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Fully well-balanced, positive and simple approximate Riemann solver for shallow water equations

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Abstract. The present work is focused on the numerical approximation of the shallow water equations. When studying this problem, one faces at least two important issues, namely the ability of the scheme to preserve the positiveness of the water depth, along with the ability to capture the stationary states. We propose here a Godunov-type method that fully satisfies the previous conditions, meaning that the method is in particular able to preserve the steady states with non-zero velocity.

Keywords: Shallow-water equations, steady states, finite volume schemes, well-balanced property, positive preserving scheme.

Mathematical subject classification: 65M08, 65M12, 76M12, 35L65.

1 Introduction

The present work is dedicated to the derivation of a numerical scheme for the well-known shallow water equations, given by

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + g\frac{h^2}{2} \right) = -gh\partial_x z \end{cases} \tag{1}$$

where $z(x)$ denotes a given smooth topography and $g > 0$ is the gravity constant. The primitive variables are the water depth h and its velocity u, which both depend on the space and time variables, respectively $x \in \mathbb{R}$ and $t \in [0, \infty)$. At time $t = 0$, we assume that the initial water depth $h(x, t = 0) = h_0(x)$ and velocity $u(x, t = 0) = u_0(x)$ are given.

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The term "shallow water" comes from the idea that, if we consider a space domain of length $L > 0$ and if we assume that the water depth $h(x, t)$ is very small compared to L for all $x \in [0, L]$ and all $t > 0$, then it is reasonable to assume that the velocity $u(x, t)$ does not depend on the water depth. Under this hypothesis, the previous model is obtained from the conservation of mass and momentum with no friction, viscosity and Coriolis forces (see [13]).

To shorten the notations, let us rewrite (1) as follows:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}, z) \tag{2}$$

where $\mathbf{U} = \begin{pmatrix} h \\ hu \end{pmatrix}$, $\mathbf{F}(\mathbf{U}) = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \end{pmatrix}$, $\mathbf{S}(\mathbf{U}, z) = \begin{pmatrix} 0 \\ -gh\partial_x z \end{pmatrix}$.

From now on, let us emphasize we do not consider dry areas (i.e. where $h = 0$). The method should then satisfy $h(x, t) > 0$ for all (x, t). We will also pay a particular attention to the steady states, governed by $\partial_x(hu) = 0$ and $\partial_x \left(hu^2 + g\frac{h^2}{2}\right) = -gh\partial_x z$. Therefore, the smooth steady states under consideration are given by

$$\begin{cases} hu = \text{constant} \\ \dfrac{u^2}{2} + g(h + z) = \text{constant.} \end{cases} \tag{3}$$

There have been a huge amount of works on this topic since the last two decades, most of them focusing on the design of numerical schemes satisfying the so-called lake at rest equilibrium. Much less indeed considered the moving water equilibria. Without any attempt to be exhaustive, on can first quote the following series of works, [1], [3], [5], [6], [10], [11], [12], [14], [16], [18], and then the following papers which are more specifically interested in moving water equilibria, namely [7], [8], [14], [20], [21], [22], [23]. We also refer the reader to the recent book [15] where a more complete list of references can be found, and to the recent papers [4] and [2] where similar techniques to the ones proposed in the present contribution are used to derive entropy-satisfying and/or fully well-balanced schemes based on simple approximate Riemann solvers (see also [9] for related issues). In the present work, we will indeed describe an easy to implement Godunov-type scheme which is positive (i.e. that preserves the positiveness of h), fully well-balanced (i.e. able to restore all the steady states) and based on the derivation of a simple approximate Riemann solver. Interestingly, this solver is however derived from a suitable linearization of a positive, entropy-satisfying and fully well-balanced scheme described in [4], which makes it far easier to implement. Numerical experiments are then proposed to illustrate the behavior of this Godunov-type method.

2 Numerical method

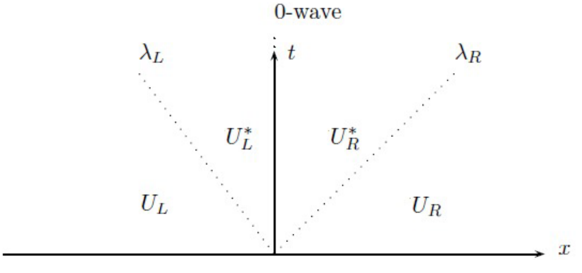
2.1 Godunov-type methods

We introduce a space step Δx and a time step Δt. We define the mesh interfaces $x_{j+1/2} = j\Delta x$, the cells $C_j = [x_{j-1/2}, x_{j+1/2})$, the cell centers x_j and the intermediate times $t^{n+1} = t^n + \Delta t$. For all $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, we compute a piecewise constant approximation of the exact solution $\mathbf{U}(x, t^n)$ and denote it $\mathbf{U}_j^n$, namely $\mathbf{U}_j^n \approx \mathbf{U}(x, t^n)$, if $x \in [x_{j-1/2}, x_{j+1/2})$. At $t = 0$, we take $\mathbf{U}_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}_0(x)\mathrm{d}x$. Now, assuming $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ to be known, we make it evolve to the next time level t^{n+1} by considering a Godunov-type scheme. This is achieved through two steps, the first one being the computation of the evolution from the initial data $(\mathbf{U}_j^n)_{j\in\mathbb{Z}}$ to time t^{n+1} according to the model, and the second one being a projection to ensure that $(\mathbf{U}_j^{n+1})_{j\in\mathbb{Z}}$ remains piecewise constant on each C_j.

Step 1: Evolution in time. The aim of this first step is to compute an approximate solution of the model at time Δt with initial condition given by $x \to \mathbf{U}(x, t^n)$. We build an approximate solution of the Riemann problem that appears at each interface $x_{j+1/2}$ and associated with the initial data

$$(\mathbf{U}(x,0), z(x)) = \begin{cases} (\mathbf{U}_L, z_L) & \text{if } x < x_{j+1/2} \\ (\mathbf{U}_R, z_R) & \text{if } x > x_{j+1/2} \end{cases},$$

where we introduced the notations $\mathbf{U}_L = \mathbf{U}_j^n$, $\mathbf{U}_R = \mathbf{U}_{j+1}^n$, $z_L = z_j$, $z_R = z_{j+1}$. The proposed approximate Riemann solver is made of three waves propagating with velocities $\lambda_L = \lambda_L(u_L, h_L) < 0, \lambda_0 = 0, \lambda_R = \lambda_R(u_R, h_R) > 0$ and separating four constant states $\mathbf{U}_L$, $\mathbf{U}_L^*$, $\mathbf{U}_R^*$ and $\mathbf{U}_R$. Let us note that the stationary wave is introduced in order to make the scheme able to preserve the moving water equilibria. This Riemann solver will be described in details in the next section. It can be already represented in the phase space (x, t) as follows:



and setting $\xi_{j+1/2} = \frac{x - x_{j+1/2}}{t}$ is thus given by

$$\mathbf{U}(x,t) = \begin{cases} \mathbf{U}_L & \text{if } \xi_{j+1/2} \leq \lambda_L \\ \mathbf{U}_L^* & \text{if } \lambda_L < \xi_{j+1/2} \leq 0 \\ \mathbf{U}_R^* & \text{if } 0 < \xi_{j+1/2} \leq \lambda_R \\ \mathbf{U}_R & \text{if } \lambda_R < \xi_{j+1/2}. \end{cases}$$

The approximate solution of the model at time Δt and with initial condition $x \to \mathbf{U}(x, t^n)$ is denoted $x \to \mathbf{U}^{n+1}(x)$ and then defined as the juxtaposition of the approximate Riemann solutions set at each interface $x_{j+1/2}$ with Δt such that the waves created at each interface do not meet with each other, namely such that

$$\Delta t \leq \frac{1}{2} \min_j \left[\min \left(\frac{\Delta x}{|\lambda_{Lj}|}, \frac{\Delta x}{\lambda_{Rj}} \right) \right]. \tag{4}$$

Step 2: Projection. The solution $x \to \mathbf{U}^{n+1}(x)$ is piecewise constant but not on the cells of the mesh. As it is customary, a piecewise constant approximate solution on each cell of the mesh is obtained by averaging $x \to \mathbf{U}^{n+1}(x)$ on the C_j, $j \in \mathbb{Z}$. This is expressed with the following update formula:

$$\mathbf{U}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{U}^{n+1}(x)\mathrm{d}x, \quad j \in \mathbb{Z}. \tag{5}$$

To conclude this section, note that $x \to \mathbf{U}^{n+1}(x)$ being piecewise constant, it can be easily proved that the exact value of (5) is also given by the formulas

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\Delta t}{\Delta x} \left((\Phi_{j+1/2} - \Phi_{j-1/2}) - g\Delta x \frac{s_{j+1/2} + s_{j-1/2}}{2} \right), \quad j \in \mathbb{Z},$$

$$\Phi(\mathbf{U}_L, \mathbf{U}_R) = \frac{\mathbf{F}_L + \mathbf{F}_R}{2} - \frac{|\lambda_L|(\mathbf{U}_L^* - \mathbf{U}_L) + |\lambda_R|(\mathbf{U}_R - \mathbf{U}_R^*)}{2}.$$

Here $\Phi_{j+1/2} = \Phi(\mathbf{U}_j, \mathbf{U}_{j+1})$ for all j and $s_{j+1/2} = s(\mathbf{U}_j, \mathbf{U}_{j+1})$ denotes an approximation of the source term $-\{h\partial_x z\}$ designed in such a way that the well-balanced property is satisfied. It will be specified in the next section.

2.2 Definition of the approximate Riemann solver

The intermediate states

$$\mathbf{U}_L^* = \begin{pmatrix} h_L^* \\ h_L^* u_L^* \end{pmatrix}, \quad \mathbf{U}_R^* = \begin{pmatrix} h_R^* \\ h_R^* u_R^* \end{pmatrix}$$

are defined in such a way that some consistency properties in the integral sense are satisfied (see [11], [12], [9], [4] for more details), which are given by the relation

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) - \Delta x\, \mathbf{S}(\mathbf{U}_L, \mathbf{U}_R) = \lambda_L(\mathbf{U}_L^* - \mathbf{U}_L) + \lambda_R(\mathbf{U}_R - \mathbf{U}_L^*), \tag{6}$$

with $\mathbf{S}(\mathbf{U}_L, \mathbf{U}_R) = (0, gs(\mathbf{U}_L, \mathbf{U}_R))^T$, together with two equilibrium relations across the stationary wave which write as follows:

$$h_L^* u_L^* = h_R^* u_R^* \tag{7}$$

$$h_L^* \frac{u_L^2}{2h_L} + g(h_L^* + z_L) = h_R^* \frac{u_R^2}{2h_R} + g(h_R^* + z_R). \tag{8}$$

These two relations can be understood as a natural h-linearization of the following ones which are used in [4] and express the continuity of the Riemann invariants (3) across the stationary discontinuity:

$$\begin{aligned} h_L^* u_L^* &= h_R^* u_R^* \\ \frac{(h_L^* u_L^*)^2}{2(h_L^*)^2} + g(h_L^* + z_L) &= \frac{(h_R^* u_R^*)^2}{2(h_R^*)^2} + g(h_R^* + z_R). \end{aligned}$$

Note that other linearizations could have been considered. The intermediate states are then uniquely defined by solving explicitly the 4×4 linear system (6)-(7)-(8) through the following relations

$$\begin{aligned} h_L^* &= \frac{(\lambda_R - \lambda_L)\big(g + \frac{u_R^2}{2h_R}\big)h^{HLL} + g\lambda_R(z_R - z_L)}{\lambda_R\big(g + \frac{u_L^2}{2h_L}\big) - \lambda_L\big(g + \frac{u_R^2}{2h_R}\big)} \\ h_R^* &= \frac{(\lambda_R - \lambda_L)\big(g + \frac{u_L^2}{2h_L}\big)h^{HLL} - g\lambda_L(z_L - z_R)}{\lambda_R\big(g + \frac{u_L^2}{2h_L}\big) - \lambda_L\big(g + \frac{u_R^2}{2h_R}\big)} \\ u_L^* &= \frac{1}{h_L^*}\Big(q^{HLL} + \frac{g \cdot \Delta x \cdot s}{\lambda_R - \lambda_L}\Big), \\ u_R^* &= \frac{1}{h_R^*}\Big(q^{HLL} + \frac{g \cdot \Delta x \cdot s}{\lambda_R - \lambda_L}\Big) \\ h^{HLL} &= \frac{\lambda_R h_R - \lambda_L h_L}{\lambda_R - \lambda_L} - \frac{1}{\lambda_R - \lambda_L}(h_R u_R - h_L u_L) \\ q^{HLL} &= \frac{\lambda_R h_R u_R - \lambda_L h_L u_L}{\lambda_R - \lambda_L} - \frac{1}{\lambda_R - \lambda_L}\Big(h_R u_R^2 + g\frac{h_R^2}{2} - h_L u_L^2 - g\frac{h_L^2}{2}\Big) \end{aligned}$$

are the values of h and q associated with the HLL Riemann solver [17].

2.3 Fully well-balanced and positivity properties

In order for the approximate Riemann solver to be fully well-balanced, we propose to define $s(\mathbf{U}_L, \mathbf{U}_R)$ at each interface by

$$\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = -\frac{h_L h_R}{\overline{h}}(z_R - z_L) + \frac{(h_R - h_L)^3}{4\overline{h}} \tag{9}$$

where $\overline{h} = \frac{h_L + h_R}{2}$. We have the following fully well-balanced property (see [4]).

Lemma. *Let $\mathbf{U}_L$ and $\mathbf{U}_R$ such that the following well-balanced relations hold,*

$$h_L u_L = h_R u_R \tag{10}$$

$$\frac{u_L^2}{2} + g(h_L + z_L) = \frac{u_R^2}{2} + g(h_R + z_R). \tag{11}$$

Then, the proposed approximate Riemann solver is stationary in the sense that $\mathbf{U}_L^* = \mathbf{U}_L$ and $\mathbf{U}_R^* = \mathbf{U}_R$. The proposed Godunov-type method is then fully well-balanced in the sense that it preserves stationary solutions of the model.

Proof. The approximate Riemann solver is stationary provided that

$$h_L^* = h_L, \quad h_L^* u_L^* = h_L u_L, \quad h_R^* = h_R, \quad h_R^* u_R^* = h_R u_R. \tag{12}$$

Let us check that such a choice of h_L^*, h_R^*, $(h_L u_L)^*$ and $(h_R u_R)^*$ actually satisfies (6)-(7)-(8). Since (6)-(7)-(8) admits an unique solution, this is indeed sufficient to prove the fully well-balanced property by uniqueness of the proposed approximate Riemann solution. Let us then assume that h_L^*, h_R^*, $(h_L u_L)^*$ and $(h_R u_R)^*$ are defined by (12).

On the first hand, the first equation of (6) then writes $h_L u_L = h_R u_R$, which is clearly satisfied by (10). On the other hand, the second equation of (6) writes

$$\left(h_R u_R^2 + g\frac{h_R^2}{2}\right) - \left(h_L u_L^2 + g\frac{h_L^2}{2}\right) - g\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = 0,$$

or

$$g\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = \frac{(h_R u_R)^2}{h_R} - \frac{(h_L u_L)^2}{h_L} + \frac{g}{2}(h_R + h_L)(h_R - h_L).$$

Thanks to (10) and setting $q = h_L u_L = h_R u_R$, we have the following equalities:

$$\begin{aligned}
g\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) &= q^2\left(\frac{1}{h_R} - \frac{1}{h_L}\right) + \frac{g}{2}(h_R + h_L)(h_R - h_L) \\
&= -\frac{q^2}{h_L h_R}(h_R - h_L) + \frac{g}{2}(h_R + h_L)(h_R - h_L) \\
&= \frac{g}{2}(h_R^2 - h_L^2) - \frac{q^2(h_R^2 - h_L^2)}{h_L h_R(h_L + h_R)}.
\end{aligned}$$

But from (11), we write $-g(z_R - z_L) = g(h_R - h_L) - \frac{q^2}{2h_L^2 h_R^2}(h_R^2 - h_L^2)$, so that

$$-\frac{q^2(h_R^2 - h_L^2)}{h_L h_R} = -2g h_L h_R(z_R - z_L + h_R - h_L).$$

We finally obtain

$$g\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = \frac{g}{2}(h_R^2 - h_L^2) - 2g\frac{h_L h_R}{h_L + h_R}(z_R - z_L + h_R - h_L)$$

$$\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = -\frac{h_L h_R}{\overline{h}}(z_R - z_L) - (h_R - h_L)\left(\frac{h_L h_R}{\overline{h}} - \overline{h}\right).$$

Since $h_L h_R - \overline{h}^2 = -(h_R - h_L)^2/4$, we end up with the relation

$$\Delta x\, s(\mathbf{U}_L, \mathbf{U}_R) = -\frac{h_L h_R}{\overline{h}}(z_R - z_L) + \frac{(h_R - h_L)^3}{4\overline{h}},$$

which is nothing but the proposed definition of $s(\mathbf{U}_L, \mathbf{U}_R)$. The second equation of (6) is thus also satisfied. At last, it is clear that (7) and (8) hold true thanks to (10) and (11), which concludes the proof.

Now, observe that while the source term $s(\mathbf{U}_L, \mathbf{U}_R)$ is expected to be null when $z_L = z_R$, it is clearly not true since

$$s(\mathbf{U}_L, \mathbf{U}_R)_{|\{z_L = z_R\}} = +\frac{1}{\Delta x}\frac{(h_R - h_L)^3}{4\overline{h}} \underset{\Delta x \to 0}{\not\to} 0,$$

except if $h_L = h_R$. Therefore a compromise has to be achieved and this is the purpose of the following discussion. Let C be a L^∞ bound of the derivative $\partial_x h$ which can be numerically implemented using a finite difference approximation:

$$C > \max_j \frac{|h_{j+1} - h_{j-1}|}{2\Delta x}. \tag{13}$$

Note that we took here a centered approximation for the sake of precision, but a forward approximation would have suited as well. Now, let us define

$$\delta h = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x \\ \operatorname{sgn}(h_R - h_L) C\Delta x & \text{otherwise} \end{cases} \tag{14}$$

and propose the following approximation for the source term

$$\Delta x \, s(\mathbf{U}_L, \mathbf{U}_R) = -\frac{h_L h_R}{\overline{h}}(z_R - z_L) + \frac{(\delta h)^3}{4\overline{h}}. \tag{15}$$

We note that for smooth solutions, we obtain the same approximation than before at least for sufficiently small Δx (the scheme then remains fully well-balanced), while when $z_L = z_R$, we now clearly have $s(\mathbf{U}_L, \mathbf{U}_R)_{|\{z_L = z_R\}} = O(\Delta x^2) \underset{\Delta x \to 0}{\longrightarrow} 0$. We thus took this definition of the source term in practice. Let us now conclude this section with the positivity property of the proposed approximate Riemann solver, which proves that the Godunov-type scheme keeps h positive (see [4]).

Lemma. *There exists $-\lambda_L > 0$ and $\lambda_R > 0$ large enough such that h_L^* and h_R^* are positive.*

Proof. Let us first assume that $-\lambda_L$ and λ_R large enough to enforce h^{HLL} to be positive. Next, we first assume $z_R - z_L > 0$ so that h_L^* is obviously positive. Concerning h_R^* we have

$$h_R^* = \frac{\left(1 + \left|\frac{\lambda_L}{\lambda_R}\right|\right)\left(g + \frac{u_L^2}{2h_L}\right) h^{HLL} + g\left|\frac{\lambda_L}{\lambda_R}\right|(z_L - z_R)}{\left(g + \frac{u_L^2}{2h_L}\right) + \left|\frac{\lambda_L}{\lambda_R}\right|\left(g + \frac{u_R^2}{2h_R}\right)}.$$

By considering $|\lambda_L/\lambda_R|$ small enough, which is always possible, we obtain $h_R^* > 0$. Similarly, if $z_R - z_L < 0$, we have $h_R^* > 0$ and we get $h_L^* > 0$ as soon as $|\lambda_R/\lambda_L|$ is fixed small enough.

3 Numerical results

We consider three different cases: the propagation of perturbations around an equilibrium state for which we compare the results with those obtained with the hydrostatic reconstruction scheme [1], and two cases where the ability of the scheme to converge to moving water equilibria when the final time goes to infinity is tested. Let us emphasize that trivial test cases starting from a moving water equilibrium as initial data are not presented since by construction the proposed scheme does preserve them and is thus exact.

3.1 Propagation of perturbations

In this test case, we perturbed a steady state solution by a pulse that splits into two opposite waves over a continuous bed. The parameters are described hereafter: the space domain is reduced to the interval [0, 2]. We work with outflow boundary conditions, and the bottom topography is defined by $z(x) = 2 + 0.25(\cos(10\pi(x - 0.5)) + 1)$ if $1.4 < x < 1.6$, and 2 otherwise. The initial data are $u(0, x) = 0$ and $h(0, x) = 3 - z(x) + \Delta h$ if $1.1 < x < 1.2$, and $3 - z(x)$ otherwise, where $\Delta h = 0.001$ is the height of the perturbation. The CFL parameter is set to 0.9. The final time is fixed at $T = 0.2$, and the space step at $\Delta x = 1/40$.

A reference solution is obtained by hydrostatic reconstruction applied to the HLL flux with a mesh of 20000 cells. We compare the results provided by our fully well-balanced scheme and the hydrostatic reconstruction, in the conditions described above. We obtain good results but the numerical diffusion turns out to be more important with the fully well-balanced scheme.

3.2 Steady flow over a bump.

The aim of this test case is to test the ability of the scheme to converge to some moving water equilibrium. Let us remind that the steady states are governed by the equations $hu = K_1$ and $\frac{u^2}{2} + g(h + z) = K_2$. To ensure that these equalities are satisfied when $t \to \infty$, we define

$$\varepsilon = (\max_x(hu) - \min_x(hu)) + (\max_x(u^2/2 + g(h + z)) - \min_x(u^2/2 + g(h + z))$$

and use ε as a stopping criterion. The time history of ε is shown for both test cases on Figure 2.

Fluvial regime. In this test case, we set $K_1 = 1$ and $K_2 = 25$, we denote $h_{eq}(x)$, $u_{eq}(x)$ the values of h and u at this equilibrium. The domain is $[-2, 2]$ and the bottom topography is defined by $z(x) = (\cos(10\pi(x + 1)) + 1)/4$ if $-0.1 \leq x \leq 0.1$ and 0 elsewhere. The CFL parameter is equal to 0.5. Initial condition is chosen out of equilibrium and given by $h = h_{eq}$ and $u = 0$. The boundary conditions are set to be

$$\begin{cases} \partial_x h(x = -2) = 0, \\ (hu)(x = -2) = K_1, \end{cases} \quad \text{and} \quad \begin{cases} h(x = 2) = h_{eq}(x = 2), \\ \partial_x(hu)(x = 2) = 0. \end{cases}$$

We let the system evolve and actually observed that the solution reaches the equilibrium previously defined within the machine precision when $t \to \infty$.

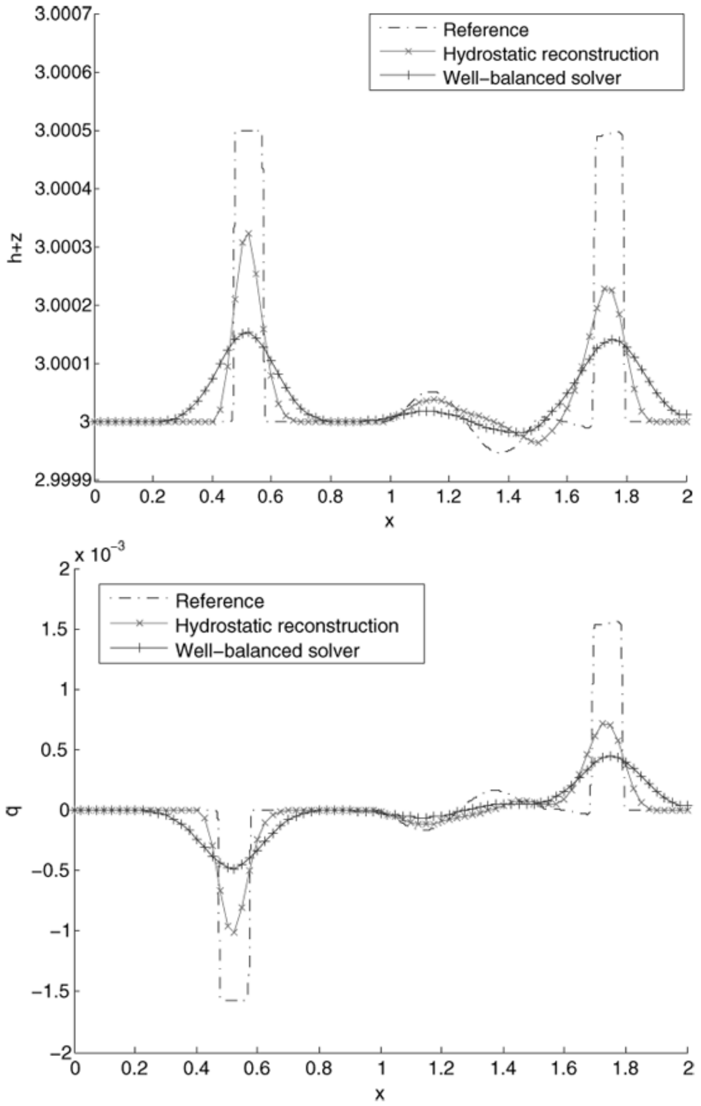


Figure 1: Propagation of perturbations: Comparison of free surfaces $h + z$ (top) and discharge q (bottom) for different schemes.

Transcritical regime without shock. In this test case, we set $K_1 = 3$, $2K_2 = 3(K_1 g)^{2/3} + g$. We used the same boundary conditions and started from the same initial condition. Here again, the moving water equilibrium were reached to the machine precision, which is a benefit from the fully well-balanced property.

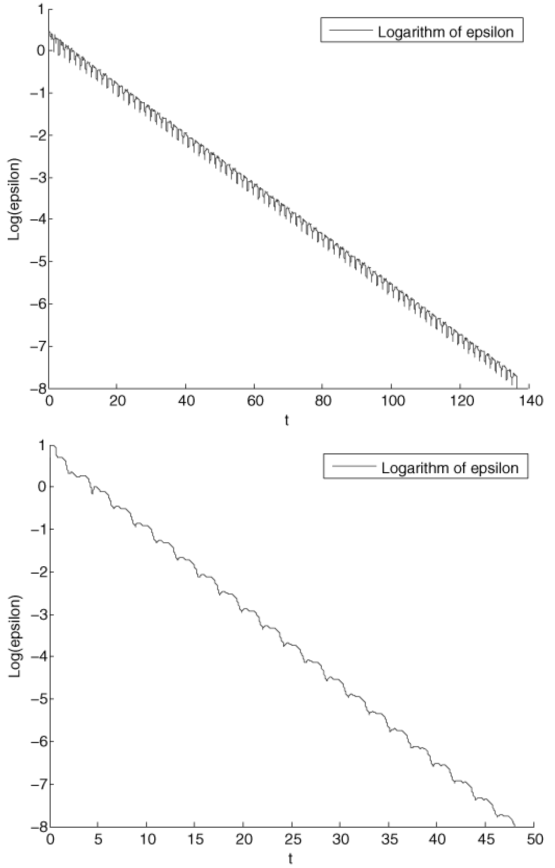


Figure 2: Time history of $\log_{10}(\varepsilon)$ for the fluvial regime (top) and transcritical regime (bottom).

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