

A Note on Total and List Edge-Colouring of Graphs of Tree-Width 3

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Abstract It is shown that Halin graphs are Δ -edge-choosable and that graphs of tree-width 3 are $(\Delta + 1)$ -edge-choosable and $(\Delta + 2)$ -total-colourable.

Keywords List edge-coloring · Total coloring · Halin graphs · Tree-width 3

Mathematics Subject Classification 05C15

1 Introduction

In this note we present some results concerning the list chromatic index $\text{ch}'(G)$ and the total chromatic number $\chi''(G)$ of graphs G of tree-width 3 (see Sect. 2 for proper definitions). One of the central open questions in the field of list colouring is known as the list colouring conjecture:

Conjecture 1 For all graphs G it holds that $\text{ch}'(G) = \chi'(G)$.

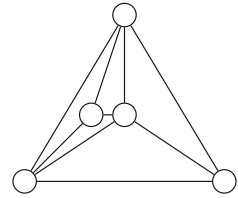
Conjecture 1 appeared for the first time in print in 1985 [3], but was, according to Alon [1], Woodall [16] and Jensen and Toft [11], suggested independently by Vizing, Albertson, Collins, Erds, Tucker and Gupta in the late seventies. If Conjecture 1 is true we have $\chi''(G) \leq \Delta(G) + 3$ for all graphs G ¹. The *total colouring conjecture* asserts a little more:

¹ If we colour the vertices of G using the colours $C = \{1, \dots, \Delta(G) + 3\}$, then for each edge there are still $\Delta(G) + 1$ colours of C available, which permits a total colouring if Conjecture 1 holds.

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Fig. 1 A graph of tree-width 3 and chromatic index 5



Conjecture 2 For all graphs G it holds that $\chi''(G) \leq \Delta(G) + 2$.

Conjecture 2 has been promoted independently by Behzad [2] and Vizing [15]. Our first result is a list version of Vizing's theorem for graphs of tree-width 3.

Theorem 1 For a graph G of tree-width 3 it holds that $\text{ch}'(G) \leq \Delta(G) + 1$.

There are graphs for which this is a sharp bound, see Fig. 1. Conjecture 2 has been proved for graphs of maximum degree at most 5 by Kostochka [12] and Rosenfeld [14]. We will use this and a slight variation of the proof of Theorem 1 to show Conjecture 2 for graphs of tree-width 3 in Sect. 3.

Theorem 2 For a graph G of tree-width 3 it holds that $\chi''(G) \leq \Delta(G) + 2$.

A Halin graph is constructed by taking a planar embedding of a tree without vertices of degree 2 and connecting all leaves of the tree with a cycle that passes around the tree in the natural cyclic order. Halin graphs have tree-width 3 [4]. In Sect. 4 we prove Conjecture 1 for Halin graphs.

Theorem 3 For a Halin graph G it holds that $\text{ch}'(G) = \Delta(G)$.

While still open in general, Conjecture 1 has been verified for some particular families of graphs. Galvin proved that $\text{ch}'(G) = \Delta(G)$ for all bipartite multigraphs G [9]. Ellingham and Goddyn used a method of Alon and Tarsi to show that every d -regular, d -edge-colourable, planar multigraph is d -edge-choosable [8]. Some years later Juvan, Mohar and Thomas showed that $\text{ch}'(G) = \Delta(G)$ holds for series parallel graphs G [10]. This class of graphs can also be characterized in terms of tree-width. Series parallel graphs have tree-width at most 2. Conversely, a graph has tree-width at most 2 if and only if every biconnected component is series parallel [5]. This is why we are interested in list edge-colouring graphs of tree-width 3. The methods presented here are extended in [13] in order to prove Conjecture 1 for graphs of tree-width 3 and a high maximum degree.

2 Graphs of Tree-Width 3 are $(\Delta(G) + 1)$ -Choosable

We will mostly use standard notation as seen in [7]. All graphs are finite and simple. The size of a graph G is $|V(G)| + |E(G)|$. H is smaller than G if its size is less than the size of G . For a graph G a tree decomposition (T, \mathcal{V}) consists of a tree T and a collection $\mathcal{V} = \{V_t; t \in V(T)\}$ of bags $V_t \subset V(G)$ such that

- $V(G) = \bigcup_{t \in V(T)} V_t$,
- for each $vw \in E(G)$ there exists a vertex $t \in V(T)$ such that $v, w \in V_t$ and

- if $v \in V_{t_1} \cap V_{t_2}$ then $v \in V_t$ for all vertices t that lie on the path connecting t_1 and t_2 in T .

A tree decomposition (T, \mathcal{V}) of G has width k , if all bags have a size of at most $k + 1$. The tree-width of G is the smallest number k for which there exists a width k tree decomposition of G . As our proofs are based on minimality it is important to mention that the graphs of tree-width at most k form a minor-closed family. We call a width k tree decomposition (T, \mathcal{V}) smooth if each bag has size $k + 1$, no two bags are identical and for each $t_1 t_2 \in E(T)$ the bags of t_1 and t_2 share exactly k vertices. A graph of tree-width k has a smooth width k tree decomposition [5]. Given a tree decomposition (T, \mathcal{V}) of G where T is rooted in some vertex $r \in V(T)$ we define the height of any vertex $t \in V(T)$ to be the distance from r to t .

An instance of list edge-colouring consists of a graph G and an assignment of lists $L : E(G) \rightarrow \mathcal{P}(\mathbb{N})$ that maps the edges of G to lists of colours $L(e)$. A function $\mathcal{C} : E(G) \rightarrow \mathbb{N}$ is called an L -edge-colouring of G , if $\mathcal{C}(e) \in L(e)$ for each $e \in E(G)$ and no two adjacent edges receive the same colour. G is said to be k -edge-choosable, if for each assignment of lists L to the edges of G , where all lists have a size of at least k , there is an L -edge-colouring of G . The list chromatic index, denoted by $\text{ch}'(G)$, is the smallest integer k for which a graph G is k -edge-choosable.

Let G be a graph and L an assignment of lists to the edges of G . For an L -edge-colouring \mathcal{C} of some subgraph $H \subset G$ we call a colour c of the list of an uncoloured edge e available, if no edge adjacent to e has already been coloured with c . The set of available colours of e is called the list of remaining colours and denoted by $L^\mathcal{C}(e)$. We can always try to colour G greedily, by iteratively colouring the edge with the smallest list of available colours with an arbitrary available colour.

For a subset of vertices $W \subset V(G)$, we denote by $G\langle W \rangle$ the graph with vertex set $W \cup N(W)$ and edge set $E(G) \setminus E(G - W)$. Let G be a graph with an assignment of lists L to the edges of G such that each list $L(vw)$ has a size of at least $\max(\deg_G(v), \deg_G(w)) + 1$. Suppose that for some proper subset of vertices $W \subset V(G)$, we can find an L -edge-colouring \mathcal{C} of the graph $G - W$. In order to extend \mathcal{C} to an L -edge-colouring of G we need to find an $L^\mathcal{C}$ -colouring of $G\langle W \rangle$. For an edge $w_1 w_2 \in E(G)$ with $w_1, w_2 \in W$ we have

$$|L^\mathcal{C}(w_1 w_2)| = |L(w_1 w_2)|. \tag{1}$$

For an edge $vw \in E(G)$ with $w \in W$ and $v \in V(G) \setminus W$, we have

$$|L^\mathcal{C}(vw)| \geq |L(vw)| - \deg_{G-W}(v) \geq \deg_{G\langle W \rangle}(v) + 1, \tag{2}$$

since $\max(\deg_G(v), \deg_G(w)) \geq \deg_G(v) = \deg_{G-W}(v) + \deg_{G\langle W \rangle}(v)$. In the proofs of the following results, we will generally assume that the size of each list is exactly the size of its given lower bound. The next theorem has already been mentioned in the introduction.

Theorem 4 (Galvin [9]) *Let G be a bipartite graph; then $\text{ch}'(G) = \Delta(G)$.*

Lemma 1 *Let G be a cycle e_1, \dots, e_n with an additional edge f that is incident exactly to the one vertex of C that e_1 and e_n share. For any assignment of lists L ,*

where each list has a size of at least 2 for all edges and the size of the list of e_1 is at least 3, there is an L -edge-colouring.

Proof If there is a colour $c \in L(e_n) \setminus L(f)$, colour e_n with c and finish greedily. This yields $L(f) = L(e_n)$ and so there is a colour $c \in L(e_1) \setminus (L(f) \cup L(e_n))$. Colour e_1 with c and finish greedily. \square

The next lemma implies Theorem 1.

Lemma 2 *Let G be a graph of tree-width at most 3 with an assignment of lists L such that each list $L(vw)$ has a size of at least $\max(\deg(v), \deg(w)) + 1$ for each edge $vw \in E(G)$. Then there is an L -edge-colouring of G .*

Proof We will assume that the lemma is wrong and obtain a contradiction. Let G be a smallest counterexample to the lemma with an assignment of lists L to the edges of G for which each $L(vw)$ has a size of at least $\max(\deg(v), \deg(w)) + 1$ for each edge $vw \in E(G)$ such that there is no L -edge-colouring of G .

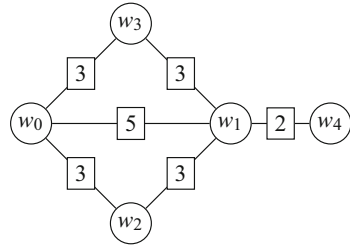
If there is a vertex $v \in V(G)$ of degree at most 2, we can find an L -edge-colouring of $G - v$ by minimality and extend this to an L -edge-colouring of G by colouring the edges adjacent to v greedily from the lists of remaining colours. Therefore we might assume that G has a minimum degree of at least 3. This implies that G has tree-width 3, since the bag of any leaf of a width 2 tree-decomposition contains a vertex of degree at most 2. Let (T, \mathcal{V}) be a smooth width 3 tree decomposition of G where T is rooted in some arbitrary vertex. Let $t \in V(T)$ be a vertex of degree at least 2 in T and of maximum height.

Suppose the neighbourhood of t contains at least two leaves $t_1, t_2 \in V(T)$. Since (T, \mathcal{V}) is smooth, there are vertices $v_1 \in V_{t_1}$ and $v_2 \in V_{t_2}$ which are uniquely in V_{t_1} and V_{t_2} . We have $\deg(v_1) = \deg(v_2) = 3$, v_1 and v_2 are not adjacent and $|N(v_1) \cup N(v_2)| \leq 4$. By minimality we can find an L -edge-colouring of the graph $G - v_1 - v_2$. If v_1 and v_2 have the same neighbourhood we can extend this to an L -edge-colouring of G by applying Theorem 4 to the bipartite graph $G(\{v_1, v_2\})$, as their lists of remaining colours have each a size of at least 3 by (2). If v_1 and v_2 have distinct neighbourhoods, colour the two edges in $G(\{v_1, v_2\})$ that are adjacent to vertices outside of $N(v_1) \cap N(v_2)$ greedily. We can extend this to an L -edge-colouring of G by applying Theorem 4 as the uncoloured edges form a 4-cycle and their lists of remaining colours have each at least size 2 by (2).

So we can assume that t is adjacent to exactly one leaf $t_0 \in V(T)$. As (T, \mathcal{V}) is smooth and t has maximum height, there are vertices $w_0 \in V_{t_0}$ and $w_1 \in V_t \cap V_{t_0}$ that appear uniquely in V_{t_0} and $V_t \cap V_{t_0}$. Let $V_t = \{w_1, w_2, w_3, w_4\}$. Since w_0 appears uniquely in the bag of the leaf t_0 , it has a degree of at most 3. So $\deg(w_0) = 3$ and we may also assume that $w_0w_4 \notin E(G)$. Further, since the neighbours of w_1 are either in V_{t_0} or V_t , w_1 has a degree of at most 4.

Now if $\deg(w_1) = 3$, we can choose an L -edge-colouring of $G - \{w_0, w_1\}$ by minimality and extend this to an L -edge-colouring of G as follows. If w_1w_4 is an edge, colour it with an arbitrary colour and apply Lemma 1 to $G(\{w_0, w_1\}) - w_0w_4$. So $N(w_1) = \{w_0, w_2, w_3\}$ and we can colour w_0w_1 with an arbitrary colour and apply Theorem 4 to the 4-cycle $G(\{w_0, w_1\}) - w_0w_1$. This yields $\deg(w_1) = 4$.

Fig. 2 The integers on the edges indicate the minimum sizes of the respective lists



We pick a final L -edge-colouring of $G - w_0 - w_1$ by minimality. In order to extend this to an L -edge-colouring of G we need to colour the edges of the graph $G \setminus \{w_0, w_1\}$ from the lists of remaining colours. This instance of list edge-colouring is shown in Fig. 2. The lower bounds on the lists of remaining colours are given by (1) and (2). Colour $w_1 w_2$ with some colour $c \in L(w_1 w_2) \setminus L(w_1 w_4)$, colour the edge $w_2 w_0$ greedily and finish as shown in Lemma 1. A contradiction. \square

We are now ready to proof Theorem 1.

Proof (of Theorem 1) Let L be an assignment of lists to the edges of G such that each list has a size of at least $\Delta(G) + 1$. Since for each edge $vw \in E(G)$ we have $\max(\deg(v), \deg(w)) + 1 \leq \Delta(G) + 1$, there is an L -edge-colouring of G by Lemma 2. \square

3 Graphs of Tree-Width 3 are $(\Delta + 2)$ -Total-Colourable

As mentioned in the introduction Conjecture 2 is true for graphs of maximum degree at most 5. In this section we handle the case where a graph has tree-width 3, and maximum degree greater than 5 and use this to proof Theorem 2.

An instance of list total colouring consists of a graph G and an assignment of lists $L : V(G) \cup E(G) \rightarrow \mathcal{P}(\mathbb{N})$ to the vertices and edges of G . A function $\mathcal{C} : V(G) \cup E(G) \rightarrow \mathbb{N}$ is called an L -total-colouring of G , if $\mathcal{C}(v) \in L(v)$ for each $v \in V(G)$, $\mathcal{C}(e) \in L(e)$ for each $e \in E(G)$, no two adjacent vertices receive the same colour, no two adjacent edges receive the same colour and no edge has the same colour as one of its ends. G is said to be k -total-choosable, if for each assignment of lists L to the vertices and edges of G , where all lists have a size of at least k , there is an L -total-colouring of G .

Let G be a graph and L an assignment of lists to the edges and vertices of G . For an L -total-colouring \mathcal{C} of some subgraph $H \subset G$ we call a colour c of the list of an uncoloured edge e available, if no edge adjacent to e and no endvertex of e has already been coloured with c . Similarly we call a colour c of the lists of an uncoloured vertex v available, if none of the edges and vertices adjacent to v have already been coloured with c . The set of available colours of an edge or vertex x is called list of remaining colours and denoted by $L^{\mathcal{C}}(x)$. We can always try to colour G greedily, by iteratively colouring the edge or vertex with the smallest list of available colours with an arbitrarily available colour.

For $\Delta \geq 1$, let G be a graph with an assignment of lists L to the edges and vertices of G such that the list of each edge vw and each vertex w has a size of at least $\Delta + 2$. Suppose that for some proper subset of vertices $W \subset V(G)$, we can find an L -total-colouring \mathcal{C} of the graph $G - W$. In order to extend \mathcal{C} to an L -total-colouring of G we need to find an $L^{\mathcal{C}}$ -colouring of $G(W)$. Since $\Delta \geq \deg_G(v) = \deg_{G-W}(v) + \deg_{G(W)}(v)$, we have for an edge $vw \in E(G)$ with $w \in W$ and $v \in V(G) \setminus W$

$$|L^{\mathcal{C}}(vw)| \geq |L(vw)| - \deg_{G-W}(v) - 1 \geq \deg_{G(W)}(v) + 1. \quad (3)$$

For any vertex $w \in W$ we have

$$|L^{\mathcal{C}}(w)| \geq |L(w)| - |N(w) \setminus W|. \quad (4)$$

The proof of the following lemma mirrors the proof of Lemma 2.

Lemma 3 *Let G be a graph of tree-width at most 3 with an assignment of lists L to the vertices and edges of G such that each list has a size of at least $\max(5, \Delta(G)) + 2$. Then G has an L -total-colouring.*

Proof We will assume the lemma is wrong and obtain a contradiction. Let G be a smallest counterexample to the lemma with an assignment of lists L to the vertices and edges of G for which each list has a size of at least $\Delta := \max(5, \Delta(G)) + 2$ such that there is no L -total-colouring of G .

If there is a vertex v of degree at most 2, we can find a L -total-colouring of $G - v$ by minimality. By (3) and (4) the lists of remaining colours of $G(\{v\})$ retain sizes large enough to extend this colouring greedily. Therefore we might assume that G has a minimum degree of at least 3. As before it follows that G has tree-width 3. Let (T, \mathcal{V}) be a smooth width 3 tree decomposition of G where T is rooted in some arbitrary vertex and let $t \in V(T)$ be a vertex of degree at least 2 in T of maximum height.

Suppose the neighbourhood of t contains at least two leaves $t_1, t_2 \in V(T)$. Since (T, \mathcal{V}) is smooth and t has maximum height, there are vertices $v_1 \in V_{t_1}$ and $v_2 \in V_{t_2}$ that are uniquely in V_{t_1} respectively V_{t_2} . We have $\deg(v_1) = \deg(v_2) = 3$, v_1 and v_2 are not adjacent and $|N(v_1) \cup N(v_2)| \leq 4$. By minimality we can find an L -total-colouring of the graph $G - v_1 - v_2$. If v_1 and v_2 have the same neighbourhood we first apply Theorem 4 to the bipartite graph induced by the edges adjacent to v_1 and v_2 . This is possible since the lists of remaining colours have sizes of at least 3 by (3). The lists of the vertices v_1 and v_2 retain at least one available colour by (4). So we can finish colouring greedily to extend this to an L -edge-colouring of G . If v_1 and v_2 have distinct neighbourhoods, colour the two uncoloured edges adjacent to vertices outside of $N(v_1) \cap N(v_2)$ greedily. We apply Theorem 4 to the bipartite graph induced by the edges between $\{v_1, v_2\}$ and $N(v_1) \cap N(v_2)$, which is possible because their lists of remaining colours have each a size of at least 2 by (3). As before colour the vertices v_1 and v_2 greedily to extend this to an L -edge-colouring of G .

Thus we can assume that t is incident to exactly one leaf $t_0 \in V(T)$. Since (T, \mathcal{V}) is smooth, there are vertices $w_0 \in V_{t_0}$ and $w_1 \in V_t \cap V_{t_0}$ that appear uniquely in V_{t_0} respectively $V_t \cap V_{t_0}$. By minimality there is an L -total-colouring \mathcal{C} of $G - w_0w_1$.

Delete the colour of w_0 from \mathcal{C} . As $\deg(w_0) = 3$ and $\deg(w_1) \leq 4$ we can finish greedily. Contradiction. \square

4 Halin Graphs are Δ -Edge-Choosable

To show the case where $\Delta(G) = 3$ of Theorem 3 we will use a result of Ellingham and Goddyn [8].

Theorem 5 *Let G be a d -regular planar graph. If G is d -edge-colourable, then G is d -edge-choosable.*

The next Lemma is a corollary to the 4 Colour Theorem, which is equivalent to the statement that every bridgeless cubic planar graph chromatic index 3. We include it for sake of completeness.

Lemma 4 *Let G be a 3-regular Halin graph. Then G is 3-edge-colourable.*

Proof We will assume that the lemma is wrong and obtain a contradiction. Consider a smallest counterexample G to the lemma with a tree T and a cycle C that passes along the leaves of T . The cycle C has at least 4 vertices, since otherwise G is the complete graph on 4 vertices and we are done. Choose an arbitrary vertex to be the root of T and let v be a vertex of maximum height among the vertices of degree 3 in T . By maximality v has exactly two neighbours v_1 and v_2 that lie on the cycle C . As C has at least 4 vertices, there are distinct vertices $v_0, v_3 \in V(C)$ to which v_1 respectively v_2 are adjacent. Let G_1 be the graph obtained from $G - v_1$ by adding the edge v_0v_2 and v_2w , where w is the third neighbour of v . G_1 is a 3-regular Halin graph smaller than G . So by minimality there is an edge-colouring \mathcal{C}_1 of G_1 using the colours $\{1, 2, 3\}$. We can extract an edge-colouring of $G - v_1 - v_2 - v$ from the edge-colouring of G_1 and extend this greedily to an edge colouring of G using only the colours $\{1, 2, 3\}$. A contradiction. \square

For a graph G with an assignment of lists L to the edges of G and $e, f \in E(G)$ we call two colours $c_1 \in L(e)$ and $c_2 \in L(f)$ compatible if $c_1 = c_2$ or if for each edge g that is adjacent to both e and f the list $L(g)$ contains at most one of the two colours c_1 and c_2 . The following lemma turns out to be quite useful in order solve instance of list edge-colourings of small graphs. The idea for the proof can be extracted from [6].

Lemma 5 *Let G be a graph with an assignment of lists L to the edges of G and $v_1v_2, w_1w_2 \in E(G)$ two edges that are not adjacent. If it holds that*

$$|L(v_1v_2)||L(w_1w_2)| > \sum_{v_iw_j \in E(G)} \left\lfloor \frac{|L(v_iw_j)|}{2} \right\rfloor \left\lceil \frac{|L(v_iw_j)|}{2} \right\rceil$$

then there are two compatible colours $c_1 \in L(v_1v_2)$ and $c_2 \in L(w_1w_2)$.

Proof If the lists of the edges v_1v_2 and w_1w_2 share a colour c we are done. Therefore assume that $L(v_1v_2) \cap L(w_1w_2) = \emptyset$. This yields that there are $|L(v_1v_2)||L(w_1w_2)|$

pairs of distinct colours (c_1, c_2) with $c_1 \in L(v_1v_2)$ and $c_2 \in L(w_1w_2)$. But an edge $v_iw_j \in E(G)$ for $1 \leq i, j \leq 2$ can contain both colours of at most $\lfloor \frac{|L(v_iw_j)|}{2} \rfloor \lceil \frac{|L(v_iw_j)|}{2} \rceil$ of those pairs. So if the above inequation holds we can find the desired two compatible colours $c_1 \in L(v_1v_2)$ and $c_2 \in L(w_1w_2)$. \square

Remark that the inequality holds if all involved lists have a size of exactly k , where k is an odd number. The next lemma implies that Halin graphs of maximum degree $\Delta \geq 4$ are Δ -edge-choosable.

Lemma 6 *Let G be a Halin graph with an assignment of lists L to the edges of G such that for each edge $vw \in E(G)$ the list $L(vw)$ has a size of at least $\max(\deg(v), \deg(w), 4)$. Then there is an L -edge-colouring of G .*

Proof We will assume that the lemma is wrong and obtain a contradiction. Consider a smallest counterexample G to the lemma with a tree T and a cycle C that passes along the leaves of T . Let $L(vw)$ be an assignment of lists to the edges $vw \in E(G)$, where each list has a size of at least $\max(\deg(v), \deg(w), 4)$ such that there is no L -edge-colouring of G . We can assume that G is not a complete graph on four vertices by Theorem 5 and hence C has at least 4 vertices. Choose an arbitrary vertex to be the root of T and let v be a vertex of maximum height among the vertices of degree at least 3 in T . By maximality v has $\deg(v) - 1$ neighbours that lie on the cycle C .

If $\deg(v) \geq 4$ then v has three neighbours w_1, w_2 and $w_3 \in V(C)$ with say $N(w_2) = \{v, w_1, w_3\}$ and $N(w_1) = \{v, w_0, w_2\}$. We denote by G_1 the graph obtained from $G - w_1$ by adding the edge w_0w_2 . Observe that G_1 is a Halin graph, smaller than G and the degrees of w_0, w_2 and v did not increase. Hence there is an L -edge-colouring \mathcal{C}_1 of G_1 by minimality (where an arbitrary list of size 4 was assigned to w_0w_2). We can extract an L -colouring of the graph $G - w_1 - w_2 - w_3$ from \mathcal{C}_1 . In order to extend this to an L -edge-colouring of G we need colour the graph $G \setminus \{w_1, w_2, w_3\}$ from the lists of remaining colours. The lower bounds on the sizes of these lists are given by (1) and (2). This instance of list edge-colouring is shown in Fig. 3. By Lemma 5 we can find two compatible colours $c_1 \in L(vw_1), c_2 \in L(w_2w_3)$. Colour the respective edges with c_1 and c_2 and finish greedily.

Thus we might assume that $\deg(v) = 3$ and so v has exactly two neighbours v_1 and $v_2 \in V(C)$. As C has at least 4 vertices, there are distinct vertices $v_0, v_3 \in V(C)$ to which v_1 respectively v_2 are adjacent. Further, v is adjacent to neither v_0 nor v_2 . Let

Fig. 3 The integers on the edges indicate the minimum sizes of the respective lists

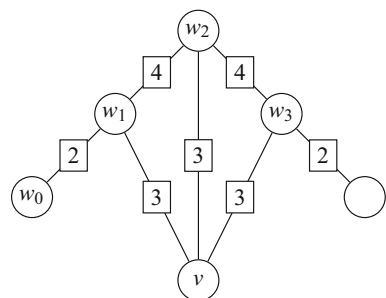


Fig. 4 The integers on the edges indicate the minimum sizes of the respective lists

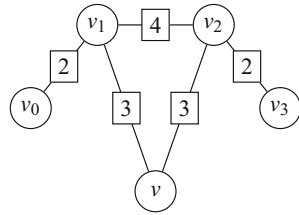
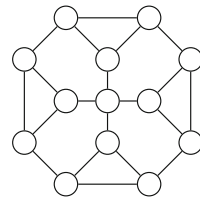


Fig. 5 This graphs shows that the lower bound of 4 is necessary



G_2 be the graph obtained from $G - v_1 - v$ by adding the edge v_0v_2 and v_2w , where w is the third neighbour of v . As before, G_2 is still a Halin graph, smaller than G and the degree of v_0 and v_2 did not increase. So by minimality there is an L -edge-colouring \mathcal{C}_2 of G_2 (where arbitrary lists of size $\max(4, \deg(w))$ was assigned to v_0v_2 and v_2w). We can extend \mathcal{C}_2 to an L -edge-colouring of G by colouring the graph $G\setminus\{v_1, v_2\}$ from the lists of remaining colours, which lower bounds are given by (1) and (2). Colour the edge between v and the vertex distinct from v_1 and v_2 greedily. The remaining instance of list edge-colouring is shown in Fig. 4. Colour the edge v_0v_1 greedily and apply Lemma 1 to the rest. A contradiction. \square

Note that the lower bound of 4 on the list size in this result is necessary. For the graph shown in Fig. 5 the lists $L(vw) = \{1, \dots, \max(\deg(v), \deg(w))\}$ for each edge vw do not permit an L -edge-colouring.

Proof (Proof of Theorem 3) Let L be an assignment of lists to the edges of G such that each list has a size of at least $\Delta(G)$. If $\Delta(G) \geq 4$, there is an L -colouring of G by Lemma 6. Otherwise G is planar, 3-regular and can be 3-edge-coloured by Lemma 4. Hence there is an L -colouring of G by Theorem 5. \square

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