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Symmetry results for positive solutions of mixed integro-differential equations

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ABSTRACT

In this paper, we study symmetry property for positive solutions of mixed integro-differential equations

$$\begin{cases} (-\Delta)_x^{\alpha_1} u + (-\Delta)_y^{\alpha_2} u = f(u) & \text{in} \;\; B_1^N(0) \times B_1^M(0), \\ u = 0 \quad \text{in} \quad (\mathbb{R}^N \times \mathbb{R}^M) \setminus (B_1^N(0) \times B_1^M(0)), \end{cases} \tag{0.1}$$

where $N,\ M \geq 1$, $x \in B_1^N(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, $y \in B_1^M(0) = \{y \in \mathbb{R}^M : |y| < 1\}$, the operator $(-\Delta)_x^{\alpha_1}$ denotes the fractional Laplacian of exponent $\alpha_1 \in (0,1)$ with respect to x, $(-\Delta)_y^{\alpha_2}$ denotes the fractional Laplacian of exponent $\alpha_2 \in (0,1)$ with respect to y. We make use of the Maximum Principle for small domain to start the moving planes to obtain the symmetry results for positive solutions.

1. Introduction

The study of radial symmetry of positive solutions to semilinear elliptic equations in bounded domains has been the concern of numerous authors along the last several decades. It was the seminal work by Gidas, Ni and Nirenberg [10] that settled this property of positive C^2-solutions for elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad B_1, \\ u = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{1.1}$$

They proved that any positive C^2-solution of (1.1) is radially symmetric and decreasing by the method of moving planes as in [20]. More later, Berestycki and Nirenberg in [3] gave a more simple proof of this result

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using a very powerful nonlinear strategy, the method of moving planes based on the Maximum Principle for small domain which is derived by the Aleksandrov–Bakelman–Pucci (ABP) estimate. More generally, if the domain is symmetric and convex with respect to a hyperplane then the solutions have the same symmetry. Related results in the whole space and exterior domains were obtained by Li [12], Reichel [15] and Sirakov [23], under the supplementary hypothesis that f is nonincreasing in a right neighborhood of zero.

During the last years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, especially, the fractional Laplacian, motivated by great applications and by important advances on the theory of nonlinear partial differential equations, see [4,6,9,14,16,21,22,24] for details. In some recent works, Guillen and Schwab in [11] proved an ABP estimate for integro-differential equations, Ros-Oton et al. obtained the Pohozaev identities in [19] and the regularities in [17,18], for more see [13]. Felmer et al. [8] provided the Maximum Principle for small domain to equations involving the fractional Laplacian and then obtained the radial symmetry of positive classical solutions for fractional elliptic equations

$$\begin{cases} (-\Delta)^{\alpha} u = f(u) & \text{in} \quad B_1, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus B_1, \end{cases} \tag{1.2}$$

using the method of moving planes as in [3,10]. The method of moving planes is applied to deal with the overdetermined fractional problems, see [7,25].

The elliptic equations with mixed integro-differential operators $(-\Delta)_x^{\alpha_1} + (-\Delta)_y^{\alpha_2}$, which is the fractional Laplacian of exponent $\alpha_1 \in (0,1)$ with respect to x and the fractional Laplacian of exponent $\alpha_2 \in (0,1)$ with respect to y, modeling diffusion sensible to the direction, are associated to Brownian and Levy–Itô processes. Barles, Chasseigne, Ciomaga and Imbert in [1,2] and Ciomaga in [5] considered the existence and the regularity of solutions of equations involving mixed integro-differential operators. Later on, Felmer and Wang studied the decay and the symmetry properties of positive solutions to the mixed integro-differential equations in the whole space. In the present paper, we are interested in the symmetry results of positive solutions for mixed integro-differential equation in a bounded domain, that is,

$$\begin{cases} (-\Delta)_x^{\alpha_1} u + (-\Delta)_y^{\alpha_2} u = f(u) & \text{in} \quad B_1^N(0) \times B_1^M(0), \\ u = 0 \quad \text{in} \quad (\mathbb{R}^N \times \mathbb{R}^M) \setminus (B_1^N(0) \times B_1^M(0)), \end{cases} \tag{1.3}$$

where N, $M \geq 1$, $x \in B_1^N(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, $y \in B_1^M(0) = \{y \in \mathbb{R}^M : |y| < 1\}$, the operators $(-\Delta)_x^{\alpha_1}$ and $(-\Delta)_y^{\alpha_2}$ are given by

$$(-\Delta)_x^{\alpha_1} u(x,y) = P.V. \int_{\mathbb{R}^N} \frac{u(x,y) - u(z,y)}{|x-z|^{N+2\alpha_1}} dz \tag{1.4}$$

and

$$(-\Delta)_y^{\alpha_2} u(x,y) = P.V. \int_{\mathbb{R}^M} \frac{u(x,y) - u(x,\tilde{z})}{|y-\tilde{z}|^{M+2\alpha_2}} d\tilde{z}, \tag{1.5}$$

for all $(x,y) \in B_1^N(0) \times B_1^M(0)$. Here $P.V.$ denotes the principal value of the integral, that for notational simplicity we omit in what follows.

Before stating our main result we make precise the notion of solution that we use in this paper. We say that a continuous function $u : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is a classical solution of equation (1.3) if $(-\Delta)_x^{\alpha_1} u$ and $(-\Delta)_y^{\alpha_2} u$ are defined at any point of $B_1^N(0) \times B_1^M(0)$, according to the definitions given in (1.4) and (1.5), and if u satisfies the equation and the external condition in a pointwise sense.

Now we are ready for our main theorem on symmetry results of positive solutions of equation (1.3). It states as follows:

Theorem 1.1. *Assume that the function $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz. If u is a positive classical solution of equation (1.3), then*

$$u(x, y) = u(|x|, |y|).$$

We prove this result using the method of moving planes based on the Maximum Principle for small domain, which is derived by Aleksandrov–Bakelman–Pucci (ABP) estimate. While for the equations with mixed integro-differential operators, ABP estimate is not available, we introduce a new way to obtain the Maximum Principle for small domain. Then we start the moving planes as in [3,8] to prove the symmetry results of positive solutions of (1.3).

The rest of the paper is organized as follows. In Section 2, we prove the Maximum Principle for small domain for equations involving mixed integro-differential operators. In Section 3, we prove Theorem 1.1 by the moving planes method based on the Maximum Principle for small domain.

2. Preliminaries

In this section, we introduce a type of Maximum Principle for small domain for our mixed type operators, which is a key tool in the proceeding of moving planes. For an open domain $\Omega \subset \mathbb{R}^{N+M}$, we denote by Ω_n the projection of Ω in the direction x and by Ω_m the projection of Ω in the direction y, that is,

$$\Omega_n = \{x \in \mathbb{R}^N : \ \exists y \in \mathbb{R}^M \text{ s.t. } (x, y) \in \Omega\} \tag{2.1}$$

and

$$\Omega_m = \{y \in \mathbb{R}^M : \ \exists x \in \mathbb{R}^N \text{ s.t. } (x, y) \in \Omega\}. \tag{2.2}$$

It is obvious that Ω_n and Ω_m are open sets in $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively and

$$\Omega \subset \Omega_n \times \Omega_m.$$

We start with the following lemma:

Lemma 2.1. *Let Ω be a bounded open set. Suppose that $h : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}^M)$ is a classical solution of*

$$\begin{cases} -(-\Delta)_x^{\alpha_1} w(x, y) - (-\Delta)_y^{\alpha_2} w(x, y) \le h(x, y), & (x, y) \in \Omega, \\ w(x, y) \ge 0, & (x, y) \in (\mathbb{R}^N \times \mathbb{R}^M) \setminus \Omega. \end{cases} \tag{2.3}$$

Then there exists $C > 0$ such that

$$-\inf_{\Omega} w \le C\|h\|_{L^\infty(\Omega)}(|\Omega_n|^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}})^{-1}, \tag{2.4}$$

where Ω_n and Ω_m are defined in (2.1) and (2.2) respectively.

Proof. The result is obvious if $\inf_\Omega w \ge 0$. Now we assume that $\inf_\Omega w < 0$, then there exists $(x_0, y_0) \in \Omega$ such that $w(x_0, y_0) = \inf_\Omega w < 0$. Combining with (2.3), we have that

$$\|h\|_{L^\infty(\Omega)} \ge h(x_0, y_0) \ge -(-\Delta)_x^{\alpha_1} w(x_0, y_0) - (-\Delta)_y^{\alpha_2} w(x_0, y_0). \tag{2.5}$$

By direct computation, we obtain that

$$\begin{aligned}
-(-\Delta)_x^{\alpha_1} w(x_0,y_0) &= \int_{\mathbb{R}^N} \frac{w(z,y_0)-w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz \\
&= \int_{\Omega_n} \frac{w(z,y_0)-w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz + \int_{\mathbb{R}^N\setminus\Omega_n} \frac{w(z,y_0)-w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz \\
&\geq -\int_{\mathbb{R}^N\setminus\Omega_n} \frac{w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz,
\end{aligned}$$

let $r = c_1|\Omega_n|^{\frac{1}{N}}$ with $c_1 > 0$ such that $|\Omega_n| = |B_r^N(x_0)|$, so, since $\frac{1}{|z-x_0|^{N+2\alpha_1}}$ is radially symmetric and positive, we have that

$$-\int_{\mathbb{R}^N\setminus\Omega_n} \frac{w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz \geq -\int_{\mathbb{R}^N\setminus B_r^N(x_0)} \frac{w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz. \tag{2.6}$$

This inequality comes of simple calculations, but as it is a key point of the proof we will carry it out for the sake of the readers. Note first, that

$$|\Omega_n \setminus B_r^N(x_0)| = |B_r^N(x_0) \setminus \Omega_n|,$$

and $i \geq s$ for

$$i := \inf_{B_r^N(x_0)\setminus\Omega_n} \frac{1}{|z-x_0|^{N+2\alpha_1}} \text{ and } s := \sup_{\Omega_n\setminus B_r^N(x_0)} \frac{1}{|z-x_0|^{N+2\alpha_1}},$$

then

$$\begin{aligned}
\int_{B_r^N(x_0)\setminus\Omega_n} \frac{1}{|z-x_0|^{N+2\alpha_1}}dz &\geq |B_r^N(x_0)\setminus\Omega_n|.i \\
&= |\Omega_n \setminus B_r^N(x_0)|.i \\
&\geq |\Omega_n \setminus B_r^N(x_0)|.s \\
&\geq \int_{\Omega_n\setminus B_r^N(x_0)} \frac{1}{|z-x_0|^{N+2\alpha_1}}dz
\end{aligned}$$

and therefore, the inequality (2.6) follows by direct calculations taking into account that

$$\mathbb{R}^N \setminus \Omega_n = (B_r^N(x_0)\setminus\Omega_n)\dot{\cup}((\mathbb{R}^N \setminus B_r^N(x_0))\setminus\Omega_n).$$

Thus we can conclude that

$$\begin{aligned}
-(-\Delta)_x^{\alpha_1} w(x_0,y_0) &\geq -\int_{\mathbb{R}^N\setminus\Omega_n} \frac{w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz \\
&\geq -\int_{\mathbb{R}^N\setminus B_r^N(x_0)} \frac{w(x_0,y_0)}{|z-x_0|^{N+2\alpha_1}}dz \\
&= -c_2 w(x_0,y_0)|\Omega_n|^{-\frac{2\alpha_1}{N}},
\end{aligned}$$

for some $c_2 > 0$. Similarly, there exists $c_3 > 0$ such that

$$-(-\Delta)_y^{\alpha_2} w(x_0, y_0) \geq -\int\limits_{\mathbb{R}^M \setminus \Omega_m} \frac{w(x_0, y_0)}{|\tilde{z} - y_0|^{M+2\alpha_2}} d\tilde{z} = -c_3 w(x_0, y_0) |\Omega_m|^{-\frac{2\alpha_2}{M}}.$$

Then we have that

$$-(-\Delta)_x^{\alpha_1} w(x_0, y_0) - (-\Delta)_y^{\alpha_2} w(x_0, y_0) \geq -c_4 w(x_0, y_0)(|\Omega_n|^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}}),$$

where $c_4 = \min\{c_2, c_3\}$. Combining with (2.5), we obtain that

$$\|h\|_{L^\infty(\Omega)} \geq -c_4 w(x_0, y_0)(|\Omega_n|^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}}).$$

Therefore,

$$-\inf_{\Omega} w = -w(x_0, y_0) \leq c_5 \|h\|_{L^\infty(\Omega)} (|\Omega_n|^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}})^{-1},$$

for some $c_5 > 0$. □

We remark that the estimate (2.4) allows us to apply the Maximum Principle for special small domains that should be narrow in x direction or in y direction, but does not allow to obtain the Maximum Principle for any small domain, since the projections $|\Omega_n|$ and $|\Omega_m|$ could be large although $|\Omega|$ is small.

As a consequence, we have the Maximum Principle for small domain, which is stated as follows:

Proposition 2.1. *Let Ω be a bounded open set and Ω_n and Ω_m be defined in (2.1) and (2.2) respectively. Suppose that $\varphi : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}^M)$ is a classical solution of*

$$\begin{cases} -(-\Delta)_x^{\alpha_1} w(x, y) - (-\Delta)_y^{\alpha_2} w(x, y) \leq \varphi(x, y) w(x, y), & (x, y) \in \Omega, \\ w(x, y) \geq 0, & (x, y) \in (\mathbb{R}^N \times \mathbb{R}^M) \setminus \Omega. \end{cases} \tag{2.7}$$

Then there is $\delta > 0$ such that whenever $|\Omega_n| \leq \delta$ or $|\Omega_m| \leq \delta$, w has to be non-negative in Ω.

Proof. Let $\Omega^- = \{(x, y) \in \Omega \mid w(x, y) < 0\}$. By (2.7), we observe that

$$\begin{cases} -(-\Delta)_x^{\alpha_1} w(x, y) - (-\Delta)_y^{\alpha_2} w(x, y) \leq \varphi(x, y) w(x, y), & (x, y) \in \Omega^-, \\ w(x, y) \geq 0, & (x, y) \in (\mathbb{R}^N \times \mathbb{R}^M) \setminus \Omega^-. \end{cases}$$

Then, using Lemma 2.1 with $h(x, y) = \varphi(x, y) w(x, y)$, there exists $C > 0$ such that

$$\|w\|_{L^\infty(\Omega^-)} = -\inf_{\Omega^-} w \leq C \|\varphi\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega^-)} (|\Omega_n|^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}})^{-1}.$$

Let $\delta > 0$ satisfying

$$C \|\varphi\|_{L^\infty(\Omega)} (\delta^{-\frac{2\alpha_1}{N}} + |\Omega_m|^{-\frac{2\alpha_2}{M}})^{-1} < 1$$

and

$$C \|\varphi\|_{L^\infty(\Omega)} (|\Omega_n|^{-\frac{2\alpha_1}{N}} + \delta^{-\frac{2\alpha_2}{M}})^{-1} < 1,$$

then if we take $|\Omega_n| \le \delta$ or $|\Omega_m| \le \delta$ we have that

$$\|w\|_{L^\infty(\Omega^-)} = 0.$$

This implies that $|\Omega^-| = 0$ and since Ω^- is open, we have that $\Omega^- = \emptyset$, completing the proof. □

3. Proof of Theorem 1.1

Theorem 3.1. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is locally Lipschitz. If u is a positive classical solution of equation (1.3), then*

$$u(x, y) = u(|x|, y).$$

Proof. For $\lambda \in (0, 1)$, we denote $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{N-1}$,

$$\begin{gathered}\Sigma_\lambda = \{(x_1, x', y) \in B_1^N(0) \times B_1^M(0) \mid x_1 > \lambda,\ (x_1, x') \in B_1^N(0)\},\\ T_\lambda = \{(x_1, x', y) \in \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}^M \mid x_1 = \lambda\},\\ u_\lambda(x_1, x', y) = u(2\lambda - x_1, x', y),\\ w_\lambda(x_1, x', y) = u_\lambda(x_1, x', y) - u(x_1, x', y).\end{gathered}$$

Step 1: We prove that if $\lambda \in (0, 1)$ is close to 1, then $w_\lambda > 0$ in Σ_λ. For this purpose, we start proving that if $\lambda \in (0, 1)$ is close to 1, then $w_\lambda \ge 0$ in Σ_λ. If we define $\Sigma_\lambda^- = \{(x, y) \in \Sigma_\lambda \mid w_\lambda(x, y) < 0\}$, then we just need to prove that if $\lambda \in (0, 1)$ is close to 1, then

$$\Sigma_\lambda^- = \emptyset. \tag{3.1}$$

By contradiction, we assume that (3.1) is not true, that is, $\Sigma_\lambda^- \neq \emptyset$. We denote

$$w_\lambda^+(x, y) = \begin{cases} w_\lambda(x, y), & (x, y) \in \Sigma_\lambda^-, \\ 0, & (x, y) \in (\mathbb{R}^N \times \mathbb{R}^M) \setminus \Sigma_\lambda^-, \end{cases} \tag{3.2}$$

$$w_\lambda^-(x, y) = \begin{cases} 0, & (x, y) \in \Sigma_\lambda^-, \\ w_\lambda(x, y), & (x, y) \in (\mathbb{R}^N \times \mathbb{R}^M) \setminus \Sigma_\lambda^- \end{cases} \tag{3.3}$$

and we observe that $w_\lambda^+(x, y) = w_\lambda(x, y) - w_\lambda^-(x, y)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^M$. We reason in a similar way as in the proof of Theorem 1.1 in [8] to obtain that for all $0 < \lambda < 1$,

$$(-\Delta)_x^{\alpha_1} w_\lambda^-(x, y) \le 0, \quad \forall (x, y) \in \Sigma_\lambda^- \tag{3.4}$$

and

$$(-\Delta)_y^{\alpha_2} w_\lambda^-(x, y) \le 0, \quad \forall (x, y) \in \Sigma_\lambda^-. \tag{3.5}$$

Then combining (3.4) with (3.5) and using the linearity of the fractional Laplacian, we have that for $(x, y) \in \Sigma_\lambda^-$,

$$\begin{aligned}(-\Delta)_x^{\alpha_1} w_\lambda^+(x, y) + (-\Delta)_y^{\alpha_2} w_\lambda^+(x, y) &\ge (-\Delta)_x^{\alpha_1} w_\lambda(x, y) + (-\Delta)_y^{\alpha_2} w_\lambda(x, y)\\ &= (-\Delta)_x^{\alpha_1} u_\lambda(x, y) - (-\Delta)_x^{\alpha_1} u(x, y) + (-\Delta)_y^{\alpha_2} u_\lambda(x, y) - (-\Delta)_y^{\alpha_2} u(x, y)\end{aligned}$$

$$= f(u_\lambda(x,y)) - f(u(x,y))$$
$$= \frac{f(u_\lambda(x,y)) - f(u(x,y))}{u_\lambda(x,y) - u(x,y)} w_\lambda^+(x,y).$$

Let us define $\varphi(x,y) = -(f(u_\lambda(x,y)) - f(u(x,y)))/(u_\lambda(x,y) - u(x,y))$ for $(x,y) \in \Sigma_\lambda^-$. Since f is locally Lipschitz, we have that $\varphi \in L^\infty(\Sigma_\lambda^-)$. Then we have that

$$-(-\Delta)_x^{\alpha_1} w_\lambda^+(x,y) - (-\Delta)_y^{\alpha_2} w_\lambda^+(x,y) \le \varphi(x,y) w_\lambda^+(x,y), \qquad (x,y) \in \Sigma_\lambda^- \tag{3.6}$$

and since $w_\lambda^+ = 0$ in $(\mathbb{R}^N \times \mathbb{R}^M) \setminus \Sigma_\lambda^-$, w_λ^+ is a continuous bounded function, by the fact that u is continuous and bounded, we may apply Proposition 2.1. Choosing $\lambda \in (0,1)$ close enough to 1 we observe that there exist $\Sigma_\lambda^N \subset B_1^N(0) \subset \mathbb{R}^N$ and $\Sigma_\lambda^M \subset B_1^M(0) \subset \mathbb{R}^M$ such that $\Sigma_\lambda^- \subset \Sigma_\lambda^N \times \Sigma_\lambda^M$ and $|\Sigma_\lambda^N|$ is small, then

$$w_\lambda = w_\lambda^+ \ge 0 \quad \text{in} \quad \Sigma_\lambda^-.$$

But this is a contradiction with our assumption so we have that

$$w_\lambda \ge 0 \quad \text{in} \quad \Sigma_\lambda.$$

In order to complete Step 1, we claim that for $0 < \lambda < 1$, if $w_\lambda \ge 0$ and $w_\lambda \not\equiv 0$ in Σ_λ, then $w_\lambda > 0$ in Σ_λ. Assuming that the claim is true, we complete the proof, since the function u is positive in $B_1^N(0) \times B_1^M(0)$ and $u = 0$ on $\partial(B_1^N(0) \times B_1^M(0))$, so that w_λ is positive in $\partial(B_1^N(0) \times B_1^M(0)) \cap \partial\Sigma_\lambda$ and then, by continuity $w_\lambda \not\equiv 0$ in Σ_λ.

Now we prove the claim. Assume that there exists $(x_0, y_0) \in \Sigma_\lambda$ such that $w_\lambda(x_0, y_0) = 0$, that is, $u_\lambda(x_0, y_0) = u(x_0, y_0)$. Then we have that

$$\begin{aligned}
&(-\Delta)_x^{\alpha_1} w_\lambda(x_0,y_0) + (-\Delta)_y^{\alpha_2} w_\lambda(x_0,y_0) \\
&\quad = (-\Delta)_x^{\alpha_1}(u_\lambda - u)(x_0,y_0) + (-\Delta)_y^{\alpha_2}(u_\lambda - u)(x_0,y_0) \\
&\quad = f(u_\lambda(x_0,y_0)) - f(u(x_0,y_0)) = 0.
\end{aligned} \tag{3.7}$$

On the other hand, defining $A_\lambda = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^N \mid x_1 > \lambda\}$, since $w_\lambda(z_\lambda, y) = -w_\lambda(z,y)$ for any $(z,y) \in \mathbb{R}^N \times \mathbb{R}^M$ and $w_\lambda(x_0, y_0) = 0$, we find

$$\begin{aligned}
(-\Delta)_x^{\alpha_1} w_\lambda(x_0,y_0) &= \int_{A_\lambda} \frac{-w_\lambda(z,y_0)}{|x_0 - z|^{N+2\alpha_1}} dz + \int_{\mathbb{R}^N \setminus A_\lambda} \frac{-w_\lambda(z,y_0)}{|x_0 - z|^{N+2\alpha_1}} dz \\
&= \int_{A_\lambda} \frac{-w_\lambda(z,y_0)}{|x_0 - z|^{N+2\alpha_1}} dz + \int_{A_\lambda} \frac{-w_\lambda(z_\lambda,y_0)}{|x_0 - z_\lambda|^{N+2\alpha_1}} dz \\
&= \int_{A_\lambda} w_\lambda(z,y_0) \left(\frac{1}{|x_0 - z_\lambda|^{N+2\alpha_1}} - \frac{1}{|x_0 - z|^{N+2\alpha_1}} \right) dz.
\end{aligned} \tag{3.8}$$

Since $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in A_\lambda$ and $w_\lambda \ge 0$ in Σ_λ, then we get

$$(-\Delta)_x^{\alpha_1} w_\lambda(x_0,y_0) \le 0. \tag{3.9}$$

We observe that $\{x_0\} \times B_1^M(0) \subset \Sigma_\lambda$ and by $w_\lambda \ge 0$ in Σ_λ and $w_\lambda(x_0,y_0) = 0$, then we have that

$$(-\Delta)_y^{\alpha_2} w_\lambda(x_0, y_0) = -\int_{\mathbb{R}^N} \frac{w_\lambda(x_0, \tilde{z})}{|y_0 - \tilde{z}|^{M+2\alpha_2}} d\tilde{z}$$
$$= -\int_{B_1^M(0)} \frac{w_\lambda(x_0, \tilde{z})}{|y_0 - \tilde{z}|^{M+2\alpha_2}} d\tilde{z} \le 0. \tag{3.10}$$

Combining (3.7), (3.9) and (3.10), we have that

$$(-\Delta)_x^{\alpha_1} w_\lambda(x_0, y_0) = 0, \qquad (-\Delta)_y^{\alpha_2} w_\lambda(x_0, y_0) = 0,$$

and by (3.8), we have that $w_\lambda(z, y_0) = 0$ for $z \in A_\lambda$, then $w_\lambda(z, y_0) = 0$ for $z \in \mathbb{R}^N$ by the first equality in (3.8). Using similar way, we obtain that

$$w_\lambda(x_0, \tilde{z}) = 0, \quad \forall \tilde{z} \in \mathbb{R}^M \quad \text{and} \quad w_\lambda(z, y_0) = 0, \quad \forall z \in \mathbb{R}^N. \tag{3.11}$$

Since $w_\lambda \not\equiv 0$ and $w_\lambda \ge 0$ in Σ_λ, then there exists $(\tilde{x}, \tilde{y}) \in \Sigma_\lambda$ such that $w_\lambda(\tilde{x}, \tilde{y}) > 0$. Let

$$A = \{(z, \tilde{y}) \in \mathbb{R}^N \times \mathbb{R}^M \text{ s.t. } z \in \mathbb{R}^N \text{ and } (z, \tilde{y}) \in \Sigma_\lambda\}$$

and

$$B = \{(\tilde{x}, \tilde{z}) \in \mathbb{R}^N \times \mathbb{R}^M \text{ s.t. } \tilde{z} \in \mathbb{R}^M \text{ and } (\tilde{x}, \tilde{z}) \in \Sigma_\lambda\}.$$

We note that A and B are the extensions in Σ_λ of the point $(\tilde{x}, \tilde{y})$ in the x-direction and y-direction respectively. Now we show that $w_\lambda > 0$ in $A \cup B$. In fact, if there is $(\tilde{x}_0, \tilde{y}_0) \in A \cup B$ satisfying $w_\lambda(\tilde{x}_0, \tilde{y}_0) = 0$, by the above argument, we have that

$$w_\lambda(z, \tilde{y}_0) = 0, \quad \forall z \in \mathbb{R}^N \quad \text{and} \quad w_\lambda(\tilde{x}_0, \tilde{z}) = 0, \quad \forall \tilde{z} \in \mathbb{R}^M,$$

that is,

$$w_\lambda = 0 \quad \text{in} \quad (\mathbb{R}^N \times \{\tilde{y}_0\}) \cup (\{\tilde{x}_0\} \times \mathbb{R}^M). \tag{3.12}$$

But we observe that $(\tilde{x}, \tilde{y}) \in (\mathbb{R}^N \times \{\tilde{y}_0\}) \cup (\{\tilde{x}_0\} \times \mathbb{R}^M)$ and $w_\lambda(\tilde{x}, \tilde{y}) > 0$, which contradicts (3.12). Thus,

$$w_\lambda > 0 \quad \text{in} \quad A \cup B. \tag{3.13}$$

We observe that $(A \cup B) \cap ((\mathbb{R}^N \times \{y_0\}) \cup (\{x_0\} \times \mathbb{R}^M)) \ne \emptyset$, then (3.13) is in contradiction with (3.11). As a consequence, we have that $w_\lambda > 0$ in Σ_λ.

Step 2: We define $\lambda_0 = \inf\{\lambda \in (0,1) \mid w_\lambda > 0 \quad \text{in} \quad \Sigma_\lambda\}$ and we prove that $\lambda_0 = 0$. Proceeding by contradiction, we assume that $\lambda_0 > 0$, then $w_{\lambda_0} \ge 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}. Thus, by the claim just proved above, we have $w_{\lambda_0} > 0$ in Σ_{λ_0}.

Next we claim that if $w_\lambda > 0$ in Σ_λ for $\lambda \in (0, 1)$, then there exists $\epsilon \in (0, \lambda)$ such that $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon = \lambda - \epsilon$. This claim directly implies that $\lambda_0 = 0$, completing Step 2.

Now we prove the claim. Let $D_\mu = \{(x, y) \in \Sigma_\lambda \mid dist((x, y), \partial\Sigma_\lambda) \ge \mu\}$ for $\mu > 0$ small. Since $w_\lambda > 0$ in Σ_λ and D_μ is compact, then there exists $\mu_0 > 0$ such that $w_\lambda \ge \mu_0$ in D_μ. By the continuity of $w_\lambda(x, y)$ with respect to λ, for $\epsilon > 0$ small enough and denoting $\lambda_\epsilon = \lambda - \epsilon$, we have that

$$w_{\lambda_\epsilon}(x, y) \ge 0 \quad \text{in} \quad D_\mu.$$

As a consequence,

$$\Sigma^-_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$$

and $|\Sigma^-_{\lambda_\epsilon}|$ is small if ϵ and μ are small. Using (3.4) and proceeding as in Step 1, we have for all $(x, y) \in \Sigma^-_{\lambda_\epsilon}$ that

$$\begin{aligned}
&(-\Delta)^{\alpha_1}_x w^+_{\lambda_\epsilon}(x, y) + (-\Delta)^{\alpha_2}_y w^+_{\lambda_\epsilon}(x, y)\\
&\quad \geq (-\Delta)^{\alpha_1}_x w_{\lambda_\epsilon}(x, y) + (-\Delta)^{\alpha_2}_y w_{\lambda_\epsilon}(x, y)\\
&\quad = (-\Delta)^{\alpha_1}_x u_{\lambda_\epsilon}(x, y) - (-\Delta)^{\alpha_1}_x u(x, y) + (-\Delta)^{\alpha_2}_y u_{\lambda_\epsilon}(x, y) - (-\Delta)^{\alpha_2}_y u(x, y)\\
&\quad = f(u_{\lambda_\epsilon}(x, y)) - f(u(x, y)),
\end{aligned}$$

then

$$-(-\Delta)^{\alpha}_x w^+_{\lambda_\epsilon}(x, y) - (-\Delta)_y w^+_{\lambda_\epsilon}(x, y) \leq \varphi(x, y) w^+_{\lambda_\epsilon}(x, y),$$

where $\varphi(x, y) = -\frac{f(u_{\lambda_\epsilon}(x,y)) - f(u(x,y))}{u_{\lambda_\epsilon}(x,y) - u(x,y)}$ is bounded, since f is locally Lipschitz.

Since $w^+_{\lambda_\epsilon} = 0$ in $(\mathbb{R}^N \times \mathbb{R}^M) \setminus \Sigma^-_{\lambda_\epsilon}$, $w^+_{\lambda_\epsilon}$ is a continuous bounded function, by the fact that u is continuous and bounded, $|\Sigma^-_{\lambda_\epsilon}|$ is small, for ϵ and μ small, Proposition 2.1 implies that $w_{\lambda_\epsilon} \geq 0$ in $\Sigma_{\lambda_\epsilon}$. Thus, since $\lambda_\epsilon > 0$ and $w_{\lambda_\epsilon} \not\equiv 0$ in $\Sigma_{\lambda_\epsilon}$, as before we have $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, completing the proof of the claim.

Step 3: By Step 2, we have that $\lambda_0 = 0$, which implies that $u(-x_1, x', y) \geq u(x_1, x', y)$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x', y) \leq u(x_1, x', y)$ for $x_1 \geq 0$ and then $u(-x_1, x', y) = u(x_1, x', y)$ for $x_1 \geq 0$. Repeating this procedure in any x-direction, we conclude that

$$u(x, y) = u(|x|, y).$$

Finally, we prove that for any given $y \in B^M_1(0)$, $u(r, y)$ is strictly decreasing in $r \in (0, 1)$. Let us consider $0 < x_1 < \widetilde{x}_1 < 1$ and let $\lambda = \frac{x_1 + \widetilde{x}_1}{2}$. Then, as proved above we have that

$$w_\lambda(x, y) > 0 \quad \text{for} \quad (x, y) \in \Sigma_\lambda.$$

Then for any given $y \in B^M_1(0)$, we obtain that

$$\begin{aligned}
0 < w_\lambda(\widetilde{x}_1, 0, \cdots, 0, y) &= u_\lambda(\widetilde{x}_1, 0, \cdots, 0, y) - u(\widetilde{x}_1, 0, \cdots, 0, y)\\
&= u(x_1, 0, \cdots, 0, y) - u(\widetilde{x}_1, 0, \cdots, 0, y),
\end{aligned}$$

that is, $u(x_1, 0, \cdots, 0, y) > u(\widetilde{x}_1, 0, \cdots, 0, y)$. Using the symmetry result of u with respect to x, we conclude that $u(r, y)$ is strictly decreasing in $r \in (0, 1)$. □

Using the same argument, we also can obtain the symmetry result on y-direction.

Theorem 3.2. *Assume that the function $f : [0, \infty) \to \mathbb{R}$ is locally Lipschitz. If u is a positive classical solution of equation (1.3), then*

$$u(x, y) = u(x, |y|).$$

Proof of Theorem 1.1. From Theorem 3.1, we obtain that the solution u is symmetric in x, that is,

$$u(x, y) = u(|x|, y).$$

Together with Theorem 3.2, we have that

$$u(x, y) = u(|x|, |y|).$$

The proof ends. □

Remark 3.1. We note that our method could be extended into the semilinear elliptic problem involving the fractional Laplacians in multiple directions, such as

$$(-\Delta)_x^{\alpha_1} + (-\Delta)_y^{\alpha_2} + (-\Delta)_z^{\alpha_3}.$$

However, when we treat with a sum of the Laplacian and a fractional Laplacian,

$$(-\Delta)_x + (-\Delta)_y^{\alpha},$$

it is not clear for us if it is possible to obtain a result similar to the Lemma 2.1 to this case. However, we believe be possible to prove a maximum principle for small domains for this case and thus the method of the moving planes used here drives us to a symmetry result. We hope treat this in future works.

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