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On Wigner transforms in infinite dimensions

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We investigate the Schrödinger representations of certain infinite-dimensional Heisenberg groups, using their corresponding Wigner transforms. [http://dx.doi.org/10.1063/1.4941328]

I. INTRODUCTION

The topic of this paper belongs to representation theory of Heisenberg groups, more specifically we investigate to what extent the square-integrability properties of the Schrödinger representations carry over to the setting of infinite-dimensional Heisenberg groups. Recall that the Schrödinger representation of the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_{2n+1}$ is a group representation

$$\pi\colon \mathbb{H}_{2n+1} \to \mathcal{B}(L^2(\mathbb{R}^n, \mathrm{d}\lambda))$$

and its corresponding Wigner transform is a unitary operator

$$\mathcal{W}\colon L^2(\mathbb{R}^n, \mathrm{d}\lambda)\bar{\otimes}\overline{L^2(\mathbb{R}^n, \mathrm{d}\lambda)} \to L^2(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{d}\mu),$$

where we denote by $\mathrm{d}\lambda$ the Lebesgue measure on $\mathbb{R}^n$, by $\mathcal{B}(L^2(\mathbb{R}^n, \mathrm{d}\lambda))$ the bounded linear operators on the complex Hilbert space $L^2(\mathbb{R}^n, \mathrm{d}\lambda)$, and by $\mathrm{d}\mu$ a suitable normalization of the measure $\mathrm{d}\lambda \otimes \mathrm{d}\lambda$ on $\mathbb{R}^n \times \mathbb{R}^n$. One way to define the Wigner transform is that of defining $\mathcal{W}(f \otimes \overline{\varphi}) \in L^2(\mathbb{R}^n \times \mathbb{R}^n, \mathrm{d}\mu)$ as a Fourier transform of the representation coefficient $(\pi(\cdot)f \mid \varphi)|_{\mathbb{R}^n\times\mathbb{R}^n\times\{0\}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n \times \{0\}, \mathrm{d}\lambda \otimes \mathrm{d}\lambda)$, recalling that $\mathbb{H}_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ as smooth manifolds.

As the translation invariance property of the Lebesgue measure plays a central role in the above discussion, it is not straightforward to replace here $\mathbb{R}^n$ by an infinite-dimensional real Hilbert space. It is customary in the infinite-dimensional analysis to try to replace the Lebesgue measure by a Gaussian measure, and this is what we will do in the present paper as well. Furthermore, the main problems that we must address are to construct the Wigner transform as a unitary operator on square-integrable functions of infinitely many variables and to realize the image of that unitary operator as an L^2-space, which amounts to determining the infinite-dimensional analogue of the measure $\mathrm{d}\mu$ from the above paragraph. In some sense, these problems form a complement to the ones addressed in our recent investigation of square-integrable families of operators.[7]

We also take this opportunity to mention the interesting recent investigation[3] of Weyl calculus and Wigner functions in infinite dimensions, leading to a very deep L^2-boundedness property of the corresponding pseudodifferential operators in Ref. 3 [Theorem 1.4]. Our approach is conducted in a completely different manner, using infinite tensor products and Fourier transforms on uniform spaces, and our main aim is to investigate infinite-dimensional versions of the specific isometry property of the Weyl calculus for an adequate space of square-integrable symbols.

The present paper is organized as follows. Section II develops an abstract framework for the study of Wigner transforms associated to unitary representations of general topological groups. The main ingredients of that framework are the Fourier transforms on uniform spaces and an

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operator calculus that involves the Banach algebra structures of the dual of the spaces of left uniformly continuous functions on topological groups. Then Section III records some computations with Gaussian functions and their Wigner transforms. In Section IV we introduce the infinite-dimensional Heisenberg groups and their Schrödinger representations for which we construct their corresponding Wigner transforms in Theorem 4.6. Finally, the Appendix records some auxiliary results on Fourier transforms on uniform spaces.

Throughout this paper, we denote by $\bar{\otimes}$ the Hilbertian scalar product and by $\mathcal{B}(\mathcal{X})$ and $\mathcal{X}'$ the spaces of all bounded linear operators and bounded linear functionals on some Banach space $\mathcal{X}$, respectively, and it will always be clear from the context if the ground field is $\mathbb{R}$ or $\mathbb{C}$.

II. OPERATOR CALCULUS ON TOPOLOGICAL GROUPS

In this section we introduce an operator calculus for unitary representations of topological groups, since it will allow us to handle in Section IV some representations of infinite-dimensional Heisenberg groups, which are Lie groups modeled on Hilbert spaces. In the case of Banach-Lie groups, the space of continuous 1-parameter subgroups and exponential map, as defined below, agree with the usual notions of Lie algebra and exponential map from the Lie theoretic setting (see Ref. 22 for extensive information in this connection).

Let G be a topological group endowed with the right uniform structure. We recall that a basis of this uniform structure is provided by the sets

$$S_V^{\lambda} = \{(x,y) \in G \times G \mid yx^{-1} \in V\},$$

where $V \in \mathcal{V}_G(\mathbf{1})$. Consider the corresponding space of mappings

$$\mathfrak{L}(G) = \{X : \mathbb{R} \to G \mid X \text{ homomorphism of topological groups}\}$$

with the structure of uniform convergence on the compact subsets of $\mathbb{R}$. Hence, a basis of this uniform structure consists of the entourages

$$S_{n,V}^{\lambda} = \{(X,Y) \in \mathfrak{L}(G) \times \mathfrak{L}(G) \mid (\forall t \in [-n,n]) \quad Y(t)X(t)^{-1} \in V\}$$

parameterized by $V \in \mathcal{V}_G(\mathbf{1})$ and $n \in \mathbb{N}$ (see Ref. 18 [Definition 2.6]).

It is easily seen that the exponential mapping

$$\exp_G : \mathfrak{L}(G) \to G, \quad \exp_G X := X(1)$$

is uniformly continuous, hence it gives rise to a unital $*$-homomorphism of C^*-algebras

$$\mathcal{UC}_b(\exp_G) : \mathcal{LUC}_b(G) \to \mathcal{UC}_b(\mathfrak{L}(G)), \quad f \mapsto f \circ \exp_G$$

with the dual map

$$\mathcal{UC}_b{}'(\exp_G) : (\mathcal{UC}_b(\mathfrak{L}(G)))' \to (\mathcal{LUC}_b(G))'.$$

We recall from Ref. 15 (see also Ref. 4 [Theorem 3.9]) that $(\mathcal{LUC}_b(G))'$ is a unital associative Banach algebra in a natural way and every continuous unitary representation $\pi : G \to \mathcal{B}(\mathcal{H}_\pi)$ gives rise to a Banach algebra representation

$$\widehat{\pi}_{\mathcal{L}} : (\mathcal{LUC}_b(G))' \to \mathcal{B}(\mathcal{H})$$

such that $(\widehat{\pi}_{\mathcal{L}}(\nu)\phi \mid \psi) = \langle \nu, (\pi(\cdot)\phi \mid \psi)\rangle$ for $\phi,\psi \in \mathcal{H}$ and $\nu \in (\mathcal{LUC}_b(G))'$. Therefore, we get a bounded linear operator

$$\widetilde{\pi}_{\mathcal{L}} : (\mathcal{UC}_b(\mathfrak{L}(G)))' \to \mathcal{B}(\mathcal{H})$$

such that the diagram

$$\begin{tikzcd}
(\mathcal{LUC}_b(G))' \arrow[r, "\widehat{\pi}_{\mathcal{L}}"] & \mathcal{B}(\mathcal{H}) \\
(\mathcal{UC}_b(\mathfrak{L}(G)))' \arrow[u, "\mathcal{UC}_b{}'(\exp_G)"] \arrow[ur, "\widetilde{\pi}_{\mathcal{L}}"'] &
\end{tikzcd}$$

is commutative.

Setting 2.1. Throughout the rest of this section, we fix a continuous unitary representation $\pi\colon G \to \mathcal{B}(\mathcal{H})$ of the above topological group G on the complex Hilbert space $\mathcal{H}$ and we assume the setting defined by the following data, where we use notation from Definition A.1:

- a uniform space Ξ and a uniformly continuous map $\theta\colon \Xi \to \mathfrak{L}(G)$,
- a locally convex space Γ such that there exists an injective continuous inclusion map $\Gamma \hookrightarrow \mathcal{M}(\mathfrak{L}(G)^{\nabla})$,
- a locally convex space $\mathcal{H}_{\Xi,\infty}$ such that there exists an injective continuous inclusion map $\mathcal{H}_{\Xi,\infty} \hookrightarrow \mathcal{H}$.

Also let $\eta_{\mathfrak{L}(G)}\colon \mathfrak{L}(G) \to (\mathfrak{L}(G)^{\nabla})^{\nabla}$ be the uniformly continuous map defined in Remark A.2.

Definition 2.2. We say that Γ and θ are *compatible* if the linear mapping

$$\mathcal{F}_{\Xi}\colon \Gamma \to \mathcal{UC}_b(\Xi), \quad \mu \mapsto \widehat{\mu} \circ \eta_{\mathfrak{L}(G)} \circ \theta$$

is well defined in the sense that it takes values in $\mathcal{UC}_b(\Xi)$ as indicated here, and is injective.

If this is the case, then we denote $Q_{\Xi} := \mathcal{F}_{\Xi}(\Gamma) \hookrightarrow \mathcal{UC}_b(\Xi)$ and endow it with the topology which makes the *Fourier transform*

$$\mathcal{F}_{\Xi}\colon \Gamma \to Q_{\Xi}$$

into a linear topological isomorphism. We then also have the linear topological isomorphism $(\mathcal{F}_{\Xi}')^{-1}\colon \Gamma' \to Q_{\Xi}'$.

Lemma 2.3. If Γ and θ are compatible, then the following conditions are equivalent:

1. *We have the well-defined continuous sesquilinear mapping*
$$\mathcal{A}^{\pi,\theta}\colon \mathcal{H}_{\Xi,\infty} \times \mathcal{H}_{\Xi,\infty} \to Q_{\Xi}, \quad (\phi,\psi) \mapsto \mathcal{A}^{\pi,\theta}_{\psi}\phi := (\pi(\exp_G(\theta(\cdot)))\phi \mid \psi).$$
2. *There exists a unique continuous sesquilinear mapping*
$$\mathcal{W}\colon \mathcal{H}_{\Xi,\infty} \times \mathcal{H}_{\Xi,\infty} \to \Gamma,$$
such that for all $\phi,\psi \in \mathcal{H}_{\Xi,\infty}$ we have $\mathcal{F}_{\Xi}(\mathcal{W}(\phi,\psi)) = (\pi(\exp_G(\theta(\cdot)))\phi \mid \psi)$.

Proof. If condition (2) is satisfied, then $\mathcal{A}^{\pi,\theta} = \mathcal{F}_{\Xi} \circ \mathcal{W}$, hence condition (1) follows since $\mathcal{F}_{\Xi}\colon \Gamma \to Q_{\Xi}$ is a linear topological isomorphism. For the same reason, it also follows that if condition (1) holds true, then (2) is satisfied. ■

Definition 2.4. Assume that Γ and θ are compatible and the equivalent conditions in Lemma 2.3 are satisfied. Then the sesquilinear map $\mathcal{W}$ is called the *Wigner transform*. The *operator calculus for π along θ* is the linear map

$$\mathrm{Op}^{\theta}\colon \Gamma' \to \mathcal{L}(\mathcal{H}_{\Xi,\infty}, \overline{\mathcal{H}}'_{\Xi,\infty})$$

defined by

$$(\mathrm{Op}^{\theta}(a)\phi \mid \psi) = \langle \underbrace{(\mathcal{F}_{\Xi}')^{-1}(a)}_{\in Q_{\Xi}'}, \underbrace{(\pi(\exp_G(\theta(\cdot)))\phi \mid \psi)}_{\in Q_{\Xi}} \rangle \tag{2.1}$$

for $a \in \Gamma'$ and $\phi,\psi \in \mathcal{H}_{\Xi,\infty}$, where $\overline{\mathcal{H}}'_{\Xi,\infty}$ denotes the space of continuous antilinear functionals on $\mathcal{H}_{\Xi,\infty}$.

Remark 2.5. In the setting of Definition 2.4 we have for all $a \in \Gamma'$ and $\phi,\psi \in \mathcal{H}_{\Xi,\infty}$,

$$(\mathrm{Op}^{\theta}(a)\phi \mid \psi) = \langle (\mathcal{F}_{\Xi}')^{-1}(a), \mathcal{A}^{\pi,\theta}_{\psi}\phi \rangle = \langle a, \mathcal{F}_{\Xi}^{-1}(\mathcal{A}^{\pi,\theta}_{\psi}\phi) \rangle = \langle a, \mathcal{W}(\phi,\psi) \rangle,$$

where the later equality follows by Lemma 2.3(2).

Definition 2.6. Assume the setting of Definition 2.4. We say that the representation π satisfies the *orthogonality relations* along the mapping $\theta\colon \Xi \to \mathfrak{L}(G)$ if the following conditions are satisfied:

1. The linear subspace $\mathcal{H}_{\infty,\Xi}$ is dense in $\mathcal{H}$.
2. There exists a continuous, positive definite, sesquilinear, inner product on Γ such that if we denote by Γ_2 the corresponding Hilbert space obtained by completion, then the sesquilinear mapping $\mathcal{W}\colon \mathcal{H}_{\Xi,\infty}\times\mathcal{H}_{\Xi,\infty}\to\Gamma$ extends to a unitary operator
$$\mathcal{W}\colon \mathcal{H}\bar{\otimes}\overline{\mathcal{H}}\to\Gamma_2 \tag{2.2}$$
which is still called the *Wigner transform*.

The above definition extends the notion of orthogonality relations introduced in Ref. 6 for representations of some infinite-dimensional Lie groups, for which Ξ is a finite-dimensional vector space.

Remark 2.7. In Definition 2.6, since the inner product on Γ is continuous, it gives rise to a continuous injective map $\overline{\Gamma}_2\hookrightarrow\Gamma'$. By using the fact that operator (2.2) is an isometry, we easily get
$$(\forall\phi,\psi\in\mathcal{H})\quad \mathrm{Op}^\theta(\mathcal{W}(\phi,\psi))=(\cdot\mid\psi)\phi.$$

III. SOME COMPUTATIONS INVOLVING GAUSSIAN MEASURES

This section records some auxiliary facts that will be needed in the proof of Theorem 4.6.

Notation 3.1. We shall use the following notation:

1. For $x=(x_1,\dots,x_m)\in\mathbb{R}^m$ we set $x'=(x_1,\dots,x_{m-1})\in\mathbb{R}^{m-1}$ if $m\ge 2$, hence
$$x=(x',x_m)\in\mathbb{R}^{m-1}\times\mathbb{R}.$$
2. For every $t>0$, we denote
$$(\forall x\in\mathbb{R})\quad \gamma_t(x)=\Big(\frac{1}{2\pi t}\Big)^{1/2}\mathrm{e}^{-\frac{x^2}{2t}},$$
so that $\gamma_t(x)\mathrm{d}x$ is the centered Gaussian probability measure on $\mathbb{R}$ with *variance* t (and mean 0).

Remark 3.2. We recall that
$$\int_{\mathbb{R}}\gamma_t(x)\mathrm{e}^{\mathrm{i}vx}\mathrm{d}x=\Big(\frac{2\pi}{t}\Big)^{1/2}\gamma_{1/t}(v)=\mathrm{e}^{-\frac{tv^2}{2}} \tag{3.1}$$
for all $v\in\mathbb{R}$ and $t>0$.

Remark 3.3. For any integer $m\ge 1$, recall the unitary operator that gives the integral kernels of operators obtained by classical Weyl calculus with L^2-symbols
$$T_m\colon L^2(\mathbb{R}^m\times(\mathbb{R}^m)')\to L^2(\mathbb{R}^m\times\mathbb{R}^m)$$
defined by
$$(T_m(a))(x,y)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}a(\frac{x+y}{2},\xi)\mathrm{e}^{\mathrm{i}\langle x-y,\xi\rangle}\mathrm{d}\xi,$$
with its inverse
$$T_m^{-1}\colon L^2(\mathbb{R}^m\times\mathbb{R}^m)\to L^2(\mathbb{R}^m\times(\mathbb{R}^m)')$$
given by
$$(T_m^{-1}(K))(x,\xi)=\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m}K(x+\frac{v}{2},x-\frac{v}{2})\mathrm{e}^{-\mathrm{i}\langle v,\xi\rangle}\mathrm{d}v, \tag{3.2}$$
for every $m\ge 1$.

Then for arbitrary $m_1, m_2 \geq 1$, there exist the natural unitary operators

$$V\colon L^2(\mathbb{R}^{m_1}\times\mathbb{R}^{m_1})\bar{\otimes}L^2(\mathbb{R}^{m_2}\times\mathbb{R}^{m_2}) \to L^2(\mathbb{R}^{m_1+m_2}\times\mathbb{R}^{m_1+m_2})$$

given by

$$(V(f_1\otimes f_2))((x_1,x_2),(y_1,y_2)) = f_1(x_1,y_1)f_2(x_2,y_2)$$

for $x_j, y_j \in \mathbb{R}^{m_j}$, $f_j \in L^2(\mathbb{R}^{m_j}\times\mathbb{R}^{m_j})$, $j = 1,2$, and

$$W\colon L^2(\mathbb{R}^{m_1}\times(\mathbb{R}^{m_1})')\bar{\otimes}L^2(\mathbb{R}^{m_2}\times(\mathbb{R}^{m_2})') \to L^2(\mathbb{R}^{m_1+m_2}\times(\mathbb{R}^{m_1+m_2})')$$

given by

$$(W(g_1\otimes g_2))((x_1,x_2),(\xi_1,\xi_2)) = g_1(x_1,\xi_1)g_2(x_2,\xi_2)$$

for $x_j \in \mathbb{R}^{m_j}$, $\xi_j \in L^2((\mathbb{R}^{m_j})')$, $f_j \in L^2(\mathbb{R}^{m_j}\times(\mathbb{R}^{m_j})')$, $j = 1,2$, and the diagram

$$\begin{tikzcd}
L^2(\mathbb{R}^{m_1}\times(\mathbb{R}^{m_1})')\bar{\otimes}L^2(\mathbb{R}^{m_2}\times(\mathbb{R}^{m_2})') \arrow[r, "W"] \arrow[d, "T_{m_1}\otimes T_{m_2}"'] & L^2(\mathbb{R}^{m_1+m_2}\times(\mathbb{R}^{m_1+m_2})') \arrow[d, "T_{m_1+m_2}"] \\
L^2(\mathbb{R}^{m_1}\times\mathbb{R}^{m_1})\bar{\otimes}L^2(\mathbb{R}^{m_2}\times\mathbb{R}^{m_2}) \arrow[r, "V"] & L^2(\mathbb{R}^{m_1+m_2}\times\mathbb{R}^{m_1+m_2})
\end{tikzcd}$$

is commutative.

The following simple lemma plays a key role in the present paper since it allows us to determine integral kernels of operators on Gaussian L^2-spaces in terms of integral kernels on Lebesgue L^2-spaces.

Lemma 3.4. The unitary operator $T_1\colon L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ defined by

$$(\forall a \in L^2(\mathbb{R}^2)) \quad (T_1(a))(x,y) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} a\Big(\frac{x+y}{2},\xi\Big)\mathrm{e}^{\mathrm{i}(x-y)\xi}\mathrm{d}\xi$$

has the property

$$(\forall t>0) \quad T_1^{-1}((\gamma_t\otimes\gamma_t)^{1/2}) = (\gamma_{t/2}\otimes\gamma_{1/8t})^{1/2}.$$

Proof. Let us denote $K = (\gamma_t\otimes\gamma_t)^{1/2}$. Then we have

$$K(x,y) = \Big(\frac{1}{2\pi t}\Big)^{1/2}\mathrm{e}^{-\frac{x^2+y^2}{4t}},$$

hence, using (3.2), one obtains

$$\begin{aligned}
(T_1^{-1}(K))(x,\xi) &= \frac{1}{2\pi t^{1/2}}\int_{\mathbb{R}}\mathrm{e}^{-\frac{1}{4t}\left((x+\frac{v}{2})^2+(x-\frac{v}{2})^2\right)}\mathrm{e}^{-\mathrm{i}v\xi}\mathrm{d}v \\
&= \frac{1}{2\pi t^{1/2}}\mathrm{e}^{-\frac{x^2}{2t}}\int_{\mathbb{R}}\mathrm{e}^{-\frac{v^2}{8t}}\mathrm{e}^{-\mathrm{i}v\xi}\mathrm{d}v \\
&= \frac{1}{2\pi t^{1/2}}\mathrm{e}^{-\frac{x^2}{2t}}(8\pi t)^{1/2}\int_{\mathbb{R}}\gamma_{4t}(v)\mathrm{e}^{\mathrm{i}v\xi}\mathrm{d}v \\
&= \Big(\frac{2}{\pi}\Big)^{1/2}\mathrm{e}^{-\frac{x^2}{2t}}\mathrm{e}^{-2t\xi^2} \\
&= \Big(\Big(\frac{1}{\pi t}\Big)^{1/2}\mathrm{e}^{-\frac{x^2}{t}}\Big(\frac{4t}{\pi}\Big)^{1/2}\mathrm{e}^{-4t\xi^2}\Big)^{1/2} \\
&= (\gamma_{t/2}(x)\gamma_{1/8t}(\xi))^{1/2},
\end{aligned}$$

where the third from last equality follows by (3.1). This completes the proof. ∎

Proposition 3.5. Let $t_1 \geq t_2 \geq \cdots > 0$ and for $m = 1,2,\ldots$ define

$$\widetilde{\gamma}_m\colon \mathbb{R}^m\to\mathbb{R}, \quad \widetilde{\gamma}_m(x_1,\ldots,x_m) := \gamma_{t_1}(x_1)\cdots\gamma_{t_m}(x_m)$$

and the corresponding measure $\mathrm{d}\widetilde{\gamma}_m \coloneqq \widetilde{\gamma}_m(x)\mathrm{d}x$. *Then for* $m \geq 2$ *the diagram*

$$\begin{array}{ccc}
L^2(\mathbb{R}^{m-1}\times(\mathbb{R}^{m-1})') & \xrightarrow{\beta} & L^2(\mathbb{R}^{m}\times(\mathbb{R}^{m})') \\
\downarrow{\scriptstyle S_{m-1}} & & \downarrow{\scriptstyle S_m} \\
L^2(\mathbb{R}^{m-1}\times(\mathbb{R}^{m-1})') & \xrightarrow{\alpha} & L^2(\mathbb{R}^{m}\times(\mathbb{R}^{m})') \\
\downarrow{\scriptstyle T_{m-1}} & & \downarrow{\scriptstyle T_m} \\
L^2(\mathbb{R}^{m-1}\times\mathbb{R}^{m-1}) & \xrightarrow{\eta} & L^2(\mathbb{R}^{m}\times\mathbb{R}^{m}) \\
\downarrow{\scriptstyle U_{m-1}} & & \downarrow{\scriptstyle U_m} \\
L^2(\mathbb{R}^{m-1}\times\mathbb{R}^{m-1},\mathrm{d}\widetilde{\gamma}_{m-1}\times\mathrm{d}\widetilde{\gamma}_{m-1}) & \xrightarrow{\iota} & L^2(\mathbb{R}^{m}\times\mathbb{R}^{m},\mathrm{d}\widetilde{\gamma}_{m}\times\mathrm{d}\widetilde{\gamma}_{m})
\end{array}$$

is commutative, where the vertical arrows are unitary operators defined by

$$(S_\ell(b))(x_1,\dots,x_\ell,\xi_1,\dots,\xi_\ell) = \frac{1}{(t_1\cdots t_\ell)^2}\cdot b(\frac{x_1}{t_1},\dots,\frac{x_\ell}{t_\ell},\frac{\xi_1}{t_1},\dots,\frac{\xi_\ell}{t_\ell})$$
$$(T_\ell(a))(x,y) = \frac{1}{(2\pi)^{l/2}}\int\limits_{\mathbb{R}^\ell} a(\frac{x+y}{2},\xi)\mathrm{e}^{\mathrm{i}\langle x-y,\xi\rangle}\mathrm{d}\xi,$$
$$(U_\ell(K))(v,w) = K(v,w)\widetilde{\gamma}_\ell(v)^{-1/2}\widetilde{\gamma}_\ell(w)^{-1/2},$$

for $\ell \in \{m-1,m\}$ *and the horizontal arrows are isometries defined by*

$$\begin{aligned}
(\beta(b))(x,\xi) &= b(x',\xi')\gamma_{1/2}(x_m)^{1/2}\gamma_{1/8t_m^2}(\xi_m)^{1/2},\\
(\alpha(a))(x,\xi) &= a(x',\xi')\gamma_{t_m/2}(x_m)^{1/2}\gamma_{1/8t_m}(\xi_m)^{1/2},\\
(\eta(K))(x,y) &= K(x',y')\gamma_{t_m}(x_m)^{1/2}\gamma_{t_m}(y_m)^{1/2},\\
(\iota(Q))(v,w) &= Q(v',w').
\end{aligned}$$

Proof. It is clear that the horizontal arrows in the diagram are isometries, while the vertical arrows are unitary operators.

Moreover, for $K \in L^2(\mathbb{R}^{m-1}\times\mathbb{R}^{m-1})$ we have

$$\begin{aligned}
(U_m(\eta(K)))(v,w) &= (\eta(K))(v,w)\widetilde{\gamma}_m(v)^{-1/2}\widetilde{\gamma}_m(w)^{-1/2}\\
&= K(v',w')\gamma_{t_m}(v_m)^{1/2}\gamma_{t_m}(w_m)^{1/2}\widetilde{\gamma}_m(v)^{-1/2}\widetilde{\gamma}_m(w)^{-1/2}\\
&= K(v',w')\widetilde{\gamma}_{m-1}(v')^{-1/2}\widetilde{\gamma}_{m-1}(w')^{-1/2}\\
&= (U_{m-1}(K))(v',w')\\
&= (\iota(U_{m-1}(K)))(v,w),
\end{aligned}$$

hence the lower part of the diagram in the statement is commutative. With regard to the middle part of the diagram, we have

$$\begin{aligned}
(\eta(T_{m-1}(a)))(x,y) &= (T_{m-1}(a))(x',y')\gamma_{t_m}(x_m)^{1/2}\gamma_{t_m}(y_m)^{1/2}\\
&= (T_{m-1}(a))(x',y')\cdot(T_1((\gamma_{t_m/2}\otimes\gamma_{1/8t_m})^{1/2}))(x_m,y_m)\\
&= (T_m(a\otimes(\gamma_{t_m/2}\otimes\gamma_{1/8t_m})^{1/2}))(x',x_m,y',y_m)\\
&= (T_m(\alpha(a)))(x,y),
\end{aligned}$$

where the second equality follows by Lemma 3.4, while the third equality is a consequence of Remark 3.3.

Finally, by using the fact that

$$(\forall t,a>0)(\forall x\in\mathbb{R})\quad a^{1/2}\gamma_t(ax) = \gamma_{t/a}(x),$$

it follows by a straightforward computation that the upper part of the diagram in the statement is commutative. ∎

Remark 3.6. In order to point out the role of the unitary operators

$$U_\ell\colon L^2(\mathbb{R}^\ell \times \mathbb{R}^\ell) \to L^2(\mathbb{R}^\ell \times \mathbb{R}^\ell, \mathrm{d}\widetilde{\gamma}_\ell \times \mathrm{d}\widetilde{\gamma}_\ell)$$

in Proposition 3.5, we note the following fact. The multiplication operator

$$V_\ell\colon L^2(\mathbb{R}^\ell) \to L^2(\mathbb{R}^\ell, \mathrm{d}\widetilde{\gamma}_\ell), \quad V_\ell f = (\widetilde{\gamma}_\ell)^{-1/2} f$$

is unitary. Furthermore, for any $K \in L^2(\mathbb{R}^\ell \times \mathbb{R}^\ell)$, if we denote the corresponding integral operator by

$$A_K\colon L^2(\mathbb{R}^\ell) \to L^2(\mathbb{R}^\ell), \quad (A_K f)(x) = \int_{\mathbb{R}^\ell} K(x,y) f(y)\mathrm{d}y,$$

then $V_\ell A_K V_\ell^{-1}\colon L^2(\mathbb{R}^\ell, \mathrm{d}\widetilde{\gamma}_\ell) \to L^2(\mathbb{R}^\ell, \mathrm{d}\widetilde{\gamma}_\ell)$ is the operator whose integral kernel is $U_\ell(K) \in L^2(\mathbb{R}^\ell \times \mathbb{R}^\ell, \mathrm{d}\widetilde{\gamma}_\ell \times \mathrm{d}\widetilde{\gamma}_\ell)$.

IV. INFINITE-DIMENSIONAL HEISENBERG GROUPS AND WIGNER TRANSFORMS

In this section we study representations of a special type of infinite-dimensional Heisenberg groups, as introduced in the following definition. A more general framework can be found, for instance, in Ref. 21.

For the sake of completeness, we recall here a few facts from Ref. 5 [Sec. 3].

Definition 4.1. If $\mathcal{V}$ is a real Hilbert space, $A \in \mathcal{B}(\mathcal{V})$ with $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in \mathcal{V}$, and $\operatorname{Ker} A = \{0\}$, then the *Heisenberg algebra* associated with the pair $(\mathcal{V}, A)$ is the real Hilbert space $\mathfrak{h}(\mathcal{V}, A) = \mathcal{V} \dotplus \mathcal{V} \dotplus \mathbb{R}$ endowed with the Lie bracket defined by $[(x_1, y_1, t_1), (x_2, y_2, t_2)] = (0, 0, (Ax_1 \mid y_2) - (Ax_2 \mid y_1))$. The corresponding *Heisenberg group* $\mathbb{H}(\mathcal{V}, A) = (\mathfrak{h}(\mathcal{V}, A), *)$ is the Lie group whose underlying manifold is $\mathfrak{h}(\mathcal{V}, A)$ and whose multiplication is defined by

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + ((Ax_1 \mid y_2) - (Ax_2 \mid y_1))/2)$$

for $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{H}(\mathcal{V}, A)$.

A. Gaussian measures and Schrödinger representations

Let $\mathcal{V}_-$ be a real Hilbert space with the scalar product denoted by $(\cdot \mid \cdot)_-$. For every vector $a \in \mathcal{V}_-$ and every symmetric, non-negative, injective, *trace-class* operator K on $\mathcal{V}_-$ there exists a unique probability Borel measure γ on $\mathcal{V}_-$ such that

$$(\forall x \in \mathcal{V}_-) \quad \int_{\mathcal{V}_-} \mathrm{e}^{\mathrm{i}(x|y)_-} \mathrm{d}\gamma(y) = \mathrm{e}^{\mathrm{i}(a|x)_- - \frac{1}{2}(Kx|x)_-}$$

(see for instance Ref. 20 [Theorem I.2.3]). We also have

$$a = \int_{\mathcal{V}_-} y \,\mathrm{d}\gamma(y) \quad \text{and} \quad Kx = \int_{\mathcal{V}_-} (x \mid y)_- \cdot (y - a)\mathrm{d}\gamma(y) \text{ for all } x \in \mathcal{V}_-,$$

where the integrals are weakly convergent, and γ is called the *Gaussian measure with the mean a and the variance K*.

Let us assume that the Gaussian measure γ is centered, that is, $a = 0$. Denote $\mathcal{V}_+ := \operatorname{Ran} K$ and $\mathcal{V}_0 := \operatorname{Ran} K^{1/2}$ endowed with the scalar products $(Kx \mid Ky)_+ := (x \mid y)_-$ and $(K^{1/2}x \mid K^{1/2}y)_0 := (x \mid y)_-$, respectively, for all $x, y \in \mathcal{V}_-$, which turn the linear bijections $K\colon \mathcal{V}_- \to \mathcal{V}_+$ and $K^{1/2}\colon \mathcal{V}_- \to \mathcal{V}_0$ into isometries. We thus get the real Hilbert spaces

$$\mathcal{V}_+ \hookrightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V}_-,$$

where the inclusion maps are Hilbert-Schmidt operators, since so is $K^{1/2} \in \mathcal{B}(\mathcal{V}_-)$. Also, the scalar product of $\mathcal{V}_0$ extends to a duality pairing $(\cdot \mid \cdot)_0\colon \mathcal{V}_- \times \mathcal{V}_+ \to \mathbb{R}$.

We also recall that for every $x \in \mathcal{V}_+$ the translated measure $\mathrm{d}\gamma(-x+\cdot)$ is absolutely continuous with respect to $\mathrm{d}\gamma(\cdot)$ and we have the Cameron-Martin formula

$$\mathrm{d}\gamma(-x+\cdot) = \rho_x(\cdot)\mathrm{d}\gamma(\cdot) \quad \text{with } \rho_x(\cdot) = \mathrm{e}^{(\cdot|x)_0 - \frac{1}{2}(x|x)_0}.$$

(This actually holds true for every $x \in \mathcal{V}_0$, by suitably defining the function $\rho_x(\cdot)$ no longer as a continuous function, but only almost everywhere; see, for instance, Ref. 20 [Lemma I.4.7 and Theorem II.3.1].)

Definition 4.2. Let $\mathcal{V}_+$ be a real Hilbert space with the scalar product $(x,y) \mapsto (x \mid y)_+$. Also let $A\colon \mathcal{V}_+ \to \mathcal{V}_+$ be a non-negative, symmetric, injective, trace-class operator. Define $\mathcal{V}_0$ and $\mathcal{V}_-$ as the completions of $\mathcal{V}_+$ with respect to the scalar products

$$(x,y) \mapsto (x \mid y)_0 := (A^{1/2}x \mid A^{1/2}y)_+$$

and

$$(x,y) \mapsto (x \mid y)_- := (Ax \mid Ay)_+,$$

respectively. Then the operator A uniquely extends to a non-negative, symmetric, injective, trace-class operator $K \in \mathcal{B}(\mathcal{V}_-)$, hence by the above observations one obtains the centered Gaussian measure γ on $\mathcal{V}_-$ with the variance K.

For the Heisenberg group $\mathbb{H}(\mathcal{V}_+,A)$, its *Schrödinger representation* $\pi\colon \mathbb{H}(\mathcal{V}_+,A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ is defined by

$$\pi(x,y,t)\phi = \rho_x(\cdot)^{1/2}\mathrm{e}^{\mathrm{i}(t+(\cdot|y)_0+\frac{1}{2}(x|y)_0)}\phi(-x+\cdot)$$

whenever $(x,y,t) \in \mathbb{H}(\mathcal{V}_+,A)$ and $\phi \in L^2(\mathcal{V}_-,\gamma)$. This is a continuous unitary irreducible representation of the Heisenberg group $\mathbb{H}(\mathcal{V}_+,A)$ by Ref. 17 [Theorems 5.2.9 and 5.2.10] and Proposition 4.4 below.

Remark 4.3. More general Schrödinger representations of infinite-dimensional Heisenberg groups are described in Ref. 21 [Proposition II.4.6] by using cocycles and reproducing kernel Hilbert spaces.

The following result is known, but we recall here from Ref. 5 [Remark 3.6] the method of proof since its constructions will be needed also for Theorem 4.6.

Proposition 4.4. The representation $\pi\colon \mathbb{H}(\mathcal{V}_+,A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ *from Definition 4.2 is irreducible.*

Proof. Let $t_1 \ge t_2 \ge \cdots$ (>0) be the eigenvalues of A counted according to their multiplicities. Since A is a self-adjoint trace-class operator and $\operatorname{Ker} A = \{0\}$, there exists an orthonormal basis $\{v_k\}_{k\ge1}$ in $\mathcal{V}_+$ such that $Av_k = t_k v_k$ for every $k \ge 1$. For every integer $n \ge 1$ let

$$\mathcal{V}_{n,+} = \operatorname{span}\{v_1,\dots,v_n\}.$$

It follows that $\dim \mathcal{V}_{n,+} < \infty$. We have

$$\mathcal{V}_{1,+} \subseteq \mathcal{V}_{2,+} \subseteq \cdots \subseteq \bigcup_{n\ge1}\mathcal{V}_{n,+} \subseteq \mathcal{V}_+$$

and $\bigcup_{n\ge1}\mathcal{V}_{n,+}$ is a dense subspace of $\mathcal{V}_+$. Let us denote by A_n the restriction of A to $\mathcal{V}_{n,+}$. Then $\mathbb{H}(\mathcal{V}_{n,+},A_n)$ is a finite-dimensional Heisenberg group, hence its Schrödinger representation $\pi_n\colon \mathbb{H}(\mathcal{V}_{n,+},A_n) \to \mathcal{B}(L^2(\mathcal{V}_{n,-},\gamma_n))$ is irreducible, as it is well known, where γ_n is the Gaussian measure on the finite-dimensional space $\mathcal{V}_{n,-}$ obtained out of the pair $(\mathcal{V}_{+,n},A_n)$ by the construction outlined at the very beginning of Definition 4.2. Note that

$$\mathbb{H}(\mathcal{V}_{1,+},A_1) \subseteq \mathbb{H}(\mathcal{V}_{2,+},A_2) \subseteq \cdots \subseteq \bigcup_{n\ge1}\mathbb{H}(\mathcal{V}_{n,+},A_n) =: \mathbb{H}(\mathcal{V}_{\infty,+},A_\infty) \subseteq \mathbb{H}(\mathcal{V}_+,A)$$

and $\mathbb{H}(\mathcal{V}_{\infty,+},A_\infty)$ is a dense subgroup of $\mathbb{H}(\mathcal{V}_+,A)$, hence the Schrödinger representation $\pi : \mathbb{H}(\mathcal{V}_+,A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ is irreducible if and only if so is its restriction $\pi|_{\mathbb{H}(\mathcal{V}_{\infty,+},A_\infty)}$.

On the other hand, if we denote by $\mathbf{1}_n$ the function that is identically equal to 1 on the orthogonal complement $\mathcal{V}_{n+1,+} \ominus \mathcal{V}_{n,+}$, then it is straightforward to check that the operator

$$L^2(\mathcal{V}_{n,-},\gamma_n) \to L^2(\mathcal{V}_{n+1,-},\gamma_{n+1}), \quad f \mapsto f \otimes \mathbf{1}_n \tag{4.1}$$

is an isometry and intertwines the representations π_n and π_{n+1}. We can thus make the sequence of representations $\{\pi_n\}_{n\geq 1}$ into an inductive system of irreducible unitary representations and then their inductive limit $\pi|_{\mathbb{H}(\mathcal{V}_{\infty,+},A_\infty)} = \operatorname{ind}_{n\to\infty} \pi_n$ is irreducible (see for instance Ref. 19). As noted above, this implies that the Schrödinger representation $\pi\colon \mathbb{H}(\mathcal{V}_+,A) \to \mathcal{B}(L^2(\mathcal{V}_-,\gamma))$ of Definition 4.2 is irreducible. ∎

Remark 4.5. Let $\mathcal{X}$ be a real locally convex Hausdorff space and denote by $\mathcal{X}'^{\,\mathrm{alg}}$ the space of all linear functionals on $\mathcal{X}$ endowed with the locally convex topology of pointwise convergence on $\mathcal{X}$. If B is any algebraic basis in $\mathcal{X}$, then every $\xi = \{\xi_b\}_{b\in B} \in \mathbb{R}^B$ corresponds to a linear functional

$$\Psi_B(\xi)\colon \mathcal{X} \to \mathbb{R}, \quad \sum_{b\in B} x_b b \mapsto \sum_{b\in B} \xi_b x_b,$$

and we get the linear topological isomorphism

$$\Psi_B\colon \mathbb{R}^B \to \mathcal{X}'^{\,\mathrm{alg}}, \quad \xi \mapsto \Psi_B(\xi),$$

where $\mathbb{R}^B$ is endowed with its natural weak topology (that is, projective limit of finite-dimensional linear spaces; see Ref. 18 [Definition A2.5]).

For later use, we also note that if the topological dual $\mathcal{X}'$ is endowed with the weak*-topology, then the following assertions hold:

1. The locally convex space $\mathcal{X}'^{\,\mathrm{alg}}$ is complete and the topological dual $\mathcal{X}'$ is a dense subspace of $\mathcal{X}'^{\,\mathrm{alg}}$. Therefore, $\mathcal{X}'^{\,\mathrm{alg}}$ is isomorphic to the completion of $\mathcal{X}'$ as a topological vector space (hence as a uniform space).
2. The mapping
$$\mathcal{UC}_b(\mathcal{X}'^{\,\mathrm{alg}}) \to \mathcal{UC}_b(\mathcal{X}'), \quad f \mapsto f|_{\mathcal{X}'}$$
is an isometric $*$-isomorphism of unital C^*-algebras.
3. If $B_0 \subseteq B$ and we consider the natural injective linear map
$$\iota_{B_0,B}\colon \mathbb{R}^{B_0} \hookrightarrow \mathbb{R}^B \xrightarrow{\Psi_B} \mathcal{X}'^{\,\mathrm{alg}},$$
then we get a mapping
$$\mathcal{UC}_b(\mathcal{X}'^{\,\mathrm{alg}}) \to \mathcal{UC}_b(\mathbb{R}^{B_0}), \quad f \mapsto f \circ \iota_{B_0,B}$$
which is a surjective $*$-homomorphism of unital C^*-algebras.

In fact, Assertion (1) follows by a result of Ref. 14; see also Refs. 11 and 16 [Solution 30]. A version of this property for general uniform spaces was established in Ref. 12. Assertion (2) follows by using the fact that the uniformly continuous functions extend uniquely to the completion; see, for instance, Ref. 9 [Chapter II, Section 3]. As for Assertion (3), we just have to note that if $p_{B,B_0}\colon \mathbb{R}^B \to \mathbb{R}^{B_0}$ denotes the natural projection, then $(p_{B,B_0} \circ (\Psi_B)^{-1}) \circ \iota_{B_0,B} = \mathrm{id}_{\mathbb{R}^{B_0}}$.

In the following statement, we use notation from Definition 4.2 and moreover we denote by $\mathfrak{S}_2(\cdot)$ the Hilbert space of Hilbert-Schmidt operators on some complex Hilbert space.

Theorem 4.6. *Let $\mathcal{V}_+$ be a separable real Hilbert space and $A\colon \mathcal{V}_+ \to \mathcal{V}_+$ be a non-negative, symmetric, injective, trace-class operator with the eigenvalues*

$$t_0 \geq t_1 \geq \cdots > 0,$$

counted according to their multiplicities. Let $\mathcal{V}_{\infty,+}$ denote the linear span of the eigenvectors of A and define

$$\theta\colon \Xi := \mathcal{V}_{\infty,+} \times \mathcal{V}_{\infty,+} \to \mathfrak{h}(\mathcal{V}_+,A), \quad (x,y) \mapsto (x,y,0).$$

Then there exist the locally convex spaces $\Gamma \hookrightarrow \mathcal{M}(\mathfrak{h}(\mathcal{V}_+, A)')$ and $\mathcal{H}_{\Xi,\infty} \hookrightarrow \mathcal{H} := L^2(\mathcal{V}_-, \gamma)$ such that the following assertions hold:

1. *Γ and θ are compatible.*
2. *The Schrödinger representation $\pi\colon \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma))$ satisfies the orthogonality relations along θ.*
3. *For the Hilbert space obtained as the completion of Γ we have*
$$\Gamma_2 \simeq L^2\Big(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \Big(\bigotimes_{j\in\mathbb{N}} \mathrm{d}\gamma_{1/2}\Big) \otimes \Big(\bigotimes_{j\in\mathbb{N}} \mathrm{d}\gamma_{1/8t_j^2}\Big)\Big) \hookrightarrow \mathcal{M}(\mathfrak{h}(\mathcal{V}_+, A)')$$
and the operator $\mathrm{Op}^\theta\colon \Gamma_2 \to \mathfrak{S}_2(L^2(\mathcal{V}_-, \gamma))$ defined in (2.1) is unitary.

Proof. We shall use the notation of Proposition 4.4. For every $k \geq 0$ we have $\|v_k\|_- = \|Av_k\|_+ = t_k$ hence, if we denote $e_k = t_k^{-1} v_k$, then $\{e_k\}_{k\geq 1}$ is an orthonormal basis in $\mathcal{V}_-$, and we use it to perform the identifications $\mathcal{V}_- \simeq \ell^2_{\mathbb{R}}(\mathbb{N})$ and $\mathcal{V}_{m,-} \simeq \mathbb{R}^m \hookrightarrow \ell^2_{\mathbb{R}}(\mathbb{N})$ for $m = 1, 2, \ldots$.

For $m \geq 1$, let us denote by
$$\mathcal{H}_{m,\infty} := \{\varphi \in L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m) \mid \pi_m(\cdot)\varphi \in C^\infty(\mathbb{H}(\mathcal{V}_{m,+}, A_m), L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m))\}$$
the space of smooth vectors for the representation
$$\pi_m\colon \mathbb{H}(\mathcal{V}_{m,+}, A_m) \to \mathcal{B}(L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m)) = \mathcal{B}(L^2(\mathbb{R}^m, \mathrm{d}\widetilde{\gamma}_m)),$$
where $\mathrm{d}\widetilde{\gamma}_m$ is the Gaussian measure on $\mathbb{R}^m$ as in Proposition 3.5. (See, for instance, Ref. 23 and the references therein for differentiability of vectors in Lie group representation theory.) It is clear that the operators of type (4.1) take $\mathcal{H}_{m,\infty}$ into $\mathcal{H}_{m+1,\infty}$. We thus get an inductive limit of Fréchet spaces continuously and densely embedded into an inductive limit of Hilbert spaces,
$$\mathcal{H}_{\Xi,\infty} := \operatorname*{ind}_{m\to\infty} \mathcal{H}_{m,\infty} \hookrightarrow \operatorname*{ind}_{m\to\infty} L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m) = L^2(\mathcal{V}_-, \gamma).$$
It is well known that for every $m \geq 1$ the representation π_m satisfies the orthogonality relations along the mapping
$$\theta|_{\mathcal{H}_{m,\infty}\times\mathcal{H}_{m,\infty}}\colon \mathcal{H}_{m,\infty} \times \mathcal{H}_{m,\infty} \to \mathfrak{h}(\mathcal{V}_{m,+}, A_m)$$
(see, for instance, Ref. 7). The corresponding unitary operator as in (2.2), denoted by
$$\mathcal{W}_m\colon L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m)\bar{\otimes}\overline{L^2(\mathcal{V}_{m,-}, \widetilde{\gamma}_m)} \to L^2(\mathbb{R}^m \times \mathbb{R}^m),$$
is given by $\mathcal{W}_m = (U_m \circ T_m \circ S_m)^{-1}$, where we use the notation of Proposition 3.5. It follows by that proposition that for the Gaussian Radon measures $\mu_1 := \bigotimes_{j\in\mathbb{N}} \mathrm{d}\gamma_{1/2}$ and $\mu_2 := \bigotimes_{j\in\mathbb{N}} \mathrm{d}\gamma_{1/8t_j^2}$ on $\mathbb{R}^{\mathbb{N}}$ we obtain a unitary operator
$$\mathcal{W} = \operatorname*{ind}_{m\to\infty} \mathcal{W}_m\colon L^2(\mathcal{V}_-, \gamma)\bar{\otimes}\overline{L^2(\mathcal{V}_-, \gamma)} \to L^2(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathrm{d}\mu_1 \times \mathrm{d}\mu_2).$$
By using Remark 4.5 we now obtain natural continuous injective linear maps
$$\begin{aligned} L^2(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathrm{d}\mu_1 \times \mathrm{d}\mu_2) &\hookrightarrow L^1(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathrm{d}\mu_1 \times \mathrm{d}\mu_2) \\ &\hookrightarrow \mathcal{M}(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}) \\ &\hookrightarrow \mathcal{M}((\mathcal{V}_+ \times \mathcal{V}_+)'^{\,\mathrm{alg}}) \\ &\simeq \mathcal{M}((\mathcal{V}_+ \times \mathcal{V}_+)') \\ &\hookrightarrow \mathcal{M}(\mathfrak{h}(\mathcal{V}_+, A)'). \end{aligned}$$
Finally, if we define
$$\Gamma := \mathcal{W}(\mathcal{H}_{\Xi,\infty} \otimes \overline{\mathcal{H}_{\Xi,\infty}})$$
endowed with the topology which makes the linear isomorphism
$$\mathcal{W}\colon \mathcal{H}_{\Xi,\infty} \otimes \overline{\mathcal{H}_{\Xi,\infty}} \to \Gamma$$
into a topological isomorphism, then it is easily seen that Γ and the mapping θ are compatible, and this concludes the proof. ∎

Remark 4.7. The infinite-dimensional pseudo-differential calculus of Refs. 1 and 2 can be recovered as a special case of the operator calculus from our Definition 2.4 for the Schrödinger representations introduced in Definition 4.2 above. Compare, for instance, Ref. 2 [Proposition 3.7].

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APPENDIX: FOURIER TRANSFORMS ON UNIFORM SPACES

In this appendix we record some results on uniform spaces. The theory of uniform spaces and uniform measures is the natural framework which allows one to study uniformly continuous functions and completions based on Cauchy sequences (see, for instance, Refs. 9 and 24). The Fourier transform in this general setting seems to be less investigated, although it is certainly well known for measures on locally convex spaces (see, for instance, Refs. 8, 20, and 10). Therefore, we take this opportunity for introducing a notion of Fourier transform on general uniform spaces which seems to be useful in the study of measures on nonlinear spaces, for instance, topological groups which may not be locally compact.

For every uniform space in this section, we assume that its topology has the Hausdorff property. Let X be any uniform space. We denote by $\mathcal{UC}$ and $\mathcal{UC}_b$ the various spaces of uniformly continuous functions and uniformly continuous bounded functions on any uniform space, respectively. Note that $\mathcal{UC}_b(X) := \mathcal{UC}_b(X,\mathbb{C})$ is a Banach space, so we may consider its dual Banach space $\mathcal{M}(X) := \mathcal{UC}_b(X)'$, which should be thought of as a space of generalized complex measures on X. We also define $\mathcal{M}^+(X)$ as the set of all $\varphi \in \mathcal{M}(X)$ satisfying $0 \le \varphi(f)$ if $0 \le f \in \mathcal{UC}_b(X)$.

Definition A.1. We define the *uniform space dual* to X as

$$X^\nabla := \mathcal{UC}(X,\mathbb{R})$$

endowed with the uniform structure of pointwise convergence.

Remark A.2. There exists a natural injective mapping

$$\eta_X\colon X \to (X^\nabla)^\nabla, \quad x \mapsto \eta_x,$$

where $\eta_x(f) = f(x)$ for every $f \in \mathcal{UC}(X,\mathbb{R})$ and $x \in X$. It is easily seen that η is uniformly continuous.

Definition A.3. The *Fourier transform* on X is the linear mapping

$$\mathcal{F}\colon \mathcal{M}(X) \to \ell^\infty(X^\nabla), \quad (\mathcal{F}\mu)(f) = \langle \mu, \mathrm{e}^{\mathrm{i}f} \rangle$$

for $f \in X^\nabla$ and $\mu \in \mathcal{M}(X)$. We will also denote $\widehat{\mu} := \mathcal{F}\mu$ for $\mu \in \mathcal{M}(X)$.

Note that $\mathcal{F}$ is a bounded linear mapping and in fact $\|\mathcal{F}\| \le 1$.

Proposition A.4. The Fourier transform $\mathcal{F}\colon \mathcal{M}(X) \to \ell^\infty(X^\nabla)$ is injective.

Proof. Let $\mu \in \mathcal{M}(X)$ with $\mathcal{F}\mu = 0$. In order to prove that $\mu = 0$, it suffices to show that for an arbitrary *real-valued* function $f \in \mathcal{UC}_b(X)$ we have $\langle \mu, f \rangle = 0$. First, recall that we have

$$\lim_{t\to 0} \frac{\mathrm{e}^{\mathrm{i}tr} - 1}{t} - \mathrm{i}r = 0$$

uniformly for r in any compact subset of $\mathbb{R}$. Therefore,

$$\lim_{t\to 0}\frac{e^{itf}-1}{t}=if$$

in $\mathcal{UC}_b(X)$, hence

$$i\langle\mu,f\rangle=\lim_{t\to 0}\frac{(\mathcal{F}\mu)(tf)-(\mathcal{F}\mu)(0)}{t}=0,$$

and we are done. ∎

Lemma A.5. If $\mu\in\mathcal{M}^+(X)$ then the following assertions hold:

1. *We have $\|\mathcal{F}\mu\|_\infty\le\langle\mu,\mathbf{1}\rangle$.*
2. *For all $f,h\in X^\nabla$ we have*

$$|(\mathcal{F}\mu)(f)-(\mathcal{F}\mu)(h)|^2\le 2\langle\mu,\mathbf{1}\rangle(\langle\mu,\mathbf{1}\rangle-\mathrm{Re}\,(\mathcal{F}\mu)(f-h)).$$

Proof. For Assertion (1), recall from Definition A.3 that $\|\mathcal{F}\|\le 1$, hence

$$\|\mathcal{F}\mu\|_\infty\le\|\mu\|=\langle\mu,\mathbf{1}\rangle,$$

where the latter equality follows since $\mu\colon\mathcal{UC}_b(X)\to\mathbb{C}$ is a positive linear functional on the C^*-algebra $\mathcal{UC}_b(X)$.

To prove Assertion (2), let $f,h\in X^\nabla$ arbitrary. By using the Cauchy-Schwartz inequality we get

$$\begin{aligned}|(\mathcal{F}\mu)(f)-(\mathcal{F}\mu)(h)|^2&=|\langle\mu,e^{if}-e^{ih}\rangle|^2\\&\le\langle\mu,|e^{if}-e^{ih}|^2\rangle\langle\mu,\mathbf{1}^2\rangle\\&=\langle\mu,\mathbf{1}\rangle\langle\mu,2-2\mathrm{Re}\,(e^{i(f-h)})\rangle\\&=2\langle\mu,\mathbf{1}\rangle(\langle\mu,\mathbf{1}\rangle-\langle\mu,\mathrm{Re}\,(e^{i(f-h)})\rangle)\\&=2\langle\mu,\mathbf{1}\rangle(\langle\mu,\mathbf{1}\rangle-\mathrm{Re}\,((\mathcal{F}\mu)(f-h))),\end{aligned}$$

where we also used the fact that $|e^{it}-e^{is}|^2=2-2\mathrm{Re}\,(e^{i(t-s)})$ for all $t,s\in\mathbb{R}$. ∎

Lemma A.6. Let X and Y be uniform spaces. Assume that D is a dense subset of X and $\mathcal{A}$ is a uniformly equi-continuous family of mappings from X into Y. Then the following uniform structures on the mappings from X into Y induce the same uniform structure on $\mathcal{A}$:

(a) *the structure of uniform convergence on the precompact subsets of X;*
(b) *the structure of pointwise convergence on X;*
(c) *the structure of pointwise convergence on D.*

Proof. See the Ascoli-Arzelà theorem on uniform spaces and its proof in Ref. 8. ∎

We now introduce the linear space of tight measures $\mathcal{M}_t(X)$ on a uniform space X (see Ref. 24 [Sec. 5.1] for more information in this connection). Namely, $\mathcal{M}_t(X)$ is the set of all linear functionals $\mu\colon\mathcal{UC}_b(X)\to\mathbb{C}$ with the property that for every net $\{f_i\}_{i\in I}$ in $\mathcal{UC}_b(X)$ with $\sup_{i\in I}\|f_i\|_\infty<\infty$ and $\lim_{i\in I}f_i=0$ uniformly on every compact subset of X one has $\lim_{i\in I}\mu(f_i)=0$. We also denote $\mathcal{M}_t{}^+(X):=\mathcal{M}_t(X)\cap\mathcal{M}^+(X)$.

Lemma A.7. Let X be any uniform space and $\mu\in\mathcal{M}_t(X)$. Then there exist $\mu_j^\pm\in\mathcal{M}_t{}^+(X)$ for $j=1,2$ with $\mu=\mu_1^+-\mu_1^-+i(\mu_2^+-\mu_2^-)$.

Proof. See Ref. 13. ∎

Proposition A.8. The following assertions hold:

1. *Let $\mu\in\mathcal{M}^+(X)$. We have $\mathcal{F}\mu\in\mathcal{UC}_b(X^\nabla)$ if and only if $\mathrm{Re}\,(\mathcal{F}\mu)$ is continuous at $0\in X^\nabla$.*
2. *If $\mu\in\mathcal{M}_t(X)$, then $\mathcal{F}\mu\in\mathcal{UC}_b(X^\nabla)$.*

Proof. Assertion (1) follows at once by Lemma A.5.

For proving Assertion (2), we see from Lemma A.7 that we may assume $\mu \in \mathcal{M}_t{}^+(X)$. Then, according to Assertion (1), it suffices to show that for every $\mu \in \mathcal{M}_t{}^+(X)$ the function $\mathrm{Re}\,(\mathcal{F}\mu)\colon X^\nabla \to \mathbb{C}$ is continuous at $0 \in X^\nabla$. To this end, let us assume that $\lim_{j\in J} f_j = 0$ in X^∇. In other words, $\{f_j\}_{j\in J}$ is a net of uniformly continuous real functions on X with $\lim_{j\in J} f_j = 0$ pointwise on X. By using Lemma A.6, we see that $\{\cos f_j\}_{j\in J}$ is a uniformly bounded net in $\mathcal{UC}_b(X)$ which converges to $1 \in \mathcal{UC}_b(X)$ uniformly on the compact subsets of X. Since $\mu \in \mathcal{M}_t(X)$, we then get

$$\lim_{j\in J} \mathrm{Re}\,(\mathcal{F}\mu)(f_j) = \lim_{j\in J} \mathrm{Re}\,\langle \mu, \mathrm{e}^{\mathrm{i}f_j}\rangle = \lim_{j\in J}\langle \mu, \cos f_j\rangle = 1.$$

Therefore, the function $\mathrm{Re}\,(\mathcal{F}\mu)$ is continuous at $0 \in X^\nabla$, and this completes the proof. ■

Remark A.9. Lemma A.5 and Proposition A.8 are straightforward extensions of some results from Ref. 8 [Section 6, no. 8].

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