

On Möbius Duality and Coarse-Graining

Thierry Huillet · Servet Martínez

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Abstract We study duality relations for zeta and Möbius matrices and monotone conditions on the kernels. We focus on the cases of families of sets and partitions. The conditions for positivity of the dual kernels are stated in terms of the positive Möbius cone of functions, which is described in terms of Sylvester formulae. We study duality under coarse-graining and show that an h -transform is needed to preserve stochasticity. We give conditions in order that zeta and Möbius matrices admit coarse-graining, and we prove they are satisfied for sets and partitions. This is a source of relevant examples in genetics on the haploid and multi-allelic Cannings models.

Keywords Duality · Möbius matrices · Coarse-graining · Partitions · Sylvester formula · Coalescence

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1 Introduction

We study zeta and Möbius duality for a finite partially ordered space (\mathcal{A}, \preceq) with special emphasis when this space is a family of sets or of partitions. We will supply

T. Huillet
Laboratoire de Physique Théorique et Modélisation, CNRS-UMR 8089 et Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, 95302 Cergy-Pontoise, France
e-mail: Thierry.Huillet@u-cergy.fr

S. Martínez (✉)
Departamento de Ingeniería Matemática, Centro Modelamiento Matemático, UMI 2807, UCHILE-CNRS, Universidad de Chile, Casilla 170-3 Correo 3, Santiago, Chile
e-mail: smartine@dim.uchile.cl

conditions in order that the dual of a nonnegative kernel P defines a nonnegative kernel Q , and study relations between these two kernels.

Section 3.3 is devoted to introducing zeta and Möbius matrices, as done in [2, 17]. We supply the product formula for the product order which serves to list several examples in a unified way.

In Sect. 3, we study zeta and Möbius duality relations. The conditions for positivity preserving are put in terms of the positive Möbius cone of functions, which is the class of positive functions having positive image under the Möbius matrix (they are called Möbius monotone in [13]). A well-known duality relation of this type is the Siegmund duality for a finite interval of integers endowed with the usual order, see [18]. In the general case we can retrieve only few of the properties of the Siegmund duality (for its properties see [1, 6, 8]), some of them only require that duality preserves positivity, other require stronger conditions, and we always put them in terms of the positive Möbius cone.

In Sect. 4, we study Sylvester formulae for sets (the well-known inclusion–exclusion relations) and for partitions. To the best of our knowledge, the Sylvester formulae to be found in Sect. 4.1.2 for partitions are new. These formulae aim at describing the positive Möbius cone and so, in principle, they can give some insight into the problem of when duality preserves positivity.

A natural question encountered in the context of zeta and Möbius duality is when a duality relation is preserved by coarse-graining, that is when we can state some type of duality for coarser observations of the processes. Thus, instead of a set, it can be observed the number of elements it contains, and instead of a partition, it can be only access to the size of its atoms. Coarse-graining duality is studied in Sect. 5, the main result being Theorem 15 where it is proven that when the coarse-graining is satisfied, it is required an h -transform in the dual kernel in order that stochasticity is preserved. In this section, we also show that the conditions for coarse-graining are fulfilled for zeta and Möbius matrices on sets and partitions.

Finally, Sect. 6 is devoted to some examples of these duality relations. In these examples, we revisit the haploid Cannings model and the multi-allelic model with constant population size (see [3, 4, 14, 15]). In [14, 15], an ancestor type process was associated to these models, and their duality was stated. We will give a set version of these models, showing they are in duality via a transpose zeta matrix and that coarse-graining duality modified by an h -transform appears in a natural way giving the hypergeometric matrix.

We point out that many of the concepts we will introduce and even some of the results we will obtain are straightforwardly defined or satisfied in a countable infinite setting. But we prefer to keep a finite framework for clarity and to avoid technicalities that can hide the meaning and interest of our results.

A previous study on zeta and Möbius duality is found in [13]. One of its results is what we called conditions (i) in Propositions 2 and 3 in Sect. 3, we give them for completeness and because they are straightforward to obtain. The main result in [13] is Theorem 2, ensuring that there exists a strong dual (see [6]) for a stochastic kernel P such that the ratio between the initial distribution and the stationary distribution is Möbius monotone but also (mainly) that time reversed process is Möbius monotone. This type of questions will not be in the focus of our work.

1.1 Notation

For a set A , $|A|$ denotes its cardinality. By I, \mathcal{A} we denote finite sets. We denote $\mathbb{S}(I) = \{J : J \subseteq I\}$ the class of subsets of I .

By N, T we mean positive integers. We set $\mathcal{I}_N = \{1, \dots, N\}$. For two integers $s \leq t$ we denote by $\mathcal{I}_t^s = \{s, \dots, t\}$ the interval of integers. In particular, $\mathcal{I}_N^0 = \{0, 1, \dots, N\}$.

For a relation \mathcal{R} defined on some set, we define $\mathbf{1}_{\mathcal{R}}$ the function which assigns a 1 when \mathcal{R} is satisfied and 0 otherwise. For a set A , $\mathbf{1}_A$ is its characteristic function, it gives value 1 for the elements belonging to A and 0 otherwise. Also we denote by $\mathbf{1}$ the 1-constant vector with the dimension of the space where it is defined.

The transpose of a matrix or a vector H is denoted by H' . The functions $g : \mathcal{A} \rightarrow \mathbb{R}$ can be identified to a column vector in $\mathbb{R}^{\mathcal{A}}$, so g' means the row vector. In particular, the characteristic function $\mathbf{1}_A$ is a column vector and $\mathbf{1}'_A$ a row vector.

2 Zeta and Möbius Matrices

This section follows the ideas developed by Rota [17]. The examples we give are well known and the product formula supplied in [2] allows to present them in a unified way.

Let \mathcal{A} be a finite set and (\mathcal{A}, \preceq) be a partially ordered space.

The zeta matrix $Z = (Z(a, b) : a, b \in \mathcal{A})$ is given by $Z(a, b) = \mathbf{1}_{a \preceq b}$. It is nonsingular and its inverse $Z^{-1} = (Z^{-1}(a, b) : a, b \in \mathcal{A})$ is the Möbius matrix. In [2] it was shown that the Möbius matrix satisfies $Z^{-1}(a, b) = \mu(a, b)\mathbf{1}_{a \preceq b}$, where for $a \preceq b$:

$$\mu(a, b) = \begin{cases} 1 & \text{if } a = b \\ -\sum_{c \in \mathcal{A}: a \preceq c \preceq b} \mu(a, c) & \text{if } a \prec b. \end{cases} \tag{1}$$

Also see [17] Sect. 3. For completeness, let us check that this matrix is the inverse of Z . We have

$$\sum_{c \in \mathcal{A}} \mathbf{1}_{a \preceq c} \mu(a, c) \mathbf{1}_{c \preceq b} = \sum_{c \in \mathcal{A}: a \preceq c \preceq b} \mu(a, c).$$

If $a = b$ then $c = a = b$ is the unique c in the sum and the above expression is 1. When $a \neq b$, in order that there exists some c in the sum we must have $a \prec b$. In this case, by definition of μ we have

$$\left(\sum_{c \in \mathcal{A}: a \preceq c \prec b} \mu(a, c) \right) + \mu(a, b) = 0.$$

so, the inverse of Z satisfies $Z^{-1}(a, b) = \mu(a, b)\mathbf{1}_{a \preceq b}$. The function $\mu(a, b)$, that only needs to be defined for $a \preceq b$, is called the Möbius function. Since $\mu(a, a) = 1$, μ is completely described once $\mu(a, b)$ is identified for $a \prec b$.

We will also consider the transpose zeta and Möbius matrices $Z' = (Z'(a, b) = \mathbf{1}_{b \leq a} : a, b \in \mathcal{A})$ and $Z'^{-1} = (Z'^{-1}(a, b) = \mu(b, a)\mathbf{1}_{b \leq a} : a, b \in \mathcal{A})$.

Two partially ordered spaces (\mathcal{A}_1, \leq_1) (\mathcal{A}_2, \leq_2) are isomorphic if there exists a bijection $\varphi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ that verifies $a \leq_1 b$ if and only if $\varphi(a) \leq_2 \varphi(b)$. If μ_1 and μ_2 are their respective Möbius functions, then $\mu_1(a, b) = \mu_2(\varphi(a), \varphi(b))$.

2.1 Product Formula

Let us introduce the product formula, as given in Theorem 3 in [17]. Let (\mathcal{A}_1, \leq_1) and (\mathcal{A}_2, \leq_2) be two partially ordered spaces with Möbius functions μ_1 and μ_2 , respectively. The product set $\mathcal{A}_1 \times \mathcal{A}_2$ is partially ordered with the product order $\leq_{1,2}$ given by: $(a_1, a_2) \leq_{1,2} (b_1, b_2)$ if $a_1 \leq_1 b_1$ and $a_2 \leq_2 b_2$. The Möbius function for the product space $(\mathcal{A}_1 \times \mathcal{A}_2, \leq_{1,2})$ results to be the product of the Möbius functions:

$$a_1 \leq_1 b_1, a_2 \leq_2 b_2 \Rightarrow \mu((a_1, a_2), (b_1, b_2)) = \mu_1(a_1, b_1)\mu_2(a_2, b_2). \tag{2}$$

The above relations are summarized in,

$$\begin{aligned} \mathbf{1}_{(a_1, a_2) \leq_{1,2} (b_1, b_2)} &= \mathbf{1}_{a_1 \leq_1 b_1} \mathbf{1}_{a_2 \leq_2 b_2}; \\ \mathbf{1}_{(a_1, a_2) \leq_{1,2} (b_1, b_2)} \mu((a_1, a_2), (b_1, b_2)) &= \mu_1(a_1, b_1) \mathbf{1}_{a_1 \leq_1 b_1} \cdot \mu_2(a_2, b_2) \mathbf{1}_{a_2 \leq_2 b_2}. \end{aligned} \tag{3}$$

Let Z_r be the zeta matrix associated to (\mathcal{A}_r, \leq_r) for $r = 1, 2$, and $Z_{1,2}$ be the zeta matrix associated to the product space $(\mathcal{A}_1 \times \mathcal{A}_2, \leq_{1,2})$. For $g_r : \mathcal{A}_r \rightarrow \mathbb{R}$ for $r = 1, 2$ define $g_1 \otimes g_2 : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathbb{R}$ by $g_1 \otimes g_2(a_1, a_2) = g_1(a_1)g_2(a_2)$. By using (3) we get

$$\begin{aligned} (Z_{1,2} g_1 \otimes g_2)((a_1, a_2)) &= (Z_1 g_1)(a_1)(Z_2 g_2)(a_2); \\ (Z_{1,2}^{-1} g_1 \otimes g_2)((a_1, a_2)) &= (Z_1^{-1} g_1)(a_1)(Z_2^{-1} g_2)(a_2). \end{aligned} \tag{4}$$

2.2 Möbius Functions for Sets

The most trivial case is $|\mathcal{A}| = 2$. Take $\mathcal{A} = \{0, 1\}$ with the usual order \leq . In this case $\mu(0, 1) = -1$. Then, the Möbius function of the product space $\{0, 1\}^I$ endowed with the product partial order \leq is

$$\mu((a_i : i \in I), (b_i : i \in I)) = (-1)^{\sum_{i \in I} (a_i - b_i)} \quad \text{when } (a_i : i \in I) \leq (b_i : i \in I). \tag{5}$$

Let I be a finite set, the class of its subsets $\mathbb{S}(I) = \{J : J \subseteq I\}$ is partially ordered by inclusion \subseteq . Since $(\mathbb{S}(I), \subseteq)$ is isomorphic to the product space $\{0, 1\}^I$ endowed with the product partial order, the Möbius function for $(\mathbb{S}(I), \subseteq)$ is

$$\forall J, K \in \mathbb{S}(I), J \subseteq K : \mu(J, K) = (-1)^{|K| - |J|}. \tag{6}$$

Its zeta matrix $Z = (Z(J, K) : J, K \in \mathbb{S}(I))$ satisfies $Z(J, K) = \mathbf{1}_{J \subseteq K}$ and the Möbius matrix Z^{-1} is given by $Z^{-1}(J, K) = (-1)^{|K|-|J|} \mathbf{1}_{J \subseteq K}$. The transpose matrices Z' and $Z^{-1'}$ satisfy $Z'(J, K) = \mathbf{1}_{K \subseteq J}$ and $Z^{-1'}(J, K) = (-1)^{|J|-|K|} \mathbf{1}_{K \subseteq J}$.

Let $T \geq 1$ be a positive integer. The study of $(\mathbb{S}(I), \subseteq)$ also encompasses the class of product of sets $\mathbb{S}(I)^T$ endowed with the product order. To describe it, denote the elements of $\mathbb{S}(I)^T$ by

$$\vec{J} = (J_t : t \in \mathcal{I}_T) \quad \text{with } J_t \subseteq I \text{ for } t \in \mathcal{I}_T.$$

Let \vec{J} and \vec{K} be two elements of $\mathbb{S}(I)^T$. The product order is $\vec{J} \subseteq \vec{K}$ if $J_t \subseteq K_t$ for $t \in \mathcal{I}_T$. The Möbius function for the product ordered space $(\mathbb{S}(I)^T, \subseteq)$ is

$$\mu(\vec{J}, \vec{K}) = (-1)^{\sum_{t \in \mathcal{I}_T} (|K_t| - |J_t|)} \quad \text{when } \vec{J} \subseteq \vec{K}. \tag{7}$$

Now note that

$$\mathbb{S}(I)^T \rightarrow \mathbb{S}(I \times \mathcal{I}_T), \quad (J_t : t \in \mathcal{I}_T) \rightarrow \bigcup_{t \in T} J_t \times \{t\}, \tag{8}$$

is a bijection that satisfies $(\vec{J} \subseteq \vec{K}) \Leftrightarrow (\bigcup_{t \in T} J_t \times \{t\} \subseteq \bigcup_{t \in T} K_t \times \{t\})$. Then, the above bijection is an isomorphism between the partially ordered spaces $(\mathbb{S}(I)^T, \subseteq)$ and $(\mathbb{S}(I \times \mathcal{I}_T), \subseteq)$. Hence, every statement for the class of sets also holds for the class of product of sets [the isomorphism between both spaces is a natural consequence of the construction done between (5) and (6)].

2.3 Möbius Functions for Partitions

Let I be a finite set and $\mathcal{P}(I)$ be the set of partitions of I . Thus, $\alpha \in \mathcal{P}(I)$ if $\alpha = \{A_t : t = 1, \dots, T(\alpha)\}$, where:

$$\begin{aligned} \forall t \in \mathcal{I}_{T(\alpha)} \quad A_t \in \mathbb{S}(I) \setminus \{\emptyset\}, \quad t \neq t' \quad A_t \cap A_{t'} = \emptyset \text{ (disjointness),} \\ \bigcup_{t \in \mathcal{I}_{T(\alpha)}} A_t = I \text{ (covering).} \end{aligned}$$

The sets A_t are called the atoms of the partition, and the number of atoms constituting the partition α is denoted by $[\alpha] = T(\alpha)$. An atom of α is often denoted by A and we write $A \in \alpha$. Since the order of the atoms plays no role we write $\alpha = \{A \in \alpha\}$.

A partition α can be defined as the set of equivalence classes of an equivalence relation \equiv_α defined by $i \equiv_\alpha j \Leftrightarrow \exists A \in \alpha$ such that $i, j \in A$. That is, two elements are in relation \equiv_α when they are in the same atom of the partition.

The set of partitions $\mathcal{P}(I)$ is partially ordered by the following order relation

$$\alpha \preceq \beta \quad \text{if } \forall A \in \alpha \exists B \in \beta \text{ such that } A \subseteq B.$$

When $\alpha \preceq \beta$ it is said that α is finer than β or that β is coarser than α .

The zeta matrix $Z = (Z(\alpha, \beta) : \alpha, \beta \in \mathcal{P}(I))$ is given by $Z(\alpha, \beta) = \mathbf{1}_{\alpha \leq \beta}$ and the Möbius matrix by $Z^{-1}(\alpha, \beta) = \mu(\alpha, \beta)\mathbf{1}_{\alpha \leq \beta}$. The Möbius function $\mu(\alpha, \beta)$ is shown to satisfy the relation

$$\mu(\alpha, \beta) = (-1)^{[\alpha]+[\beta]} \prod_{B \in \beta} (\ell_B^\alpha - 1)! \quad \text{for } \alpha < \beta,$$

where $\ell_B^\alpha = |\{A \in \alpha : A \subseteq B\}|$ is the number of atoms of α contained in B , see [5] p. 36.

3 Zeta and Möbius Duality

We will study duality relations for zeta and Möbius matrices and the conditions for positivity in terms of what we call Möbius positive cones. Here, \mathcal{A} is the set of indexes and as assumed it is finite.

3.1 Duality

Let $P = (P(a, b) : a, b \in \mathcal{A})$ be a positive matrix, that is each entry is non-negative, and $H = (H(a, b) : a, b \in \mathcal{A})$ be a matrix. Then, $Q = (Q(a, b) : a, b \in \mathcal{A})$ is said to be a H -dual of P if it satisfies

$$HQ' = PH. \tag{9}$$

We usually refer to P and Q as kernels, and Q is said to be the dual kernel. Duality relation (9) implies $HQ'^n = P^nH$ for all $n \geq 0$. If H is nonsingular the duality relation (9) takes the form

$$Q' = H^{-1}PH. \tag{10}$$

One is mostly interested in the case when P is substochastic (that is nonnegative and satisfying $P\mathbf{1} \leq \mathbf{1}$) or stochastic (nonnegative and $P\mathbf{1} = \mathbf{1}$) and one looks for conditions in order that Q is nonnegative and, when this is the case, one seeks to know when Q is substochastic or stochastic.

Now, let $h : \mathcal{A} \rightarrow \mathbb{R}_+$ be a non-vanishing function and D_h be the diagonal matrix given by $D_h(a, a) = h(a)$ for $a \in \mathcal{A}$. Its inverse is $D_h^{-1} = D_{h^{-1}}$.

Lemma 1 *Let $h : \mathcal{A} \rightarrow \mathbb{R}_+$ be a non-vanishing function. We have:*

$$HQ' = PH \Leftrightarrow H_h Q'_{h^{-1},h} = PH_h \quad \text{with } H_h := HD_h^{-1} \quad \text{and } Q_{h^{-1},h} := D_h^{-1}QD_h. \tag{11}$$

Assume $h > 0$. Then, $Q \geq 0$ implies $Q_{h^{-1},h} \geq 0$ and

$$(Q_{h^{-1},h}\mathbf{1} = \mathbf{1} \Leftrightarrow Qh = h) \quad \text{and} \quad (Q_{h^{-1},h}\mathbf{1} \leq \mathbf{1} \Leftrightarrow Qh \leq h). \tag{12}$$

Proof All relations are straightforward. For instance (12) follows from $Q_{h^{-1},h}\mathbf{1} = \mathbf{1}$ if and only if $QD_h\mathbf{1} = D_h\mathbf{1}$, which is $Qh = h$. A similar argument proves the second relation with \leq . □

The matrix $Q_{h^{-1},h}$ is called the h -transform of Q . So, it is a H_h -dual of P . When $h > 0$ and $Q \geq 0$, the matrix $Q_{h^{-1},h}$ is stochastic if and only if h is a right eigenvector of Q with eigenvalue 1.

If P and Q are substochastic matrices then the duality has the following probabilistic interpretation in terms of their associated Markov chains. Let $X = (X_n : n \geq 0)$ and $Y = (Y_n : n \geq 0)$ be the associated Markov chains and T^X and T^Y be their lifetimes. Let \mathbb{P}_a^X and \mathbb{P}_b^Y be the laws of the chains starting from the states a and b , respectively, and \mathbb{E}_a^X and \mathbb{E}_b^Y be their associated mean expected values. Let ∂^X and ∂^Y be the coffin states of X and Y , respectively, then $X_n = \partial^X$ for $n \geq T^X$ and $Y_n = \partial^Y$ for $n \geq T^Y$. We make the extension

$$H(\partial^X, b) = 0 = H(a, \partial^Y) = H(\partial^X, \partial^Y).$$

Then, the duality relation (9) is equivalent to

$$\forall a, b \in \mathcal{A} \quad \forall n \geq 0 : \quad \mathbb{E}_a^X(H(X_n, b)) = \mathbb{E}_b^Y(H(a, Y_n)).$$

This notion of duality was introduced in [12] in a very general framework and developed in several works, see [6, 8, 11, 14] and references therein.

3.2 Möbius Positive Cones

We will study duality relations for zeta and Möbius matrices and set conditions for positivity in terms of the following classes of nonnegative functions

$$\mathcal{F}_+(\mathcal{A}) = \{g \in \mathbb{R}^{\mathcal{A}} : Z^{-1}g \geq 0\} \text{ and } \mathcal{F}'_+(\mathcal{A}) = \{g \in \mathbb{R}^{\mathcal{A}} : Z^{-1'}g \geq 0\},$$

Note that both sets are convex cones, we call them positive Möbius cones (of functions). We have

$$\mathcal{F}_+(\mathcal{A}) = \{g \in \mathbb{R}^{\mathcal{A}} : Z^{-1}g \geq 0\} \text{ and } \mathcal{F}'_+(\mathcal{A}) = \{g \in \mathbb{R}^{\mathcal{A}} : Z^{-1'}g \geq 0\}.$$

For showing the first expression, we only have to prove that if $Z^{-1}g \geq 0$ then $g \geq 0$. This follows straightforward from the non-negativity of Z ,

$$\forall a \in \mathcal{A} : \quad g(a) = \sum_{b:b \geq a} (Z^{-1}g)(b).$$

The second expression is shown similarly. In [13] the functions in $\mathcal{F}_+(\mathcal{A})$ and $\mathcal{F}'_+(\mathcal{A})$ are called Möbius monotone and the argument we just gave is the Proposition 2.1 therein. We also define

$$\mathcal{F}(\mathcal{A}) = \mathcal{F}_+(\mathcal{A}) - \mathcal{F}_+(\mathcal{A}) = \{g_1 - g_2 : g_1, g_2 \in \mathcal{F}_+(\mathcal{A})\} \text{ and}$$

$$\mathcal{F}'(\mathcal{A}) = \mathcal{F}'_+(\mathcal{A}) - \mathcal{F}'_+(\mathcal{A}).$$

For every $a \in \mathcal{A}$ the function $\mathbb{R}^{\mathcal{A}} \rightarrow \mathbb{R}, g \rightarrow Z^{-1}g(a)$ is linear. Hence, a simple consequence of the additivity gives

$$\forall g_1, g_2 \in \mathcal{F}_+(\mathcal{A}), a \in \mathcal{A} \Rightarrow Z^{-1}g_1(a) \leq Z^{-1}(g_1 + g_2)(a); \tag{13}$$

$$\forall g_1, g_2 \in \mathcal{F}'_+(\mathcal{A}), a \in \mathcal{A} \Rightarrow Z^{-1'}g_1(a) \leq Z^{-1'}(g_1 + g_2)(a). \tag{14}$$

3.3 Duality with Zeta and Möbius Matrices

We will give necessary and sufficient conditions in order that zeta and Möbius duality, as well as their transpose, preserve positivity (these conditions appear as (i) in the propositions). Also we give stronger sufficient conditions having stronger implications on the monotonicity of kernels (these conditions appear as (ii) in the propositions).

As said, zeta and Möbius duality were already studied in [13] and in this reference conditions (i) of Propositions 2 and 3 are also found. We supply them for completeness and since they are straightforward.

In the sequel, we will introduce a notation for the rows and columns of a matrix. For $P = (P(a, b) : a, b \in \mathcal{A})$ we denote by $P(a, \bullet)$ its a -th row and by $P(\bullet, b)$ its b -th column, that is

$$P(a, \bullet) : \mathcal{A} \rightarrow \mathbb{R}, c \rightarrow P(a, c) \text{ and } P(\bullet, b) : \mathcal{A} \rightarrow \mathbb{R}, c \rightarrow P(c, b).$$

3.3.1 Duality with the Zeta Matrix

Assume the kernel Q is the Z -dual of the positive kernel P , so $Q' = Z^{-1}PZ$ holds. Hence,

$$\begin{aligned} Q(a, b) &= \sum_{c \in \mathcal{A}} \sum_{d \in \mathcal{A}} Z^{-1}(b, c)P(c, d)Z(d, a) = \sum_{c: b \leq c} \mu(b, c) \sum_{d: d \leq a} P(c, d) \\ &= \sum_{c: b \leq c} \mu(b, c) \left(\sum_{d: d \leq a} P(\bullet, d) \right) (c) = Z^{-1} \left(\sum_{d: d \leq a} P(\bullet, d) \right) (b). \end{aligned} \tag{15}$$

Proposition 2 Assume $P \geq 0$. (i) We have

$$Q \geq 0 \Leftrightarrow \forall a \in \mathcal{A} : \sum_{d: d \leq a} P(\bullet, d) \in \mathcal{F}_+(\mathcal{A}). \tag{16}$$

When this condition holds the following implication is satisfied,

$$(P(c, d) > 0 \Rightarrow c \leq d) \text{ implies } (Q(c, d) > 0 \Rightarrow d \leq c). \tag{17}$$

(ii) Assume for all $d \in \mathcal{A}$ we have $P(\bullet, d) \in \mathcal{F}_+(\mathcal{A})$. Then $Q \geq 0$ and for all b the function $Q(a, b)$ is increasing in a , that is

$$\forall b \in \mathcal{A}, a_1 \preceq a_2 \Rightarrow Q(a_1, b) \leq Q(a_2, b). \tag{18}$$

Proof The equivalence (16) is straightforward from equality (15). To show relation (17) we use the equality

$$Q(a, b) = \sum_{(c,d): b \preceq c, d \preceq a} \mu(b, c)P(c, d).$$

Since we are assuming P only charges couples (c, d) such that $c \preceq d$ then the previous sum is with respect to the set $\{(c, d) : b \preceq c; d \preceq a; c \preceq d\}$. So, if this set is nonempty we necessarily have $b \preceq a$.

(ii) The first statement follows from (i) and the fact that $\mathcal{F}_+(\mathcal{A})$ is a cone. For proving (18) we note that $a_1 \preceq a_2$ implies $\{d : d \preceq a_2\} \supseteq \{d : d \preceq a_1\}$. Then,

$$\sum_{d \preceq a_2} P(\bullet, d) = \sum_{d \preceq a_1} P(\bullet, d) + g \text{ with } g = \sum_{d: d \preceq a_2, d \not\preceq a_1} P(\bullet, d).$$

Then, from the hypothesis made in (ii) we get $g \in \mathcal{F}_+(\mathcal{A})$. Hence, (13) and (15) give (18). □

Remark 1 Assume condition (16) is satisfied and that (\mathcal{A}, \preceq) has a global maximum and a global minimum, denoted, respectively, by a_{\max} and a_{\min} . Then, the hypothesis $(P(c, d) > 0 \Rightarrow c \preceq d)$ in (17) assumes in particular that a_{\max} is an absorbing point for P because $P(a_{\max}, b) = 0$ for all $b \neq a_{\max}$. The property that it implies, $(Q(c, d) > 0 \Rightarrow d \preceq c)$, says in particular that a_{\min} is an absorbing point for Q because $Q(a_{\min}, d) = 0$ for all $d \neq a_{\min}$. In the case $(\mathcal{A}, \preceq) = (\mathbb{S}(I), \subseteq)$ we have $a_{\max} = I$ and $a_{\min} = \emptyset$ and when $(\mathcal{A}, \preceq) = (\mathcal{P}(I), \subseteq)$ we have $a_{\max} = \{I\}$ and $a_{\min} = \{\{i\} : i \in I\}$.

Remark 2 Under hypothesis (2), condition (18) implies that if Q is stochastic then for comparable indexes the rows of Q are equal.

3.3.2 Duality with the Transpose Zeta Matrix

Let the kernel Q be the Z' -dual of the positive kernel P , so $Q' = Z'^{-1}PZ'$ is satisfied. Hence,

$$Q(a, b) = \sum_{c \in \mathcal{A}} \sum_{d \in \mathcal{A}} Z^{-1}(c, b)P(c, d)Z(a, d) = Z^{-1'} \left(\sum_{d: a \preceq d} P(\bullet, d) \right) (b). \tag{19}$$

Proposition 3 (i) *We have*

$$Q \geq 0 \Leftrightarrow \forall a \in \mathcal{A} : \sum_{d:a \leq d} P(\bullet, d) \in \mathcal{F}'_+(\mathcal{A}). \tag{20}$$

When this condition holds the following implication is satisfied

$$(P(c, d) > 0 \Rightarrow d \leq c) \text{ implies } (Q(c, d) > 0 \Rightarrow c \leq d). \tag{21}$$

(ii) *Assume for all $d \in \mathcal{A}$ we have $P(\bullet, d) \in \mathcal{F}'_+(\mathcal{A})$. Then $Q \geq 0$ and for all b the function $Q(a, b)$ is increasing in a , that is*

$$\forall b \in \mathcal{A}, a_2 \leq a_1 \Rightarrow Q(a_1, b) \geq Q(a_2, b). \tag{22}$$

Proof It is entirely similar as the one of Proposition 2. □

Similar notes as Remarks 1 and 2 can be made.

The conditions in part (i) of Propositions 2 and 3 ensuring positivity of Q are the same as the ones in [13].

3.3.3 Duality with the Möbius Matrix

Assume Q is the Z^{-1} -dual of the positive kernel P , so $Q' = ZPZ^{-1}$ is satisfied. This is

$$Q(a, b) = \sum_{c \in \mathcal{A}} \sum_{d \in \mathcal{A}} Z(b, c)P(c, d)Z^{-1}(d, a) = Z^{-1}' \left(\sum_{c:b \leq c} P(c, \bullet) \right) (a). \tag{23}$$

Proposition 4 *Assume $P \geq 0$.*

(i) *We have*

$$Q \geq 0 \Leftrightarrow \forall b \in \mathcal{A} : \sum_{c:b \leq c} P(c, \bullet) \in \mathcal{F}'_+(\mathcal{A}). \tag{24}$$

If this condition holds we have that (17) is satisfied.

(ii) *Assume for all $c \in \mathcal{A}$ we have $P(c, \bullet) \in \mathcal{F}'_+(\mathcal{A})$. Then $Q \geq 0$ and:*

(ii1) *$Q(a, b)$ is decreasing in b , that is*

$$\forall a \in \mathcal{A}, b_1 \leq b_2 \Rightarrow Q(a, b_1) \geq Q(a, b_2); \tag{25}$$

(ii2) *If P is stochastic and irreducible then its invariant distribution ρ satisfies $\rho \in \mathcal{F}'_+(\mathcal{A})$;*

(ii3) *If Q is stochastic and irreducible then its invariant distribution $\hat{\rho}$ is decreasing that is: $b_1 \leq b_2 \Rightarrow \hat{\rho}(b_1) \geq \hat{\rho}(b_2)$.*

Proof The proof of (i), the first statement in (ii) and (ii1) are similar to the proof of Proposition 2.

- (ii2) The invariant distribution $\rho = (\rho(a) : a \in \mathcal{A})$ satisfies $\rho' = \rho'P$, so in our notation $\rho = \sum_{a \in \mathcal{A}} \rho(a)P(a, \bullet)$. From our hypothesis we have that $P(a, \bullet) \in \mathcal{F}_+(\mathcal{A})$ for all $a \in \mathcal{A}$; since $\mathcal{F}_+(\mathcal{A})$ is a cone we get the result.
- (ii3) Since $\widehat{\rho} = \sum_{a \in \mathcal{A}} \widehat{\rho}(a)Q(a, \bullet)$ the property is derived from property (ii1).

□

A similar note as Remark 1 can be made. Duality with the Möbius matrix is a special case of duality with non-positive matrices. For a study considering other non-positive duality matrices see [19].

3.3.4 Duality with the Transpose Möbius Matrix

Assume Q is $Z^{-1'}$ -dual of the positive matrix P , so $Q' = Z'PZ^{-1'}$ is satisfied. Then,

$$Q(a, b) = \sum_{c \in \mathcal{A}} \sum_{d \in \mathcal{A}} Z(c, b)P(c, d)Z^{-1}(a, d)Z^{-1} \left(\sum_{c:c \leq b} P(c, \bullet) \right) (a). \tag{26}$$

Proposition 5 Assume $P \geq 0$. (i) We have

$$Q \geq 0 \Leftrightarrow \forall b \in \mathcal{A} : \sum_{c:c \leq b} P(c, \bullet) \in \mathcal{F}_+(\mathcal{A}). \tag{27}$$

When this condition holds, relation (21) is satisfied.

(ii) Assume for all $c \in \mathcal{A}$ we have $P(c, \bullet) \in \mathcal{F}_+(\mathcal{A})$. Then $Q \geq 0$ and:

(ii1) $Q(a, b)$ is increasing in b , this is

$$\forall a \in \mathcal{A}, b_1 \leq b_2 \Rightarrow Q(a, b_1) \leq Q(a, b_2); \tag{28}$$

(ii2) If P is stochastic and irreducible then its invariant distribution ρ satisfies $\rho \in \mathcal{F}_+(\mathcal{A})$;

(ii3) If Q is stochastic and irreducible then its invariant distribution $\widehat{\rho}$ is increasing.

Proof The proof of (i), the first statement in (ii) and (ii1) are similar to the proof of Proposition 2, and the parts (ii2) and (ii3) are shown in a similar way as (ii2) and (ii3) in Proposition 4. □

A similar note as Remark 1 can be made.

4 Möbius Positive Cones and Sylvester Formulae for Sets and Partitions

4.1 Sylvester Formulae

As already fixed I is a finite set. Let $(\mathcal{X}, \mathcal{B})$ be a measurable space and $(X_i : i \in I) \subseteq \mathcal{B}$ be a finite class of events. The σ -algebra $\sigma(X_i : i \in I)$ generated by $(X_i : i \in I)$ in \mathcal{X} , is the class of finite unions of the disjoint sets

$$\bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right), \quad J \subseteq I. \tag{29}$$

When $J = \emptyset$ the above set is $\mathcal{X} \setminus \bigcup_{i \in I} X_i$ because $\bigcap_{i \in \emptyset} X_i = \mathcal{X}$.

Since all we shall do only depends on $\sigma(X_i : i \in I)$, in the sequel we only consider the measurable space $(\mathcal{X}, \sigma(X_i : i \in I))$. When we say $(\mathcal{X}, \sigma(X_i : i \in I))$ is a measurable space we mean $(X_i : i \in I)$ is a family of subsets of \mathcal{X} and $\sigma(X_i : i \in I)$ is the σ -algebra generated by them.

4.1.1 Sylvester Formula for Sets and Product of Sets

Let $(\mathcal{X}, \sigma(X_i : i \in I))$ be a measurable space. Let ν be a finite measure or finite signed measure on $(\mathcal{X}, \sigma(X_i : i \in I))$. Sylvester formula is $\nu(\mathcal{X} \setminus \bigcap_{i \in I} X_i) = \sum_{L \subseteq I} (-1)^{|L|} \nu(\bigcap_{i \in L} X_i)$. Let $J \in \mathbb{S}(I)$ be fixed and consider $\mathcal{X}' = \bigcap_{i \in J} X_i$ and $X'_i = X_i \cap \mathcal{X}'$ for $i \in I \setminus J$. We have $\mathcal{X}' \setminus X'_k = \bigcap_{i \in J} X_i \setminus \bigcap_{i \in J \cup \{j\}} X_i$ for all $j \in I \setminus J$. Then, $\bigcap_{i \in I \setminus J} \mathcal{X}' \setminus X'_i = \bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right)$, and Sylvester formula gives

$$\forall J \in \mathbb{S}(I) : \nu \left(\bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right) \right) = \sum_{L: L \supseteq J} (-1)^{|L| - |J|} \nu \left(\bigcap_{i \in L} X_i \right). \tag{30}$$

We will write the above formula in terms of the Möbius matrix for sets.

Proposition 6 *The measurable spaces $(\mathcal{X}, \sigma(X_i : i \in I))$ and $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$ are isomorphic by:*

$$\Psi : \sigma(X_i : i \in I) \rightarrow \mathbb{S}(\mathbb{S}(I)) : \forall J \in \mathbb{S}(I), \Psi \left(\bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right) \right) = \{J\}. \tag{31}$$

For the other elements of the algebras we impose that Ψ preserves disjoint unions; thus Ψ is an isomorphism of algebras.

For every finite (respectively signed) measure ν defined on $(\mathcal{X}, \sigma(X_i : i \in I))$ the (respectively signed) measure $\nu^* = \nu \circ \Psi^{-1}$ on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$ is given by:

$$\forall J \in \mathbb{S}(I) : \nu^*(\{J\}) = \nu \left(\bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right) \right).$$

Under the isomorphism (31) we have

$$\Psi \left(\bigcap_{i \in J} X_i \right) = \{K : K \supseteq J\}; \tag{32}$$

$$\forall i \in I : \Psi(X_i) = \{J : i \in J\}; \tag{33}$$

$$v^*({J}) = \sum_{L:L \supseteq J} (-1)^{|L|-|J|} v\left(\bigcap_{i \in L} X_i^*\right) = \sum_{L:L \supseteq J} (-1)^{|L|-|J|} \left(\sum_{K:K \supseteq L} v^*({K})\right). \tag{34}$$

Proof Let $X_i^* = \Psi(X_i)$ be the image of X_i under this isomorphism, so $\bigcap_{i \in J} X_i^* = \Psi\left(\bigcap_{i \in J} X_i\right)$. Since

$$\bigcap_{i \in J} X_i = \bigcup_{K \supseteq J} \left(\bigcap_{i \in K} X_i \setminus \left(\bigcup_{L:L \supseteq K, L \neq K} \bigcap_{i \in L} X_i\right)\right).$$

the isomorphism gives (32). Then

$$X_i^* = \bigcup_{J:i \in J} \bigcap_{j \in J} X_j^* = \bigcup_{J:i \in J} \{K : K \supseteq J\} = \{J : i \in J\},$$

so (33) is shown. Then, (34) follows from Sylvester formula (30). □

Note that (34) is equivalent to $v^*({J}) = (Z^{-1}(Zv^*))({J})$ when $v^* = (v^*({J}) : J \in \mathbb{S}(I))$ is written as a column vector. Hence, Sylvester formula (30) is equivalent to the fact that the Möbius function for the class of subsets ordered by inclusion is $(-1)^{|L|-|J|}$ for $J \subseteq L$.

As noted, the isomorphism given in (8) guarantees that a similar Sylvester formula can be stated for product of sets. Let us give this formula explicitly. Let $T \geq 1$ be a positive integer. The product space $\mathbb{S}(I)^T$ was endowed with the product order also denoted by \subseteq , the elements of $\mathbb{S}(I)^T$ are written $\vec{J} = (J_t : t \in \mathcal{I}_T)$, and in general, we use the notions supplied in Sect. 2.2. Similarly to Proposition 6 we have:

Proposition 7 *The measurable spaces $(\mathcal{X}, \sigma(X_{i,t} : (i, t) \in I \times \mathcal{I}_T))$ and $(\mathbb{S}(I)^T, \mathbb{S}(\mathbb{S}(I)^T))$ are isomorphic by $\Psi : \sigma(X_{i,t} : i \in I, t \in \mathcal{I}_T) \rightarrow \mathbb{S}(\mathbb{S}(I)^T)$, where*

$$\forall \vec{J} \in \mathbb{S}(I)^T : \Psi\left(\bigcap_{t \in \mathcal{I}_T} \bigcap_{i \in J_t} X_{i,t} \setminus \left(\bigcup_{\vec{L}:\vec{L} \supseteq \vec{J}, \vec{L} \neq \vec{J}} \bigcap_{t \in \mathcal{I}_T} \bigcap_{i \in L_t} X_{i,t}\right)\right) = \{\vec{J}\}; \tag{35}$$

and on the other elements of the algebras we impose Ψ preserves the disjoint unions, so Ψ is an isomorphism of algebras.

For every finite (respectively signed) measure v defined on $(\mathcal{X}, \sigma(X_{i,t} : i \in I, t \in \mathcal{I}_T))$ the (respectively signed) measure $v^* = v \circ \Psi^{-1}$ on $(\mathbb{S}(I)^T, \mathbb{S}(\mathbb{S}(I)^T))$ is

$$\forall \vec{J} \in \mathbb{S}(I)^T : v^*(\vec{J}) = v\left(\bigcap_{t \in \mathcal{I}_T} \bigcap_{i \in J_t} X_{i,t} \setminus \left(\bigcup_{\vec{L}:\vec{L} \supseteq \vec{J}, \vec{L} \neq \vec{J}} \bigcap_{t \in \mathcal{I}_T} \bigcap_{i \in L_t} X_{i,t}\right)\right).$$

Under the isomorphism (35) we have:

$$\begin{aligned} \Psi \left(\bigcap_{t \in \mathcal{I}_T} \bigcap_{i \in J_t} X_{i,t} \right) &= \{ \vec{K} : \vec{K} \supseteq \vec{J} \}; \\ \forall (i, t) \in I \times \mathcal{I}_T : \Psi(X_{i,t}) &= \{ \vec{J} : i \in J_t \}; \\ v^*(\{\vec{J}\}) &= \sum_{\vec{L} : \vec{L} \supseteq \vec{J}} (-1)^{\sum_{t \in \mathcal{I}_T} (|L_t| - |J_t|)} \left(\sum_{\vec{K} : \vec{K} \supseteq \vec{L}} v^*(\{\vec{K}\}) \right). \end{aligned} \tag{36}$$

For any finite set I the algebra $\sigma(X_i : i \in I)$ is generated by the $2^{|I|}$ sets defined by (29) (they could be less if some intersections are empty, but for this discussion assume this does not happen). Since the isomorphism of algebras must preserve the number of generating elements, a Sylvester formula can be written with spaces having cardinality of the type 2^N (as $\mathbb{S}(I)$), and this formula retrieves the Möbius matrix for sets (similarly for product of sets). For partitions, this way is useless because the cardinality of $\mathcal{P}(N)$ does not belong to the class of numbers 2^N , except for some exceptional cases.

4.1.2 Sylvester Formula for Partitions

We seek for a Sylvester formula for partitions that allows to retrieve the Möbius matrix for partitions (instead of for sets as in the previous Section).

As noted, any measurable space $(\mathcal{X}, \sigma(X_i : i \in I))$ has $2^{|I|}$ generating elements defined by (29). Then, no natural isomorphism of algebras can be established with a measurable space of the type $(\mathcal{P}(I'), \mathbb{S}(\mathcal{P}(I')))$ for some I' , because the cardinality $|\mathcal{P}(I')|$ is the Bell number $B_{|I'|}$ which in general is not of the type 2^N . So, we require to define an algebra by using other constructive mechanisms. The basis for this construction is given by the following relation:

$$\text{For } J \in \mathbb{S}(I), \alpha \in \mathcal{P}(I) \text{ we denote } J \vdash \alpha \text{ if } \exists A \in \alpha \text{ such that } J \subseteq A. \tag{37}$$

Let $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ be a measurable space and $(X_J : J \in \mathbb{S}(I))$ be a family of sets indexed by $\mathbb{S}(I)$. We define $\sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I))$ as the σ -algebra of sets generated by the elements

$$\bigcap_{J \vdash \alpha} X_J \setminus \left(\bigcup_{\gamma : \gamma \supseteq \alpha, \gamma \neq \alpha} \bigcap_{J \vdash \gamma} X_J \right), \quad \alpha \in \mathcal{P}(I). \tag{38}$$

That is, the elements of $\sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I))$ are all the finite unions of the sets defined in (38).

On the other hand note that every partition $\alpha \in \mathcal{P}(I)$ satisfies

$$\{\alpha\} = \{\beta : \beta \supseteq \alpha\} \setminus \left(\bigcup_{\gamma : \gamma \supseteq \alpha, \gamma \neq \alpha} \{\beta : \beta \supseteq \gamma\} \right). \tag{39}$$

Proposition 8 *The measurable spaces $(\mathcal{X}, \sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I)))$ and $(\mathcal{P}(I), \mathbb{S}(\mathcal{P}(I)))$ are isomorphic by*

$$\Psi : \sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I)) \rightarrow \mathbb{S}(\mathcal{P}(I)) : \forall \alpha \in \mathcal{P}(I),$$

$$\Psi \left(\bigcap_{J \vdash \alpha} X_J \setminus \left(\bigcup_{\gamma : \gamma \succeq \alpha, \gamma \neq \alpha} \bigcap_{J \vdash \gamma} X_J \right) \right) = \{\alpha\}, \tag{40}$$

and we impose it preserves the disjoint unions, so Ψ is an isomorphism of algebras. For every finite (respectively signed) measure ν defined on $(\mathcal{X}, \sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I)))$ the finite (respectively signed) measure $\nu^* = \nu \circ \Psi^{-1}$ on $(\mathcal{P}, \mathbb{S}(\mathcal{P}(I)))$ is given by:

$$\forall \alpha \in \mathcal{P}(I) : \nu^*(\{\alpha\}) = \nu \left(\bigcap_{J \vdash \alpha} X_J \setminus \left(\bigcup_{\gamma : \gamma \succeq \alpha, \gamma \neq \alpha} \bigcap_{J \vdash \gamma} X_J \right) \right). \tag{41}$$

Moreover, under the isomorphism (40) we have

$$\Psi \left(\bigcap_{J \vdash \alpha} X_J \right) = \{\beta : \beta \succeq \alpha\}; \tag{42}$$

$$\forall J \in \mathbb{S}(I) : \Psi(X_J) = \{\alpha : J \vdash \alpha\}. \tag{43}$$

Proof Let us prove (42). From the isomorphism (40) and by setting $X_J^* = \Psi(X_J)$ we get

$$\bigcap_{J \vdash \alpha} X_J^* = \Psi \left(\bigcap_{J \vdash \alpha} X_J \right) = \{\beta : \beta \succeq \alpha\}.$$

Then,

$$X_J^* = \bigcup_{\alpha : J \vdash \alpha} \bigcap_{K : K \vdash \alpha} X_K^* = \bigcup_{\alpha : J \vdash \alpha} \{\beta : \beta \succeq \alpha\}.$$

Now use,

$$\bigcap_{J \vdash \alpha} \{\beta : \beta \succeq \alpha\} = \{\beta : \forall J, J \vdash \alpha \Rightarrow J \vdash \beta\}$$

to get (43). □

Let us now give the Sylvester formula in this setting. We recall the Möbius function μ defined in (1).

Proposition 9 *Let ν be a finite measure or a finite signed measure on the measurable space $(\mathcal{X}, \sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I)))$. Then,*

$$\nu \left(\bigcap_{J \vdash \alpha} X_J \setminus \left(\bigcup_{\beta: \beta \geq \alpha, \beta \neq \alpha} \bigcap_{J \vdash \beta} X_J \right) \right) = \sum_{\beta: \beta \geq \alpha} \mu(\alpha, \beta) \nu \left(\bigcap_{J \vdash \beta} X_J \right). \tag{44}$$

Proof Let $\nu^* = \nu \circ \Psi^{-1}$ be given by (41). Since the finite measure or signed measure spaces $(\mathcal{X}, \sigma^{\mathcal{P}}(X_i : i \in \mathbb{S}(I)), \nu)$ and $(\mathcal{P}, \mathbb{S}(\mathcal{P}(I)), \nu^*)$ are isomorphic we get that (44) is equivalent to

$$\nu^* \left(\bigcap_{J \vdash \alpha} X_J^* \setminus \left(\bigcup_{\beta: \beta \geq \alpha, \beta \neq \alpha} \bigcap_{J \vdash \beta} X_J^* \right) \right) = \sum_{\beta: \beta \geq \alpha} \mu(\alpha, \beta) \nu^* \left(\bigcap_{J \vdash \beta} X_J \right).$$

So, it is equivalent to

$$\nu^*({\alpha}) = \sum_{\beta: \beta \geq \alpha} \mu(\alpha, \beta) \left(\sum_{\gamma: \gamma \geq \beta} \nu^*({\gamma}) \right), \tag{45}$$

which is exactly $\nu^*({\alpha}) = (Z^{-1}(Z\nu^*))({\alpha})$ when $\nu^* = (\nu^*({\alpha} : \alpha \in \mathbb{S}(I)))$ is written as a column vector. Hence, the result is shown. \square

4.2 Möbius Positive Cones for Sets and Partitions

Below we describe the Möbius positive cones $\mathcal{F}_+(\mathcal{A}), \mathcal{F}'_+(\mathcal{A}), \mathcal{F}(\mathcal{A}) = \mathcal{F}_+(\mathcal{A}) - \mathcal{F}'_+(\mathcal{A})$ and $\mathcal{F}'(\mathcal{A}) = \mathcal{F}'_+(\mathcal{A}) - \mathcal{F}_+(\mathcal{A})$ by using Sylvester formulae for the class of subsets and the set of partitions.

4.2.1 Möbius Positive Cones for Sets

Proposition 10 *We have that $g \in \mathcal{F}_+(\mathbb{S}(I))$ (respectively $g \in \mathcal{F}(\mathbb{S}(I))$) if and only if there exists a finite measure (respectively a finite signed measure) ν^g defined on the measurable space $(\mathcal{X}, \sigma(X_i : i \in I))$ that satisfies*

$$\forall J \in \mathbb{S}(I) : g(J) = \nu^g \left(\bigcap_{i \in J} X_i \right). \tag{46}$$

In this case,

$$Z^{-1}g(J) = \nu^g \left(\bigcap_{i \in J} X_i \setminus \left(\bigcup_{L: L \supseteq J, L \neq J} \bigcap_{i \in L} X_i \right) \right). \tag{47}$$

Note that $g(\emptyset) = \nu^g(\mathcal{X})$ and $Z^{-1}g(\emptyset) = \nu^g(\mathcal{X} \setminus \bigcup_{i \in I} X_i)$ because $\bigcap_{i \in \emptyset} X_i = \mathcal{X}$.

Moreover, if $g \in \mathcal{F}_+(\mathbb{S}(I))$ (respectively $g \in \mathcal{F}(\mathbb{S}(I))$) the finite (respectively signed) measure $\nu^{g^*} = \nu^g \circ \Psi^{-1}$ defined on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$ satisfies

$$\forall J \in \mathbb{S}(I) : \nu^{g^*}(\{J\}) = Z^{-1}g(J) \text{ and } g(J) = \sum_{K:K \supseteq J} \nu^{g^*}(\{K\}).$$

The function $g \rightarrow \nu^{g^*}$ defined from $\mathcal{F}(\mathbb{S}(I))$ into the space of finite signed measures on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$, is linear and sends $\mathcal{F}_+(\mathbb{S}(I))$ into the space of finite measures on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$.

Proof Assume there exists a finite measure ν^g defined on $(\mathcal{X}, \sigma(X_i : i \in I))$ such that g satisfies (46). If ν^g is a measure, the expression on the right hand side of (47) is nonnegative because it is the measure of some event. We use (30) to state the equality in (47). Then, $Z^{-1}g(J) \geq 0$ for all $J \in \mathbb{S}(I)$, so $g \in \mathcal{F}_+(\mathbb{S}(I))$. If ν^g is a signed measure we find $g \in \mathcal{F}(\mathbb{S}(I)) = \mathcal{F}_+(\mathbb{S}(I)) - \mathcal{F}_+(\mathbb{S}(I))$.

Conversely, if $g \in \mathcal{F}_+(\mathbb{S}(I))$ we have $Z^{-1}g \geq 0$, so we can define a measure ν^{g^*} on $\mathbb{S}(I)$ by the nonnegative weights $\nu^{g^*}(\{J\}) = Z^{-1}g(J)$ for $J \in \mathbb{S}(I)$. By using that Ψ is an isomorphism and the equality

$$\{J\} = \bigcap_{i \in J} X_i^* \setminus \left(\bigcup_{L:L \supseteq J, L \neq J} \bigcap_{i \in L} X_i^* \right),$$

we conclude that the measure $\nu^g = \nu^{g^*} \circ \Psi$ satisfies (47). Also, from the shape of Z we get that $g(J) = Z(Z^{-1}g)(J) = \sum_{K:K \supseteq J} \nu^{g^*}(\{K\})$ for all $J \in \mathbb{S}(I)$. Then g satisfies (46). The linearity property $g \rightarrow \nu^{g^*}$ is a consequence of the linearity of Z^{-1} and the final statement on positivity of this application follows straightforwardly. □

Proposition 11 We have $g \in \mathcal{F}'_+(\mathbb{S}(I))$ (respectively $g \in \mathcal{F}'(\mathbb{S}(I))$) if and only if there exists a finite measure (respectively a finite signed measure) ν_g defined on a measurable space $(\mathcal{X}, \sigma(X_i : i \in I))$ that satisfies

$$\forall J \in \mathbb{S}(I) : g(J) = \nu_g\left(\bigcap_{i \in J^c} X_i\right).$$

In this case,

$$Z^{-1}g(J) = \nu_g\left(\bigcap_{i \in J^c} X_i \setminus \left(\bigcup_{L:L \supseteq J^c, L \neq J^c} \bigcap_{i \in L} X_i \right)\right).$$

Note that $g(I) = \nu_g(\mathcal{X})$ and $Z^{-1}g(I) = \nu_g(\mathcal{X} \setminus \bigcup_{i \in I} X_i)$. For each $g \in \mathcal{F}'_+(\mathbb{S}(I))$ (respectively $g \in \mathcal{F}'(\mathbb{S}(I))$) the finite (respectively signed) measure $\nu_g^* = \nu_g \circ \Psi^{-1}$ on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$ satisfies

$$\forall J \in \mathbb{S}(I) : \nu_g^*(J) = Z^{-1'}g(J^c) \text{ and } g(J) = \sum_{K:K \supseteq J^c} \nu_g^*(\{K\}).$$

The function $g \rightarrow v_g^*$ defined from $\mathcal{F}'(\mathbb{S}(I))$ into the space of finite signed measures on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$, is linear and sends $\mathcal{F}'_+(\mathbb{S}(I))$ into the space of finite measures on $(\mathbb{S}(I), \mathbb{S}(\mathbb{S}(I)))$.

Proof Define $\widehat{g}(J) = g(J^c)$, $J \in \mathbb{S}(I)$. We have

$$Z^{-1'}g(J) = \sum_{K^c:K^c \subseteq J} (-1)^{|J|-|K^c|}g(K^c) = \sum_{K:K \supseteq J^c} (-1)^{|K|-|J^c|}\widehat{g}(K) = Z^{-1}\widehat{g}(J^c).$$

Hence, the result is a straightforward consequence of Proposition 10 applied to \widehat{g} . \square

4.2.2 Möbius Positive Cones for Partitions

Consider the Möbius positive cones $\mathcal{F}_+(\mathcal{P}(I))$, $\mathcal{F}'_+(\mathcal{P}(I))$ and the spaces $\mathcal{F}(\mathcal{P}(I)) = \mathcal{F}_+(\mathcal{P}(I)) - \mathcal{F}_+(\mathcal{P}(I))$, $\mathcal{F}'(\mathcal{P}(I)) = \mathcal{F}'_+(\mathcal{P}(I)) - \mathcal{F}'_+(\mathcal{P}(I))$. We shall describe them as we did in Propositions 10 and 11. But we will only write the statement for the cones $\mathcal{F}_+(\mathcal{P}(I))$ and $\mathcal{F}(\mathcal{P}(I))$. A similar statement can be written for $\mathcal{F}'_+(\mathcal{P}(I))$ and $\mathcal{F}'(\mathcal{P}(I))$, analogously as we did in Proposition 11.

Proposition 12 $g \in \mathcal{F}_+(\mathcal{P}(I))$ (respectively $g \in \mathcal{F}(\mathcal{P}(I))$) if and only if there exists a finite (respectively signed) measure v^g defined on a measurable space $(\mathcal{X}, \sigma^{\mathcal{P}}(X_J : J \in \mathbb{S}(I)))$ that satisfies

$$\forall \alpha \in \mathcal{P}(I) : g(\alpha) = v^g\left(\bigcap_{J:J \vdash \alpha} X_J\right). \tag{48}$$

In this case,

$$Z^{-1}g(\alpha) = v^g\left(\bigcap_{J:J \vdash \alpha} X_J \setminus \left(\bigcup_{\gamma:\gamma \geq \alpha, \gamma \neq \alpha} \bigcap_{J:J \vdash \gamma} X_J\right)\right). \tag{49}$$

For each $g \in \mathcal{F}_+(\mathcal{P}(I))$ the finite (respectively signed) measure $v^{*g} = v^g \circ \psi^{-1}$ defined on $(\mathcal{P}(I), \mathbb{S}(\mathcal{P}(I)))$ satisfies

$$\forall \alpha \in \mathcal{P}(I) : v^{*g}(\{\alpha\}) = Z^{-1}g(\alpha) \text{ and } g(\alpha) = \sum_{\beta:\beta \geq \alpha} v^{*g}(\{\beta\}).$$

The function $g \rightarrow v^{*g}$ defined from $\mathcal{F}(\mathcal{P}(I))$ into the space of finite signed measures of $(\mathcal{P}(I), \mathbb{S}(\mathcal{P}(I)))$, is linear and sends $\mathcal{F}_+(\mathcal{P}(I))$ into the space of finite measures on $(\mathcal{P}(I), \mathbb{S}(\mathcal{P}(I)))$.

Proof Assume there exists a finite (respectively signed) measure v^g defined on $(\mathcal{X}, \sigma^{\mathcal{P}}(X_J : J \in \mathbb{S}(I)))$ such that g satisfies (48). Relation (49) is a consequence of formula (44). Since this formula is equivalent to (45), when v^g is a measure we have $Z^{-1}g(\alpha) \geq 0$ for all $\alpha \in \mathcal{P}(I)$ and so $g \in \mathcal{F}_+(\mathcal{P}(I))$. When v^g is a signed measure we find $g \in \mathcal{F}(\mathcal{P}(I))$.

Now, let $g \in \mathcal{F}_+(\mathcal{P}(I))$, so $Z^{-1}g \geq 0$. We take the construction of Proposition 9. We define the measure ν^{g^*} on $\mathcal{P}(I)$ by the nonnegative weights $\nu^{g^*}(\{\alpha\}) = Z^{-1}g(\alpha)$ for $\alpha \in \mathcal{P}(I)$. By using $\bigcap_{J:J \vdash \alpha} X_J^* = \{\beta : \beta \geq \alpha\}$ and

$$\{\alpha\} = \bigcap_{J:J \vdash \alpha} X_J^* \setminus \left(\bigcup_{\gamma:\gamma \geq \alpha, \gamma \neq \alpha} \bigcap_{J:J \vdash \gamma} X_J^* \right),$$

we get that $\nu^g = \nu^{g^*} \circ \Psi$ satisfies (49) (where Ψ is defined in (40)). From the shape of Z we find

$$\forall \alpha \in \mathcal{P}(I) : g(\alpha) = Z(Z^{-1}g)(\alpha) = \sum_{\beta:\beta \geq \alpha} \nu^{g^*}(\{\beta\}).$$

Then g satisfies (48). The linearity property $g \rightarrow \nu^{g^*}$ is a consequence of the linearity of Z^{-1} and the final statement on positivity of this application follows straightforwardly. □

5 Coarse-Graining

5.1 Conditions for Coarse-Graining

As assumed \mathcal{A} is a finite set. In this paragraph, we do not require that it is partially ordered. Let \sim be an equivalence relation on \mathcal{A} and denote by $\tilde{\mathcal{A}}$ the set of equivalence classes and by $\tilde{a} = \{b \in \mathcal{A} : b \sim a\} \in \tilde{\mathcal{A}}$ the equivalence class containing a . As always the equivalence classes are used, either as elements of $\tilde{\mathcal{A}}$ or as subsets of \mathcal{A} . At each occasion, it will be clear from the context in which of the two meanings we will be using them.

A function $f : \mathcal{A} \rightarrow \mathbb{R}$ is compatible with \sim if $a \sim b$ implies $f(a) = f(b)$. In this case $\tilde{f} : \tilde{\mathcal{A}} \rightarrow \mathbb{R}, \tilde{a} \rightarrow \tilde{f}(\tilde{a}) = f(a)$ is a well-defined function.

A matrix $H = (H(a, b) : a, b \in \mathcal{A})$ is said to be compatible with \sim if for any function $f : \mathcal{A} \rightarrow \mathbb{R}$ compatible with \sim the function Hf is also compatible with \sim , that is $a_1 \sim a_2$ implies $Hf(a_1) = Hf(a_2)$. Since the set of compatible functions is a linear space generated by the characteristic functions of the sets, we get that H is compatible with \sim if and only if it verifies the following condition,

$$\forall a_1 \sim a_2, \forall \tilde{b} \in \tilde{\mathcal{A}} : H\mathbf{1}_{\tilde{b}}(a_1) = H\mathbf{1}_{\tilde{b}}(a_2),$$

being $\mathbf{1}_{\tilde{b}}$ the characteristic function of the set $\tilde{b} \subseteq \mathcal{A}$. Thus, H is compatible with \sim if it satisfies the conditions known as those of coarse-graining,

$$\forall a_1 \sim a_2, \forall \tilde{b} \in \tilde{\mathcal{A}} : \sum_{c \in \tilde{b}} H(a_1, c) = \sum_{c \in \tilde{b}} H(a_2, c). \tag{50}$$

Note that $\mathbf{1} = \sum_{\tilde{b} \in \tilde{\mathcal{A}}} \mathbf{1}_{\tilde{b}}$. So, if H is compatible with \sim we must necessarily have $\sum_{c \in \mathcal{A}} H(a_1, c) = \sum_{c \in \mathcal{A}} H(a_2, c)$ when $a_1 \sim a_2$. Hence, we have proven:

Lemma 13 Assume H is compatible with \sim . Then, the coarse-graining matrix $\tilde{H} = (\tilde{H}(\tilde{a}, \tilde{b}) : \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}})$ given by

$$\forall \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}} : \tilde{H}(\tilde{a}, \tilde{b}) = \sum_{c \in \tilde{b}} H(a, c),$$

is well defined and for every $f : \mathcal{A} \rightarrow \mathbb{R}$ compatible with \sim it holds

$$\forall \tilde{a} \in \tilde{\mathcal{A}} : \tilde{H} \tilde{f}(\tilde{a}) = Hf(a).$$

Note that if H_1 and H_2 are two matrices indexed by $\mathcal{A} \times \mathcal{A}$ compatible with \sim then $H_1 + H_2$ and $H_1 H_2$ are compatible with \sim . For the sum this is a consequence of property (50). For the product of matrices this property is also straightforward: let $f : \mathcal{A} \rightarrow \mathbb{R}$ be a function compatible with \sim , then $H_2 f$ is function compatible with \sim and so $H_1 H_2 f$ is also compatible with \sim , proving that $H_1 H_2$ is compatible with \sim .

Now, we claim that if H is nonsingular and H and H^{-1} are both compatible with \sim , then \tilde{H} is nonsingular and its inverse \tilde{H}^{-1} satisfies $\tilde{H}^{-1} = \widetilde{H^{-1}}$, that is

$$\forall \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}} : \tilde{H}^{-1}(\tilde{a}, \tilde{b}) = \widetilde{H^{-1}}(\tilde{a}, \tilde{b}) = \sum_{c \in \tilde{b}} H^{-1}(a, c).$$

In fact since H and H^{-1} are compatible with \sim we get that for all $f : \mathcal{A} \rightarrow \mathbb{R}$ compatible with \sim ,

$$\forall \tilde{a} \in \tilde{\mathcal{A}} : \widetilde{H^{-1}}(\tilde{H} \tilde{f})(\tilde{a}) = \widetilde{H^{-1} H f}(\tilde{a}) = H^{-1}(Hf)(a) = \tilde{a}.$$

Note that for all equivalence relation \sim the unit vector $\mathbf{1}$ is compatible with \sim . In the following result we exploit this fact. We denote by $\tilde{\mathbf{1}}$ the unit vector with the dimension of $\tilde{\mathcal{A}}$.

Lemma 14 If P is compatible with \sim then the coarse-graining matrix \tilde{P} preserves positivity, stochasticity and substochasticity, that is

$$P \geq 0 \Rightarrow \tilde{P} \geq 0; \quad P \mathbf{1} = \mathbf{1} \Rightarrow \tilde{P} \tilde{\mathbf{1}} = \tilde{\mathbf{1}}; \quad P \mathbf{1} \leq \mathbf{1} \Rightarrow \tilde{P} \tilde{\mathbf{1}} \leq \tilde{\mathbf{1}}. \tag{51}$$

Proof The positivity is straightforward from the definition of \tilde{P} . On the other hand since $\mathbf{1}$ is compatible with \sim , from $\tilde{P} \tilde{\mathbf{1}}(\tilde{a}) = P \mathbf{1}(a)$ we get the last two relations in (51). □

Theorem 15 Assume the duality relation $Q' = H^{-1} P H$ is satisfied. Let \sim be an equivalence relation on \mathcal{A} such that the matrices H, H^{-1} and P are compatible with \sim . Then, $\tilde{Q} = (\tilde{Q}(\tilde{a}, \tilde{b}) : \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}})$ given by

$$\tilde{Q}(\tilde{a}, \tilde{b}) = \sum_{c \in \tilde{a}} Q(c, b), \tag{52}$$

is a well-defined matrix. It satisfies $Q \geq 0 \Rightarrow \tilde{Q} \geq 0$ and the following duality relation holds,

$$\tilde{Q}' = \tilde{H}^{-1} \tilde{P} \tilde{H}. \tag{53}$$

For every strictly positive vector $\tilde{h} : \tilde{\mathcal{A}} \rightarrow \mathbb{R}_+$ the following duality relation holds

$$\tilde{Q}'_{\tilde{h}^{-1}, \tilde{h}} = \tilde{H}_{\tilde{h}}^{-1} \tilde{P} \tilde{H}_{\tilde{h}} \text{ where } \tilde{H}_{\tilde{h}} = \tilde{H} D_{\tilde{h}}^{-1} \text{ and } \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = D_{\tilde{h}}^{-1} \tilde{Q} D_{\tilde{h}}, \tag{54}$$

and positivity is preserved: $Q \geq 0 \Rightarrow \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \geq 0$.

For $\tilde{h} : \tilde{\mathcal{A}} \rightarrow \mathbb{R}_+$ defined by

$$\tilde{h}(\tilde{a}) = |\tilde{a}| = |\{c \in \mathcal{A} : c \sim a\}|, \tag{55}$$

the duality (54) preserves stochasticity and substochasticity of Q ,

$$(Q\mathbf{1} = \mathbf{1} \Rightarrow \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \tilde{\mathbf{1}} = \tilde{\mathbf{1}}) \text{ and } (Q\mathbf{1} \leq \mathbf{1} \Rightarrow \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \tilde{\mathbf{1}} \leq \tilde{\mathbf{1}})$$

Hence, if the kernels P and Q are stochastic (respectively substochastic) then the kernels \tilde{P} and $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}}$ are stochastic (respectively substochastic).

Proof From the hypotheses we get that $H^{-1}PH$ is compatible with \sim . Hence $Q' = H^{-1}PH$ is compatible with \sim , and so Q satisfies

$$\forall b_1 \sim b_2, \forall \tilde{a} \in \tilde{\mathcal{A}} : \sum_{c \in \tilde{a}} Q(c, b_1) = \sum_{c \in \tilde{a}} Q(c, b_2).$$

Hence \tilde{Q} given by (52) is well defined on $\tilde{\mathcal{A}}$. Let us show (53). We must prove $\tilde{Q}'(\tilde{a}, \tilde{b}) = (\tilde{H}^{-1} \tilde{P} \tilde{H})(\tilde{a}, \tilde{b})$ for all $\tilde{a}, \tilde{b} \in \tilde{\mathcal{A}}$. This relation is implied by the equality $\sum_{c \in \tilde{b}} Q(c, a) = \sum_{c \in \tilde{b}} (H^{-1}PH)(a, c)$ for all $a \in \mathcal{A}, \tilde{b} \in \tilde{\mathcal{A}}$, and this last relation is fulfilled because the duality relation (10) is $Q(c, a) = (H^{-1}PH)(a, c)$ for all $a, c \in \mathcal{A}$.

From Lemma 14 it follows that coarse-graining preserves positivity, stochasticity and substochasticity of P . On the other hand, by definition, we have that $Q \geq 0$ implies $\tilde{Q} \geq 0$.

Let $\tilde{h} : \tilde{\mathcal{A}} \rightarrow \mathbb{R}$ be a non-vanishing vector. From (11) we have that duality relation (53) implies duality relation (54) for any non-vanishing vector \tilde{h} . So, for \tilde{h} strictly positive we get the implication $Q \geq 0 \Rightarrow \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = D_{\tilde{h}}^{-1} \tilde{Q} D_{\tilde{h}} \geq 0$.

Now we define $h : \mathcal{A} \rightarrow \mathbb{R}$ by $h(a) = \tilde{h}(\tilde{a})$. The duality relation $Q' = H^{-1}PH$ implies (11) which is

$$Q'_{h^{-1}, h} = H_h^{-1} P H_h \text{ where } H_h = H D_h^{-1} \text{ and } Q_{h^{-1}, h} = D_h^{-1} Q D_h.$$

On the other hand the diagonal matrices D_h and D_h^{-1} preserve \sim and their coarse-graining matrices are $\tilde{D}_h = D_{\tilde{h}}$ and $\tilde{D}_h^{-1} = D_{\tilde{h}}^{-1}$. Then

$$\begin{aligned} (\tilde{H}_h^{-1} \tilde{P} \tilde{H}_h)(\tilde{a}, \tilde{b}) &= \sum_{c \in \tilde{b}} (H_h^{-1} P H_h)(a, c) = \sum_{c \in \tilde{b}} Q'_{h^{-1}, h}(a, c) \\ &= \sum_{c \in \tilde{b}} D_h Q' D_h^{-1}(a, c). \end{aligned}$$

By the same argument and by definition of $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}}$ and \tilde{Q} we get

$$\tilde{Q}'_{\tilde{h}^{-1}, \tilde{h}}(\tilde{a}, \tilde{b}) = D_{\tilde{h}} \tilde{Q}' D_{\tilde{h}}^{-1}(\tilde{a}, \tilde{b}) = \sum_{c \in \tilde{b}} D_h Q' D_h^{-1}(a, c).$$

So, duality relation (54) is satisfied: $\tilde{Q}'_{\tilde{h}^{-1}, \tilde{h}} = \tilde{H}_h^{-1} \tilde{P} \tilde{H}_h$. Hence, (12) implies that $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \tilde{\mathbf{1}} = \tilde{\mathbf{1}}$ is satisfied if and only if $\tilde{Q} \tilde{h} = \tilde{h}$, so if and only if \tilde{h} is a right eigenvector of \tilde{Q} with eigenvalue 1. Let us check that \tilde{h} defined in (55) is such an eigenvector.

We have

$$\tilde{Q} \tilde{h}(\tilde{a}) = \sum_{\tilde{b} \in \tilde{\mathcal{A}}} \tilde{Q}(\tilde{a}, \tilde{b}) \tilde{h}(\tilde{b}) = \sum_{\tilde{b} \in \tilde{\mathcal{A}}} \sum_{c \in \tilde{a}} Q(c, b) \tilde{h}(\tilde{b}) = \sum_{\tilde{b} \in \tilde{\mathcal{A}}} |\tilde{b}| \left(\sum_{c \in \tilde{a}} Q(c, b) \right).$$

Since $\sum_{c \in \tilde{a}} Q(c, d)$ does not depend on $d \in \tilde{b}$ we get $|\tilde{b}| \left(\sum_{c \in \tilde{a}} Q(c, b) \right) = \sum_{d \in \tilde{b}} \sum_{c \in \tilde{a}} Q(c, d)$ and so

$$\tilde{Q} \tilde{h}(\tilde{a}) = \sum_{\tilde{b} \in \tilde{\mathcal{A}}} \sum_{d \in \tilde{b}} \sum_{c \in \tilde{a}} Q(c, d) = \sum_{c \in \tilde{a}} \left(\sum_{d \in \mathcal{A}} Q(c, d) \right). \tag{56}$$

So, if Q is stochastic we obtain $\sum_{d \in I} Q(c, d) = 1$ for all $c \in I$ and we deduce

$$\tilde{Q} \tilde{h}(\tilde{a}) = \sum_{c \in \tilde{a}} 1 = |\tilde{a}| = \tilde{h}(\tilde{a}).$$

We have shown that stochasticity is preserved: $Q \mathbf{1} = \mathbf{1} \Rightarrow \tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \tilde{\mathbf{1}} = \tilde{\mathbf{1}}$.

The proof that substochasticity is also preserved is entirely similar. In fact the above arguments show the equivalence $(\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} \tilde{\mathbf{1}} \leq \tilde{\mathbf{1}}) \Leftrightarrow (\tilde{Q} \tilde{h} \leq \tilde{h})$. Now, Q substochastic means $\sum_{d \in \mathcal{A}} Q(c, d) \leq 1$ for all $c \in \mathcal{A}$. We replace it in (56) to obtain $\tilde{Q} \tilde{h}(\tilde{a}) \leq \tilde{h}(\tilde{a})$ for \tilde{h} given by (55). Therefore, the result is shown. \square

As it is clear from the above computations, in general $Q \mathbf{1} = \mathbf{1}$ (respectively $Q \mathbf{1} \leq \mathbf{1}$) does not imply $\tilde{Q} \tilde{\mathbf{1}} = \tilde{\mathbf{1}}$ (respectively $\tilde{Q} \tilde{\mathbf{1}} \leq \tilde{\mathbf{1}}$). But it does when the function $\tilde{h}(\tilde{a}) = |\tilde{a}|$ is constant, because in this case $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = \tilde{Q}$.

A precision is required on transpose matrices and coarse-graining. When the transpose matrix H' is compatible with \sim , the matrix \tilde{H}' denotes its coarse-graining matrix. So,

$$\widetilde{H}'(\widetilde{a}, \widetilde{b}) = \sum_{c \in \widetilde{b}} H'(a, c) = \sum_{c \in \widetilde{b}} H(c, a).$$

If H is also compatible with \sim then \widetilde{H}' is the transpose of the coarse-graining matrix \widetilde{H} . In general the matrices \widetilde{H}' and \widetilde{H} are not equal. In fact

$$\widetilde{H}'(\widetilde{a}, \widetilde{b}) = \widetilde{H}(\widetilde{b}, \widetilde{a}) = \sum_{c \in \widetilde{a}} H(b, c).$$

Therefore, we must take care in the notations. Thus, when \widetilde{H}' is nonsingular the matrix \widetilde{H}'^{-1} is its inverse and if \widetilde{H} is nonsingular then $\widetilde{H}'^{-1} = \widetilde{H}^{-1'}$ is the inverse of the matrix \widetilde{H}' . In general the matrices \widetilde{H}'^{-1} and $\widetilde{H}^{-1'}$ are not equal. But as noted, when these inverses exist we have the equalities

$$\widetilde{H}'^{-1} = \widetilde{H}^{-1'} \quad \text{and} \quad \widetilde{H}^{-1} = \widetilde{H}'^{-1'}$$

5.2 Coarse-Graining Product Formula

Let $(\mathcal{A}_r, \preceq_r)$ be a partially ordered space with Möbius functions μ_r , for $r = 1, 2$. Recall that the Möbius function for the product space $(\mathcal{A}_1 \times \mathcal{A}_2, \preceq_{1,2})$ is given by $\mu((a_1, a_2), (b_1, b_2)) = \mu_1(a_1, b_1)\mu_2(a_2, b_2)$ when $a_1 \preceq_1 b_1, a_2 \preceq_2 b_2$ [see (2)]. Let Z_r be the zeta matrix associated to $(\mathcal{A}_r, \preceq_r)$ for $r = 1, 2$ and $Z_{1,2}$ be the zeta matrix associated to the product space $(\mathcal{A}_1 \times \mathcal{A}_2, \preceq_{1,2})$.

Let \sim_1 and \sim_2 be two equivalence relations on \mathcal{A}_1 and \mathcal{A}_2 respectively. Then the product relation $\sim_{1,2}$ defined on $\mathcal{A}_1 \times \mathcal{A}_2$ by $(a_1, a_2) \sim_{1,2} (b_1, b_2)$ if $a_1 \sim_1 b_1$ and $a_2 \sim_2 b_2$, is an equivalence relation. From definition we get $(\widetilde{a_1}, \widetilde{a_2}) = \widetilde{a_1} \times \widetilde{a_2}$ for all $(a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$.

Proposition 16 *If Z_r is compatible with \sim_r for $r = 1, 2$, then $Z_{1,2}$ is compatible with the product equivalence relation $\sim_{1,2}$ and the coarse-graining matrix is given by*

$$\widetilde{Z}_{1,2}(\widetilde{(a_1, a_2)}, \widetilde{(b_1, b_2)}) = \widetilde{Z}_1(\widetilde{a_1}, \widetilde{b_1}) \cdot \widetilde{Z}_2(\widetilde{a_2}, \widetilde{b_2}). \tag{57}$$

Also, if Z_r^{-1} is compatible with \sim_r for $r = 1, 2$, then $Z_{1,2}^{-1}$ is compatible with $\sim_{1,2}$ and

$$\widetilde{Z}_{1,2}^{-1}(\widetilde{(a_1, a_2)}, \widetilde{(b_1, b_2)}) = \widetilde{Z}_1^{-1}(\widetilde{a_1}, \widetilde{b_1}) \cdot \widetilde{Z}_2^{-1}(\widetilde{a_2}, \widetilde{b_2}). \tag{58}$$

Similar statements and formulae can be stated for the transpose zeta and Möbius matrices.

Proof Assume $(a_1, a_2) \sim (a'_1, a'_2)$. From the product formula (3) we get

$$\{(c_1, c_2) \in \widetilde{b_1} \times \widetilde{b_2} : (c_1, c_2) \preceq_{1,2} (a_1, a_2)\} = \{c_1 \in \widetilde{b_1} : c_1 \preceq_1 b_1\} \times \{c_2 \in \widetilde{b_2} : c_2 \preceq_2 b_2\}.$$

Then,

$$Z_{1,2} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a_1, a_2) = Z_1 \mathbf{1}_{\tilde{b}_1}(a_1) Z_2 \mathbf{1}_{\tilde{b}_2}(a_2). \tag{59}$$

[Also see (4)]. From our hypothesis we have $a'_r \sim_r a_r$ implies $Z_r \mathbf{1}_{\tilde{b}_r}(a_r) = Z_r \mathbf{1}_{\tilde{b}_r}(a'_r)$ for $r = 1, 2$. Therefore,

$$(a'_1, a'_2) \sim_{1,2} (a_1, a_2) \Rightarrow Z_{1,2} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a_1, a_2) = Z_{1,2} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a'_1, a'_2).$$

We have proven that the zeta matrix $Z_{1,2}$ is compatible with $\sim_{1,2}$. Relation (59) gives (57).

Now assume the Möbius matrices Z_r^{-1} is compatible with \sim_r so $Z_r^{-1} \mathbf{1}_{\tilde{b}_r}(a_r) = Z_r^{-1} \mathbf{1}_{\tilde{b}_r}(a'_r)$ in the above setting, for $r = 1, 2$. Also from the product formulae (3) and (4) we obtain

$$Z_{1,2}^{-1} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a_1, a_2) = Z_1^{-1} \mathbf{1}_{\tilde{b}_1}(a_1) Z_2^{-1} \mathbf{1}_{\tilde{b}_2}(a_2). \tag{60}$$

Then,

$$(a'_1, a'_2) \sim_{1,2} (a_1, a_2) \Rightarrow Z_{1,2}^{-1} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a_1, a_2) = Z_{1,2}^{-1} \mathbf{1}_{\tilde{b}_1 \times \tilde{b}_2}(a'_1, a'_2),$$

proving that the Möbius matrix $Z_{1,2}^{-1}$ is compatible with $\sim_{1,2}$. Equality (60) gives (58). □

5.3 Coarse-Graining on Zeta and Möbius Matrices on Sets and Partitions

Let (\mathcal{A}, \leq) be a partially ordered space with Möbius function μ . Let \sim be an equivalence relation on \mathcal{A} . By definition, the zeta matrix Z is compatible with \sim if and only if we have

$$a_1 \sim a_2 \Rightarrow (\forall \tilde{b} \in \tilde{\mathcal{A}} : |\{c \in \tilde{b} : a_1 \leq c\}| = |\{c \in \tilde{b} : a_2 \leq c\}|). \tag{61}$$

Similarly, Z' is compatible with \sim if and only if

$$a_1 \sim a_2 \Rightarrow (\forall \tilde{b} \in \tilde{\mathcal{A}} : |\{c \in \tilde{b} : c \leq a_1\}| = |\{c \in \tilde{b} : c \leq a_2\}|), \tag{62}$$

When the previous conditions hold we get

$$\tilde{Z}(\tilde{a}, \tilde{b}) = |\{c \in \tilde{b} : a \leq c\}|, \quad \tilde{Z}'(\tilde{a}, \tilde{b}) = |\{c \in \tilde{b} : c \leq a\}|. \tag{63}$$

Hence, a sufficient condition for having zeta and Möbius compatibility with \sim is the following one.

Proposition 17 *Assume for all couples $a_1, a_2 \in \mathcal{A}$ with $a_1 \sim a_2$ there exists a bijection $\pi : \mathcal{A} \rightarrow \mathcal{A}$ such that:*

$$\pi(a_1) = a_2; \tag{64}$$

$$c \leq d \Leftrightarrow \pi(c) \leq \pi(d) \text{ (that is } \pi \text{ is an automorphism of } (\mathcal{A}, \leq)); \tag{65}$$

$$\forall \tilde{b} \in \tilde{\mathcal{A}} : \pi(\tilde{b}) = \tilde{b} \text{ and } \pi : \tilde{b} \rightarrow \tilde{b} \text{ is a bijection.} \tag{66}$$

Then, Z, Z', Z^{-1} and Z'^{-1} are compatible with \sim .

Proof The conditions imply

$$\forall \tilde{b} \in \tilde{\mathcal{A}} : \pi(\{c \in \tilde{b} : a_1 \leq c\}) = \{c \in \tilde{b} : a_2 \leq c\}, \quad \pi(\{c \in \tilde{b} : c \leq a_1\}) = \{c \in \tilde{b} : c \leq a_2\}.$$

Then, (61) and (62) are satisfied, so Z and Z' are compatible with \sim . Since property (65) ensures that π is an isomorphism of (\mathcal{A}, \leq) into itself, then the Möbius function satisfies $\mu(c, d) = \mu(\pi(c), \pi(d))$ for all $c, d \in \mathcal{A}$. Hence,

$$\begin{aligned} \sum_{c \in \tilde{b}} Z^{-1}(a_1, c) &= \sum_{c \in \tilde{b}, a_1 \leq c} \mu(a_1, c) = \sum_{\pi(c) \in \tilde{b}, \pi(a_1) \leq \pi(c)} \mu(\pi(a_1), \pi(c)) \\ &= \sum_{c \in \tilde{b}, a_2 \leq c} \mu(a_2, c). \end{aligned}$$

Similarly for Z'^{-1} . Then, the result is shown. □

Remark 3 Assume that the following property holds for all $a' \sim a$ and $b' \sim b$:

$$\mu(a, b) = \mu(a', b') \text{ and } (a \leq b \Rightarrow a' \leq b').$$

Then, $\tilde{a} \lesssim \tilde{b} \Leftrightarrow a \leq b$ is a well-defined order relation in $\tilde{\mathcal{A}}$. Moreover, $\mu(\tilde{a}, \tilde{b}) = |\tilde{b}| \mu(a, b)$ is the Möbius function for $(\tilde{\mathcal{A}}, \lesssim)$. We have $\tilde{Z}(\tilde{a}, \tilde{b}) = \mathbf{1}_{\tilde{a} \lesssim \tilde{b}}$ and $\tilde{Z}^{-1}(\tilde{a}, \tilde{b}) = \mathbf{1}_{\tilde{a} \lesssim \tilde{b}} \mu(\tilde{a}, \tilde{b})$. When the above properties are satisfied, they also hold for the product equivalence relation and the product order.

In the sequel, I is a finite set and $N = |I|$ denotes its cardinality, so whenever needed we can assume $I = \mathcal{I}_N$.

5.3.1 Coarse-Graining on zeta and Möbius Matrices on Sets and Product of Sets

On $\mathcal{A} = \mathbb{S}(I)$ consider the equivalence relation \sim given by $J \sim K$ if $|J| = |K|$. In this case the set of equivalence classes admits the following identification $\widehat{\mathbb{S}}(I) = \mathcal{I}_N^0$ where $\mathcal{I}_N^0 = \{0, \dots, N\}$.

Proposition 18 *The matrices Z, Z^{-1}, Z' and Z'^{-1} are all compatible with \sim . For $j, k \in \mathcal{I}_N^0$ the (j, k) -entry of the coarse-graining matrices are:*

$$\begin{aligned} \tilde{Z}(j, k) &= \binom{N-j}{k-j} \mathbf{1}_{j \leq k}; & \widetilde{Z^{-1}}(j, k) &= \binom{N-j}{k-j} (-1)^{k-j} \mathbf{1}_{j \leq k}; \\ \tilde{Z}'(j, k) &= \binom{j}{k} \mathbf{1}_{k \leq j}; & \widetilde{Z'^{-1}}(j, k) &= \binom{j}{k} (-1)^{j-k} \mathbf{1}_{k \leq j}. \end{aligned} \tag{67}$$

Proof Let us check that the hypotheses of Proposition 17 are satisfied. Let $J, K \in \mathbb{S}(I)$ be such that $|J| = |K|$. Let $\widehat{\pi} : I \rightarrow I$ be a bijection satisfying $\widehat{\pi}(J) = K$. Since $\mu(L, M) = (-1)^{|M|-|L|}$ when $L \subseteq M$, it is easy to see that $\pi : \mathbb{S}(I) \rightarrow \mathbb{S}(I)$ defined by $\pi(L) = M$ (as elements) if and only if $\widehat{\pi}(L) = M$ (as sets), is a bijection satisfying

the hypotheses of Proposition 17. Then Z, Z^{-1}, Z' and Z'^{-1} are compatible with \sim .

Let $j, k \in \mathcal{I}_N^0$ and $J \in \mathbb{S}(I)$ with $j = |J|$. When $k \geq j$ we have $|\{L \in \mathbb{S}(I) : J \subseteq L, |L| = k\}| = \binom{N-j}{k-j}$. Also $\mu(J, L) = (-1)^{k-j}$ for any $L \supseteq J$ with $|L| = k$. This gives the first two equalities in (67). On the other hand if $k \leq j$ then $|\{L \in \mathbb{S}(I) : L \subseteq J, |L| = k\}| = \binom{j}{k}$ and $\mu(L, J) = (-1)^{j-k}$ for any $L \subseteq J$ with $|L| = k$. This gives the last two equalities in (67). This finishes the proof. \square

Let us consider the product space $\mathbb{S}(I)^T$ endowed with the product order noted by \subseteq . By the isomorphism (8), all the relations and formulae obtained for the class of sets continue to hold for the class of product of sets. Nevertheless, let us give explicitly the coarse-graining relations. On the class of product of sets, we consider the equivalence relation $\vec{J} \sim \vec{K}$ if $|J_t| = |K_t|$ for all $t \in \mathcal{I}_T$. Recall that the Möbius function of $(\mathbb{S}(I)^T, \subseteq)$ is $\mu(\vec{J}, \vec{K}) = (-1)^{\sum_{t \in \mathcal{I}_T} (|K_t| - |J_t|)} \mathbf{1}_{\vec{J} \subseteq \vec{K}}$. From Propositions 18 and 16, we get that the zeta matrix Z and the Möbius matrix satisfy the coarse-graining relations with respect to \sim .

The set of equivalence classes $\widetilde{\mathbb{S}(I)^T}$ is naturally identified with $(I_N^0)^T$ which is endowed with the product partial order \leq . The elements of $(I_N^0)^T$ are written $\vec{j} = (j_t : t \in \mathcal{I}_T)$ and so, $\vec{j} \leq \vec{k}$ when $j_t \leq k_t \forall t \in \mathcal{I}_T$. If $\vec{j} \leq \vec{k}$ we denote

$$\binom{\vec{k}}{\vec{j}} = \prod_{t \in \mathcal{I}_T} \binom{k_t}{j_t}.$$

With this notation the coarse-graining matrices are

$$\begin{aligned} \tilde{Z}(\vec{j}, \vec{k}) &= \left(\prod_{t \in \mathcal{I}_T} \binom{N - j_t}{k_t - j_t} \right) \mathbf{1}_{\vec{j} \leq \vec{k}}; \\ \widetilde{Z^{-1}}(\vec{j}, \vec{k}) &= \left(\prod_{t \in \mathcal{I}_T} \binom{N - j_t}{k_t - j_t} \right) (-1)^{\sum_{t \in \mathcal{I}_T} (k_t - j_t)} \mathbf{1}_{\vec{j} \leq \vec{k}}; \\ \tilde{Z}'(\vec{j}, \vec{k}) &= \binom{\vec{j}}{\vec{k}} \mathbf{1}_{\vec{k} \leq \vec{j}}; \\ \widetilde{Z'^{-1}}(\vec{j}, \vec{k}) &= \binom{\vec{j}}{\vec{k}} (-1)^{\sum_{t \in \mathcal{I}_T} (j_t - k_t)} \mathbf{1}_{\vec{k} \leq \vec{j}}. \end{aligned}$$

We note that for the classes of sets and product of sets the conditions in Remark 3 are satisfied.

5.3.2 Coarse-Graining on Zeta and Möbius Matrices on Partitions

Recall we can assume $I = \mathcal{I}_N$. Let us define the decompositions of N in an additive way. We set

$$\mathcal{E}_N = \{\eta := \{e_s : s \in \mathcal{I}_T\} : T \geq 1, e_s \geq 1 \forall s \in \mathcal{I}_T, \sum_{s \in \mathcal{I}_T} e_s = N\}.$$

Note that every $\eta \in \mathcal{E}_N$ is a multiset with elements $e_s \in \mathcal{I}_N$ and with at most T repetitions. The specificity is that the sum of the elements of $\eta \in \mathcal{E}_N$ is N .

Let $[\eta] = T$ be the number of elements (including repetitions) of the multiset η . Let $\kappa = \{k_r : r \in \mathcal{I}_R\}$ be another element in \mathcal{E}_N , we put

$$\eta \preceq \kappa \Leftrightarrow T \geq R \text{ and } \exists \theta : \mathcal{I}_T \rightarrow \mathcal{I}_R \text{ onto such that } \sum_{s \in \mathcal{I}_T : \theta(s)=l} e_s = k_r \forall r \in \mathcal{I}_R.$$

For every partition $\alpha = \{A_t : t \in \mathcal{I}_{[\alpha]}\} \in \mathcal{P}(I)$ we denote by $\langle \alpha \rangle = \{|A_t| : t \in \mathcal{I}_{[\alpha]}\}$ the multiset of the cardinal numbers of its atoms and call it the *skeleton* of the partition. We have $\langle \alpha \rangle \in \mathcal{E}_N$ and $[\langle \alpha \rangle] = [\alpha]$.

On $\mathcal{P}(I)$ we denote by $\alpha \sim \beta$ the equivalence relation $\langle \alpha \rangle = \langle \beta \rangle$.

Let us compute the number of partitions in $\mathcal{P}(I)$ that has a certain skeleton. For $\eta = \{e_s : s \in \mathcal{I}_T\} \in \mathcal{E}_N$ define the equivalence relation $\hat{=}_\eta$ on \mathcal{I}_T by $s_1 \hat{=}_\eta s_2$ if $e_{s_1} = e_{s_2}$. Let \hat{s} be the equivalence class of s for the relation $\hat{=}_\eta$, so $|\hat{s}|$ is the number of its elements. Denote by \hat{I}^η the set of equivalent classes. Define

$$\#(\eta) := \binom{N}{\eta} \left(\prod_{\hat{s} \in \hat{I}^\eta} |\hat{s}|! \right)^{-1} \text{ with } \binom{N}{\eta} := \frac{N!}{\prod_{s \in \mathcal{I}_T} e_s!}.$$

We have that $\#(\eta) = |\{\alpha \in \mathcal{P}(I) : \langle \alpha \rangle = \eta\}|$ is the number of different elements of $\mathcal{P}(I)$ whose skeleton is η , see equality (1) in [2]. We recall that for a partition α and an atom $C \in \gamma$ of a coarser partition γ , we denoted by ℓ_C^α the number of atoms of α contained in C ,

Proposition 19 *The matrices Z, Z^{-1}, Z' and $Z^{-1'}$ are all compatible with \sim and the coarse-graining matrices $\tilde{Z} = (\tilde{Z}(\eta, \kappa) : \eta, \kappa \in \mathcal{E}_N)$ and $\tilde{Z}^{-1} = (\tilde{Z}^{-1}(\eta, \kappa) : \eta, \kappa \in \mathcal{E}_N)$ satisfy:*

$$\begin{aligned} \tilde{Z}(\eta, \kappa) &= |\{\gamma : \langle \gamma \rangle = \langle \delta \rangle, \alpha \preceq \gamma\}| \mathbf{1}_{\eta \preceq \kappa} \text{ for } \langle \alpha \rangle = \eta, \langle \delta \rangle = \kappa; \\ \tilde{Z}^{-1}(\eta, \kappa) &= \left(\sum_{\gamma : \langle \gamma \rangle = \langle \delta \rangle, \gamma \preceq \alpha} (-1)^{|\alpha|+|\gamma|} \prod_{C \in \gamma} (\ell_C^\alpha - 1)! \right) \mathbf{1}_{\eta \preceq \kappa} \text{ for } \langle \alpha \rangle = \eta, \langle \delta \rangle = \kappa. \end{aligned} \tag{68}$$

Proof Let α, β be a pair of equivalent partitions in $\mathcal{P}(I)$, so $\alpha \sim \beta$. We will construct a permutation $\pi : \mathcal{P}(I) \rightarrow \mathcal{P}(I)$ that satisfies the properties (64), (65) and (66) of Proposition 17, then the result will follow.

We denote $T := [\alpha] = [\beta]$. Let us fix an order to the atoms of α , we denote by $\alpha^o = (A_t : t \in \mathcal{I}_T)$ the ordered sequence. Since $\langle \alpha \rangle = \langle \beta \rangle$ we can fix an order $\beta^o = (B_t : t \in \mathcal{I}_T)$ of the atoms of β in such a way that $|A_t| = |B_t|$ for $m \in \mathcal{I}_T$. We fix two permutations $\varphi_\alpha : I \rightarrow I$ and $\varphi_\beta : I \rightarrow I$ that satisfy

$$\forall t \in \mathcal{I}_T : \varphi_\alpha(t) \in A_t \Leftrightarrow \varphi_\beta(t) \in B_t.$$

Note that $\varphi = \varphi_\beta \circ \varphi_\alpha^{-1}$ is also a permutation of I . We extend this permutation to the class of partitions, we define $\pi : \mathcal{P}(I) \rightarrow \mathcal{P}(I)$ by

$$\gamma = \{C_t : t \in \mathcal{I}_T\} \rightarrow \pi(\gamma) = \{D_t : t \in \mathcal{I}_T\}$$

where the partition $\pi(\gamma)$ is given by the equivalence relation

$$i \equiv_{\pi(\gamma)} j \Leftrightarrow \pi^{-1}(i) \equiv_\gamma \pi^{-1}(j).$$

Since π is defined by a pointwise permutation φ in I , it follows straightforwardly that π satisfies (65). Also note that $\pi(\alpha) = \beta$, so (64) holds. It is also clear from the definition of π that it preserves the skeletons, that is $\langle \gamma \rangle = \langle \pi(\gamma) \rangle$. Then (66) is satisfied.

Hence, from Proposition 17 we get that Z, Z^{-1}, Z' and Z^{-1}' are compatible with \sim . The expression for \tilde{Z} is the first equality in (63). On the other hand,

$$\begin{aligned} \tilde{Z}^{-1}(\langle \alpha \rangle, \langle \delta \rangle) &= \sum_{\gamma: \langle \gamma \rangle = \langle \delta \rangle} Z^{-1}(\alpha, \gamma) = \sum_{\gamma: \langle \gamma \rangle = \langle \delta \rangle, \gamma \leq \alpha} \mu(\alpha, \gamma) \\ &= \sum_{\gamma: \langle \gamma \rangle = \langle \delta \rangle, \gamma \leq \alpha} (-1)^{|\alpha|+|\gamma|} \prod_{C \in \gamma} (\ell_C^\alpha - 1)!. \end{aligned}$$

Hence, the equalities in (68) are satisfied. Similar expressions can be found for \tilde{Z}' and \tilde{Z}'^{-1} . □

6 Examples

We will revisit the Cannings haploid and multi-allelic discrete population model with constant population size. The Cannings haploid discrete population model with constant population size [3,4] was introduced as a model encompassing the models of Wright-Fisher [20], Moran [16], Kimura [10] and Karlin and McGregor [9]. The multi-allelic model was introduced and studied in Gladstien and Möhle in [7, 15]. In [14, 15] an ancestor type process was associated to the haploid and the multi-allelic models, and their duality was stated. We will provide a set version of these models and prove they are in duality via a transpose zeta matrix. The coarse-graining of the set model gives the Cannings model and the zeta transposed duality becomes an hypergeometric duality.

6.1 Haploid Cannings Model

The Canning haploid discrete population model with constant population size ([3,4]) was studied in a duality perspective in [14]. There it was introduced an ancestor type model which was proven to be in duality with the former one via an hypergeometric matrix.

Here, we construct an evolution model on the class of subsets of a fixed finite set whose coarse-graining is the Cannings haploid model. We also construct an ancestor type model on the family of sets which is in transpose zeta duality with the former one. The coarse-graining version of these kernels are the Cannings model and its ancestor type model, and the transpose zeta matrix becomes the hypergeometric matrix.

Let I be a finite set, denote by $\widehat{\mathcal{P}}(I)$ the class of indexed partitions of I defined by: $(J_i : i \in I) \in \widehat{\mathcal{P}}(I)$ if

$$\forall i \in I J_i \in \mathbb{S}(I), \quad \forall i \neq j \quad J_i \cap J_j = \emptyset, \quad \bigcup_{i \in I} J_i = I.$$

Let $(\Omega, \mathbb{B}, \mathbb{P})$ be a probability space and $\nu : \Omega \rightarrow \widehat{\mathcal{P}}(I), \omega \rightarrow \nu(\omega)$ be a random element. Consider a collection of independent equally distributed random elements $(\nu^n : n \in \mathbb{Z})$ with the law of ν . The elements indexed by nonnegative integers will serve to construct the haploid forward process, and the elements with negative indexes will be at the basis of the definition of the backward process.

Now, we select a fixed allele and consider the set of individuals having this allele. In time n , this set is called X_n (so $I \setminus X_n$ is the set of individuals having the another allele). The evolution of the process $(X_n : n \in \mathbb{N})$ with values in $\mathbb{S}(I)$ is given by

$$X_{n+1} = \bigcup_{i \in X_n} \nu_i^{n+1}.$$

The process $(X_n : n \in \mathbb{N})$ is a Markov chain with stochastic transition matrix $P = (P(J, K) : J, K \in \mathbb{S}(I))$ given by

$$P(J, K) = \mathbb{P}(X_{n+1} = K \mid X_n = J) = \mathbb{P}\left(\bigcup_{i \in J} \nu_i = K\right). \tag{69}$$

This chain is called the forward process.

Now we consider the process $(Y_n : n \in \mathbb{N})$ with values in $\mathbb{S}(I)$ and defined recursively by

$$\left(\bigcup_{i \in Y_{n+1}} \nu_i^{-(n+1)} \supseteq Y_n\right) \text{ and } \left(\forall L \subseteq Y_{n+1}, L \neq Y_{n+1} : \bigcup_{i \in L} \nu_i^{-(n+1)} \not\supseteq Y_n\right).$$

Let us see that $Y_n \in \mathbb{S}(I)$ defines uniquely $Y_{n+1} \in \mathbb{S}(I)$. The existence of Y_{n+1} follows from $\bigcup_{i \in I} \nu_i^{-(n+1)} \supseteq Y_n$. In fact, if for all proper subset L of I we have $\bigcup_{i \in L} \nu_i^{-(n+1)} \not\supseteq Y_n$, then $Y_{n+1} = I$. If there exists some proper subset L_0 such that $\bigcup_{i \in L_0} \nu_i^{-(n+1)} \supseteq Y_n$, then we apply the above argument to the proper subsets of L_0 , and we continue up to the moment when we find a subset satisfying the requirements of Y_{n+1} . The uniqueness is a consequence of the disjointedness: $L \cap L' = \emptyset$ implies $(\bigcup_{i \in L} \nu_i^{-(n+1)}) \cap (\bigcup_{i \in L'} \nu_i^{-(n+1)}) = \emptyset$. By definition, Y_{n+1} can be seen as the set of ancestors of Y_n .

We have that $(Y_n : n \in \mathbb{N})$ is a Markov chain with stochastic transition matrix $Q = (Q(J, K) : J, K \in \mathbb{S}(I))$ given by

$$Q(J, K) = \mathbb{P}(Y_{n+1} = K \mid Y_n = J) = \mathbb{P}\left(\left(\bigcup_{i \in K} v_i \supseteq J\right) \text{ and } \left(\forall L \subseteq K, L \neq K : \bigcup_{i \in L} v_i \not\supseteq J\right)\right).$$

(The fact that Q is stochastic is a consequence of the fact that $Y_n \in \mathbb{S}(I)$ determines $Y_{n+1} \in \mathbb{S}(I)$). Define $\mathcal{X}_i = \{v_i \subseteq J^c\}$. We have

$$\bigcap_{i \in K^c} \mathcal{X}_i = \left\{ \bigcup_{i \in K^c} v_i \subseteq J^c \right\}.$$

Since $\bigcup_{i \in I} v_i = I$ and the sets $(v_i : i \in I)$ are disjoint, we deduce

$$\bigcap_{i \in K^c} \mathcal{X}_i = \left\{ \bigcup_{i \in K} v_i \supseteq J \right\}.$$

Hence

$$\begin{aligned} & \bigcap_{i \in K^c} \mathcal{X}_i \setminus \left(\bigcup_{L: L \subseteq K, L \neq K} \left(\bigcap_{i \in L^c} \mathcal{X}_i \right) \right) \\ &= \left\{ \bigcup_{i \in K} v_i \supseteq J \right\} \setminus \left(\bigcup_{L: L \subseteq K, L \neq K} \left\{ \bigcup_{i \in L} v_i \supseteq J \right\} \right). \end{aligned}$$

By the Sylvester formula we get,

$$\begin{aligned} Q(J, K) &= \sum_{L \subseteq K} (-1)^{|K|-|L|} \mathbb{P}\left(\bigcup_{i \in L} v_i \supseteq J\right) \\ &= \sum_{L \subseteq K} (-1)^{|K|-|L|} \left(\sum_{M: M \supseteq J} \mathbb{P}\left(\bigcup_{i \in L} v_i = M\right) \right) \\ &= \sum_{L \subseteq K} (-1)^{|K|-|L|} \left(\sum_{M: M \supseteq J} P(L, M) \right). \end{aligned}$$

We can check that equality (19) is satisfied, then the kernel Q is the Z' -dual (transpose zeta dual) of P , that is $Q' = Z'^{-1} P Z'$ is satisfied where the Z' matrix is given by $Z'(J, K) = \mathbf{1}_{K \subseteq J}$.

Now assume the law of ν is invariant under permutation of I , this means for all permutation $\pi = (\pi_i : i \in I)$ of I we have

$$\forall (J_i : i \in I) \in \widehat{\mathcal{P}}(I) : \mathbb{P}(v_i = J_i : i \in I) = \mathbb{P}(v_{\pi(i)} = J_i : i \in I). \tag{70}$$

As in Sect. 5.3.1 let us take on $\mathbb{S}(I)$ the equivalence relation given by the cardinality, $J \sim K$ if $|J| = |K|$. Recall $\mathcal{I}_N^0 = \{0, \dots, N\}$ is identified with the set of equivalence classes. Let us check that P satisfies the coarse-graining conditions. For $m \in \mathcal{I}_N^0$ set $\Gamma^*(m) = \{L \subseteq I : |L| = m\}$. We must verify that,

$$|J| = |K| \Rightarrow \forall m \in \mathcal{I}_N^0 : \sum_{L \in \Gamma^*(m)} P(J, L) = \sum_{L \in \Gamma^*(m)} P(K, L). \tag{71}$$

Let π be any permutation of I such that $\pi(J) = K$. We have that $\pi : \Gamma^*(m) \rightarrow \Gamma^*(m)$, $L \rightarrow \pi(L)$, is a bijection. From (69) and (70) we have

$$P(J, L) = \mathbb{P}\left(\bigcup_{i \in J} v_i = L\right) = \mathbb{P}\left(\bigcup_{i \in J} v_{\pi(i)} = L\right) = \mathbb{P}\left(\bigcup_{i \in K} v_i = L\right) = P(K, L).$$

Hence $\sum_{L \in \Gamma^*(m)} P(J, L) = \sum_{L \in \Gamma^*(m)} P(K, L)$, and so (71) is satisfied.

The coarse-graining matrix $\tilde{P} = (\tilde{P}(i, j : i, j \in \mathcal{I}_N^0))$ satisfies

$$\text{For } |J| = i : \tilde{P}(i, j) = \sum_{L:|L|=j} P(J, L) = \sum_{L:|L|=j} \mathbb{P}\left(\bigcup_{i \in J} v_i = L\right).$$

Let us show \tilde{P} is the transition matrix of the forward process for the haploid model of Cannings ([3]). Let

$$\widehat{\mathcal{E}}_N = \{\vec{e} = (e_1, \dots, e_N) \in (I_N^0)^N : \sum_{i=1}^N e_i = N\}.$$

Define the random element $|v| : \Omega \rightarrow \widehat{\mathcal{E}}_N$, $\omega \rightarrow |v(\omega)|$, that is $|v(\omega)|_i = |v_i(\omega)|$ is the number of elements of the set $v_i(\omega)$. Note that $\sum_{i \in I} |v_i(\omega)| = N$ because $v(\omega)$ is an indexed partition.

Since the law of v is invariant by permutations, see (70), we get that the law of $|v|$ is exchangeable, that is for all permutation π of \mathcal{I}_N it is satisfied

$$\forall \vec{e} \in \widehat{\mathcal{E}}_N : \mathbb{P}(|v_{\pi(i)}| = e_i, i \in \mathcal{I}_N) = \mathbb{P}(|v_i| = e_i, i \in \mathcal{I}_N).$$

On the other hand, we have

$$\mathbb{P}\left(\sum_{l=1}^i |v_l| = j\right) = \sum_{J:|J|=j} \mathbb{P}\left(\bigcup_{l=1}^i v_l = J\right).$$

Hence, the coarse-graining kernel \tilde{P} satisfies

$$\tilde{P}(i, j) = \mathbb{P}\left(\sum_{l=1}^i |v_l| = j\right).$$

Then \tilde{P} is the kernel of the forward process of the haploid model of Cannings. Denote $H = Z'$. Let us compute $\tilde{H} = \tilde{Z}'$ in this coarse-graining setting. Let $i, j \in \mathcal{I}_N^0$, take J be such that $|J| = i$, we have

$$\tilde{H}(i, j) = \sum_{L:|L|=j} Z'(J, L) = \sum_{L:|L|=j} \mathbf{1}_{L \subseteq J} = |\{L : |L| = j, L \subseteq J\}| = \binom{i}{j} \mathbf{1}_{i \geq j}.$$

In this case, the function of (55) is $\tilde{h}(j) = |\{L \subseteq I : |L| = j\}| = \binom{N}{j}$ for $j \in \mathcal{I}_N^0$. Then $\tilde{H}_{\tilde{h}} = \tilde{H} D_{\tilde{h}}^{-1}$ satisfies

$$\tilde{H}_{\tilde{h}}(i, j) = \frac{\binom{i}{j}}{\binom{N}{j}} \mathbf{1}_{i \geq j}.$$

An easy computation gives,

$$\tilde{H}_{\tilde{h}}^{-1}(i, j) = (-1)^{i-j} \binom{i}{j} \binom{N}{i} \mathbf{1}_{i \geq j}.$$

Therefore, Theorem 15 ensures that the matrix $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = D_{\tilde{h}}^{-1} \tilde{Q} D_{\tilde{h}}$ is a stochastic matrix that satisfies

$$\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}}^t = \tilde{H}_{\tilde{h}}^{-1} \tilde{P} \tilde{H}_{\tilde{h}}.$$

So, it is the $\tilde{H}_{\tilde{h}}$ -dual of \tilde{P} , see (54).

The matrix $\tilde{H}_{\tilde{h}}$, called the hypergeometric matrix, was firstly introduced in [14] as a dual kernel between the forward process and the backward process of the haploid model of Cannings. As said, as a consequence of our results, the transition matrix of the backward process is given by $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = D_{\tilde{h}}^{-1} \tilde{Q} D_{\tilde{h}}$. In [14], it is proven that this transition matrix also satisfies

$$\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}}(i, j) = \frac{\binom{N}{j}}{\binom{N}{i}} \sum_{(l_1, \dots, l_j) \in (\mathcal{I}_N^0)^j : \sum_{r=1}^j l_r = i} \mathbb{E} \left(\prod_{r=1}^j \binom{|v_r|}{l_r} \right).$$

6.2 Multi-allelic Cannings Model

Here, we construct a multi-allelic model on the product class of subsets of a fixed finite set. We also construct an ancestor type process on the family of sets which is in transpose zeta duality with the former one. We show that the coarse-graining of these

models are the multi-allelic Cannings model as introduced in [7, 15] and the associated ancestor type process defined in [15]. The coarse-graining of the transpose zeta matrix becomes a generalized hypergeometric matrix.

Let I be a finite set and T be the number of types. We assume $T \geq 2$. Consider two different classes of product of sets:

$$\widehat{\mathcal{P}}^{(T)}(I) = \{ \vec{J} := (J_t : t \in \mathcal{I}_T) : \forall t J_t \in \mathbb{S}(I), t \neq t' J_t \cap J_{t'} = \emptyset, \bigcup_{t \in \mathcal{I}_T} J_t = I \};$$

$$\widehat{\mathcal{D}}^{(T)}(I) = \{ \vec{J} = (J_t : t \in \mathcal{I}_T) : \forall t J_t \in \mathbb{S}(I), t \neq t' J_t \cap J_{t'} = \emptyset \}.$$

That is, the elements $(J_t : t \in \mathcal{I}_T) \in \widehat{\mathcal{D}}^{(T)}(I)$ do not necessarily cover I (they satisfy $\bigcup_{t \in \mathcal{I}_T} J_t \subseteq I$). Note that $\widehat{\mathcal{P}}^{(T)}(I) \subseteq \mathbb{S}(I)^T$ and $\widehat{\mathcal{D}}^{(T)}(I) \subseteq \mathbb{S}(I)^T$.

As before, $(\Omega, \mathbb{B}, \mathbb{P})$ is a probability space and $v : \Omega \rightarrow \widehat{\mathcal{P}}(I), \omega \rightarrow v(\omega)$ is a random element. Consider a collection of independent equally distributed random elements $(v^n : n \in \mathbb{Z})$ with the law of v . The elements indexed by a nonnegative n will serve to construct the forward process, and the elements with negative n will serve to define the backward process.

Let us define the process $(X_n : n \in \mathbb{N})$ with values in $\widehat{\mathcal{P}}^{(T)}(I)$. The t -coordinate of X_n is noted by $(X_n)_t$. The process is given by,

$$\forall t \in \mathcal{I}_T : (X_{n+1})_t = \bigcup_{i \in (X_n)_t} v_i^{n+1}.$$

The process $(X_n : n \in \mathbb{N})$ is well defined in $\widehat{\mathcal{P}}^{(T)}(I)$, that is $X_0 \in \widehat{\mathcal{P}}^{(T)}(I)$ implies $X_n \in \widehat{\mathcal{P}}^{(T)}(I)$ for all $n \in \mathbb{N}$, because v takes values in $\widehat{\mathcal{P}}(I)$.

The process $(X_n : n \in \mathbb{N})$ is a Markov chain with stochastic transition matrix $P = (P(\vec{J}, \vec{K}) : \vec{J}, \vec{K} \in \widehat{\mathcal{P}}^{(T)}(I))$ given by

$$P(\vec{J}, \vec{K}) = \mathbb{P}(X_{n+1} = \vec{K} \mid X_n = \vec{J}) = \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\bigcup_{i \in J_t} v_i = K_t \right) \right). \tag{72}$$

This chain is called the forward process.

Now, we define the backward process $(Y_n : n \in \mathbb{N})$ which will take values in $\widehat{\mathcal{D}}^{(T)}(I)$. The t -coordinate of Y_n will be denoted by $(Y_n)_t$. To define the process it is useful to use the product order on $\mathbb{S}(I)^T : \vec{L} \subseteq \vec{M}$ when $L_t \subseteq M_t$ for $t \in \mathcal{I}_T$. We define Y_{n+1} from Y_n by:

$$\left(\bigcap_{t \in \mathcal{I}_T} \bigcup_{i \in (Y_{n+1})_t} v_i^{-(n+1)} \supseteq (Y_n)_t \right) \text{ and}$$

$$\left(\forall \vec{L} \subseteq Y_{n+1}, \vec{L} \neq Y_{n+1} : \bigcup_{t \in \mathcal{I}_T} \left(\bigcup_{i \in L_t} v_i^{-(n+1)} \not\supseteq (Y_n)_t \right) \right).$$

In this case, it is not guaranteed that for all $Y_n \in \widehat{\mathcal{D}}^{(T)}(I)$ there exists some $Y_{n+1} \in \widehat{\mathcal{D}}^{(T)}(I)$ satisfying the above requirements. But when it exists, it is uniquely defined because of the disjointedness property: $L \cap L' = \emptyset$ implies $(\bigcup_{i \in L} v_i^{-(n+1)}) \cap (\bigcup_{i \in L'} v_i^{-(n+1)}) = \emptyset$.

The random set Y_{n+1} can be thought as the set of ancestors of Y_n . The process $(Y_n : n \in \mathbb{N})$ is a Markov chain that can lose mass. Its evolution is given by the (substochastic) transition matrix $Q = (Q(\vec{J}, \vec{K}) : \vec{J}, \vec{K} \in \widehat{\mathcal{D}}^{(T)}(I))$ defined by

$$Q(\vec{J}, \vec{K}) = \mathbb{P}(Y_{n+1} = \vec{K} \mid Y_n = \vec{J}) \\ = \mathbb{P} \left(\left(\forall t \in \mathcal{I}_T \bigcup_{i \in K_t} v_i \supseteq J_t \right) \text{ and } (\forall \vec{L} \subseteq Y_{n+1}, \vec{L} \neq \vec{K}, \exists t \in \mathcal{I}_T : \bigcup_{i \in L_t} v_i \not\supseteq J_t) \right).$$

Let us relate both kernels Q and P . To this purpose, it is convenient to define $A_{i,t} = \{v_i \subseteq J_t^c\}$. As before, from $\bigcup_{i \in I} v_i = I$ and the disjointedness of the sets $(v_i : i \in I)$ we get

$$\bigcap_{i \in K_t^c} A_{i,t} = \left\{ \bigcup_{i \in K_t} v_i \supseteq J_t \right\}.$$

Let us consider

$$\mathcal{X}_{\vec{K}}^{(\vec{J})} = \bigcap_{t \in \mathcal{I}_T} \left(\bigcap_{i \in K_t^c} A_{i,t} \right) = \bigcap_{t \in \mathcal{I}_T} \left(\left\{ \bigcup_{i \in K_t} v_i \supseteq J_t \right\} \right). \tag{73}$$

Hence, relation (73) and the product order allows to write,

$$Q(\vec{J}, \vec{K}) = \mathbb{P} \left(\mathcal{X}_{\vec{K}}^{(\vec{J})} \setminus \left(\bigcup_{\vec{L} \subseteq \vec{K}, \vec{L} \neq \vec{K}} \mathcal{X}_{\vec{L}}^{(\vec{J})} \right) \right).$$

Note that

$$\mathbb{P}(\mathcal{X}_{\vec{K}}^{(\vec{J})}) = \sum_{\vec{M} : \vec{M} \supseteq \vec{J}} P(\vec{K}, \vec{M}).$$

By the Sylvester formula (36) for product of sets we get

$$Q(\vec{J}, \vec{K}) = \sum_{\vec{L} : \vec{L} \subseteq \vec{K}} (-1)^{\sum_{t \in \mathcal{I}_T} (|K_t| - |L_t|)} \left(\sum_{\vec{M} : \vec{M} \supseteq \vec{J}} P(\vec{L}, \vec{M}) \right).$$

Hence, equality (19) is satisfied, then $Q' = Z'^{-1}PZ'$ holds with Z' given by $Z'(\vec{J}, \vec{K}) = \mathbf{1}_{\vec{K} \subseteq \vec{J}}$. That is, the kernel Q is the Z' -dual (transpose zeta dual) of P .

Now, assume the law of ν is invariant under permutation of I , so (70) is satisfied. In $\mathbb{S}(\mathcal{A})^T$ we define $|\vec{J}| = (|J_t| : t \in \mathcal{I}_T)$ and we endow $\mathbb{S}(\mathcal{A})^T$ with the equivalence relation $\vec{J} \sim \vec{K}$ if $|\vec{J}| = |\vec{K}|$. Let us check that P satisfies the coarse-graining conditions. Fix $N = |I|$, let

$$\widehat{\mathcal{E}}_N^{(T)} = \{\vec{e} = (e_1, \dots, e_T) \in (\mathcal{I}_N^0)^T : \sum_{t \in \mathcal{I}_T} e_t = N\}.$$

For all $\vec{e} \in \widehat{\mathcal{E}}_N^{(T)}$ we define

$$\Gamma^*(\vec{e}) = \{\vec{L} \in \widehat{\mathcal{P}}^{(T)}(I) : |\vec{L}| = \vec{e}\}.$$

We must verify that,

$$|\vec{J}| = |\vec{K}| \Rightarrow \forall \vec{e} \in \widehat{\mathcal{E}}_N^{(T)} : \sum_{\vec{L} \in \Gamma^*(\vec{e})} P(\vec{J}, \vec{L}) = \sum_{\vec{L} \in \Gamma^*(\vec{e})} P(\vec{K}, \vec{L}).$$

Let π be any permutation of I such that $\pi(J_t) = K_t$ for all $t \in \mathcal{I}_T$, this permutation exists because the elements of \vec{J} are disjoint sets, as well as those of \vec{K} . We have that $\pi : \Gamma^*(\vec{e}) \rightarrow \Gamma^*(\vec{e}), \vec{L} \rightarrow \pi(\vec{L})$, is a bijection. From (72) and (70) we have

$$\begin{aligned} P(\vec{J}, \vec{L}) &= \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\bigcup_{i \in J_t} v_i = L_t \right) \right) = \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\bigcup_{i \in J_t} v_{\pi(i)} = L_t \right) \right) \\ &= \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\bigcup_{i \in K_t} v_i = L_t \right) \right) = P(\vec{K}, \vec{L}). \end{aligned}$$

Hence $\sum_{\vec{L} \in \Gamma^*(\vec{e})} P(\vec{J}, \vec{L}) = \sum_{\vec{L} \in \Gamma^*(\vec{e})} P(\vec{K}, \vec{L})$, and so (71) is satisfied.

The coarse-graining matrix $\tilde{P} = (\tilde{P}(\vec{d}, \vec{e}) : \vec{d}, \vec{e} \in \widehat{\mathcal{E}}_N^{(T)})$ is such that for all $\vec{d}, \vec{e} \in \widehat{\mathcal{E}}_N^{(T)}$ and every \vec{J} that satisfies $|\vec{J}| = \vec{d}$,

$$\tilde{P}(\vec{d}, \vec{e}) = \sum_{\vec{L} \in \Gamma^*(\vec{e})} P(\vec{J}, \vec{L}) = \sum_{\vec{L} \in \Gamma^*(\vec{e})} \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\bigcup_{i \in J_t} v_i = L_t \right) \right). \tag{74}$$

Let us show \tilde{P} is the transition matrix of the forward process for the multi-allelic model in [7, 15]. Recall the random element $|\nu| : \Omega \rightarrow \widehat{\mathcal{E}}_N^{(T)}, \omega \rightarrow |\nu(\omega)|$, so $|\nu(\omega)|_i = |\nu_i(\omega)|$. As pointed out, since the law of ν is invariant under permutations, the law

of $|\nu|$ is exchangeable. From exchangeability and relation (74), the coarse-graining matrix \tilde{P} satisfies for all pair $\vec{d}, \vec{e} \in \hat{\mathcal{E}}_N^{(T)}$,

$$\tilde{P}(\vec{d}, \vec{e}) = \mathbb{P} \left(\bigcap_{t \in \mathcal{I}_T} \left(\sum_{l=\Delta_{t-1}+1}^{\Delta_t} |\nu|_l = e_t \right) \right)$$

where $\Delta_t = \sum_{s=1}^t d_s$ for $t \in \mathcal{I}_T$ and $\Delta_0 = 0$.

Then \tilde{P} is the kernel of the forward process of the multi-allelic model in [7, 15]. Let $H = Z'$. Let us compute the dual matrix $\tilde{H} = \tilde{Z}'$ in this coarse-graining setting. Let $\vec{d}, \vec{e} \in \hat{\mathcal{E}}_N^{(T)}$ and \vec{J} be such that $|\vec{J}| = \vec{d}$, we have

$$\begin{aligned} \tilde{H}(\vec{d}, \vec{e}) &= \sum_{\vec{L} \in \Gamma^*(\vec{e})} Z'(\vec{J}, \vec{L}) = \sum_{\vec{L} \in \Gamma^*(\vec{e})} \mathbf{1}_{\vec{L} \subseteq \vec{J}} = |\{\vec{L} \in \Gamma^*(\vec{e}), \vec{L} \subseteq \vec{J}\}| \\ &= \prod_{t \in \mathcal{I}_T} \binom{d_t}{e_t} \mathbf{1}_{\vec{d} \geq \vec{e}} := \binom{\vec{d}}{\vec{e}} \mathbf{1}_{\vec{d} \geq \vec{e}}. \end{aligned}$$

In this case the function \tilde{h} of (55) is given by,

$$\forall \vec{e} \in \hat{\mathcal{E}}_N^{(T)} : \tilde{h}(\vec{e}) = |\Gamma^*(\vec{e})| = \frac{N!}{\prod_{t \in \mathcal{I}_T} e_t!} = \binom{N}{\vec{e}}. \tag{75}$$

Then $\tilde{H}_{\tilde{h}} = \tilde{H} D_{\tilde{h}}^{-1}$ satisfies

$$\tilde{H}_{\tilde{h}}(\vec{d}, \vec{e}) = \frac{\binom{\vec{d}}{\vec{e}}}{\binom{N}{\vec{e}}} \mathbf{1}_{\vec{d} \geq \vec{e}}. \tag{76}$$

Therefore, Theorem 15 ensures that the matrix $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}} = D_{\tilde{h}}^{-1} \tilde{Q} D_{\tilde{h}}$ is substochastic (because Q is) and satisfies

$$\tilde{Q}'_{\tilde{h}^{-1}, \tilde{h}} = \tilde{H}_{\tilde{h}}^{-1} \tilde{P} \tilde{H}_{\tilde{h}},$$

that is, is the $\tilde{H}_{\tilde{h}}$ -dual of \tilde{P} , see (54).

We note that the coefficients of (76) are exactly the same as those appearing in expression (8) in [15]. Hence, by coarse-graining, we have retrieved the result proven in [15] that a dual kernel between the forward process and the backward process of the multi-allelic model of Cannings is given by (76). In [15], it is supplied several formulae for $\tilde{Q}_{\tilde{h}^{-1}, \tilde{h}}$, in particular see its Proposition 3.

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