ISA Transactions 60 (2016) 82–88

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Research Article

Boundedness of the solutions for certain classes of fractional differential equations with application to adaptive systems

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ARTICLE INFO

Article history:
Received 30 June 2015
Received in revised form
1 September 2015
Accepted 9 November 2015
Available online 28 November 2015
This paper was recommended for publication by Prof. Y. Chen

Keywords:
Fractional calculus
Fractional differential equations
Error models
Adaptive systems

ABSTRACT

This paper presents the analysis of three classes of fractional differential equations appearing in the field of fractional adaptive systems, for the case when the fractional order is in the interval $\alpha \in (0,1]$ and the Caputo definition for fractional derivatives is used. The boundedness of the solutions is proved for all three cases, and the convergence to zero of the mean value of one of the variables is also proved. Applications of the obtained results to fractional adaptive schemes in the context of identification and control problems are presented at the end of the paper, including numerical simulations which support the analytical results.

1. Introduction

Fractional calculus relates to the calculus of integrals and derivatives of orders that may be real or complex, and it has become very popular due to its demonstrated applications in numerous fields of science and engineering [1].

We can mention the control field, where innumerable control strategies [2] including adaptive control schemes [3–7] have been generalized using fractional operators. The success of fractional operators in the control field is because they allow increased flexibility in the design and adjustment of the controller, obtaining in that way controlled systems with better performance as compared with integer order schemes.

Regarding the identification field, the nature of many complex systems makes that they can be more accurately modeled using fractional differential equations. Among many examples, we can mention several systems that have been modeled using fractional differential equations with the Caputo definition, see for instance [8–10]. In that sense, the systems to be controlled/identified can now be described for fractional differential equations as well.

We can find in the literature many excellent books and works related to the analysis of fractional differential equations, such as [11,1,12], and many others that address the fractional differential equations in a more applied way [13,14]. However, still there are many specific fractional differential equations that have not been analyzed. Since there are real problems which are in the form of this specific fractional differential equations, these are open problems that need to be eventually solved.

This paper presents the analysis of three specific fractional differential equations, when the order of the fractional derivatives α is in the interval $(0,1)$ and the Caputo definition for fractional derivatives is used. The boundedness of the solutions is analytically proved, as well as the convergence to zero of the mean value of the squared norm of one of the variables. The application of the presented results to adaptive schemes in the context of identification and control is presented, and numerical simulations are given, which support the analytical results.

The paper is organized as follows: Section 2 presents some basic concepts about fractional calculus. A new lemma is presented in this section as well, which proves the convergence to zero of the mean value of a non-negative function, based on the boundedness of its fractional integral. Section 3 introduces the three fractional differential equations analyzed in this study, with the corresponding proof of the boundedness of the variables and conclusions about the evolution along the time of some of them. Section 4 introduces the application of the results in Section 3 to

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http://dx.doi.org/10.1016/j.isatra.2015.11.013
0019-0578/

adaptive schemes, together with some illustrative numerical simulations, which support the analytical results. Finally, Section 5 presents the main conclusions of the work.

2. Basic concepts

This section presents some basic concepts of fractional calculus and some properties of fractional operators that are used along the paper.

2.1. Fractional calculus

Fractional calculus studies integrals and derivatives of orders that can be any real or complex numbers [1]. The Riemann–Liouville fractional integral is one of the main concepts of fractional calculus, and is presented in Definition 1.

Definition 1 (*Riemann–Liouville fractional integral [1]*).

$$I_{a+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau,\quad t>a,\ R(\alpha)>0 \tag{1}$$

where $\Gamma(\alpha)$ corresponds to the Gamma Function [1].

There are some alternative definitions regarding fractional derivatives. Definition 2 corresponds to the fractional derivative according to Caputo, which is the one most frequently used in engineering problems and the one used in this paper.

Definition 2 (*Caputo fractional derivative [1]*).

$$^{C}D_a^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau \tag{2}$$

where $t>a$, $n-1<\alpha<n$, $n\in\mathbb{Z}^{+}$.

The following lemma will be useful for proving the boundedness of solutions of fractional differential equations in Section 3, and was reported in [15,16].

Lemma 3 (*Duarte-Mermoud et al. [15]*). *Let $x(t)\in\mathbb{R}^n$ be a vector of differentiable functions. Then, for any time instant $t\ge t_0$, the following relationship holds:*

$$\tfrac{1}{2}\,{}^{C}D_{t_0}^{\alpha}\left(x^T(t)Px(t)\right)\le x^T(t)P\,{}^{C}D_{t_0}^{\alpha}x(t),\quad \forall\alpha\in(0,1] \tag{3}$$

where $P\in\mathbb{R}^{n\times n}$ is a constant, square, symmetric and positive definite matrix.

2.2. Evolution of a function with bounded fractional integral of order $\alpha\in(0,1)$

In what follows, a new lemma is proposed and proved, which will be useful in establishing conclusions on the evolution of some solutions of fractional differential equations in Section 3.

Lemma 4. *Let $x(\cdot):\mathbb{R}^{+}\to\mathbb{R}$ be a bounded nonnegative function. If there exists some $\alpha\in(0,1]$ such that*

$$\frac{1}{\Gamma(\alpha)}\int_{t_0}^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau<M,\quad \forall t\ge t_0,\ \text{with } M\in(0,\infty) \tag{4}$$

then

$$\lim_{t\to\infty}\left[t^{\alpha-\varepsilon}\frac{\int_{t_0}^t x(\tau)\,d\tau}{t}\right]=0,\quad \forall\varepsilon>0 \tag{5}$$

Proof. Let us start with the fact that the fractional integral is bounded $\forall t\ge t_0$, then it can be written as

$$\int_{t_0}^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau<M\Gamma(\alpha),\quad \forall t\ge t_0 \tag{6}$$

Multiplying expression (6) by $t^{-\varepsilon},\varepsilon>0$ and applying limit algebra

$$\lim_{t\to\infty}\left[\frac{1}{t^{\varepsilon}}\int_{t_0}^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau\right]<\lim_{t\to\infty}\left[\frac{M\Gamma(\alpha)}{t^{\varepsilon}}\right]=0 \tag{7}$$

Since $x(t)$ is nonnegative we can state that

$$\lim_{t\to\infty}\left[\frac{1}{t^{\varepsilon}}\int_{t_0}^t \frac{x(\tau)}{t^{1-\alpha}}\,d\tau\right]\le\lim_{t\to\infty}\left[\frac{1}{t^{\varepsilon}}\int_{t_0}^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau\right] \tag{8}$$

and from (7) and (8) we can write

$$\lim_{t\to\infty}\left[\frac{1}{t^{\varepsilon}}\int_{t_0}^t \frac{x(\tau)}{t^{1-\alpha}}\,d\tau\right]=0 \tag{9}$$

Expression (9) can be rewritten as

$$\lim_{t\to\infty}\left[t^{\alpha-\varepsilon}\frac{\int_{t_0}^t x(\tau)\,d\tau}{t}\right]=0 \tag{10}$$

and this completes the proof.□

As can be seen, Lemma 4 does not allow concluding that function $x(t)$ converges to zero, although it assures that its mean value do converge to zero, with a convergence rate greater than $t^{-(\alpha-\varepsilon)}$.

3. Analysis of certain classes of fractional differential equations

In what follows, three kinds of fractional order differential equations (FODE) will be analyzed. The boundedness of the solutions is proved in all three cases, as well as certain characteristics of the evolution along the time for some of them.

3.1. Fractional order differential equations of Class 1

One of the parametrization that appears very often in many real problems has the form

$$\begin{aligned}
&y(t)=k_p\eta^T(t)u(t)+\xi(t)u_1(t)\\
&{}^{C}D_{t_0}^{\alpha}\eta(t)=-\gamma\ \mathrm{sgn}(k_p)y(t)u(t)\quad \alpha\in(0,1]\\
&{}^{C}D_{t_0}^{\alpha}\xi(t)=-\gamma_1 y(t)u_1(t)\quad \alpha\in(0,1]
\end{aligned} \tag{11}$$

where $k_p\in\mathbb{R}$ is an unknown constant with known sign, $\gamma,\gamma_1\in\mathbb{R}^{+}$, $y(t):\mathbb{R}^{+}\to\mathbb{R}$, $\eta(t):\mathbb{R}^{+}\to\mathbb{R}^n$, $u(t):\mathbb{R}^{+}\to\mathbb{R}^n$ are assumed to be bounded, $\xi(t):\mathbb{R}^{+}\to\mathbb{R}$ and $u_1(t):\mathbb{R}^{+}\to\mathbb{R}$ are assumed to be bounded.

Boundedness of the solution $\eta(t),\xi(t)$ and also of $y(t)$ is always required in these problems, and that is why the analysis of (11) is so attractive.

Lemma 5 (*Boundedness of solutions for FODE of Class 1*). *Let us consider the FODE defined in (11) with the assumption that $\eta(t), \xi(t)$ are differentiable. Then it can be assured that*

- *$\eta(t), \xi(t), y(t)$ remain bounded $\forall t \geq t_0$.*
- *The mean value of $y^2(t)$ converges to zero as $t \to \infty$.*

Proof. Before starting the demonstration, we would like to mention that although the case $\alpha = 1$ is included in this proof, it was already solved in [17].

Let us start the proof with the fact that, since it is assumed that $\eta(t), \xi(t)$ are differentiable, then using Lemma 3 with $P = I_{n \times n}$, we can write the following inequality:

$$ {}^{C}D_{t_0}^{\alpha}\left[\frac{|k_p|}{2\gamma}\eta^T(t)\eta(t)+\frac{1}{2\gamma_1}\xi^2(t)\right] \leq \frac{|k_p|}{\gamma}\eta^T(t)\,{}^{C}D_{t_0}^{\alpha}\eta(t)+\frac{1}{\gamma_1}\xi(t)\,{}^{C}D_{t_0}^{\alpha}\xi(t) \tag{12} $$

Using expressions (11) in the right hand side of inequality (12), then it can be written that

$$ {}^{C}D_{t_0}^{\alpha}\left[\frac{|k_p|}{2\gamma}\eta^T(t)\eta(t)+\frac{1}{2\gamma_1}\xi^2(t)\right] \leq -y^2(t) \tag{13} $$

Applying the fractional integral of order α to expression (13) it follows that

$$ \frac{|k_p|}{2\gamma}\eta^T(t)\eta(t)+\frac{1}{2\gamma_1}\xi^2(t)-\frac{|k_p|}{2\gamma}\eta^T(t_0)\eta(t_0)-\frac{1}{2\gamma_1}\xi^2(t_0) \leq -I_{t_0}^{\alpha}y^2(t) \tag{14} $$

Since $I_{t_0}^{\alpha}y^2(t) \geq 0, \forall t \geq t_0$, and $\gamma, \gamma_1, |k_p| > 0$, then

$$ \frac{|k_p|}{2\gamma}\eta^T(t)\eta(t)+\frac{1}{2\gamma_1}\xi^2(t) \leq \frac{|k_p|}{2\gamma}\eta^T(t_0)\eta(t_0)+\frac{1}{2\gamma_1}\xi^2(t_0) \tag{15} $$

Considering bounded initial values for $\eta(t_0)$, $\xi(t_0)$, then expression (15) implies that $\eta(t), \xi(t)$ remain bounded $\forall t \geq t_0$.

Since $u(t), u_1(t)$ are assumed to be bounded in this problem, then using the equation of $y(t)$ in (11), it can be concluded that $y(t)$ remains bounded too.

Regarding the convergence to zero of the mean value of $y^2(t)$, the following analysis can be made. From expression (14), using the fact that $\eta(t), \xi(t)$ are bounded, then it can be concluded that $I_{t_0}^{\alpha}y^2(t) < \infty$. Then we can apply Lemma 4 and conclude that

$$ \lim_{t\to\infty}\left[t^{\alpha-\varepsilon}\frac{\int_{t_0}^{t}y^2(\tau)\,d\tau}{t}\right]=0, \quad \forall \varepsilon > 0 \tag{16} $$

that is to say, we can assure that the mean value of $y^2(t)$ converges to zero when $t \to \infty$, and this concludes the proof.□

3.2. Fractional order differential equations of Class 2

In the case of FODE (11), $y(t)$ is a linear combination of $\eta(t), \xi(t)$, $u(t)$ and $u_1(t)$. However, very often the evolution of $y(t)$ is described not by the class of equation in (11), but for a fractional differential equation using the Caputo fractional derivative as well. This is precisely the case we study in this subsection, as can be seen from (17),

$$ \begin{aligned} {}^{C}D_{t_0}^{\alpha}y(t) &= Ay(t)+k_p\, b\,\eta^T(t)u(t) \\ {}^{C}D_{t_0}^{\alpha}\eta(t) &= -\gamma\,\mathrm{sgn}(k_p)y^T(t)P\,b\,u(t) \quad \alpha \in (0,1] \end{aligned} \tag{17} $$

where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, $b \in \mathbb{R}^n$, $\eta(t):\mathbb{R}^+ \to \mathbb{R}^m$, $y(t):\mathbb{R}^+ \to \mathbb{R}^n$, $u(t):\mathbb{R}^+ \to \mathbb{R}^m$ and is assumed to be bounded, $P \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix that satisfies the equation $A^TP+PA=-Q<0$ (with $Q \in \mathbb{R}^{n\times n}$ being positive definite), $k_p \in \mathbb{R}$ is an unknown constant, whose sign is known, and $\gamma \in \mathbb{R}^+$.

In what follows, the boundedness of $\eta(t), y(t)$ is proved in Lemma 6, as well as the convergence to zero of the mean value of $\|y(t)\|^2$.

Lemma 6 (*Boundedness of solutions for FODE of Class 2*). *Let us consider the FODE defined in (17) with the assumption that $\eta(t), y(t)$ are differentiable. Then it can be assured that*

- *$\eta(t), y(t)$ remain bounded $\forall t \geq t_0$.*
- *The mean value of $\|y(t)\|^2$ converges to zero when $t \to \infty$.*

Proof. As it was mentioned in the case of the fractional differential equations of Class 1, we would like to mention that although the case $\alpha = 1$ is included in this proof, it was already solved in [17].

Let us start the proof with the fact that, since $y(t), \eta(t)$ are assumed to be differentiable, then we can use Lemma 3 and write the following inequality:

$$ \begin{aligned} {}^{C}D_{t_0}^{\alpha}\left[y^T(t)Py(t)+\frac{|k_p|}{\gamma}\eta^T(t)\eta(t)\right] &\leq 2y^T(t)P\,{}^{C}D_{t_0}^{\alpha}y(t) \\ &+\frac{2|k_p|}{\gamma}\eta^T(t)\,{}^{C}D_{t_0}^{\alpha}\eta(t) \end{aligned} \tag{18} $$

Using (17) in (18), using also the fact that $A^TP+PA=-Q<0$ and applying the fractional integral of order α to the resulting expression we obtain

$$ y^T(t)Py(t)+\frac{|k_p|}{\gamma}\eta^T(t)\eta(t) \leq y^T(t_0)Py(t_0)+\frac{|k_p|}{\gamma}\eta^T(t_0)\eta(t_0) \tag{19} $$

Considering bounded initial values for $y(t_0), \eta(t_0)$, expression (19) implies that $y(t), \eta(t)$ remain bounded $\forall t \geq t_0$.

Regarding the convergence to zero of $\|y(t)\|^2$, using the fact that $\eta(t), y(t)$ remain bounded, in a similar way to that used in the case of fractional differential equations of Class 1, it can be concluded here that $I_{t_0}^{\alpha}\|y(t)\|^2 < \infty$.

Then using Lemma 4 we can assure that

$$ \lim_{t\to\infty}\left[t^{\alpha-\varepsilon}\frac{\int_{t_0}^{t}\|y(\tau)\|^2\,d\tau}{t}\right]=0, \quad \forall \varepsilon > 0 \tag{20} $$

that is to say, for the FODE of Class 2 (17), the mean value of $\|y(t)\|^2$ converges to zero as $t \to \infty$, and this concludes the proof.□

3.3. Fractional order differential equations of Class 3

Fractional differential equations of Class 3 have the same structure as fractional differential equations of Class 2, with the difference that the entire vector $y(t):\mathbb{R}^+ \to \mathbb{R}^n$ is not used in the fractional differential equation of $\eta(t)$, but an algebraic combination of their components $y_1(t):\mathbb{R}^+ \to \mathbb{R}$ is used instead. The structure of FODE of Class 3 is presented in the following:

$$ \begin{aligned} {}^{C}D_{t_0}^{\alpha}y(t) &= Ay(t)+b\eta^T(t)u(t) \\ y_1(t) &= k_p\, c^Ty(t) \\ {}^{C}D_{t_0}^{\alpha}\eta(t) &= -\gamma\,\mathrm{sgn}(k_p)y_1(t)u(t), \quad \alpha \in (0,1] \end{aligned} \tag{21} $$

where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $\eta(t):\mathbb{R}^+ \to \mathbb{R}^m$, $y(t):\mathbb{R}^+ \to \mathbb{R}^n$, $u(t):\mathbb{R}^+ \to \mathbb{R}^m$ are assumed to be bounded, $y_1(t):\mathbb{R}^+ \to \mathbb{R}$, $k_p \in \mathbb{R}$ is an unknown constant but with known sign and $\gamma \in \mathbb{R}^+$. Besides, positive definite matrices $P = P^T \in \mathbb{R}^{n\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$ exist such that

$$ A^TP+PA=-QPb=c \tag{22} $$

Let us now analyze the boundedness of $\eta(t), y(t)$, as well as the convergence to zero of the mean value of $\|y(t)\|^2$.

Lemma 7 (*Boundedness of solutions for FODE of Class 3*). *Let us consider the FODE defined in (21) with the assumption that $\eta(t), y(t)$ are differentiable. Then it can be assured that*

- *$\eta(t), y(t)$ remain bounded $\forall t \geq t_0$.*
- *The mean value of $\|y(t)\|^2$ converges to zero when $t \to \infty$.*

Proof. As in the two previous classes of fractional differential equations, the following proof is valid for $\alpha \in (0,1]$. However, we must mention that the particular case $\alpha = 1$ was already analyzed in [17].

Let us start the proof using the assumption that $y(t), \eta(t)$ are differentiable functions. Then Lemma 3 allows writing the following inequality:

$$ {}^C D_{t_0}^{\alpha}\left[y^T(t)Py(t)+\frac{1}{\gamma|k_p|}\eta^T(t)\eta(t)\right] \leq 2y^T(t)P\,{}^C D_{t_0}^{\alpha}y(t) + \frac{2}{\gamma|k_p|}\eta^T(t)\,{}^C D_{t_0}^{\alpha}\eta(t) \quad (23)$$

Using (21) in (23), using also the expression for $y_1(t)$ and the expressions in (22), and applying the fractional integral of order α to the resulting expression, it follows that

$$ y^T(t)Py(t)+\frac{1}{\gamma|k_p|}\eta^T(t)\eta(t) \leq y^T(t_0)Py(t_0)+\frac{1}{\gamma|k_p|}\eta^T(t_0)\eta(t_0) \quad (24)$$

Considering bounded initial values $\eta(t_0), y(t_0)$, expression (24) implies that $y(t), \eta(t)$ remain bounded $\forall t \geq t_0$.

Now, in the same way we made for the two previous fractional differential equations, it can be proved here that $I_{t_0}^{\alpha}\|y(t)\|^2 < \infty$, using the fact that $y(t), \eta(t)$ are bounded. Then using Lemma 4 we can assure that

$$ \lim_{t\to\infty}\left[t^{\alpha-\varepsilon}\,\frac{\int_{t_0}^{t}\|y(\tau)\|^2\,d\tau}{t}\right] = 0, \quad \forall \varepsilon > 0 \quad (25)$$

that is to say, for the FODE of Class 3 (21), the mean value of $\|y(t)\|^2$ converges to zero as $t \to \infty$, and this concludes the proof.□

As the reader can note, only the convergence to zero of the mean value of $y^2(t)$ was proved in Lemma 5, and the convergence to zero of the mean value of $\|y(t)\|^2$ was proved in the cases of Lemmas 6 and 7. However, the convergence to zero of the signal $y^2(t)$ or $\|y(t)\|^2$ was not mentioned in any of the cases. Currently, this is a topic under investigation.

Remark 8. It is important to point out that for the case of $\alpha = 1$, Eqs. (11), (17) and (21) correspond to the well known Error Models 1, 2 and 3, respectively [17]. Therefore, for the case of $\alpha \in (0,1)$ these equations could be understood as the fractional order versions of Error Models 1, 2 and 3, respectively. That is the reason why results are important to analyze these types of FODE. When presenting the applications in Section 4 we will elaborate a little bit more about these concepts.

4. Applications to adaptive schemes in the context of identification and control problems

Generally speaking, adaptive systems refer to the identification/control of partially known systems. Several adaptive algorithms have been developed in the past for stable identification/control of integer order systems with unknown parameters [17–20], using adaptive laws described by integer order differential equations as well.

The introduction of fractional operators has also reached the adaptive schemes [3–7], describing the systems to be controlled/identified and using adaptive laws also described by fractional differential equations. The analytical proof of the boundedness of the signals in the schemes, however, is not solved in most of the cases, due to the lack of tools and/or methodologies. Using the results proposed in this paper, the boundedness of the signals in many fractional adaptive schemes can now be proved, as we will see in the following.

4.1. An identification scheme

Let us assume that a plant to be identified has the following algebraic form:

$$ x_p(t) = \theta^{*T}\omega(t) \quad (26)$$

where $x_p \in \mathbb{R}$, the vector of constant parameters $\theta^* \in \mathbb{R}^n$, is unknown and $\omega(t) \in \mathbb{R}^n$ is a vector of available signals, and is assumed to be bounded. The aim in this problem is to estimate the unknown parameter vector θ^*.

To that extent, let us construct an estimator of the form

$$ \hat{x}_p(t) = \theta^T(t)\omega(t) \quad (27)$$

where $\theta(t) \in \mathbb{R}^n$ is the vector of the estimated parameters.

Let us define the identification error as $e(t) = \hat{x}_p(t) - x_p(t)$ and the parameter error as $\phi(t) = \theta(t) - \theta^*$. Then, subtracting (27) and (26) we obtain the following equation for the identification error:

$$ e(t) = \phi^T(t)\omega(t) \quad (28)$$

From (11) we can select

$$ {}^C D_{t_0}^{\alpha}\phi(t) = {}^C D_{t_0}^{\alpha}\theta(t) = -e(t)\omega(t) \quad (29)$$

then the pair of Eqs. (28) and (29) has the same structure than the FODE of Class 1 studied in this paper, for the particular case when $e_2, \xi = 0$.

In that way, since $\omega(t)$ is assumed to be bounded, under the assumption that $\phi(t)$ is differentiable, it can be assured that $e(t)$, $\phi(t)$ remain bounded $\forall t \geq 0$, and that the mean value of the squared identification error $e^2(t)$ converges asymptotically to zero.

Let us briefly consider a particular example, where $\theta^* = [5\ 2]^T$, $\omega(t) = [\sin x_p\ \ \cos u(t)]^T$ and $u(t)$ is a unit step. Initial values for the estimated parameters are $\theta(0) = [4\ 4]^T$ and the fractional order used for the adaptive law is $\alpha = 0.7$. Simulations were performed using the NID block of the Ninteger toolbox [21], developed for Matlab/Simulink. The Crone approximation of order 10 was used to implement the fractional operator, with a frequency interval of [0.01,1000].

Fig. 1 shows the evolution of the identification error $e(t)$ and the norm of the parameter error $\|\phi(t)\|$. As can be seen, the output error and the norm of the parameter error remain bounded, as it was expected from the analytical results. Besides, it can be seen that the output error converges to zero, as well as the norm of the parameter error, although it was not analytically proved. Additional comments about the convergence of the parameter error will be given at the end of this section.

We may note that in this particular example, the value of $\alpha = 0.7$ was selected only for illustrative purposes. In a real application, this is a design parameter, that is to say, the selection of the order to be used is a decision of the designer. This decision of course will depend of the specific application, the control goals, etc. One option to choose this parameter could be an optimization problem, if it is possible to apply, as it was done in [3] using genetic algorithms.

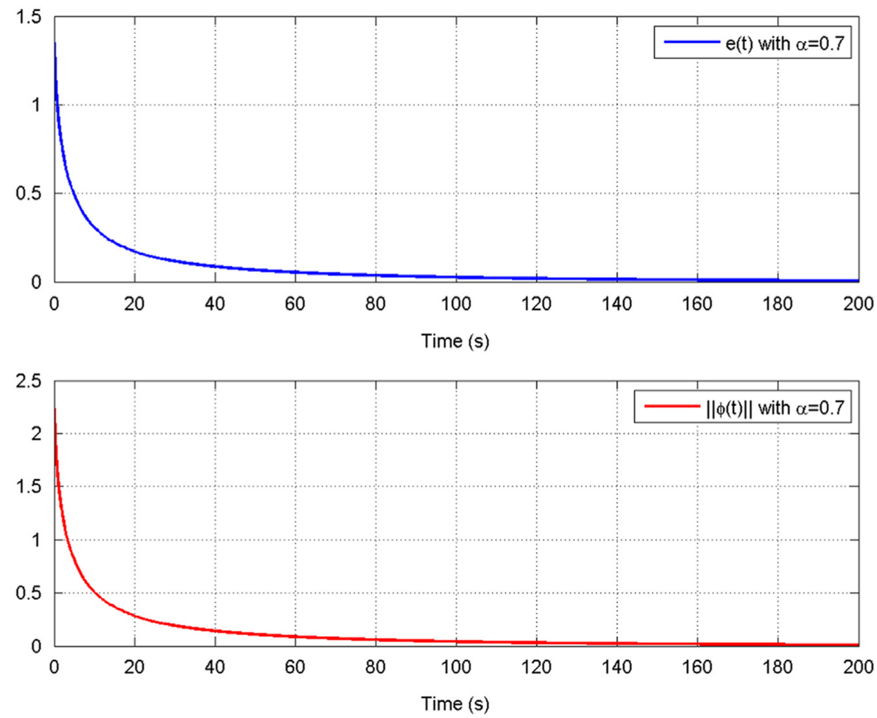

Fig. 1. Evolution of the identification error and the norm of the parameter error in the identification scheme, when the input signal corresponds to a unit step.

Remark 9 (*Fractional Error Model 1*). Although the example presented here is for an identification scheme, we can assure, based on the analysis from Section 3, that for any adaptive scheme with the structure (28), (29), the output error $e(t)$ and the parameter error $\phi(t)$ will remain bounded $\forall t \geq t_0$, under the hypothesis that $\phi(t)$ is differentiable and $\omega(t)$ is bounded. Besides, the mean value of the squared output error will converge asymptotically to zero.

The pair of Eqs. (28) and (29) could be seen as a Fractional Order Error Model 1 (FOEM-1).

Note that error models have been completely studied in the integer order case [17,22–25]. They are particularly attractive because they provide a common framework for the analysis of many adaptive schemes.

4.2. A fractional order model reference adaptive control scheme (FOMRAC)

Let us consider a fractional order linear time-invariant plant to be controlled, given by

$$^{C}D^{\alpha}_{t_0} x_p(t) = A_p x_p(t) + b_p u(t) \tag{30}$$

where $A_p \in \mathbb{R}^{n \times n}$ is an unknown constant matrix, $b_p \in \mathbb{R}^n$ is a known constant vector, the pair (A_p, b_p) is controllable, $x_p(t) \in \mathbb{R}^n$ is assumed to be accessible, $u(t) \in \mathbb{R}$ is the control input to be defined and the fractional order $\alpha \in (0,1)$.

Let a reference model be given by

$$^{C}D^{\alpha}_{t_0} x_m(t) = A_m x_m(t) + b_m r(t) \tag{31}$$

where $A_m \in \mathbb{R}^{n \times n}$ is a known Hurwitz constant matrix, $b_m \in \mathbb{R}^n$ is a known constant vector which satisfies $b_m k = b_p$ for some $k \in \mathbb{R}$, and $r(t) \in \mathbb{R}$ is a given uniformly bounded piecewise-continuous reference input. It is assumed that $x_m(t)$, for all $t \geq t_0$, represents the desired trajectory for $x_p(t)$. The aim here is to control the plant so that all the signals remain bounded and ideally $\lim_{t \to \infty} \| x_p(t) - x_m(t) \| = 0$.

Let us choose the control input as

$$u(t) = \theta^T(t) x_p(t) + k\, r(t) \tag{32}$$

where $\theta(t) \in \mathbb{R}^n$ is a vector consisting of adjustable parameters and it is further assumed that a constant vector θ^* exists such that

$$A_p + b_p \theta^{*T} = A_m \tag{33}$$

Defining the control error as $e(t) = x_p(t) - x_m(t)$, the fractional differential equation describing the evolution of the output error can be expressed as

$$^{C}D^{\alpha}_{t_0} e(t) = A_m e(t) + b_p \phi^T(t) x_p(t), \quad \alpha \in (0,1) \tag{34}$$

where $\phi(t) = \theta(t) - \theta^*$.

From (17) we can select

$$^{C}D^{\alpha}_{t_0} \phi(t) = {}^{C}D^{\alpha}_{t_0} \theta(t) = -e^T(t) P b_p x_p(t) \tag{35}$$

then we can see that the pair of Eqs. (34) and (35) has the same structure than the FODE of Class 2 analyzed in Section 3. For that reason, we can assure that the output error $e(t)$ and the parameter error $\phi(t)$ remain bounded $\forall t \geq t_0$. We can also assure that the mean value of the squared norm of the output error $\| e(t) \|^2$ will converge to zero when $t \to \infty$. Note that in this problem k_p is considered known, and that is why it does not appear in the equations.

Let us show a particular example, considering

$$A_p = \begin{bmatrix} -4 & 1 \\ -3 & -1 \end{bmatrix}, \quad b_p = b_m = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_m = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

For this case, matrices $P = I_{2 \times 2}$ and $Q = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right]$ exist such that $A_m^T P + P A_m = -Q$. Also, a vector $\theta^* = [3\ \ -1]^T$ exists such that (33) holds and $k = 1$ in (32).

Fig. 2 shows the evolution of the norm of the output error $\| e(t) \|$ and the norm of the parameter error $\| \phi(t) \|$, when the reference signal $r(t)$ corresponds to a unit step. The initial values used in the simulations correspond to $\theta(0) = [2\ 0]^T$, $x_p(0) = [0\ 1]^T$ and $x_m(0) = [1\ 2]^T$, and the fractional order used is $\alpha = 0.8$.

As can be seen from Fig. 2, the norm of the output error $\| e(t) \|$ and the norm of the parameter error $\| \phi(t) \|$ remain bounded $\forall t \geq 0$, as it was expected from the analysis above. Although the

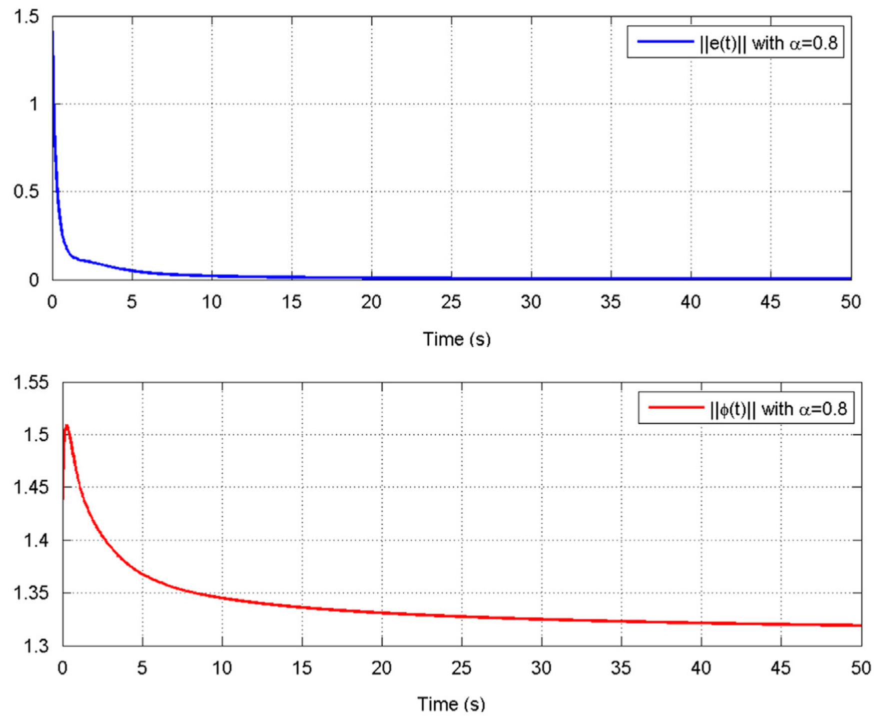

Fig. 2. Evolution of the norm of the output error and the norm of the parameter error in the FOMRAC scheme, when the reference signal corresponds to a unit step.

numerical example presented here uses $\alpha = 0.8$, the same results were observed for any other fractional order $\alpha \in (0,1)$ used.

Besides the boundedness of the signals, it can be seen from Fig. 2 that the norm of the output error converges to zero. Although only the convergence of the mean value of $\|e(t)\|^2$ was analytically proved, the convergence of $\|e(t)\|$ was observed as well, for every simulation study we made in the FOMRAC problem. The analytical proof of this fact is currently under investigation.

Regarding the convergence of the norm of the parameter error $\|\phi(t)\|$, it can be seen from Fig. 2 that it does not converge to zero. Since this is a control scheme, only the convergence to zero of $e(t)$ is required. Nevertheless, some facts regarding the convergence of the parameter error are presented at the end of this section.

Remark 10 (*Fractional Error Model 2*). Although the example presented here is for a specific FOMRAC scheme, we can assure, based on the analysis from Section 3, that for any adaptive scheme with the structure (34), (35), the output error $e(t)$ and the parameter error $\phi(t)$ will remain bounded $\forall t \geq t_0$, under the hypothesis that $\phi(t), e(t)$ are differentiable. Besides, the mean value of the squared norm of the output error will converge asymptotically to zero.

The pair of Eqs. (34), (35) could be seen as a Fractional Order Error Model 2 (FOEM-2).

4.3. Other structures in fractional adaptive schemes

In previous subsections we referred to what could be seen as FOEM-1 and FOEM-2, stating the fact that they allow representing many adaptive schemes. Besides FOEM-1 and FOEM-2, we can identify at least two more structures in the context of adaptive schemes, which could be seen as two more fractional error models.

Remark 11 (*Fractional Error Model 3*). Fractional Error Model 3 (FOEM-3) has the same structure than FOEM-2, with the difference that the entire error vector $e(t):\mathbb{R}^+ \to \mathbb{R}^n$ is not accessible, but only an algebraic combination of their components $e_1(t):\mathbb{R}^+ \to \mathbb{R}$ is available. This error model usually arises when we can only measure the output of the plant to be controlled or identified, and for that reason FOEM-3 is applicable to a much wider class of problems than FOEM-2. Structure of FOEM-3 has the form in (36):

$$
\begin{aligned}
&{}^{C}D_{t_0}^{\beta} e(t) = Ae(t) + b\phi^T(t)\omega(t), \quad \beta \in (0,1)\\
&e_1(t) = k_p\ h^T e(t)\\
&{}^{C}D_{t_0}^{\alpha} \phi(t) = -\gamma\ \mathrm{sgn}(k_p) e_1(t)\omega(t), \quad \alpha \in (0,1)
\end{aligned}
\tag{36}
$$

where $A \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix, $b \in \mathbb{R}^n$, the pair (A, b) is controllable, $h \in \mathbb{R}^n$ and the pair (h^T, A) is observable. Besides, $\theta^* \in \mathbb{R}^m$ is the ideal (true) parameter, which is assumed to be unknown, $\theta(t):\mathbb{R}^+ \to \mathbb{R}^m$ is the adjustable parameter that estimates θ^* and $\phi(t) = \theta(t) - \theta^*:\mathbb{R}^+ \to \mathbb{R}^m$ is the parametric error. $e(t):\mathbb{R}^+ \to \mathbb{R}^n$ is the error vector, $\omega(t):\mathbb{R}^+ \to \mathbb{R}^m$ is the input signal to the error model, and $e_1(t):\mathbb{R}^+ \to \mathbb{R}$ is the output error, which is accessible. $k_p \in \mathbb{R}$ is an unknown constant but with known sign and $\gamma \in \mathbb{R}^+$ is the adaptive gain, which is assumed to be scalar and constant in this study. However, it is possible to use either scalar or matrix adaptive gains which are constant or time varying (see [17]).

As might be expected, having no access to $e(t)$ implies that more stringent conditions must be imposed on the transfer function between $\phi^T(t)\omega(t)$ and $e_1(t)$. In that sense, we assume that positive definite matrices $P = P^T \in \mathbb{R}^{n\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$ [17] exist such that

$$
A^T P + PA = -QPb = h \tag{37}
$$

As can be seen from (36), the structure of FOEM-3 coincides with the FODE of Class 3 studied in this paper. In that way, under the hypothesis that $e(t), \phi(t)$ are differentiable, we can assure that $e(t), \phi(t)$ remain bounded $\forall t \geq t_0$, and also that the mean value of $\|e(t)\|^2$ converges asymptotically to zero.

In many adaptive schemes, the parametrization of the identification/control models do not lead to any of the three fractional error models presented previously. One case we can mention is when we want to implement a model reference adaptive controller for a plant with relative degree $n^* \geq 2$. In this case, additional signals must be included in the scheme (filtering the control input and the plant output) and auxiliary and augmented errors must be included as well (see [17] for details). In that way, the equation representing the augmented error has the form of the first equation in (11), and Lemma 5 allows the use of fractional adaptive laws to adjust the parameters.

For cases like this one, the Fractional Error Model 4 arises.

Remark 12 (*Fractional Error Model 4*). Fractional Error Model 4 (FOEM-4) arises when vector $e(t)$ is not accessible, and condition (37) does not hold neither. This occurs mainly in those cases in which, to achieve the identification or control, additional signals must be introduced in the adaptive scheme. The equations describing FOEM-4 are stated in (38):

$$\begin{aligned}
&\varepsilon_1(t) = e_1(t) + k(t)e_2(t)\\
&e_1(t) = k_p\ W(s)\phi^T(t)\omega(t), \quad \beta \in (0,1)\\
&e_2(t) = \theta^T(t)W(s)I\omega(t) - W(s)\theta^T(t)\omega(t)\\
&{}^C D_{t_0}^{\alpha}\phi(t) = -\gamma_1\ \mathrm{sgn}(k_p)\varepsilon_1(t)\zeta(t), \quad \alpha \in (0,1)\\
&{}^C D_{t_0}^{\alpha}\psi(t) = -\gamma_2\varepsilon_1(t)e_2(t), \quad \alpha \in (0,1)\\
&\zeta(t) = W(s)I_m\omega(t)
\end{aligned} \tag{38}$$

where $e_1(t): \mathbb{R}^+ \to \mathbb{R}$ is the output error, $e_2(t): \mathbb{R}^+ \to \mathbb{R}$ is the auxiliary error, $k_p \in \mathbb{R}$ is an unknown constant, with known sign, $\varepsilon_1(t): \mathbb{R}^+ \to \mathbb{R}$ is the augmented error, $k(t): \mathbb{R}^+ \to \mathbb{R}$ is an adjustable parameter that estimates $\frac{1}{k_p}$, the transfer function $W(s)$ is asymptotically stable, $\theta^* \in \mathbb{R}^m$ is the ideal (true) parameter, which is assumed to be unknown, $\theta(t): \mathbb{R}^+ \to \mathbb{R}^m$ is the adjustable parameter that estimates θ^*, $\phi(t) = \theta(t) - \theta^*: \mathbb{R}^+ \to \mathbb{R}^m$ is the parameter error, $\psi(t) = k(t) - \frac{1}{k_p}: \mathbb{R}^+ \to \mathbb{R}$ and $\omega(t): \mathbb{R}^+ \to \mathbb{R}^m$ is the input signal to the error model. $\zeta(t): \mathbb{R}^+ \to \mathbb{R}^m$ is assumed to be bounded and $\gamma_1, \gamma_2 \in \mathbb{R}$ correspond to the adaptive gains, which are assumed to be constant in this study. Either scalar or matrix, as well as constant or time varying adaptive gains can also be used.

Equations in (38) can be put in the form of FODE of Class 1 studied in this paper (see [17] for the equivalent integer order case). In that way, under the assumption that $\phi(t), \psi(t)$ are differentiable and $\omega(t)$ is bounded, it can be assured that in FOEM-4 $\phi(t), \psi(t)$ remain bounded, as well as the rest of the signals in the scheme. In the same way, it can be assured that the mean value of the squared augmented error ${\varepsilon_1}^2$ converges asymptotically to zero.

Remark 13. In most of the adaptive schemes we analyzed, we assumed that $\omega(t)$ is bounded. When this condition cannot be assured, then a normalized version of the information signal $\frac{\omega(t)}{1+\omega^T(t)\omega(t)}$ can be used instead in adaptive laws, and the corresponding boundedness of the solutions can be proved in the same way.

Remark 14. In the study of the classic error models, the convergence to zero of the parameter error $\phi(t)$ was shown to be dependent of a property of the information signal $\omega(t)$. This property is referred to as persistent excitation, and it has been thoroughly studied for classic error models [17,24].

In general, it was observed from simulation studies performed in this work that in Fractional Error Models the persistent excitation condition is related to the spectral density of the information signal $\omega(t)$, same as in the integer order case. However, for some particular cases of $\omega(t)$, no conclusive results can be observed from simulations.

The concept of fractional order persistent excitation condition has not been addressed yet in the published literature. Given the importance of this concept in adaptive schemes, this is a topic currently under investigation.

5. Conclusions

In this paper, the analysis of three classes of fractional order differential equations using the Caputo definition when $\alpha \in (0,1)$ has been presented. The analysis allows proving boundedness of the solutions and also the convergence to zero of the mean value of the squared norm of some variables of the FODE.

The application of these theoretical results to adaptive schemes was addressed, introducing the concept of fractional order error models. The results presented in this paper allow proving boundedness of signals in many different adaptive schemes, appearing in adaptive control and system identification fields.

Acknowledgments

The results reported in this paper have been financed by CONICYT-Chile, under the Basal Financing Program FB0809 "Advanced Mining Technology Center", FONDECYT Project 1150488, "Fractional Error Models in Adaptive Control and Applications"; and FONDECYT 3150007, "Postdoctoral Program 2015".

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