjust their assortment decisions on a periodic basis by taking turns, always reacting optimally to their competitors' last observed offering (i.e., retailers behave myopically, and adjust their decisions periodically without anticipating the competitors' reaction). More generally, starting from arbitrary offerings (A_1^0, A_2^0), suppose that retailer n offers in period t the assortment

$$A_n^t \in \begin{cases} a_n(E_m(A_m^{t-1})) & \text{if } t = 2k + n \text{ for some } k \in \mathbb{N} \\ \{A_n^{t-1}\} & \sim . \end{cases} \tag{7}$$

The next result establishes that such a best-response process always converges to an equilibrium outcome. Moreover, starting from a monopolistic setting, the result establishes convergence to the equilibrium outcome that Pareto-dominates all others, as described in Proposition 2.

COROLLARY 2. *Let (A_1^0, A_2^0) be arbitrary initial assortments. There exists $\tilde{t} < \infty$ such that the tatônement process $\{(A_1^t, A_2^t) : t \geq 1\}$ described in Equation (7) is such that (A_1^t, A_2^t) is a pure-strategy equilibrium for all $t \geq \tilde{t}$. Moreover, if $A_2^0 = \emptyset$, for $t \geq \tilde{t}$, the equilibrium (A_1^t, A_2^t) Pareto-dominates all others.*

The result above, whose proof can be found in Appendix S1, follows from Proposition 1. In particular, we show that if $E_2(A_2^0) \geq E_2(A_2^2)$, then the recursive application of Proposition 1(ii) implies that the sequences of total attractiveness of the assortments offered by both retailers (excluding A_0^1, which does not affect the sequence) will be non-increasing, that is, $E_n(A_n^t) \geq E_n(A_n^{t+1})$, $t \geq 1$, $n = 1,2$. Similarly, if $E_2(A_2^0) \leq E_2(A_2^2)$ then the sequences of attractiveness will be non-decreasing. This observation, together with the finiteness of the product sets, implies convergence to an equilibria outcome. With regard to the second statement in the result, it follows from the fact that the attractiveness offered by a monopolist is below that offered by a Pareto-dominant equilibrium, and that a Pareto-dominant equilibrium minimizes (among equilibria) the attractiveness offered by both retailers. From the above, one can envision a Pareto-dominant equilibrium as arising naturally as the outcome of an iterative best-response process that starts in the absence of competition.

3.2.2. Competitive Outcome vs. Monopolist and Centralized Solutions. Let (A_1^{comp}, A_2^{comp}) denote an equilibrium that pareto-dominates all others. Let (A_1^{cent}, A_2^{cent}) denote an optimal pair of assortments to offer by a central planner, that is,

$$(A_1^{cent}, A_1^{cent}) \in \underset{(A_1,A_2) \in \mathcal{A}_1 \times \mathcal{A}_2}{\operatorname{argmax}} \{\pi_1(A_1, A_2) + \pi_2(A_2, A_1)\}.$$

Finally, let A_n^* denote an optimal assortment for a monopolist that does not face any competition, that is,

$$A_n^* \in \underset{A_n \in \mathcal{A}_n}{\operatorname{argmax}} \{\pi_n(A_n, \emptyset)\}.$$

We have the following result, which we state without proof.

COROLLARY 3. *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.*

(i) $E_n(A_n^*) \leq E_n(A_n^{comp})$.
(ii) $E_1(A_1^{cent}) + E_2(A_2^{cent}) \leq E_1(A_1^{comp}) + E_2(A_2^{comp})$.

In particular, (i), which follows from Proposition 1, implies that a retailer operating as a monopolist with some fixed capacity (i.e., with its competitor offering no products) will increase her/his offerings in terms of attractiveness when a competitor enters the market. (ii) establishes that competing retailers will jointly offer a broader offering, resulting in a higher probability of purchase, relative to a setting in which decisions are coordinated by a central planner aiming to maximize total profits, and facing similar capacity constraints. To see this, note that the central planner achieves higher profits than those achieved by any of the retailers in a potential equilibrium; a close inspection of the proof of Proposition 1(ii) reveals that the attractiveness of the solution to the assortment maximization in Equation (2b) is non-increasing in the level λ; this implies that the attractiveness of the products offered by the central planner would never be higher than the joint attractiveness of the products offered by the competing retailers in equilibrium.

3.3. The Case of Both Exclusive and Common Products

We now turn to the case when retailers may offer the same products in their respective assortments, that is, when $\mathcal{S}_1 \cap \mathcal{S}_2$ is not empty. Our next result shows that an equilibrium is guaranteed to exist when retailers do not face display constraints.

THEOREM 2 (EQUILIBRIUM EXISTENCE WITH AMPLE CAPACITY). *Suppose that $C_n = |\mathcal{S}_n|$ for $n = 1,2$. Then an equilibrium always exists.*

The proof of this result is constructive: we establish that the tatônement process described in section 3.2 converges to an equilibrium provided that initially both retailers offer all common products. In addition, it is possible to establish that the tatonement process (7) with $A_1^0 \in a_1(0)$ and $A_2^0 = \emptyset$ is guaranteed to converge to an equilibrium. Thus, as in section 3.2.1, one can envision such a limit equilibrium as arising naturally as the outcome of an iterative best-response process that starts in the absence of competition. However, in this setting, such an equilibrium does not necessarily Pareto-dominate all others.

It is possible to find alternative conditions that will ensure the existence of an equilibrium. For example, it is possible show that conditions 1 and 2 below each ensure existence of an equilibrium.

CONDITION 1. Monotonic margins: $v_{n,i} \geq v_{n,i+1}$, and $p_{n,i} - c_{n,i} \geq p_{n,i+1} - c_{n,i+1}$ for all $i \in \mathcal{S}_n$, $n = 1,2$, and $v_{n,i} > v_{m,i}$ for all $i \in B_n \cap \mathcal{S}_m$, for some $B_n \in \mathcal{P}_n$, $n = 1,2$. Here, $\mathcal{P}_n$ is the set of "popular assortments" defined as (see, e.g., Kök et al. 2008):

$$\mathcal{P}_n := \{\{\sigma_n(1), \cdots, \sigma_n(C_n)\} \\ : \sigma_n \text{ is a permutation of } \mathcal{S}_n \text{ s.t. } v_{n,\sigma_n(1)} \geq \cdots \geq v_{n,\sigma_n(|\mathcal{S}_n|)}\}.$$

CONDITION 2. Equal margins: $p_{n,i} - c_{n,i} = r_n$, for all $i \in \mathcal{S}_n$, $n = 1,2$, where r_1 and r_2 are given positive constants, and $\mathcal{S}_1 = \mathcal{S}_2$, $C_1 = C_2$, and $v_{1,i} = v_{2,i}$ for all $i \in \mathcal{S}_1$.

3.3.1. The Impact of Display Constraints. In general, when common products are available and display constraints are present, the structural results of the previous section fail to hold, as we illustrate through the following example.

EXAMPLE 2 (NON-EXISTENCE OF EQUILIBRIUM IN PURE STRATEGIES). Consider a setting with two retailers, each having access to the same three products $\mathcal{S}_1 = \mathcal{S}_2 = \{1, 2, 3\}$, and with display capacities $C_1 = 2$, $C_2 = 1$. Suppose that prices and costs are uniform across products and retailers and given by $p_{n,1} = p_{n,2} = p_{n,3} = p > 1$ and $c_{n,1} = c_{n,2} = c_{n,3} = p - 1$ for $n = 1,2$, and that the remaining parameters are such that $v_{n,1} = 1.1$, $v_{n,2} = 1.01$, $v_{n,3} = 1$. Table 1 depicts the rewards for each retailer for feasible pairs of assortment decisions (A_1, A_2).

One can verify that no equilibrium exists. Intuitively, the latter stems from the fact that retailer 1, with a capacity of 2, will always prefer to incorporate in its assortment the product that retailer 2 is offering, while retailer 2 prefers to offer a product not offered by the competitor. Recalling the discussion following the definition of $\theta_{n,i}(\lambda)$ in Equation (3), the current example illustrates how a product gains in appeal (measured by $\theta_{n,i}(\lambda)$) when offered by the competitor. In this setting, this prevents the possibility of an equilibrium.

The focus above was on equilibria in pure strategies. Since each retailer has a finite number of alternatives *the assortment game will always admit a mixed-strategy equilibrium* (see, e.g., Fudenberg and Tirole 1991, section 1.3.1): in the setting of Example 2, one can check that retailer 1 offering $A_1 = \{1, 2\}$ with probability 0.51 and $A_1 = \{1, 3\}$ with probability 0.49, and retailer 2 offering $A_2 = \{2\}$ with probability 0.35 and $A_2 = \{3\}$ with probability 0.65 constitutes a mixed-strategy Nash equilibrium.

In addition to the above, even when an equilibrium in pure strategies exists, the presence of common products may also preclude the existence of a Pareto-

Table 1 Illustration of Non-Existence of Equilibrium

	A_1					
	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}
A_2						
{1}	(0.262, 0.262)	(0.354, 0.325)	(0.355, 0.323)	(0.177, 0.502)	(0.177, 0.500)	(0.268, 0.489)
{2}	(0.325, 0.354)	(0.251, 0.251)	(0.336, 0.332)	(0.162, 0.516)	(0.246, 0.511)	(0.168, 0.500)
{3}	(0.323, 0.355)	(0.332, 0.336)	(0.250, 0.250)	(0.243, 0.513)	(0.161, 0.516)	(0.166, 0.502)

Each entry in the table corresponds the profit of retailer 1 and retailer 2 as a function of the assortments selected for the setup in Example 2.

dominant equilibrium, as highlighted by the following example.

EXAMPLE 3 (EXPONENTIAL NUMBER OF NON-PARETO-DOMINATED EQUILIBRIA). Consider the following setup. Suppose that the set of available products is common to both retailers ($\mathcal{S}_1 = \mathcal{S}_2$) and has $S = 2C$ elements, where $C_1 = C_2 = C$, that all products are priced at the same price p, that their marginal cost is zero, and that other parameters are such that $v_{1,i} = v_{2,i}$ for all $i \in \mathcal{S}$, and

$$v_{n,1} > v_{n,2} > \ldots > v_{n,S}, \quad \text{and} \quad v_{n,1} < \frac{3}{2} v_{n,S}.$$

Define $A^* := \mathcal{P}_1$ and suppose that $3E_1(A^*) \leq 1$. This condition corresponds to assuming that the maximum share any retailer can achieve (under any scenario) is below 25%. Under the setup above, we show that if retailer 2 offers an arbitrary selection of products A_2, then the best response of retailer 1 is to offer the set of C products with the highest $v_{n,i}$'s in $\mathcal{S}_1 \backslash A_2$. Recalling (2), retailer 1 solves for the maximal λ such that

$$\max_{A_1 \subseteq \mathcal{S}} \Big\{ E_1(A_1 \backslash A_2)(p - \lambda) + E_1(A_1 \cap A_2) \left(\frac{p}{2} - \lambda\right) - \lambda E_2(A_2 \backslash A_1) \Big\} \geq \lambda.$$

Given any assortment offered by retailer 2, A_2, the revenue of retailer 1, λ, is bounded by the revenue of a monopolist (with display capacity C), i.e., $\lambda \leq p(E_1(A^*))(1 + E_1(A^*))^{-1}$. This, in conjunction with the market share condition above, implies that $\lambda \leq p/4$.

For any products $j \in \mathcal{S}_1 \backslash A_2$ and $j' \in A_2$, given that $v_{2,j'} < (3/2)v_{2,j}$, it will always be the case that $v_{2,j}(p - \lambda) \geq (3/4)v_{2,j}p > v_{2,j'}p/2$. Hence $\theta_{2,j}(\lambda) > \theta_{2,j'}(\lambda)$ and the best response of retailer 1 to A_2 will never include any product in A_2. In addition, since $\lambda \leq p/4$, retailer 1 will always include C products in $\mathcal{S}_1$ not offered by the competitor.

Given the above, one can verify that any pair of assortments (A_1, A_2) that belongs to the set

$$\{A_1 \subseteq \mathcal{S}, A_2 = \mathcal{S} \backslash A_1 : \{A_1\} = C\}$$

is an equilibrium. It is also possible to see that one can choose the $v_{n,i}$'s so that all equilibria yield different profits to the retailers and are non-Pareto dominated. The cardinality of the set above is $\binom{2C}{C}$. This illustrates that in general, even when prices are uniform across products, the number of non-Pareto-dominated equilibria may be exponential in the capacities of the retailers in contrast with what was observed in the case of exclusive products.

4. Joint Assortment and Price Competition: Main Results

We now turn attention to the case in which in addition to assortment decisions, retailers also set prices for the products they offer. We follow a parallel exposition to that of section 3 by separating the analysis for the case of exclusive products and that of both exclusive and common products.

We assume throughout this section that prices are restricted to be greater or equal than $c_{n,i} + r_{n,i}$ for any product i and retailer n, where $r_{n,i}$ is a minimal margin imposed by the product manufacturer. This assumption reflects the fact that minimal prices are commonly imposed directly or indirectly by manufacturers through, for example, a Manufacturer's suggested retail price (MSRP).

When price is an additional lever, various forms of competition may arise. We focus on the case in which assortments and prices are selected simultaneously by the firms. Thus, best responses are computed as unilateral deviations in both assortment and prices.

4.1. The Case of Exclusive Products

We start with the case of retailers having only exclusive products, that is, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. It turns out that equilibrium prices can be related to assortment selections through the profit attained by each retailer. This is, given fixed assortment selections, each firm solves a classical multi-product pricing problem with constraints on the prices. In particular, one can show that retailer n will set the price of product i (if offered) to

$$p^*_{n,i} = c_{n,i} + \max\left\{\frac{1}{\alpha_{n,i}} + \lambda_n, r_{n,i}\right\}, \tag{8}$$

where λ_n is the equilibrium profit retailer n achieves (which will be shown to be well defined in the proof of Theorem 3). Relationship (8) can be seen to be an expression of *equal margins* across offered products, with the modifications to account for the differentiated minimal margins imposed. Margins, adjusted to differences in price sensitivities, will be equal provided that the retailer's profit is relatively large compared to the minimum margins. Variants of such a property have previously appeared in various related settings; see, for example, Anderson et al. (1992).

Following the analysis in section 3.2, and using the observation above, one can show that the best response of a retailer still depends only on the competitor's offered attractiveness (which now depends on the competing assortment and prices). In particular, retailer n's expected profit per customer as a function of the competing attractiveness level e is given by Equation (5), but considering that

$$\theta_{n,i}(\lambda) = (p^*_{n,i}(\lambda) - c_{n,i} - \lambda)\exp\{\mu_{n,i} - \alpha_{n,i}p^*_{n,i}(\lambda)\}.$$

Thus, one has that the best response correspondence $a_n(e)$, whose elements are now assortment and price vector pairs, is given by

$$a_n(e) := \underset{A \in \mathcal{A}_n}{\operatorname{argmax}} \left\{ \sum_{i \in A} \theta_{n,i}(\lambda_n(e)) \right\} \times \{p^*_n(\lambda_n(e))\}.$$

In such a setting, in which prices are set by the firms, one may establish parallel results to Proposition 1 and Theorem 1, which were established when prices were exogenously fixed.

Theorem 3 (equilibrium existence and pareto dominance). *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Then, there always exists a pure-strategy Nash equilibrium in assortments and prices. Moreover, there is always one equilibrium that Pareto-dominates all others.*

To establish this result, we first establish that prices are uniquely defined by assortment selections. We then show that similar monotonicity properties as those established in Proposition 1 hold in the current setting in which prices may be adjusted. These properties in turn imply the existence of a pure-strategy Nash equilibrium and that one equilibrium Pareto-dominates all others. Moreover, such properties can be used to extend the comparison of outcomes under competitive vs. a monopolist or a centralized setting to obtain a result similar to that in Corollary 3.

From Equations (3) and (8) one observes that $\theta_i > 0$ for every product $i \in \mathcal{S}_n$. This, in turn, implies that retailers will always use their full capacity in any equilibrium. This stands in contrast with the case in which prices were exogenously set. In particular, in the absence of display constraints, all products are offered and the equilibrium is unique. The next result, which we state without proof, formalizes this.

Corollary 4. *Suppose that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. In equilibrium, retailers always use their full capacity. In particular, in the absence of display constraints, there exists a unique equilibrium in which firm n offers all products in $\mathcal{S}_n, n = 1, 2$.*

When no minimum margins are imposed by the manufacturers and price sensitivity is the same across products and retailers, it is possible to establish the existence of a unique equilibrium. The analysis of best responses in such a setting has already appeared in the literature (see, e.g., Misra 2008); however, no equilibrium results were provided. The present approach establishes existence and uniqueness of an equilibrium, but also illustrates along the way the general applicability of the framework we use to, for example, assortment-only competition.

4.2. The Case of Both Exclusive and Common Products

When retailers may offer the same products, equilibrium prices depend not only on the profit obtained by each retailer but also on the competitor's offering.

As in the case of competition with only exclusive products, one can show that for given fixed assortment selections and competitor's prices, retailer n will set the price of an exclusive product following Equation (8). Now, for a common product i offered by both retailers, it is possible to show that one can, without loss of generality for the equilibrium outcome in terms of profits, restrict attention to the following candidate equilibrium prices

$$p^*_{n,i} := \max\left\{\min\left\{\frac{\mu_{n,i} - \mu_{m,i}}{\alpha_{n,i}} + \frac{\alpha_{m,i}}{\alpha_{n,i}}(c_{m,i} + r_{m,i}), c_{n,i} + \frac{1}{\alpha_{n,i}} + \lambda_n\right\}, c_{n,i} + r_{n,i}\right\},$$

where λ_n is the equilibrium profit retailer n achieves. The difference between the expression above and that in Equation (8) follows from the fact whenever possible, retailers will undercut the competitor's prices so as to capture the full market for a product. In this regard, it is implicit in the formula above that retailers price marginally below the price that makes a customer indifferent between buying from either retailer as long as it yields a positive profit.

We note that if the minimal margins are sufficiently high, then the retailers will face a problem where prices are effectively fixed. Thus, the results in section 3.3 imply that it is not possible to ensure existence of an equilibrium under when common products are available in conjunction with display constraints. We show, however, that as in the analysis of assortment competition with fixed prices, a equilibrium is guaranteed to exist when retailers do not face display constraints.

The next result ensures equilibria existence is guaranteed in the absence of display constraints

Proposition 4. *In the assortment and price competition setting with common products, suppose retailers do not face display constraints. Then, there always exists a pure-strategy Nash equilibrium.*

As in the case of simultaneous competition with only exclusive products, the result above rests on the fact that a best response to any offering involves offering all products. Thus, any equilibrium in the pricing game where all products are offered is also an equilibrium in the simultaneous assortment and pricing competition setting.

5. Extensions and Additional Challenges

The present study has analyzed an assortment game in which firms face display constraints and consumer demand is driven by a MNL. We have characterized equilibrium behavior, showing that significant structure is present in such a problem. The approach taken to analyze the game is fairly general and has been extended to cases in which prices are endogenous.

5.1. Other Forms of Competition or Operational Constraints

In the present study, we have studied simultaneous assortment and price competition. Another interesting direction to analyze is the impact of the type of competition (simultaneous vs. sequential) on the type of outcomes one observes. Similarly, one might test the flexibility of the approach to incorporate new operational constrains. For example, one can show that most results in our analysis hold when the display constraint is replaced by a similar constraint of the type $|A_n| \geq C_n$, that might arise in setting in which managers must fulfill minimum assortment *diversity* requirements.

5.2. Modeling Customer Choice with Common Products

The current study has highlighted that the presence of common products may lead to significant differences in equilibrium behavior. The possibility of common products has been assumed away in most of the literature. For example, when the choice model is a nested Logit in which customers first select a retailer, the product utility shocks are assumed to be independent once a retailer has been chosen. While such assumptions are appropriate for settings in which consumers do not search across retailers due to loyalty or costs of search, it becomes inappropriate if consumers perform some data collection before selecting a product to purchase. In particular, it appears that common products ought to be treated differently than exclusive products.

5.2.1. Extension of the Basic Model. The present study has analyzed one case in this spectrum in which the product utility shock is identical across retailers. Nonetheless, the proposed framework provides enough flexibility to incorporate many variations of the base setting. For example, one could accommodate the hypothesis that products that are offered by more than one retailer become more attractive to consumers. For that, one would consider a formulation in which

$$\delta_{n,i} := \mathbf{1}\{v_{n,i} > v_{m,i}\} + \beta \mathbf{1}\{v_{n,i} = v_{m,i}\},$$

where $0.5 \leq \beta \leq 1$ reflects the potential increase in product attractiveness when it is offered by multiple firms. In such a setting, $\beta = 0.5$ corresponds to the setting analyzed in this study, and $\beta = 1$ reduces to the case of exclusive-only products. One can show that the results in sections 3 and 2 continue to hold in such a setting, after minor modifications to their proofs. (Note that cases with exclusive-only products are not affected by this modification.)

An important avenue for future research is to further one's understanding of customer choice behavior in the face of both common and exclusive products and to understand the implications of such behavior on equilibrium outcomes.

5.2.2. The Case of Nested Logit Demand. As pointed out in Remark 2, our demand model is such that it is possible for a retailer to capture the whole market for a product offered by both retailers. This feature follows from the fact that idiosyncratic shocks to utility are independent of the retailer. Let us analyze the case where such shocks depend on the retailer as well, that is,

$$U_{n,i}(t) := \mu_{n,i} - \alpha_{n,i} p_{n,i} + \xi^t_{n,i}, \quad n = 1, 2.$$

(We assume that ξ^t_0 does not depend of the retailer, as it represents an outside alternative).

As mentioned in section 2, assuming that $\{\xi^t_{n,i} : n = 1, 2, i \in \mathcal{S}_n, \ t \geq 1\}$ are i.i.d. random variables a standard Gumbel distribution would recover MNL demand at the expense of eliminating the possibility that customers recognize that common products are identical. Let us consider then the case of nested demand, where consumers first select the product to buy, and then the retailer to buy from. In terms of the utility specification above, assume that $(\xi^t_{n,i} : n = 1, 2, i \in \mathcal{S})$ has a GEV distribution of the form

$$\mathbb{P}\{\xi^t_{n,i} \leq x_{n,i}, n = 1, 2, i \in \mathcal{S}\} = \exp\left\{ - \sum_{i \in \mathcal{S}} \left(e^{-x_{1,i}/\gamma_i} + e^{-x_{2,i}/\gamma_i} \right)^{\gamma_i} \right\},$$

where $\gamma_i \in [0, 1]$ relate to the importance of a product choice over retailer choice. This gives rise to a nested Logit demand (see e.g., Train 2002), where in a first stage customers choose a product according to an aggregate measure of the utility they perceive from buying from the retailers offering such a product, and in a second stage they choose the retailer according to a traditional MNL model. In particular, the probability that a customer elects to purchase product $i \in A_n$ from retailer n, $q_{n,i}$ is

$$q_{n,i}(A_n, p_n, A_m, p_m) = q_i^1(A_n, p_n, A_m, p_m) \cdot q_{n,i}^2,$$

where

$$q_i^1(A_n, p_n, A_m, p_m) := \frac{\tilde{v}_i}{1 + \sum_{j \in A_n \cup A_m} \tilde{v}_j},$$
$$q_{n,i}^2 := \frac{v_{n,i}^{1/\gamma_i}}{v_{n,i}^{1/\gamma_i} + v_{m,i}^{1/\gamma_i}},$$

where $\tilde{v}_i := (v_{1,i}^{1/\gamma_i} + v_{1,i}^{1/\gamma_i})^{\gamma_i}$. In this model, products capture a larger market share when offered by both retailers, however the split between the retailers depends on the value of γ_i (the case of $\gamma_i = 0$ corresponds to our base definition of common products, and the case of $\gamma_i = 1$ corresponds to the case of exclusive products). Note that the modification above only affects settings with common products: for an exclusive product $i \in \mathcal{S}_n$, one has that $\tilde{v}_i = v_{n,i}$ and $q_{n,i}^2 = 1$, thus recovering the models of sections 3.2 and 4.1.

One can show that the setting above corresponds to an extension of our base model in which

$$\delta_{n,i} = v_{n,i}^{1/\gamma_i - 1}\left(v_{1,i}^{1/\gamma_i} + v_{2,i}^{1/\gamma_i}\right)^{\gamma_i - 1}.$$

With this equivalence at hand, one can show that the results in section 3.3 for assortment-only competition continue to hold under nested Logit demand, and only minor modifications to their proofs are required. For the case of assortment and price competition in section 4.2, while the results continue to hold, a different set of proof techniques is required. In particular, while a closed-form expression for the optimal price $p_{n,i}^*$ is not available, the discontinuities in the payoff functions introduced by the original definition of $\delta_{n,i}$ now disappear, allowing to prove a parallel result to Proposition 4 using standard fixed-point results.

5.3. Heterogeneous Customers: The Case of Mixed Logit Demand

While it has been shown that under mixed Logit demand, under some conditions and for fixed assortments, existence and uniqueness of an equilibrium in prices can be guaranteed (see Allon et al. 2013), the assortment problem becomes (theoretically) intractable, as highlighted in Rusmevichientong et al. (2014) where it is shown that the monopolist's problem is in general NP-Hard, even in the absence of display capacities. One may also show that structural properties presented here will not hold for such models.

Table 2 Illustration of Non-Existence of Equilibrium for the Case of Multiple Segments

	A_2	
	{3}	{4}
A_1		
{1}	(0.897, 0.575)	(0.985, 0.571)
{2}	(0.900, 0.657)	(0.977, 0.680)

Each entry in the table corresponds the profit of retailer 1 (row) and retailer 2 (column) as a function of the assortments selected.

Consider, for example, a setting with assortment-only competition and exclusive products, where a fraction ρ_n of the consumers are loyal to retailer n, meaning that they only consider products offered by retailer n when making the purchase decision, $n = 1,2$. In this setting the expected profit per customer for retailer n, $\tilde{\pi}_n(A_n, A_m)$, is given by

$$\tilde{\pi}_n(A_n, A_m) = \rho_n \pi_n(A_n, \emptyset) + (1 - \rho_n - \rho_m)\pi_n(A_n, A_m).$$

This is a special instance of mixed Logit demand. We provide below an example with exclusive products in which an equilibrium fails to exist for the assortment competition game. Consider a setting with two retailers, each with a capacity of $C_1 = C_2 = 1$; retailer 1 has access to $\mathcal{S}_1 = \{1, 2\}$ while retailer 2 has access to $\mathcal{S}_2 = \{3, 4\}$. Parameters are set such that $v_{1,1} = 9.8$, $v_{1,2} = 4.6$, $v_{2,3} = 9.1$, $v_{2,4} = 6.5$, $p_{1,1} - c_{1,1} = 0.4$, $p_{1,2} - c_{1,2} = 0.2$, $p_{2,3} - c_{2,3} = 0.9$, and $p_{2,4} - c_{2,4} = 0.8$. In addition, consider $\rho_1 = \rho_2 = 0.3$ and $\beta_1 = \beta_2 = 0$. Table 2 depicts the rewards for each retailer for feasible pairs of assortment decisions (A_1, A_2). One can verify that no equilibrium exists.

We have shown that the analysis in our study extends to the case of nested Logit demand, and that it does not for the case of mixed Logit demand. Generalizing the class of models for which assortment games are amenable to analysis is an important theoretical direction of research.

Acknowledgments

The authors thank Awi Federgruen, Kinshuk Jerath, Rajeev Kohli, Andres Musalem and Ricardo Montoya for insightful discussions and comments that helped improve the manuscript. The second author's research is supported in part by the National Science Foundation [Grant CMMI-1233441], the Chilean Millennium Institute of Complex Engineering Systems (ICM: P05-004-F FIN. ICM-FIC), Chile's national fund for scientific and technological development [Grant FONDECYT 11140261], and the Business Intelligence Research Center (CEINE) at the University of Chile.

References

Allon, G., A. Fedegruen, M. Pierson. 2013. Price competition under multinomial Logit demand functions with random coefficients. *Manage. Sci.* **59**(8): 1817–1835.

Anderson, S., A. de Palma. 2006. Market performance with multiproduct firms'. *J. Indust. Econ.* **54**(1): 95–124.

Anderson, S. P., A. de Palma, J.-F. Thisee. 1992. *Discrete Choice Theory of Product Differentiation*. MIT Press, Cambridge, MA.

Aydin, G., J. K. Ryan. 2000. 'Product line selection and pricing under the multinomial logit choice model', Working paper.

Ben-Akiva, M., S. R. Lerman. 1985. *Discrete Choice Analysis: Theory and Application to Travel Demand*. MIT Press, Cambridge, MA.

Bernstein, F., A. Federgruen. 2004. A general equilibrium model for industries with price and service competition. *Oper. Res.* **52**(6): 868–886.

Cachon, G. P., A. G. Kök. 2007. Category management and coordination in retail assortment planning in the presence of basket shopping consumers'. *Manage. Sci.* **53**(6): 934–951.

Cachon, G. P., C. Terwiesch, Y. Xu. 2008. On the effects of consumer search and firm entry in a multiproduct competitive market. *Market. Sci.* **27**(3): 461–473.

Caro, F., V. M. de Albéniz, P. Rusmevichientong. 2014. The assortment packing problem: Multiperiod assortment planning for short-lived products'. *Manage. Sci.* **60**(11): 2701–2721.

Chen, K. D., W. H. Hausman. 2000. Technical note: Mathematical properties of the optimal product line selection problem using choice-based conjoint analysis', *Manage. Sci.* **46**(2): 327–332.

Coughlan, A. T., G. Shaffer. 2009. Research note: Price-matching guarantees, retail competition, and product-line assortment. *Market. Sci.* **28**(3): 580–588.

Dantzig, G., W. Blattner, M. Rao. 1967. Finding a cycle in a graph with minimum cost to time ratio with applications to a ship routing problem. P. Rosenstiehl, ed. *Theory of Graphs International Symposium*. Dunod, Paris, Gordon and Breach, New York, 77–84.

Draganska, M., D. C. Jain. 2006. Consumer preferences and product-line pricing strategies: An empirical analysis. *Mark. Sci.* **25**(2): 164–174.

Draganska, M., M. Mazzeo. K. Seim. 2009. Beyond plain vanilla: Modeling pricing and product assortment choices. *Quant. Mark. Econ.* **7**(2): 105–146.

Dukes, A. J., T. Geylani, K. Srinivasan. 2009. Strategic assortment reduction by a dominant retailer. *Mark. Sci.* **28**(2): 309–319.

Fisher, M. L., R. Vaidyanathan. 2009. 'An algorithm and demand estimation procedure for retail assortment optimization', Working paper, University of Pennsylvania.

Fudenberg, D., J. Tirole. 1991. *Game Theory*. MIT Press, Cambridge, MA.

Gallego, G., R. Wang. 2014. Multi-product price optimization and competition under the nested logit model with product-differentiated price sensitivities'. *Oper. Res.* **62**(2): 450–461.

Gallego, G., G. Iyengar, R. Phillips, A. Dubey. 2004. 'Managing exible products on a network', CORC Technical report TR-2004-01.

Gallego, G., W. T. Huh, W. Kang, R. Phillips. 2006. Price competition with the attraction demand model: Existence of unique equilibrium and its stability'. *Manuf. Serv. Oper. Manag.* **8**(4): 359–375.

Goyal, V., R. Levi. D. Segev. 2009. 'Near-optimal algorithms for the assortment planning problem under dynamic substitution and stochastic demand', Working paper, MIT Sloan.

Guadagni, P. M., J. D. Little. 1983. A logit model of brand choice calibrated on scanner data. *Mark. Sci.* **2**(3): 203–238.

Hopp, W. J., X. Xu. 2008. A static approximation for dynamic demand substitution with applications in a competitive market. *Oper. Res.* **56**(3): 630–645.

Immorlica, N., A. T. Kalai, B. Lucier, A. Moitra, A. Postlewaite, M. Tennenholtz. 2011. Dueling algorithms. *Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing*, STOC '11. ACM, New York, NY, 215–224.

Iyer, G., D. Kuksov. 2012. Competition in consumer shopping experience'. *Mark. Sci.* **31**(6): 913–933.

Kök, A. G., Y. Xu. 2011. Optimal and competitive assortments with endogenous pricing under hierarchical consumer choice models. *Manage. Sci.* **57**(9): 1546–1563.

Kök, A. G., M. Fisher, R. Vaidyanathan. 2008. Assortment planning: Review of literature and industry practice. N. Agrawal, S. A. Smith, eds. *Retail Supply Chain Management: Quantitative Models and Empirical Studies*. Springer Science + Business Media, LLC, New York, NY, **122**(6): 99–153.

Konovalov, A., Z. Sándor. 2009. On price equilibrium with multiproduct firms. *Econ. Theor.* (published online) **44**(22): 271–292.

Kuksov, D., J. M. Villas-Boas. 2010. When more alternatives lead to less choice'. *Mark. Sci.* **29**(3): 507–524.

Maddah, B., E. K. Bish. 2007. Joint pricing, assortment, and inventory decisions for a retailers product line. *Nav. Res. Log.* **54**: 315–330.

Mahajan, S., G. van Ryzin. 2001. Stocking retail assortments under dynamic consumer substitution. *Oper. Res.* **49**(3): 334–351.

Megiddo, N. 1979. Combinatorial optimization with rational objective functions'. *Math. Oper. Res.* **4**(4): 414–424.

Miranda-Bront, J. J., I. Mendez-Diaz, G. Vulcano. 2009. A column generation algorithm for choice-based network revenue management. *Oper. Res.* **57**(3): 769–784.

Misra, K. 2008. 'Understanding retail assortments in competitive markets', Working paper, London Business School.

Rooderkerk, R. P., H. J. van Heerde, T. H. A. Bijmolt. 2013. Optimizing retail assortments. *Mark. Sci.* **32**(5): 699–715.

Rusmevichientong, P., Z. Shen, D. Shmoys. 2010. Dynamic assortment optimization with a multinomial logit choice model and capacity constraint'. *Oper. Res.* **58**(6): 1666–1680.

Rusmevichientong, P., D. B. Shmoys, C. Tong, H. Topaloglu. 2014. Assortment optimization under the multinomial logit model with random choice parameters'. *Prod. Oper. Manag.* **23**(11): 2023–2039.

van Ryzin, G., S. Mahajan. 1999. On the relationship between inventory costs and variety benefits in retail assortments. *Manage. Sci.* **45**(11): 1496–1509.

Smith, S. A., N. Agrawal. 2000. Management of multi-item retail inventory systems with demand substitution. *Oper. Res.* **48**(1): 50–64.

Symeonidis, G. 2009. Asymmetric multiproduct firms, profitability and welfare'. *Bull. Econ. Res.* **61**(2): 139–150.

Thomadsen, R. 2012. Seeking an expanding competitor: How product line expansion can increase all firms' profits. *J. Mark. Res.* **49**(3): 349–360.

Train, K. 2002. *Discrete Choice Methods with Simulation*. Cambridge University Press, Cambridge, UK.

Vives, X. 2000. *Oligopoly Pricing: Old Ideas and New Tools*. MIT Press, Cambridge, MA.

Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix S1: Proofs.