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DYNAMIC EQUILIBRIUM IN LIMIT ORDER MARKETS: ANALYSIS OF DEPTH DISCLOSURE AND LIT FRAGMENTATION

TESIS PARA OPTAR AL GRADO DE MAGISTER EN CIENCIAS DE LA INGENIERÍA MENCIÓN MATEMÁTICAS APLICADAS

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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RESUMEN DE LA TESIS PARA OPTAR AL GRADO DE MAGISTER EN
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POR: RODRIGO IGNACIO ORELLANA ALARCÓN
FECHA: 2016
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Se desarrolla un modelo dinámico a tiempo contínuo que permite el comercio de múltiples mercados financieros interconectados, organizados como *limit order markets*, en el cual agentes endógenamente toman decisiones óptimas para maximizar el valor esperado de sus ganacias. Los agentes toman sus decisiones considerando incentivos propios, condiciones de mercado, potenciales decisiones de negociación futuras y diferentes estrategias adoptadas por otros agentes.

Se concentra el estudio al análisis de divulgación de profundidad y la fragmentación en el contexto de múltiples mercados. Se prueban tres escenarios principales: (i) un único mercado *Transparente*, (ii) un único mercado *Opaco* y (iii) un mercado múltiple interconectado entre una bolsa Transparente y una Opaca que comercian el mismo activo. Los resultados principales indican que, en el contexto de un único mercado, la divulgación de profundidad genera una competencia que incrementa el suministro de liquidez y, en consecuencia, reduce el spread, el ruido de mercado e incrementa la profundidad en los precios más competitivos y en el volumen total del libro. Los agentes con una valoración privada absoluta positiva del activo incrementan sus ganancias a costa de los agentes sin valoración privada, al disminuir sus costos de espera y aumentar sus ganancias por transacción. Estos beneficios son amplificados en el contexto de múltiples mercados debido a las restricciones para transar que generan una competencia más agresiva. Se encuentra que hay un flujo de liquidez hacia la componente Transparente debido a los agentes multi mercados proveedores de liquidez, lo cual reduce el spread e incrementa las profundidades. Para mantenerse atractivos, los agentes en la Bolsa Opaca también entran en competencia, lo cual reduce el spread y ruido de mercado en esta bolsa de similar manera. Los agentes multi mercados demandantes de liquidez son los que presentan el mejor rendimiento de todos, principalmente al reducir significativamente los tiempos de sus ejecuciones.

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We develop a dynamic model in continuous time to simulate multi markets trading. Traders make endogenously sequential optimal decision to maximize their expected payoffs across different limit order markets, taking into account intrinsic incentives, markets conditions, potential future trading decisions and different strategies adopted by other agents.

We focus our study in depth disclosure and lit fragmentation, and test three main scenarios: (i) a single Lit Market, (ii) a single Opaque Market and (iii) Multi Markets interconnected with both Lit and Opaque venues trading a single common asset. Our main results indicate that, in a single market environment, depth disclosure generates a competition that increments liquidity supply and as a consequence, reduces spread, microstructure noise and increases depth at best quotes and total depth of the book. Agents with a positive absolute private valuation of the asset increases their benefits at the expense of agents without a private valuation, by decreasing their waiting costs and increasing their money transfer. These benefits are amplified in a multi market environment due to trading restrictions that generates more aggressive competition. We find a liquidity flow to the Lit venue given by multi market liquidity suppliers, that reduces the spread and increases depths. To stay in competence agents in the Opaque Venue enter the competition as well, reducing spread and microstructure noise in that exchange too. Multi market liquidity demanders with the possibility to trade in both venues have the best performance of all agents, due to a significant reduction in their execution time.

Hay dos panes. Usted se come dos. Yo ninguno. Consumo promedio: un pan por persona.
- Nicanor Parra

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Le dedico este trabajo a mis padres, hermano y familia por su infinito apoyo y amor incondicional durante todo este tiempo. A Vannia simplemente por ser la mejor. A mis amigos de la vida por su enorme paciencia. A mis compañeros y amigos de universidad por las batallas libradas. A mis amigos de folclore por mantener vivas mis raíces. A los financieros por el tremendo apoyo en esta investigación. A los compañeros de pega por toda la buena onda en esta última etapa. A quienes compartimos desde lo más cotidiano. No es necesario nombrarles, todos ustedes saben a quienes me refiero. Les agradezco por ser parte de mi vida.

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1. Introduction

In the last decade, traders have experienced a growth in the opportunities to trade simultaneously in multiple exchanges with different characteristics, which has triggered changes in financial regulation in USA (the Regulation National Market System, RegNMS) and Europe (the Market in Financial Instruments Directive, MiFID), encouraging the proliferation of new trading venues. Indeed, trading activity in USA is dispersed across 11 exchanges and approximately 44 alternative trading systems [1]. On the other hand, several electronic platforms such as BATS Chi-X and Turquoise have spurred competition in Europe. These facts open the debate about the benefits and drawbacks of market fragmentation.

In this recent growth, an important characteristic to consider is the transparency of the markets. Hasbrouck (2007) defines transparency as a market attribute that refers to how much information market participants (and potential participants) possess about the trading process. Electronic markets that communicate in real time and displays the bids and offers of buyers and sellers, depths at those quotes (pre-trade transparency) and trades already executed (post-trade transparency) are considered highly transparent. These markets are often called *lit markets*. On the contrary, markets that display lower amount of information are called *opaque markets* or even *dark markets*, depending on the amount of information provided. For instance, dealer markets have no publicly visible bids or offers nor any trade reporting, as well as dark pools, which do not show any prices. Instead, dark pools only exhibit the direction of trades, i.e., buying or selling. According to Fidessa (2012) data-source, the cumulative volume transacted on the European dark venues has increased more than ten fold in the past years and continues a stable positive trend.

Another important characteristic of less transparent markets is the trading mechanism, sometimes known as *dark trading*. In a Dark Pool or Dark Market, at any point in time, dark liquidity (i.e. unexecuted outstanding orders) can be available only to buyers *or* to sellers, but not in both sides. Dark liquidity availability depend on the imbalance of the queue at the trading price, which is usually the spread midpoint of another market. If there is a sell side imbalance, then only incoming buy orders can be executed immediately against the resting dark orders, and incoming sell orders would either be placed in the queue at the respective price or re-routed to a different market, depending on the market properties. The same applies to a buy side imbalance. In contrast, an Opaque Market, as long as there are resting outstanding orders on both sides, both buyers and sellers can instantly execute their trades against the best prices, although they cannot see the volumes available. Examples of Dark

[1] See May 11, 2015 Public Statement 'U.S. Equity Market Structure: Making Our Markets Work Better for Investors' U.S. Securities and Exchange Commission (SEC).

Markets are ITG Posit, ASX Centre Point, ITG MatchNow (after Oct. 2012), Instinet VWAP Cross, Turquoise Midpoint, and midpoint dark order types by exchanges such as TSX, Chi-X, Nasdaq, BATS, and DirectEdge. Examples of Opaque Markets are ITG MatchNow (before Oct. 2012), Alpha Intraspread (before Oct. 2012), Instinet CBX, Turquoise Integrated, Credit Suisse CrossFinder, Goldman Sachs Sigma X, Deutsche Bank Super X, Citi Match, and UBS PIN.

In spite of this financial tendency, in recent financial literature there have been limited attempts to model dynamically these multiple markets. The main reason is due to the difficulties of characterizing the dynamic trading decision mechanism including the market microstructure features, as levels of transparency, which endogenously affects agents' behaviour. Thus, previous works present some restrictive assumptions in the market microstructure setups to make models analytically tractable. [2] Our main goal is to fill this gap in the literature by developing a dynamic model in continuous time for multiple limit order markets, considering most relevant microstructure characteristics of financial exchanges, in particular, different levels of transparency.[3]. This perspective allow us to analyse the impact of different transparency regimes in terms of the trading behaviour of market participants and market quality.

In our dynamic model, agents make endogenously sequential optimal decisions in continuous time, to maximize their expected payoffs, taking into account markets conditions (which depends on the transparency of the markets), potential future trading decisions, and different strategies adopted by other agents. We model two limit order markets with a single and common financial asset. Modelling two markets in continuous time and characterize them with different levels of transparency are our main contributions in relation to similar models found in literature. Also, we consider that the underlying asset has a fundamental value v_t that follows a compound Poisson process with fixed jump size and can be interpreted as the present value of the future cash flows. As in real limit order markets, limit orders books are characterized by a set of discretized prices at which traders can submit orders that respect time and price priority for execution. Another important contribution in our work is that we consider restricted and multimarket agents, the former can only trade in some determined venue and the latter have the faculty to trade in both venues. Despite this, all agents are risk neutral, uninformed, [4] arrive randomly following Poisson processes and decide whether to put an order (limit or market order) to buy or sell one share of the asset [5] in one of two markets, depending on their trading restrictions. Another characteristic of the model is that agents may revisit the markets multiple times to revise or modify previous trading decisions. However, due to cognition limits, agents cannot immediately modify their previous orders after a change in the market conditions (i.e. they cannot monitor the market all the time). Therefore, a trader re-enters the market multiple times, according to a Poisson process as well, until he executes and leaves the market forever.

In order to consider heterogeneity across traders in terms of trading motivations, each

[2]From an analytical perspective, we include in appendix a little discussion about a possibly extension to multi markets of the model provided by Rosu (2009) to expose the main difficulties.

[3]For better understanding of trading mechanisms in financial markets see Hasbrouck (2007),Parlour and Seppi (2008) and Iori and Kovaleva (2015).

[4]This means that agents do not know the current fundamental value

[5]This is a limitation in our model since we can not consider institutional traders who usually trade in volumes. This extension is proposed for future work.

one has a private valuation for the asset denoted by α drawn from a distribution F_α. [6] For instance, traders with $\alpha > 0$ have a positive valuation for the asset and therefore they are more likely to be buyers. Analogously, traders with $\alpha < 0$ are more likely to be sellers. Agents with $\alpha = 0$ do not have private valuation for the asset and therefore they are willing to be buyers or sellers depending on trading possibilities.

Optimal decisions of agents are markovian, since they depend on previous trading history only. Formally, the trading game is a Bayesian game, and the proper solution concept here is Markov-Perfect Bayesian Equilibrium, as Maskin and Tirole (2001) point out. Unfortunately, analytic solution is not possible. For that reason, we obtain the equilibrium numerically using the algorithm introduced by Pakes and McGuire (2001), originally proposed for industrial organization problems with sequential decisions. This algorithm provides a Markov-Perfect Equilibrium which has been successfully implanted into dynamics models for limit order markets by Goettler et al. (2009), Goettler et al. (2005) and Bernales and Daoud (2013), although without considering the possibility to trade in multi markets nor different transparency levels. This algorithm will allow us to simulate a trading history in different scenarios.

We consider transparency levels as follows: Lit Markets display bid and ask quotes, their respective depth, buy and sell volumes and the last transaction executed, i.e., full pre-trade and post-trade transparency, and Opaque Markets display only bid and ask quotes, i.e., that is minimal pre-trade transparency. Thus, we test three main scenarios: (i) single Lit Market, (ii) single Opaque Market and (iii) a multi market with both Lit and Opaque Markets sharing the same financial asset. Thereby, we can measure the impact of transparency on market fragmentation in terms of welfare and determine which traders benefits most when they can trade in both venues. Furthermore, we can study market quality considering several measures of liquidity such as quoted spread, effective spread, depth at and away the quotes, and we can evaluate the effect of microstructure noise on all scenarios. Finally, we can understand traders' behavior considering main strategies they select, given their own characteristic and incentives.

We find that, in a consolidated single market, depth disclosure provide a more trustworthy place to trade that increases liquidity supply which in turn causes a decrease in the bid-ask spread, an increment in depth at and away best quotes and also reduces microstructure noise. Agents with intrinsic motives to trade take advantage of this increment in liquidity provision. Since these agents are more impatient, they are more likely to pay immediacy costs in order to get a quick trade. Hence, more orders in the book allow them to reduce their waiting costs and increase their money transfer leading to better gains. These effects are amplified with lit fragmentation in the multi market context. Trading restriction generates a more aggressive competition that improves market quality mainly in the Lit Venue, as Gresse (2015) recently found based on data from eight exchanges and a trade facility for a large sample of LSE- and Euronext-listed equities. We find a liquidity flow toward that exchange, which is consistent with Moinas (2010). Multi market liquidity demander with the benefit to trade in both venues have the best performance, given by a significantly improvement in their execution quality, mainly due to a decrement in their execution times, which is consistent with the evidence found by Chung and Chuwonganant (2009) in their study of the effect of pre-trade

[6] Private value represents idiosyncratic motives for trade (such as wealth shocks, hedging needs, tax exposures, differences in investment horizons, among others.

transparency on liquidity and execution quality using data before and after the introduction of SuperMontage for NASDAQ-listed securities.

Our study can be related to the microstructure literature regarding market fragmentation in comparison with market consolidation. Theoretical work of Mendelson (1987) shows that fragmentation reduces the expected quantity traded, reduces gains from trade, contrary to consolidation of order flow, that can reduce spreads and thus improve liquidity. Pagano (1989) considers a two-period stock market economy and shows that in absence of differential transaction costs, all order flow concentrates in a single market (the more liquid one), unless the two markets are identical. Chowdhry and Nanda (1991) develop a model with a single asset that is traded at multiple locations and show that a fragmented market can exist as equilibrium. Biais (1993) presents a framework to compare centralized limit order markets with fragmented dealer markets. Madhavan (1995) based on a dealer market model of Glosten and Milgrom (1985), shows that heterogeneity of traders allow the existence of fragmented markets which do not necessary gravitate to a single market and also shows that fragmentation results in violations of price efficiency. Hendershott and Mendelson (2000) study competition between dealer markets and crossing networks (where agents trade directly with one another). They find that crossing networks exhibit both positive and negative externalities. Parlour and Seppi (2003) present a microstructure model of competition for order flow between a pure limit order market and a hybrid market and conclude that coexistence is possible and competition between exchanges can increase as well as reduce liquidity. Foucault and Menkveld (2008) extend the framework of Parlour and Seppi (2003) to study competition between two pure limit order markets and shows that positive global externalities can be obtained from fragmentation, although this effect could differ throughout each trading venue. They also test their model empirically studying the entry of the London Stock Exchange into the Dutch equity market with the launch of EuroSETS (an electronic limit order market).

In relation to lit fragmentation, which is the study of fragmentation due to transparency differences in markets, the closer to our analysis is Iori and Kovaleva (2015), in which consider a double auction market model that extends the framework of Chiarella et al. (2009). They study the effect of pre-trade quote transparency on spread, price discovery and liquidity in an artificial limit order market. Their main finding is that restriction of displayed quote depth improves market quality in multiple dimensions: it reduces average transaction costs, maintains higher liquidity and moderate volatility, balances the limit order book, and enhances price discovery. Also, Moinas (2010) considers a sequencial model of three periods and analyzes the rationale for the submission of hidden limit orders, and compares opaque and transparent limit order books, although she considers traders informed with some probability and large orders. Nonetheless, finds that pre-trade opacity improves market liquidity and the welfare of participants.

There are several empirical studies that compare market quality before and after entry or consolidation of markets. The evidence on market fragmentation has mixed conclusions. For example, related to reduction in fragmentation, Arnold et al. (1999) find that merging exchanges attract order flow and experience a decrease in bid-ask spreads. Amihud et al. (2003) study the reduction in fragmentation of Tel Aviv Stock Exchange and find that stock liquidity improves. Gajewski and Gresse (2007) compare the fragmented hybrid order-driven segment of the London Stock Exchange with the centralised electronic order book of Euronext, and find that spreads in the centralized order book are lower. On the other hand, Boehmer and Boehmer (2004) examine the impact of NYSE entry in the market for exchange traded funds (ETFs) and find improvement in market quality in the entire market and also in individual market centres. Hengelbrock and Theissen (2009) examine the market entry of Turquoise and reveal results somewhat ambiguous but pointing in the direction of positive impacts on market quality. O'Hara and Ye (2011) study how fragmentation is affecting market quality in US equity markets and find that fragmentation does not harm liquidity. However, they relate a specific feature of US equity market and state 'our results support the conclusion that while US equity markets are spatially fragmented, they are, in fact, virtually consolidated into single market with many points of entry', which explains why positive externalities that stems from consolidated trading are observed in US equity markets.

Regarding empirical lit fragmentation literature, Foley and Putnins (2015) dissagregate dark trading in two types that theory suggest should have different effects: one sided dark trading, which is more alike to a Dark Pool, and two sided dark trading, which is closer to our work. Their main finding is that two-sided dark trading, in moderate levels, is beneficial to liquidity and informational efficiency. It tends to lower quoted, effective and realized spreads, reduces price impact measures of illiquidity, and makes prices closer to the random walk that is expected under informational efficiency. Based on data from eight exchanges and a trade reporting facility for a large sample of LSE- and Euronext-listed equities, Gresse (2015) investigates how lit and dark market fragmentation affects liquidity. Neither dark trading, nor fragmentation between lit order books, is found to harm liquidity. Duong et al. (2014) investigates the influence of improved pre-trade transparency, although in term of quotes disclosure. They consider data from two natural experiments: when the Sydney Futures Exchange increased the limit order book disclosure from the best bid and ask level to the best three price levels in 2001 and from the best three to the best five price levels in 2003. The evidence shows that the limit order book contains information on future return and volatility. The findings of their study highlight a diminishing benefit of additional limit order book disclosure. Chen and Tseng (2015) study a new information disclosure mechanism carried out by the Taiwan Stock Exchange (TWSE) on its closing call auction. The closing call auction is a mechanism designed to determine the closing price for each trading day. They find that, in entering new orders, both individual and institutional investors become more aggressive after the market becomes partially transparent.

This thesis is organized as follows. Section 2 presents the model in detail. Section 3 include the results and analyse them from different perspectives. Section 4 concludes. Finally, details of the analytical approach as well as numerical implementations are left in Appendix, among others.

2. Model

We develop a dynamic continuous-time model of two pure limit order markets[7], with a single financial asset that can be traded in both venues. The model is an asynchronous dynamic trading game in which there are risk-neutral agents, who arrive at the market randomly. Traders have one share to trade and can submit limit orders or market orders. They also can revise and modify their unexecuted limit orders multiple times before execution. In the following, we provide a complete description of the model.

2.1. Model description

Limit order books. Each market $m \in \{1, 2\}$ has a respective order book. As in real limit order markets, the limit order book relative to the market m at time t, denoted $L_{m,t}$, is described by a discrete set of feasible prices, denoted as $\{p^i_m\}^{\infty}_{i=-\infty}$, where the tick size, d_m is the distance between any two consecutive prices. There are backlogs of outstanding orders to buy or to sell in the market m, $l^i_{m,t}$, at prices p^i_m. A positive (negative) number in $l^i_{m,t}$ denotes buy (sell) orders, and it represents the depth at price p^i_m. Therefore, for the book $L_{m,t}$, the bid price is $B(L_{m,t}) = \sup\{p^i_j | l^i_{m,t} > 0\}$, while the ask price is $A(L_{m,t}) = \inf\{p^i_j | l^i_{m,t} < 0\}$, and if the order book is empty on the buy side or on the sell side, $B(L_{m,t}) = -\infty$ or $A(L_{m,t}) = \infty$.[8]

Each limit order book respects the price and time priorities for the execution of limit orders. Buy (sell) orders at higher (lower) prices are executed first, and limit orders submitted earlier have priority in the queue when they have the same price. In addition, when a trader submits an order, the order price identifies whether the order is a market order or a limit order. This means that an order to buy (sell) at a price above (below) the ask (bid) price is executed immediately at the ask (bid) price; and thus this order is a market order.

Transparency. Each market have a transparency level that traders consider in their decisions. Lit Markets display the bid and ask prices, the depths at those quotes, total buy and sell side depths and the price and direction (buy or sell) of the last transaction executed, that is, full pre-trade and post-trade transparency. On the other hand, Opaque Markets display only bid and ask prices. We emphasize the word *opaque*, since it is not

[7]Note that single market is a particular configuration of multi markets, when all trading is restricted to the first market.

[8]We use a similar notation to Goettler et al. (2009) regarding the microstructure features of the model for the dynamic order book market.

completely *dark*, because we allow displaying bid and ask prices, which is the minimum pre-trade transparency. This framework can be extended to Dark Markets, in particular, Dark Pools, allowing agents to trade at the midpoint of the spread of a centralized limit order book, but this is proposed for future work.

Asset. The financial asset has a fundamental value v_t at the time t, and follows a Poisson process at rate λ_v. Whenever a change occurs, the fundamental value rises or reduces its value in a fixed amount σ, each with the same probability. Differently to Goettler et al. (2009), when a change occurs we do not assume that the fundamental value coincides with one feasible price to trade, therefore the fixed amount σ does not depend on the tick sizes d_m, $m \in \{1, 2\}$. Traders do not know the fundamental value of the asset at time t, instead, they observe the value after a little lag $v_{t-\Delta t}$ and they have to estimate the actual value by a learning process looking at the market conditions repetitions.[9]

Traders. They are risk-neutral agents, who arrive at the market randomly following a Poisson process with intensity λ. We split agents in restricted and multimarket from an exogenous probability distribution F_R. Restricted agents can only trade in their respective venue, and multimarket agents can trade in both venues. Each agent can trade one share[10] and has to select the best action given the current state of economy. To this end a trader has to make four main trading decisions after arriving at the market:

i) If it is a multimarket agent, to choose between $L_{1,t}$ or $L_{2,t}$ to trade.
ii) To buy or to sell the asset.
iii) To set the price at which he will submit the order, which implies the decision to submit a market order or a limit order, depending on whether the price is above or below the quotes

Despite the fact that traders arrive following a Poisson process with intensity λ, the submission rate is different as agents can re-enter and modify their standing orders, which depends endogenously on the market conditions at the time t, as well as the number of agents.

Traders can re-enter the market and monitor their previous unexecuted limit orders. However, agents can not immediately modify their previous limit orders after a change in the market conditions, mainly due to cognition limits. Traders re-enter at the market according to Poisson processes with intensity λ_r and have to make additional trading decisions after re-entering the market:

i) Whether to cancel an unexecuted limit order or retain the order without changes.
ii) If he decides to cancel and modify an unexecuted limit order and if he is a multimarket agent, he has to choose the market, else he is restricted to some market.
iii) To choose the type of order, buy or sell, and its price.

Thus, agents have to take into account the possibility of re-entry in the utility maximization problem of their decision.

[9] Look at the Appendix for more details on this learning process.

[10] We can include additional shares per agent in the trading decision. However, similarly to Goettler et al. (2009) Goettler et al. (2009), we assume one share per trader to make the model computationally tractable.

Once a trader submits a limit order, he remains part of the trading game by revising her order until it is executed; however, after execution the trader exits the market forever. Consequently, there are a random number of active market participants at each instant who are monitoring their previous limit orders.

Traders have to pay a cancellation fee c_m when they cancel an unexecuted submitted limit order in $L_{m,t}$, $m \in \{1, 2\}$. In the case of a re-entry, a trader can leave the order without changes, which has the benefit of keeping her priority time in the respective queue and avoiding a cancellation fee.[11] The negative side of leaving an order in the book is that the asset value could move in directions that affect future payoffs. For instance, in the scenario of a growth in the asset value, some limit sell orders could be priced too low, and a quick trader could make profits from the difference. This possibility represents an implicit transaction cost of being 'picked off' when the fundamental value change unexpectedly after limit orders have been submitted. Conversely, when the asset value decreases, a sell limit order has the risk of not resulting in a trade since could be priced too high. To take into account the risk that a limit order may not result in a trade, we include a cost of 'delaying' by a discount rate ρ, which is common for both markets and reflects the cost of not executing immediately. This 'delaying' cost does not represent the time value of money; instead ρ reflects opportunity costs and the cost of monitoring the market until a limit order is executed. Thus, the payoffs of order executions are discounted back to the order submission time at rate ρ.

Heterogeneity. Each trader has a private value α for the asset, which is drawn from a discrete distribution F_α and known before making any trading decision. The private value can be interpreted as a personal valuation of the asset and it is constant to each agent. This private value gives additional heterogeneity to the different agents in the dynamic trading game. For instance, traders with zero private value (and hence with no intrinsic benefits to trade) are indifferent in taking either side of the market and hence maximize their benefits depending on the available trading possibilities; consequently they are likely liquidity suppliers since they will probably submit limit orders[12]. Conversely, traders with higher absolute values in their intrinsic benefits to trade are likely to be liquidity demanders and will probably submit market orders.

2.2. The traders' dynamic maximization problem

Formulation. Suppose that a trader arrives at the trading game and observes state s of the market. Let the entry time be equal to t. The state s that a given trader observes includes:

i) A set of variables of limit order books $L_{1,t}$ and $L_{2,t}$ that result from previous trading activity in each market, such as quotes, depths, last transaction, depending on the transparency of each book and the restriction of the agent: restricted agents can

[11] It is important to point out that the order priority could have changed, depending on the shape of the book, which should be taken into account in the decision to cancel and re-submit.

[12] Traders with zero private value are equivalent to liquidity suppliers as in Jovanovic and Menkveld (2011) Jovanovic and Menkveld (2011).

only see their respective book information and multimarket agents observes both book information;

ii) The lagged fundamental value of the asset $v_{t-\Delta t}$;

iii) His private value α and delay rate ρ;

iv) The status of her previous action in the case that the trader has already submitted an earlier limit order to any of the markets, which includes the original submission price, the book where the order was submitted, the current priority in the book, and if the order was a buy or a sell.

Define $\Gamma(s)$ as the set of possible of actions that a trader can take given the state s (e.g. to send an order to the limit order book $L_{1,t}$ or $L_{2,t}$, to cancel and submit a new order or to wait in an outstanding limit order, among others). Formally, we define an action as $\tilde{a} = (\tilde{b}, \tilde{x}, \tilde{p}, \tilde{q})$ where $\tilde{b}$ is a variable representing the optimal limit order book chosen, $\tilde{p}$ is the price of the order, $\tilde{x}$ is $+1$ or -1 when the order is a buy or sell, and $q \geq 0$ the priority associated to this order, which is determined by $\tilde{p}, \tilde{x}$ and $L_{\tilde{b},t}$, and $q = 0$ meaning a market order (because it is an instantaneous execution, it has no priority). Let $\eta(t|\tilde{a}, s)$ be the probability that an order is executed at time t given that the trader takes the action $\tilde{a} \in \Gamma(s)$ when he faces the state s. It is important to notice that $\eta(\cdot)$ incorporates all possible future states and strategic actions adopted by other traders until t. For notational convenience, we do not include the strategies in the formulas. If the decision $\tilde{a}$ is the submission of a market order, $\eta(0|\tilde{a}, s) = 1$, while $\eta(h|\tilde{a}, s)$ converges asymptotically to zero when the trader decides to submit a limit order with a price far away from the fundamental value (not an aggressive order). In addition, let $\gamma(v|t, s)$ be the density function of v at time t given the state s. Therefore, the expected value of an order that is executed prior to a re-entry at time t_r is:

$$\pi(t_r, \tilde{a}, s) = \int_0^{t_r} \int_{-\infty}^{\infty} e^{-\rho t} \left((\alpha + v - \tilde{p})\tilde{x} \right) \cdot \eta(t|\tilde{a}, s) \cdot \gamma(v|t, s) dv dt \tag{1}$$

Here, $(\alpha + v - \tilde{p})\tilde{x}$ is the instantaneous payoff of the order where $\tilde{p}$ is the submission price which is part of the decision $\tilde{a}$; while $\tilde{x}$ is also a component of the decision $\tilde{a}$ and reflects whether the trader decides to submit a buy order ($\tilde{x} = 1$), to submit a sell order ($\tilde{x} = -1$). This payoff is transformed to a present value at the rate ρ which is the cost of 'delaying' previously defined in this section.

Let $R(\cdot)$ be the probability distribution of the re-entry time which is exogenous and follows an exponential distribution at rate λ_r. In addition, let $\psi(s_{t_r}|\tilde{a}, s, t_r)$ be the probability that the state s_{t_r} takes place at time t_r given the previous state s and the action $\tilde{a}$, which also includes all potential states and strategic decisions followed by other traders until t_r. Therefore, the value to an agent of arriving at the state s, $V(s)$, is given by the Bellman equation of the trader's optimization problem:

$$V(s) = \max_{\tilde{a} \in \Gamma(s)} \int_0^{\infty} \left[\pi(t_r, \tilde{a}, s) + e^{-\rho t_r} \int_{s_{t_r} \in \mathcal{S}} (V(s_{t_r}) - \tilde{z}_{s_{t_r}} c_m) \cdot \psi(s_{t_r}|\tilde{a}, s, t_r) ds_{t_r} \right] dR(t_r) \tag{2}$$

where $\tilde{z}_{s_{t_r}} = 1$ if the optimal decision in the state s_{t_r} is a cancellation and $\tilde{z}_{s_{t_r}} = 0$ in any other case, $m \in \{1, 2\}$ is an indicator for the market selected previously (i.e., $L_{1,t}$ or $L_{2,t}$, respectively), and $\mathcal{S}$ is the set of possible states in re-entries. The first term is defined in equation (3); while the second term reflects the subsequent payoff in the case of re-entries.

Equilibrium. In the trading game, each time a trader arrives in the market, he chooses an action that maximizes his expected discounted utility given the state he observes. This is what equation (2) represents. Optimal strategies in this model are state-dependent. Since states are composed of previous events, the decision is Markovian. Formally, the trading game is a Bayesian game. Traders have privately known utilities from trade (since a trader's α is unknown to other traders) and a private estimation of the fundamental value v_t. As Maskin and Tirole (2001) point out, the proper solution concept is Markov perfect Bayesian equilibrium, which requires traders to play dynamically optimal strategies on each entry into the market, given their current beliefs. We focus on stationary, symmetric equilibria, in which each type of trader chooses the same stategy, and this strategy does not depend on the time at which the trader arrives at the market.

Since this equation cannot be solved analytically, we develop an algorithm inspired in Pakes and McGuire (2001) to obtain a numerical approximation of the equilibrium. In our algorithm, traders start with beliefs about payoffs to all feasible actions, and update these beliefs when they take an action and observe its realized payoff. A key step in updating their beliefs over different actions is determining the expected fundamental value of the asset, since they have a little lag. In simple words, when the learning differences have dropped down to a certain threshold, we consider that the algorithm has converged, and we run the model for a while to get the trading data that is going to be analysed. Details of the learning process and convergence criteria are left in Appendix.

Despite of this algorithmic approach, we analyse the same problem from another perspective in order to find an analytical solution. Our study found some difficulties that deviate from the purpose of this thesis, hence, the continuation of these interesting problem is left for future work. Formulation details and advances of this approach are left Appendix.

2.3. Numerical parameterization of the trading game

For our simulations, we set parameters taking into account the existing empirical literature. For instance, in single market scenarios, we follow parameters proposed by Goettler et al. (2009), while in multi markets, we adjust some parameters to make both scenarios properly comparable.

Single Market Parameterization

- We allow agents to trade only in the book $L_{1,t}$.
- We normalize the rate of the Poisson process for new trader arrivals, λ, to one. This means that a unit of time in our simulations represent the average time between new trader arrivals. On average, a trader reenters the market after four unit of time, thus the rate of the Poisson process for reentries, is, $\lambda_r = \frac{1}{4}$. This reentry rate is assumed constant across all traders.
- A tick in the simulation corresponds to one-eighth of a dollar.
- We assume that the distribution of private value F_α is discrete with support $\{-8, -4, 0, 4, 8\}$ and cumulative distribution $\{0.15, 0.35, 0.65, 0.85, 1.0\}$. This distribution is based in fin-

ding of Hollifield et al. (2006) who estimate the distributions of private values for three stocks on the Vancouver exchange.

- Innovations in the fundamental value of the asset, v, occur according to a Poisson process at rate $\lambda_v = \frac{1}{8}$, which means that the fundamental value jumps, on average every eight units of model time. Whenever v changes, it increases or decreases in a fixed amount $\sigma = 1$, each with the same probability.
- We set the lag $\Delta t = 8$ units of time of the model. This means that the lagged fundamental value may contain on average one jump of difference with respect to the actual fundamental value, given the intensity λ_v.
- We assume $\rho = 0.05$. We experimented with lower values and found results qualitatively similar.
- When a trader submits a limit order at time t, the price is restricted in a range $[v_t - k, v_t + k] \cap \mathcal{P}$, where $\mathcal{P}$ is the set of feasible prices. This assumption is made for computational tractability, since we need finite set of prices. The amount k is chosen large enough that it does not affect equilibrium. For the simulations we choose $k = 6$.[13]
- The information displayed by the Book is defined previously depending if it is Lit or Opaque Market.
- We set the cancellation fee to zero. This allow us to explore the trading process with less frictions.

Multi Markets Parameterization

- We set L_{1_t} as the Lit Market and $L_{2,t}$ as the Opaque Market.
- We adjust the arrival rate $\lambda = 2$ to inject liquidity with the same intensity to each venue in comparison with single market.
- Finally, we set trading restriction of agents F_R as a discrete distribution with support $\{1, 2, 3\}$ and cumulative distribution $\{0.4, 0.8, 1.0\}$. Values 1 and 2 represent Lit and Opaque restricted agents respectively, and value 3 represent multimarket agents. Hence, the arrival rates of different agents are the intensity λ defined previously multiplied by the respective probabilities. For instance, the arrival rate of an opaque restricted agent with a private valuation of 4 is $0.20 \times 0.4 \times \lambda$.

Since our model implementation is flexible enough, we can test both single scenarios and the multi markets scenario.[14]. Certainly, multi markets scenarios involve higher levels of complexity in comparison with single market. On the one hand, the state space is extremely increased if consider two markets instead of one. On the other hand, we need to track the evolution of two limit order books in continuous time, which represents an additional difficulty for that scenario. We provide more details about the model implementation for two markets in Appendix.

[13]We experimented with higher values of k and noticed that traders rarely send orders so far from the fundamental value.

[14]Indeed, many more scenarios can be tested, for instance, we can consider different market features between the two markets or different traders characteristics.

3. Results

This section presents the results derived from our simulations of the model with the numerical parameterization presented in previous section. We simulate two scenarios with a single limit order market, one highly transparent (henceforth, Lit market) and the other with the minimal pre-trade transparency (henceforth, Opaque market). Also, we simulate a scenario with both markets simultaneously (henceforth, Multi markets). We organize main results in three sections. Section 3.1 starts with a general perspective of the three scenarios by analysing market quality measures to get a notion of the characteristics of each market. Section 3.2 then analyse how traders behave in those context and how their behaviour relate to the market characteristic, taking into consideration that traders *make* those markets. Finally, section 3.3 integrates previous results to explain the welfare of participants in all scenarios.

3.1. Market Quality

What is the impact of depth disclosure and lit fragmentation in the characteristics of markets? In this section we present the impact of transparency on market quality by presenting measures that allow us to analyse market liquidity across each exchange. Table 1 shows different liquidity measures, such as bid-ask spread, the effective spread, average number of limit orders at the ask quote (total and effective traded), average number of limit orders on the sell side of the book (total and effective traded) and microstructure noise. The quoted spread is calculated by observing the market every 10 units of time whenever bid and ask exist. The effective spread is calculated with all transactions as mean of $|p - m|$, where p is the transaction price, and m is the midpoint between the bid and ask quotes.

Observation 1

(i) *In single markets, depth disclosure improves market liquidity. It lowers bid-ask spread, increase depth at best quotes and total depth.*

(ii) *In multi markets, lit fragmentation improves market liquidity in the same manner, particularly in the Lit Venue.*

We start presenting our main findings by looking at the characteristic of the markets to see how they change when there are differences in the information provided to market agents. This will give us a general perspective of the effects of depth disclosure and lit fragmentation

Table 1: This table shows different market quality measures for all scenarios. First columns are for single markets scenarios, and second for the multi market scenario, presented by each venue. Measures are Bid-Ask Spread, Effective Spread, Number of limit orders at the ask (total and effectively traded), Number of limit orders on the sell side of the book (total and effectively traded) and absolute mean and standard deviation of microstructure noise. Each column is a different scenario, the first represent single market and the second, multi markets. The mean and standard deviation of microstructure noise are calculated with transactions as transaction price (p_t) minus the fundamental value (v_t) in ticks. All market quality measures are determined as mean of 20 million market new entries in equilibrium for one of the two market. Since both markets have identical characteristic, we do not need to report all measures for the second market. Standard errors for all Market quality measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

Single Market		Both Markets	
Lit Market	Opaque Market	Lit Venue	Opaque Venue
Bid-ask spread			
1.530	2.027	1.459	1.305
Effective spread			
0.681	0.870	0.608	0.529
N. of limit orders at the ask			
1.742	1.316	1.959	1.272
N. of limit orders at the ask (effectively traded)			
0.543	0.631	0.525	0.366
N. of limit orders on the sell side of the book			
5.719	3.246	7.468	4.207
N. of limit orders on the sell side of the book (effectively traded)			
1.093	1.083	1.178	1.264
Microstructure noise: Mean $\|v_t - p_t\|$			
0.938	1.015	0.904	0.861
Microstructure noise: Std. Dev. $(v_t - p_t)$			
1.263	1.335	1.228	1.184

that we will further disentangle in next section, by deepening in trading behaviour. Table 1 reports different market quality measures for all scenarios. In single market case, depth disclosure improves market liquidity by a decrease in bid-ask spread from 2.027 ticks to 1.530 ticks from the Opaque case to the Lit case respectively, and effective spread from 0.870 to 0.681. It also shows that there is liquidity supply increment which is reflected in the number of limit orders at the ask, where it grows from 1.316 to 1.742, and also in the number of limit orders at the sell side of the book where it rises from 3.246 to 5.719.

To understand this, we consider that the determinants of the spread are mainly three: (i) arrival rates, (ii) information about the fundamental value and (iii) immediacy costs. First, even though arrival λ and returning λ_r intensities are the same for both scenarios, the fact that in the Lit Market agents are accumulating at the sell side means that more agents are *active* and monitoring the market, hence the overall arrival rate (which considers all waiting and upcoming new agents) is greater than the Opaque Market. This contributes to a narrow spread in the Lit Market. Remember that the model is symmetric, hence similar results are found in depth at the bid quote and the buy side of the limit order book. Second, the information about the fundamental value does not generates differences since all agents have the same lag to observe the fundamental value. Furthermore, transparency does not play much of a role in estimating the real fundamental value. This will be discussed in the next observation. Finally, the last component to analyse is immediacy costs. A decrement in spread can talk about a reduction in immediacy cost, which is the difference between transaction price and fundamental value. This relates to agents strategies when submitting orders, so that this important issue will be discussed with more details in next section, that it is about trading behaviour.

These results give lights that depth information has some value to agents that could make them have more confidence in the exchange and motivate a competition that improves liquidity on the exchange.

In the case of multi markets, lit fragmentation improves market liquidity even more, considering the fact that there are trading restrictions that enhances a more aggressive competition than in single markets case. Here there are restricted agents, who are obliged to trade in their exchange and that do not know there is another exchange, and there are multi market agents who monitor both exchanges and can submit orders where they choose and perhaps switch later. In this context, bid-ask spread narrows to 1.459 in the Lit Venue and 1.305 in the Opaque Venue, while the effective spread narrows to 0.608 for the Lit Venue and 0.529 to the Opaque one. In this environment, restricted agents have to enter into an extra liquidity competition that do not happen in a single market scenario, because in those scenarios liquidity is distributed among themselves. Now, they have to compete with the multi market agents, hence restricted agents have to place more aggressive orders that narrow the spread a little more. We can see that the Lit Venue primarily absorbs liquidity, given that the number of limit orders at the sell side of the book is much greater in the Lit Venue than the Opaque Venue, with values of 7.468 and 4.207, respectively. The number of limit orders at the ask quote is greater in the Lit Venue as well, with a value of 1.959 versus 1.272 on the Opaque Venue. Lit Venue absorbs liquidity from the multi markets agents due to depth information availability and lit fragmentation, and reduces spread in the Lit Venue for the same reasons of the single market case, hence restricted agents in the Opaque Venue have to enter into the liquidity competition and supply liquidity as well, therefore the spread in their venue narrows too. Evidence of this is found by Gresse (2015) who finds that lit fragmentation improves spread across markets and locally on the primary exchange, and also by Moinas (2010) who shows that opacity improves market liquidity in a centralized limit order book.

Observation 2

(i) *Neither depth disclosure nor lit fragmentation affects the fundamental value estimations of agents.*

In previous observation we mentioned that one of the components of spread is the information about the fundamental value. Remember that all agents are uninformed about the fundamental value and they can only see a lagged value of it, hence they have to guess they real value when taking a decision. This estimation takes in consideration market variables of the exchange, hence one may think that a transparent exchange provides a better notion. Table 2, shows that neither depth disclosure nor lit fragmentation affects how agents estimate the fundamental value. Value are very similar for all regimes. This is consistent with results found in Goettler et al. (2009) who finds similar values for single market scenarios. Furthermore, they use a regression of belief updates to find that the more significant market parameters are bid and ask prices, which is the only information provided in Opaque exchanges. Hence, no significant differences are found and we refuse the possibility that information of fundamental value has any determinant effect in spread movement.

Finally, all of this market quality considerations reflects in another measure that captures somehow the accuracy of trading: microstructure noise.

Table 2: This table reports the mean and standard deviation of error in estimation of the fundamental value for all agents separated by markets and scenarios. We numerically find the equilibrium for each regime and then simulate a further 20 million entries into market, considering both new and returning agents. These entries form the data on which this table is based. For each entry or reentry to the market by any agent, we construct the expectation of v_t following the updating process, which we include in Appendix. The error is defined as the expected value minus the actual value, in ticks.

Single Market		Both Markets		
Lit Market	Opaque Market	Lit Venue Restricted	Opaque Venue Restricted	Lit & Opaque Unrestricted
Mean deviation of errors in estimations				
0.000	0.000	-0.001	0.000	-0.002
Standard deviation of errors in estimations				
1.001	1.002	0.994	0.995	0.995

Observation 3

(i) *Microstructure noise decreases with more transparency in single market. Decrement is greater in multi markets.*

The microstructure characteristic of financial markets may induce frictions that make the transaction price p_t deviate from the fundamental value v_t. Therefore, the transaction price can be written as the fundamental value v_t plus microstructure noise ξ_t, hence $p_t = v_t + \xi_t$. In a perfect scenario without frictions, the transaction price should be identical to the fundamental value, so ξ_t should be zero, but in real markets with frictions it can be an important component of prices. For instance, in our model, we consider discrete prices given a ticksize strictly positive, we have traders with intrinsic motives to trade who are willing to trade at prices even lower (or higher) than the fundamental value for obtain their private valuation for the asset, we also include some cognition limits on market participants who cannot respond to changes in market conditions immediately or that they are uninformed about the real fundamental value and have to guess its value, among others, which adds more frictions to the model.

In Table 1, we calculate the absolute mean and standard deviation of microstructure noise to introduce a measure of the trading frictions present in the market. We observe that both, absolute mean and standard deviation of microstructure decrease their value when more information is disclosed in a consolidated market. For instance, in single markets, the mean decreases from 1.083 ticks in the Opaque Market to 0.938 ticks in the Lit Market. Madhavan (1995) suggest that less information is impounded in prices. In Observation 1 we saw that depth disclosure enhances liquidity competition, which is reflected in tighter spread, among other. This competition, together with market transparency, also provides an idea about the fundamental value of the underlying asset and moves transactions more close to the real value of the asset, which coincides with a decrement in microstructure noise. In multi markets the effect of liquidity competition is higher due to the trading restrictions and so it is the noise decrement. Since there is an increase of liquidity provision in the Lit Venue, this more aggressive competence decreases microstructure noise to 0.904. On the other hand, restricted agents in the Opaque Venue in order to stay in the liquidity competition, place more aggressive orders too which result to be more close to the fundamental value, decreasing microstructure noise to 0.861 ticks.

So far, we have analysed and discussed depth disclosure and lit fragmentation in terms of market quality. We have seen that transparency positively affects market liquidity. In the next section, we focus our analysis into agents behaviour in order to get a further insight on their strategies and how that relate with results until now.

3.2. Trading Behaviour

How do traders optimally act? Who act as liquidity supplier? Who as liquidity demander? Do traders submit more aggressive orders in more transparent market? How does prior questions change with lit fragmentation? The optimal strategies of traders involve high complexity, because they take into account large amounts of information and because they have many options, even with less transparency. On the one hand, traders can submit market orders and execute immediately since these orders do not have associated waiting costs, however, traders usually have to pay an immediacy cost per a quick trade, given by the difference between the fundamental value and the respective quote. On the other hand, traders can submit limit orders, which have associated waiting costs according to a delay rate ρ. A trader who submits a limit order plays the role of 'speculator', in the sense that he is not certain about future changes in the fundamental value, which could lead to unexpected gains. With an accurate limit order, agents can get better terms from trade, but also there is an inherent risk of being picked off when the fundamental value moves in an adverse direction. For instance, if the fundamental value increases, some of the outstanding limit sell orders could be priced too low and other agents could make profits from the difference, which can lead to place less aggressive prices to get protection behind other orders, and reduce the picking off risk.

Table 3 presents some metric for trading behaviour, such as percentage of limit orders executed per trader, probability of being picked off after submitting a limit order, average number of limit order submitted per trader, average number of limit orders cancellation per trader, the probability of submitting a limit sell at the ask quote, and some measures of time as average time to execution and average time for execution of limit orders. Since the model is symmetric, we focus our analysis on the sell side of the market, because similar results are obtained for the buy side of the market.

Observation 4

(i) *In single markets, information disclosure enhances trading activity in agents.*

(ii) *In multi markets, lit fragmentation generates a liquidity competition supported by trading restrictions. This competition increases the picking off risk.*

In previous section we showed how depth disclosure and lit fragmentation affect positively the liquidity provision in markets. This is naturally related with trading behaviour of agents. For instance, in single markets, depth disclosure enhances trading activity. This happens because depth informations gives light and expectation about the value of the asset that is being traded. I would like to explain this by a simpler analogy that i would like to call "the bar effect". For the sake of this analogy, let's consider the spread fixed. Suppose Rod likes to hang out sometimes to have some fun, and he wants to go to a bar on a Friday night. Rod goes to a street where there is Bar A that he doesn't know much about a priori, but he notices that the Bar A has transparent windows and he clearly sees that there is a lot of people inside and also sees the waiting line for tickets for that Bar. This information motives him to want to go inside, because he might find the party he wanted, even though he might have to wait in a line to buy an entrance ticket. Now, suppose Rod goes to another street near and sees Bar B with fuzzy windows that do not allow him to deduce if there are people inside or not. Despite of this, Rod might decide to try luck, enter Bar B and find a big party o perhaps an empty place. The fact that Rod does not have the same information than Bar A, makes the scenario of Bar B less attractive. In Bar A he has more certainty that he might get his wanted party time (although maybe he would have had to wait to get a ticket). But neither this is *assured*, it is only *more likely*. Even though this analogy does not contemplate all the characteristics of the model, in some sense it contributes to the notion that the Lit Market provides a more appetizing place to trade. This can be seen in Table 3, where the number of submitted limit order increase from 1.523 to 2.668 from the Opaque Market case to the Lit one. Since more orders are being submitted, the orders volume increases which affects execution time and execution probability. For instance, the time between the instant in which a trader arrives and the execution of the limit order increases from 8.406 to 16.294, and the percentage of limit orders executed decreases from from 32.824 % to 18.743 %. Finally, another effect of the increment in liquidity in a single Lit Market context, is that the probability of submitting an order at the ask price lowers to 23.83 % with respect to 26.00 % in the Opaque Market, because the competence for that quote increases.

Table 3: Trading behavior. This table shows different measures of trader behavior for different scenarios, such as, the percentage of limit orders executed among all limit orders submitted, the probability of being 'picked-off' after submitting a limit order, the number of limit orders submitted per trader, the number of limit order cancellations per trader, the average time between the instant in which a trader arrives and his execution (in time units of our model), the time between the instant in which a trader arrives and the execution of his limit order (in time units of our model) and the probability of submitting a limit sell order at the ask price (i.e., an aggressive limit sell order). Since the model is symmetric on both sides of the book it is not necessary to also report the probability of submitting a limit buy order at the bid price. The probability of being 'picked-off' is calculated with executed limit orders: we take the number of limit sell (buy) orders that are executed when their execution price is below (above) the fundamental value of the asset, which is divided by all the limit orders executed in the market. All trading behavior measures are determined as mean of 20 million market new entries in equilibrium. Standard errors for all trader behavior measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario. Results of the multi market scenario are presented on each venue.

Single Market		Both Markets		
Lit Market	Opaque Market	Lit Venue Restricted	Opaque Venue Restricted	Lit & Opaque Unrestricted
Percentage of limit orders 'executed'				
18.74 %	32.82 %	16.59 %	1.29 %	9.59 %
Prob. of being picked-off after submitting a limit order				
27.23 %	27.94 %	32.80 %	31.31 %	32.04 %
Number of limit orders 'submitted'				
2.668	1.523	3.144	42.71	3.563
Number of limit order cancellations				
2.168	1.023	2.622	41.625	3.198
Time between the instant in which a trader arrives and her execution				
9.878	4.928	14.305	177.525	16.777
Time between the instant in which a trader arrives and the execution of her limit order				
16.294	8.406	23.529	245.23	37.111
Prob. of submitting a limit sell order at the ask price				
23.83 %	26.00 %	20.94 %	1.40 %	15.56 %

In multi markets, lit fragmentation generates another competition due to trading restrictions, in particular, with the presence of multi market agents. In the previous section we saw that the Lit Venue increases the order volume significantly and that the presence of multi market agents force restricted agents to supply more liquidity to stay in competence, in particular in the Opaque Venue, where there is less of it. This increment can be seen in the number of submitted limit orders where the Lit Venue rises up to 3.144 while in the Opaque Venue increases up to 42.171. Since agents are accumulating near the fundamental value, where the spread is theoretically centred in, any exogenous change in that value have repercussions in agents' positions. Remember that we modelled jumps of the fundamental value as a Compound Poison Process with intensity λ_v, hence time between jumps is exponentially distributed. If the fundamental value moves in an adverse way, agents have to cancel their outstanding orders and move backwards, in terms of aggressiveness, to be less exposed. If it moves in the other way, even thought could be favourable, agents have to cancel their outstanding order and move forward, in terms of aggressiveness, to stay in competitive prices. This behaviour can be seen in the number of cancellations in the Lit Venue with a value of 2.622, while the Opaque Venue has an average of 41.625. This is why the liquidity supply increment in Opaque Venue is not being reflected in the volume of the book. Remember that in Observation 1, we saw that the number of limit orders on the sell side of the book is greater in the Lit Venue rather than the Opaque Venue. This particular behaviour also affects execution probability and expected execution time. Since more agents are submitting orders, the probability that a limit order is going to be executed lowers down to 16.59 % in the Lit Venue, while in the Opaque Venue it goes down to 1.29 %, also multi markets agents have a low value of 9.59 % on average. On the other hand, regarding to expected execution time, the time between the instant in which and agent arrives and her execution is 14.305 in the Lit Venue while in the Opaque Venue it significantly rises up to 177.525. These values consider both limit and market orders, with the latter being instantaneous. If we focus only in outstanding limit order that execute due to another agent's market order, values increase up to 23.529 and 245.23 for Lit and Opaque respectively. Hence, the Lit Venue provides a faster exchange to trade, which is an incentive for multi market to choose that market. In fact, execution times for multi market agents are more close to the Lit Venue values rather than the Opaque ones.

Since agents cannot modify their orders instantaneously, some times they cannot change their orders on time and are more susceptible to being picked off at an inconvenient price. This is represented in the probability of being picked off after submitting a limit order, where it grows to 32.43 % on average. Hence, more competence in this context also imply more trading risk. In the same line, another effect that intensifies, is the probability of placing an order at best quote, where it decreases to 20.94 % in the Lit Venue and 1.40 % in the Opaque one, while the multi market agents have an average value of 15.56 %. The value in the Opaque Venue can be both odd and surprising, but the number of limit order submission in that venue really makes it difficult to agents to post and order at the best quote.

Observation 5

(i) *Agents with no intrinsic motives to trade act as liquidity supplier.*

(ii) *Agents with intrinsic motives to trade act as liquidity demander.*

(iii) *Both characteristics are amplified in multi markets.*

In the previous observation, we analysed somehow the overall behaviour of agents. But, are there significant differences between them? To disentangle this question, we first start with the strategies for different agents in terms of their private valuation of the asset α and their trading restrictions. In the following, we report the average behaviour of traders by these types.

Supplying liquidity to markets requires posting limit orders on them and demanding liquidity requires sending market orders. To examine liquidity provision strategies we consider Table 4. This table shows the percentage of submitted orders differentiated by type: submitted limit orders (that did not end in an execution), executed limit orders and market orders [15]. This values are separated for each absolute private value and scenario. According to the values, it is clear that agents with no intrinsic motives to trade act as liquidity suppliers and the rest as liquidity demander. In single market, agents with a private value of zero, prefer the limit order on average 96.3 % versus 3.6 % the market order, considering both regimes of transparency. Agents with an absolute private valuation of 8 act as liquidity demander. They prefer the market order in mean 61.8 % versus 38.1 % the limit order, also considering both regimes. On the other hand, agents with an absolute private valuation of 4, $|\alpha| = 4$, share both behaviour, although in this context they prefer more the limit order 75.0 % against to 25.0 %, approximately. The same applies in multi markets for all agents, although there is an increment in their preferences. For instance, agents with private value zero, increase their preference for limit order up to 99.0 % approximately, while agents with absolute private value 8, increase their preference for market order up to 66 % approximately. This is consistent with the competition analysis discussed previously.

Differences between limit and market orders makes reference to agents' patience. Agents with no intrinsic motive to trade do not have an *urgency* on trading, while agents with intrinsic motives to trade *have* urgency to trade in order to realize or consolidate their private valuation of the asset. But, how does this affect them? This patience issue motives the next observation.

Observation 6

(i) *Depth disclosure makes agents with intrinsic motives to trade have better trading attributes at the expense of agents that do not have intrinsic motives.*

(ii) *The effect is intensified with lit fragmentation.*

All traders type have different incentives in the trading process that make them to play specific roles in the market. As we analysed in the previous observation, agents with $\alpha = 0$ supply liquidity to the market, agents with extreme valuation ($|\alpha| = 8$) are more likely to be liquidity demander, and agents with $|\alpha| = 4$ sometimes behave like both types making an equilibrium between liquidity supply and demand given market conditions. We linked this behaviour with agents' patience, because particularly in the case of liquidity demander, they are willing to take more risk in their orders. Hence, it is relevant to analyse this attributes in order to deepen in agents' strategies. This attributes involve probability of being picked

[15] We split limit orders in executed and unexecuted for analysis purposes, but remember that agents can only decide whether to submit a market order or a limit order. In the case of a limit order, they do not know when that order is going to be executed. In this particular analysis we consider the total number of limit orders submitted, that is, we consider the sum of executed and unexecuted limit orders.

Table 4: Strategies per trader type. This table reports the proportion of market orders submitted by each trader type as percentage of all market orders submitted, the proportion of limit orders submitted by each trader type as percentage of all limit orders submitted, and proportion of non order by each trader type as percentage of all traders that choose not an order in equilibrium. All percentages are determined as mean of 20 million market new entries in equilibrium. Standard errors for all trader strategies are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario. Results of the multi market scenario are presented as mean of the overall trading in both venues.

Single Market		Private Value $\|\alpha\|$		
		0	4	8
Lit Market	L.O. Subm.	87.8 %	50.8 %	21.2 %
	L.O. Exec.	9.5 %	25.8 %	13.7 %
	M.O.	2.5 %	23.2 %	64.9 %
		100 %	100 %	100 %
Opaque Market	L.O. Subm.	77.4 %	49.7 %	25.4 %
	L.O. Exec.	17.7 %	25.1 %	15.9 %
	M.O.	4.8 %	25.1 %	58.8 %
		100 %	100 %	100 %

Both Markets		Private Value $\|\alpha\|$		
Restricted		0	4	8
Lit Venue	L.O. Subm.	89.8 %	48.4 %	23.4 %
	L.O. Exec.	8.5 %	26.8 %	15.1 %
	M.O.	1.5 %	24.6 %	61.3 %
		100 %	100 %	100 %
Opaque Venue	L.O. Subm.	99.2 %	54.2 %	29.4 %
	L.O. Exec.	0.5 %	27.7 %	18.5 %
	M.O.	0.1 %	18.0 %	52.0 %
		100 %	100 %	100 %

Both Markets		Private Value $\|\alpha\|$		
Unrestricted		0	4	8
Lit Venue	L.O. Subm.	48.8 %	18.8 %	4.4 %
	L.O. Exec.	4.7 %	9.8 %	2.7 %
	M.O.	1.5 %	23.1 %	40.4 %
		55.1 %	51.8 %	47.5 %
Opaque Venue	L.O. Subm.	42.8 %	14.8 %	4.6 %
	L.O. Exec.	1.7 %	8.0 %	2.8 %
	M.O.	0.1 %	25.2 %	44.8 %
		44.8 %	48.1 %	52.4 %
Joint Venues	L.O. Subm.	91.7 %	33.6 %	9.0 %
	L.O. Exec.	6.5 %	17.9 %	5.6 %
	M.O.	1.7 %	48.4 %	85.2 %
		100 %	100 %	100 %

off, expected execution time and execution prices. Table 5 presents these values separated by private value and scenarios. In this table we focus on the sell side of the market since the buy side is analogue, as we have mentioned before in other results.

The idea is that an impatient agent wants a quick trade, hence he is willing to sacrifice price in order to gain (or reduce) time. Consider a trader who submits a market sell at price p when the fundamental value is $v_{t'}$. This market order is matched with its respective counterpart that is an outstanding limit buy order, submitted previously by another agent. The trader who places the market order, sells the asset at price p when it has a value $v_{t'}$, thus he obtains $p - v_{t'}$ and the trade is instantaneous. On the other hand, the trader who submitted the limit order, obtain $v_{t'} - p$ but waited for that trade. A higher amount $p - v_{t'}$, signifies an improvement in the terms of trade for the agent who submits the market sell, and as a consequence worse terms of trade for the agent who submitted the limit buy.

In the single market case, we have seen that depth disclosure motives an improvement in liquidity and that suppliers are primary agents with private value zero. In this table we can see that these agents have to wait more in order to get an execution, from 14.30 to 30.87 units of time, also they are a little more exposed due to competition, hence their probability of being picked off increases from 11.36 % to 14.35 %. Besides, in supplying liquidity they also compete with private value $|\alpha| = 4$, hence the price at which they execute diminishes too, from 0.82 to 0.75. Agents with $|\alpha| > 0$ take advantage of this situation. Since more liquidity is available, they can reduce their execution time from 1.95 to 1.64, increase their execution price from -0.62 to -0.55 and be less exposed when trying with a limit order, hence their probability of being picked off reduces from 64.97 % to 50.63 % [16]. The results for agents $|\alpha| = 4$ are similar but less. Hence, this results tell us that agents with a positive private valuation of the asset prefer Lit Markets, and that agents without a private motive to trade do better in an Opaque Market.

[16]It is important to notice that agents with $|\alpha| = 8$ usually submit orders at prices below the fundamental value (in the sell side case), because in doing so they still have positive income due to their private valuation of the asset. Also, they can execute faster. That is why their probability are the highest.

Table 5: Traders' behavior differentiated by private value. This table shows statistics of traders differentiated by private values both single market and multi market scenarios. We report time between the instant in which a trader arrives and his execution (as Time to execution), price of submitted limit sell orders and mean price of all executed sell orders, market and limit sell (since the model is symmetric we do not need to report the price of buy orders). All trader behavior measures are determined as mean of 20 million market entries in equilibrium. Standard errors for all trader behavior measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each case.

			Prob. of being picking off			Time to execution			Price of ex. sell orders		
			Private Value $\|\alpha\|$			Private Value $\|\alpha\|$			Private Value α		
			0	4	8	0	4	8	0	-4	-8
Single Market	Lit Market	-	14.35 %	35.92 %	50.63 %	30.87	3.52	1.64	0.75	-0.15	-0.55
	Opaque Market	-	11.36 %	35.57 %	64.97 %	14.30	3.54	1.95	0.82	-0.17	-0.62
Both Markets	Lit Venue	Restricted	35.72 %	29.93 %	30.42 %	45.58	3.05	1.55	0.63	-0.18	-0.51
	Opaque Venue	Restricted	40.22 %	25.79 %	22.89 %	591.47	3.44	1.80	0.70	-0.10	-0.40
	Lit & Opaque	Unrestricted	35.04 %	29.41 %	29.11 %	55.52	2.52	1.53	0.61	-0.20	-0.42

On the other hand, as we have seen repeatedly, the effects in multi market intensifies significantly. On the one hand, agents with zero private value have higher probability of being picked off, up to 36 % on average, higher execution time, above at least of 45.58, and diminishes the execution price to 0.64 approximately. On the other hand, agents with a private value of $|\alpha| = 8$ decrease their probability to mean 27 %, their execution time to 1.62 and increase the execution price to -0.44, all values approximated. There are no news in this results, beside intensification, at least in terms of difference in private valuation of the underlying asset. The important thing here is to analyse differences in trading restrictions, which triggers a competition that cannot be seen in a single market environment. First, we can notice the difference between restricted agents in the Lit Venue and the Opaque Venues. Restricted agents in the Lit Venue execute faster than the others restricted, for all private values. This is a difference with respect to the single market case, where liquidity suppliers, $\alpha = 0$, are faster in the Opaque Venue. Here, these agents have the worst expected execution time with a value of 591.47. Remember that these agents enter into a competition for liquidity that flow mainly to the Lit Venue, hence they increase the number of order submissions, but, unfortunately, this strategy leads to way slower executions. However, this strategy makes them have better terms of trade as seen in the price of all executed sell orders, where the Opaque Venue has a value of 0.70 versus the Lit Venue with a value of 0.63. Despite of this, the execution time advances us that payments for these agents will not be very good, question that will be seen in the next section. On the other hand, agents with a positive motive to trade take advantage of the incoming liquidity flow in the Lit Venue. This can be seen in that they trade faster in the Lit Venue rather than the single market case, with an average value of 1.67 and at higher prices of -0.45 on average for the former case, versus single 1.79 and -0.58 respectively to the latter case. In comparison between Lit Venue and Opaque Venue, liquidity demander execute faster in the more transparent one for the same reason, although at a lower price. The opposite occur in the Opaque Venue, where they execute slower but at better prices. The overall effect reflected in their welfare will be seen in the next section.

Finally, there are the multi market agents, who have the virtue of trading in both venues.

This allow them to track both venues and exploit better trading options. In this sense, agents who benefits most are those who have an urgency in trading, that is, agents with a positive absolute private value, $|\alpha| > 0$, since they are looking for quick trades and they have two best quotes to choose, they can only do better versus a single best quote. This agents send mainly market orders that executes instantaneously, and by doing so, they 'skip' the queue of all liquidity suppliers that are waiting for a trade. Hence, these agents have the fastest expected execution time, with a value of 1.53 similar with the Lit Venue case, and also one of the highest price of executed sell orders with a value of -0.42, similar to the Opaque Venue case. That is to say, multi markets take the best in terms trading attributes, where in speed they prefer the Lit venue, which is the faster exchange, and in terms of execution price they prefer the Opaque Venue, which is the better exchange to catch favourable prices. On the other hand, agents without urgency in trading, $\alpha = 0$, which are liquidity suppliers, have to put in the queue when submitting a limit order, hence they can't exploit benefits in terms of expected execution time, nonetheless, they prefer the Lit Venue which gives them an execution time of 55.52. However, they can and do exploit benefits in terms of execution price. For this, they choose the Opaque Venue which gives them a value of with 0.70. Thus, it is clear that multi market agents use the better attributes of both exchanges to improve their strategies.

Observation 7

(i) *Multi market liquidity suppliers change more often to the Lit Venue rather than the Opaque Venue.*

In the previous observation we saw that multi market liquidity suppliers take the best from both venues. In the Lit one they can execute faster and in the Opaque venue they can get better execution prices. But, which one do they prefer? We have a notion in terms of market quality, where we found an increment in the liquidity flow to the Lit Venue, hence we have lights that this venue will be chosen the most. To answer this, we see Table 4, which we used to show that agents without motives to trade are more likely to be liquidity suppliers. In this table, we present the proportion orders chosen by each type of trader. In the case of multi market agents, we split the proportion between both exchanges and we can see that the number of orders is higher in the Lit Venue rather than the Opaque Venue. This happen for all types of orders. Hence it is clear that these agents prefer that venue. To complement this we present Table 6, where we present the proportion of agents who switch from one exchange to the another. Remember that agents who submit an order can return and change their outstanding order. They can resubmit to the same venue or switch to the other. We count the agents who change from the Lit Venue to the Opaque and vice versa. Results show that the higher value is given in the more transparent venue, where agent who move are 6.1 % versus 0.7 % who switch to the Opaque Venue. This could tell us something about depth information value, as we expressed in the "bar effect" analogy. These agents might prefer the certainty versus uncertainty. To check this, we use regressions to test the market conditions variables in the number of limit orders send by these agents.

Observation 8

(i) *Multi market liquidity suppliers value depth information.*

Table 6: Average proportion of traders who switch from one market to another. First and second rows represent the proportion who switch to the other venue. Hence, Lit Venue row represent the proportion of agents who switch to Dark Venue, and vice-versa in Dark Venue row. Third row represent the proportion of agents who *ever* switch to another venue. This percentage is determined as mean of 20 million market entries in equilibrium. Standard errors are small enough since we use a large number of simulated events.

	Private Value $\|\alpha\|$			Total
	0	4	8	
Lit Venue	6.1 %	1.4 %	0.3 %	5.1 %
Opaque Venue	0.7 %	1.1 %	0.3 %	0.7 %
Joint Venues	1.2 %	1.3 %	0.3 %	1.3 %

(ii) *Multi market liquidity demander value best quotes and last transaction direction information.*

We consider a simple linear OLS regression for limit orders submission of multi market agents. Since successive observations from the market will exhibit serial dependence, we consider observations of five minutes intervals which include trader entries and reentries. The dependent variable in the regression is the number of limit buy order submissions in the five minute interval. The independent variables are the market conditions at the end of the interval, which include all parameters observed by multi market liquidity suppliers. These parameters are bid and ask for both Lit and Opaque Venues, and for the Lit Venue only we add the depth on either side at best quotes and the total side, and the last transaction price and direction (i.e., +1 if the transaction resulted from a market buy, and -1 if it resulted from a market sell). For all price variables, we use the price minus the last observed value of v. We only consider value where the books are non-empty on both sides of the respective market. We separate the dependent variable in those orders submitted to the Lit Venue, to the Opaque Venue and also we test for the Joint Venues. The results of the regression are shown Table 7.

Table 7: OLS Regression for liquidity supplier. We test what market conditions affect agents strategies in their prefered orders.

	Joint Venues			Lit Venue			Opaque Venue		
Independent Variable	Coeff.	t-stat.	p-val	Coeff.	t-stat.	p-val	Coeff.	t-stat.	p-val
Constant	4.2093	205.6668	0	2.005	152.3899	0	2.2042	160.2637	0
Lit - Bid	0.0306	5.0137	0	-0.0243	-6.1927	0	0.055	13.3849	0
Lit - Ask	0.0549	8.8816	0	0.0046	1.1627	0.2449	0.0503	12.1041	0
Lit - Bid Depth	**0.1273**	**27.5621**	**0**	**0.102**	**34.3411**	**0**	**0.0253**	**8.1622**	**0**
Lit - Ask Depth	-0.1029	-22.1615	0	-0.0045	-1.511	0.1308	-0.0984	-31.5324	0
Lit - Buy Side Depth	**0.2766**	**165.7216**	**0**	**0.1999**	**186.2863**	**0**	**0.0767**	**68.3958**	**0**
Lit - Sell Side Depth	0.0635	37.2263	0	-0.0199	-18.1297	0	0.0834	72.7391	0
Lit - Last Trans. Price	-0.1414	-26.8931	0	-0.1012	-29.931	0	-0.0402	-11.3856	0
Lit - Las Trans. Direction	0.0073	1.753	0.0796	-0.0113	-4.2403	0	0.0186	6.665	0
Opaque - Bid	-0.006	-0.6711	0.5022	0.0041	0.7081	0.4789	-0.0102	-1.676	0.0937
Opaque - Ask	-0.0394	-4.4073	0	0.0561	9.7449	0	-0.0955	-15.8807	0

We confirm that the more relevant variables are the depth of the bid quote and depth of the total buy side, as we expected. Furthermore, in a comparison between the exchanges, the coefficient is greater for the Lit Venue. Since the model is symmetric, similar results can be found for the sell side. These results mean that a change in depth on the Lit Venue affects positively the number of limit orders submission in that Venue from multi market liquidity supplier. This supports the evidence and results previously found and confirm the value that liquidity suppliers give to the Lit exchange.

We also test the market orders for multi market liquidity demander in a similar fashion than previous regression, but the difference is that here we count the number of market buy order submissions at the end of each interval. We find that more relevant variables that agents consider when submitting a market buy order are, on the hand, the direction of the last transaction, which is more relevant for the Lit Venue in particular. Although we did not highlight it, the last transaction price is also a relevant component. Together these two value provide liquidity demander a reference of Lit exchange trading. If the last transaction looks convenient, they will probably want to replicate it. On the other hand, the ask quote is also a relevant component, mainly in the Opaque Venue. We could say that these agent find more convenient prices in the Opaque Venue rather than in Lit Market, possibly the late reaction of liquidity suppliers in the Opaque Venue. This is evidence is also found in Table 4, were it can be seen that multi market liquidity suppliers prefer the Opaque Venue for market orders 44.8 % versus the Lit Venue which have a 40.4 % of preference, but we can agree that the difference is not very big. The importance of this regression is that depth information is not as valued as these parameters, and exhibit the differences in the motives of trade between liquidity demander and suppliers. This is consistent with the evidence find by Valenzuela and Zer (2013).

Table 8: OLS Regression for liquidity demander. We test what market conditions affect agents strategies in their prefered orders.

	Joint Venues			Lit Venue			Opaque Venue		
Independent Variable	Coeff.	t-stat.	p-val	Coeff.	t-stat.	p-val	Coeff.	t-stat.	p-val
Constant	0.5002	126.3323	0	0.2872	105.2503	0	0.213	73.7998	0
Lit - Bid	-0.0319	-26.9805	0	-0.0232	-28.5287	0	-0.0087	-10.0405	0
Lit - Ask	0.0099	8.2687	0	0.0019	2.2593	0.0239	0.008	9.2074	0
Lit - Bid Depth	-0.0245	-27.397	0	-0.0143	-23.2758	0	-0.0101	-15.5784	0
Lit - Ask Depth	0.0067	7.4208	0	-0.0069	-11.0693	0	0.0135	20.6461	0
Lit - Buy Side Depth	0.004	12.2355	0	0.0018	8.2017	0	0.0021	9.031	0
Lit - Sell Side Depth	-0.003	-9.1763	0	-0.0028	-12.3282	0	-0.0002	-0.9327	0.351
Lit - Last Trans. Price	0.0247	24.2505	0	0.0123	17.4781	0	0.0124	16.7433	0
Lit - Last Trans. Direction	**0.0385**	**48.0743**	**0**	**0.0492**	**89.0041**	**0**	**-0.0106**	**-18.1996**	**0**
Opaque - Bid	-0.0392	-22.4551	0	0.0014	1.1936	0.2326	-0.0406	-31.934	0
Opaque - Ask	**0.0559**	**32.3051**	**0**	**0.0184**	**15.3819**	**0**	**0.0376**	**29.7752**	**0**

Finally, in this section we have disentangled the strategies of agents which support the evidence found in market quality measures. Now, it remains to see and analyse how these elements translate into welfare values of agents and markets. In the next section we focus and conclude the study with welfare measures.

3.3. Welfare

Do pre-trade and post-trade transparency improve global welfare? Do connection and competition of different markets improve global welfare? Which traders' type take advantages of transparency and multi markets dynamics? We examine and link how quality of markets and agents' strategies reflect on welfare for both single and the multi markets scenarios to answer the previous questions.

Consider a trader with a private value α and delaying discount rate ρ who arrives to the market at time t and executes at time t' as consequence of a market order submission or the execution of his previous limit order. Then the gross payoff or profit for the trader is calculated as: [17]

$$\Pi = \tilde{x} \cdot (\alpha + v_{t'} - \tilde{p}) \cdot e^{-\rho(t'-t)} \tag{3}$$

where $\tilde{x} = 1$ if the order was a buy and $\tilde{x} = -1$ if the order was a sell, $v_{t'}$ is the fundamental value of the asset at the time of execution t' and $\tilde{p}$ is the price of the transaction.

In order to understand different elements of trading profits, we start by decomposing the agents' payoffs to analyse gains and losses from the trading process, similar to Bernales and Daoud (2013). Define $\Delta t = t' - t$. For convenience, because most result are presented from this side, suppose that $x = -1$, i.e. the order executed was a limit or market sell. Hence, we can rewrite expression (3) as:

$$\Pi = -\alpha + \alpha(1 - e^{-\rho \Delta t}) + (p - v_{t'})e^{-\rho \Delta t} \tag{4}$$

where we define,

- Private value: α
- Waiting Cost: $\alpha(1 - e^{-\rho \Delta t})$
- Money Transfer: $(p - v_{t'})e^{-\rho \Delta t}$

We can consider the buy side, i.e. $x = 1$ and rewrite the discounted realized payoff expression (4) for a buy order in a similar fashion.

In general, a trader cannot execute immediately to gain his intrinsic private value α. Instead, traders have to wait, either because there is a lack of liquidity, poor market conditions or because the optimal action is to submit a limit order or retain a previous one. This waiting cost is reflected by $\alpha(e^{-\rho \Delta t} - 1)$[18]. Additionally, when a execution occurs, the trader gains (or losses) some money product of the difference between the transaction price p and the fundamental value at the moment of the trade $v_{t'}$, that can be discounted back. Consequently, this is a money transfer and it is expressed by $(v_{t'} - p)e^{-\rho \Delta t}$.

Observation 9

(i) *In single markets, depth disclosure decreases waiting cost and increases money transfer*

[17] We already discuss en the previous section that $\tilde{x} \cdot (\alpha + v_{t'} - \tilde{p}.)$ is the instantaneous payoff, which is discounted back by multiplying $e^{-\rho(t'-t)}$ to obtain the gross payoff.

[18] The expression for the waiting cost is inspired in Hollifield et al. (2006)

for liquidity demander agents. In opposition, liquidity suppliers decrease their money transfer.

(ii) *In multi markets, unrestricted agents decrease their waiting cost the most and increase their money transfer, at the expense of restricted ones.*

Table 8 presents the average waiting cost and average money transfer [19] for each transaction in the market. Let's start our analysis with waiting cost. Notice first that this analysis can be done to liquidity demander or agents with intrinsic motives to trade $|\alpha| > 0$, since suppliers nullify this value due to their zero private value, hence let's consider $\alpha = -8$. Secondly, notice that this value is explicitly related with expected execution time in the following manner:

$$\Delta t \searrow 0 \Leftrightarrow e^{-\rho \Delta t} \nearrow 1 \Leftrightarrow -8 \cdot (1 - e^{-\rho \Delta t}) \nearrow 0^-$$

This means that faster executions lead to lower waiting costs. We have seen previously that the Lit Venue have faster execution times, hence, it is directly to verify that Lit Market in the single case and the Lit Venue in the multi market case have values closer to 0^-, which means lower waiting costs of -0.141 and -0.159 respectively. It is relevant, though, to highlight that the advantage of multi market agents generates the lowest waiting cost value of -0.055. The same goes to agents with $|\alpha| = 4$, which behave as both supplier and demander. These agents have a mean value -0.355 on average in single market case and restricted multi market case, but when they are multi market agents, their waiting cost reduce to -0.167, which is almost a half part. Hence, we can clearly see how transparency affects agents in terms of this component.

Now, let's continue with the second part. Money transfer relates to both expected execution time and transaction price, hence, we can do a similar analysis, but we will have to fix some terms in order to understand the overall effect. First, we can consider the fundamental value fixed, because it does not depend on any market or agent attribute. Second, let's consider the execution time fix, we have the following relation:

$$0 \nearrow p \Leftrightarrow 0 \nearrow (p - v_t) \cdot e^{-\rho \Delta t}$$

This means that higher execution price for sell orders generate more money transfer for the agent. On the other hand, if we fix the transaction price and analyse the execution time, we have the following relation:

$$\Delta t \searrow 0 \Leftrightarrow e^{-\rho \Delta t} \nearrow 1 \Leftrightarrow 0 \nearrow (p - v_t) \cdot e^{-\rho \Delta t}$$

[19]Note that in Table 8, the column labelled 'Total' does not report a zero as result, because the difference $v_{t'} - p$ is discounted back at time Δt, where Δt is different for the trader who submit the market order and the trader who submit the corresponding limit order which is being matched with the market order, since new arrivals are asynchronous. Instead, if we were considering the instantaneous money transfer (i.e., not discounted back) it should be report a zero as result.

In this case, this means that faster executions lead to higher money transfer. This relations are quite simple, the interesting point is that both notions are combined and we can see the overall effect. In the case of liquidity suppliers this value correspond to their welfare, since the waiting cost is zero due to their private value. For liquidity demander this means the cost to get have a quick execution, hence, the important value is the number, not the sign. Liquidity suppliers in the single market case, increase their execution time and decrease their price of executed sell orders, hence, it is direct to verify that decrease their welfare from 0.573 in the Opaque Market to 0.348 in the Lit Market. When they are in the multi market environment, we saw that on average all agents increase their execution time and reduce their execution price even more, being the restricted agents in the Opaque Venue the more affected due to an huge expected time. In this case, they have welfare values of 0.205, 0.027 and 0.174 for Lit restricted, Opaque restricted and multi market agents respectively. Notice that multi market does not have the best welfare. This is one of our main findings and it is explained in the following manner. Liquidity suppliers play the game of submitting limit orders. Multi market liquidity suppliers can monitor both venues and can perceive good opportunities, but when they submit their orders the probability to gain time and price priority is very low, since they are competing with restricted liquidity suppliers, hence *they will be mostly behind other restricted agents* and they won't be able to capitalize their trading benefit. We emphasize the word *mostly*, because sometimes they will be able to take advantage of this, but on average it is more difficult specially with the increase of liquidity supply. Since they prefer mainly the Lit Venue, the welfare of multi market agents resembles more to the Lit restricted agents, but it can't be greater.

Table 9: Average payoff, waiting cost and money transfer per trader differentiated by private value (all three measures in ticks). Payoffs are determined as in Table 1, while waiting costs and money transfers are described equation (4). The first row reports single market scenarios. In the second we report the multi markets scenario. The average payoffs, waiting cost and money transfer per trader are determined as mean of payoffs over 20 million market new entries in equilibrium. Standard errors for measures are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

			Waiting cost per trader				Money transfer per trader			
			Private Value $\|\alpha\|$			Total	Private Value $\|\alpha\|$			Total
			0	4	8		0	4	8	
Single Market	Lit Market	-	0.000	-0.364	-0.141	-0.188	0.348	-0.133	-0.542	-0.111
	Opaque Market	-	0.000	-0.348	-0.175	-0.192	0.573	-0.161	-0.608	-0.075
Both Markets	Lit Venue	Restricted	0.000	-0.320	-0.159	-0.176	0.205	-0.164	-0.500	-0.154
	Opaque Venue	Restricted	0.000	-0.391	-0.222	-0.223	0.027	-0.088	-0.396	-0.146
	Lit & Opaque	Unrestricted	0.000	-0.167	-0.055	-0.083	0.174	-0.195	-0.417	-0.151

The story for liquidity demander is very different, because *they can and do exploit time and price priority*. These agents play the game of submitting market order which executes instantaneously and skip the waiting time in the queue of liquidity suppliers. We already saw that depth disclosure and lit fragmentation makes more transparent market faster, and that generates a reduction in waiting costs. But, in money transfer we also have execution prices involved. In Table 5, we saw that the in the case of single markets, the Lit Market is faster and also provide better transaction prices for liquidity demander, hence it is direct to conclude that the Lit Market have better money transfer. Table 8 show that for these agents, the Lit Market provide a value of -0.542 versus -0.608 in the Opaque Market. On the other hand, in the multi market context, Table 5 show that Lit Venue is faster at the expense of lower execution prices, and that the opposite occur in the Opaque Venue, that is, it is a slower exchange but provide better execution prices. The combined effect can be seen in Table 8, where the money transfer value and gives the Opaque venue as the victor in this battle with a value of -0.396 versus -0.500 in the Lit Venue. But then a similar battle arises. The Lit Venue offers lower waiting cost but worse money transfer and the Opaque Venue have higher waiting cost but better money transfer, which is better then? To answer to this question lies in the welfare of agents which joins all this components together.

Observation 10

(i) *In single markets scenarios, depth disclosure generates a gain transfer from traders with no intrinsic motives to trade to those who have it.*

(ii) *In multi markets, this transfer in gains is intensified. Further, multi market liquidity demanders have the best performance.*

This observation is result of all previous analysis discussed from market quality and trading behaviour, and it is based in welfare values presented in Table 9. Welfare was presented in equation (3) and express in a single number the average gains of each agent on each regime studied. In the case of single markets, we can see a gain transfer from agents with no intrinsic motives to trade, $\alpha = 0$, to agents who have a positive intrinsic motive to trade, $|\alpha| > 0$, in particular, agents with a absolute private valuation of 8, $|\alpha| = 8$. We have seen that depth disclosure improves liquidity competition, that liquidity suppliers increase their waiting cost because of that competition and that agents with an extreme valuation of the asset take advantage of this situation by decreasing both waiting cost and improving money transfer. All of this is translated in the welfare loss of liquidity suppliers from 0.573 to 0.348 from the Opaque Market to the Lit Market regime. On the other hand agents with $|\alpha| = 8$ increase their welfare from 7.217 to 7.317, respectively. Also, we have seen that agents with $|\alpha| = 4$ behave as both supplier and demander, hence their welfare is compensated by the gains of their demander component versus the loss of their supplier component and remains almost equal in both regimes. For the Opaque Market, these agents have a welfare 3.491 while in the Lit Market it increase but only to 3.503, that it a 0.012 difference. Something similar occur with the market welfare, even though there is a decrease in welfare from the Opaque Market with a value of 3.733 to 3.700 in the Lit one, the difference is not as much as the transfer in gains between supplier and demander. [20] Hence, the question of which market is better depends on the type agent who ask it. A liquidity supplier will do better in an Opaque Market, while a liquidity demander will gain more in the Lit Market.

In the case of multi market the transfer in gain amplifies, but the agents gain more in the opposite exchange than in single market case. That is, liquidity suppliers now do better in the Lit Venue and liquidity demander do better in the Opaque Venue. This is explained by

[20] Also, remember that the equilibrium is obtained by agents decisions who are sequential. They do not have a general sight of what is going on in markets. Hence, global welfare measure could be better in another equilibrium.

the liquidity competition that arise given trading restrictions and the multi market agents factor and that we have widely discussed in previous observations. We have already seen that liquidity suppliers in the Opaque Venue increase significantly their expected execution time and that generates a the enormous loss in their welfare up to 0.027 versus 0.205 in the Lit Venue. In the case of liquidity demander, we have pending a reply. We saw that the Lit Venue has lower waiting costs but worse money transfer, and the other way around for the Opaque Venue, but which one is better? Here we observe that the Opaque Venue have better welfare of 7.382 versus 7.341 in the Lit Venue. This means that restricted demander in the Opaque Venue take advantage of the order submissions provided by liquidity suppliers. Hence, we have encountered an opposite result with respect to the single market case. Here, liquidity suppliers do better in the Lit Venue while liquidity demander do better in the Opaque Venue. This change in preferences is given by the liquidity flow provided by multi market agents. This oblige suppliers to modify their behaviour and enter into a different and more aggressive competence that increase their picking of risk and, in consequence, they are the more affected, even the multi market ones.

Table 10: Welfare per trader differentiated by private value: This table shows the welfare for each trader type calculated as the average payoff in ticks. Since the model is symmetric, we combine positive and negative values of α with the same absolute value. Each row of the table is a different scenario. The first row reports single market scenarios, in both regimes. In the second we report multi markets scenario separated on each Lit and Dark venue and the overall resulting market. The average payoffs are determined as mean of payoffs over 20 million market new entries in equilibrium following equation (3). Standard errors of average payoffs are small enough since we use a large number of simulated events. The Markov equilibrium is obtained independently for each scenario.

			Private Value $\lvert\alpha\rvert$			Total
			0	4	8	
Single Market	Lit Market	-	0.348	3.503	7.317	3.700
	Opaque Market	-	0.573	3.491	7.217	3.733
Both Markets	Lit Venue	Restricted	0.205	3.517	7.341	3.670
	Opaque Venue	Restricted	0.027	3.521	7.382	3.632
	Lit & Opaque	Unrestricted	0.174	3.638	7.528	3.764
	Joint Venues	-	0.127	3.542	7.395	3.673

We have seen that multi market liquidity supplier cannot capitalize their benefit of monitoring and trading in both venues, because they are subject to time and price priorities. Hence they will mainly be above restricted agents. But what happens with multi market liquidity demander? In previous observation we saw that these agents have a good money transfer but not as good as opaque restricted liquidity demander. Where they are most noteworthy is in execution speed. In table Table 8 we noticed that these agents have the lowest waiting cost by far, and this makes them have the best welfare of all agents with a value of 7.528. Hence, this implies that this agents significantly leverage their advantage of monitoring and trading in both venues. This affects also agents with $|\alpha| = 4$ which behave both as suppliers and demander. Their demander component also take advantage of this queue jumping and price detection, increasing their welfare up to 3.638 versus 3.520 on average for their restricted counterparties.

Again, the question of which venue is better depends on which type of agents ask. A liquidity supplier may choose the Lit Venue, while a liquidity demander may choose the

Opaque venue. Furthermore, is it worth to be multi market? The answer is clear, if you are a liquidity supplier who are more likely to be patient and play with limit orders rather than market orders, then you probably won't be able to take advantage of being multi market. On the other hand, if you are an impatient agent who are willing to consolidate a positive private valuation of the traded asset, you may surely enjoy to be multi market agent, because you will have to wait less to find better prices.

Observation 11

(i) *In multi markets, lit fragmentation reduce global welfare.*

So far, we have mentioned and discussed about welfare of different agents types, separated by private value and trading restrictions. We consider previous results as our main findings because differences are clear and consistent with their different strategies. But, what about the global welfare of the multi market scenario? To answer this, first notice in Table 10 that in the single market case, the Opaque Market have a better welfare rather than Lit Market. This could be counter intuitive since we have seen that depth disclosure enhances competition, but we have to remember that agents are uninformed about the fundamental value and even though they estimate the actual fundamental value, their optimal decisions could be different if they *were* informed. This does not mean that agents choose badly, on the contrary, they do the best they can given their restrictions. [21] In this sense, we can see that trading restrictions affects global welfare even more with lit fragmentation, because the optimal action for a restricted agent could be in another venue (from an omnipotent point of view). Still, restricted agents decide optimally given their restrictions. Table 10 shows that the multi market scenario has a lower global welfare of 3.673, but this value is an average of restricted and multi market agents. The former have the worse values and since these agents are the more, they move the mean value down, while the latter having no restrictions in trading they can place their optimal orders and have the best average welfare of 3.764, but they are few.

All of these welfare results can be interesting from a policy maker point of view, for instance, to define a cost for being multi market. The flexibility of the algorithm will allow analyse and test this possibility among others in future works.

[21] We tested the single market scenarios with fully informed agents and obtained a global welfare of 3.746 for the Lit Market and 3.744 for the Opaque. This mean that depth disclosure does not affect significantly the global welfare of market, instead it affect the inner distribution of welfare among different agents types, hence the interesting analysis is in the welfare of agents.

4. Conclusion

We provide a lithesome framework using a dynamic model in continuous time to explore different features in limit order markets. In this work, we focus our analysis in depth disclosure and the connection between a Lit Market and an Opaque Market when a small group are able to trade in both exchanges. Our algorithmic implementation generates a complete evolution of the limit order books tested. The multi market scenario represents the major contribution in our study. Consequently, we can analyse the effect of lit fragmentation in multiple markets on several aspects as market quality, trading behaviour and welfare, which can be useful for academic and regulatory purposes.

Our simulations allow us to show that in a single consolidated market, depth disclosure provide a more trustworthy place to trade that increases liquidity supply by agents without a private valuation of the asset. This liquidity competition reduces spread, increment depth at and away best quotes and reduces microstructure noise. Agents with intrinsic motives to trade take advantage of this liquidity provision allowing them to reduce their waiting costs and increase their money transfer leading to better gains.

In the multi markets context, we find that lit fragmentation improves market quality in the Lit Venue, or the primary exchange as Gresse (2015) recently found. This is given by a liquidity flow to the Lit Venue, which is supported by Moinas (2010), that is provided by multi market liquidity suppliers. Trading restrictions motive a more aggressive competition that reduces spread and microstructure noise in both exchanges. The beneficiaries are agents with intrinsic motives to trade, who are more impatient and find a reduction in their immediacy costs. Between these agents, those who have the benefit to trade in both market have the best performance, improving significantly their execution quality, similarly with Chung and Chuwonganant (2009).

Future research could consider other interesting financial trends. For example:

- The effect of high-frequency traders.
- Compare exchanges with different tick sizes.
- Agents with different level of information.
- Trading more than one share.
- The effects of market opening and close in traders behaviour.

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Appendix

1. Analytical approach

We study an analytical approach to the problem in order to explore the difficulties and also to raise and development that it can be continued in the future. We start with the framework developed by Rosu (2009), in which presents a model of price formation in an order-driven transparent market, where fully strategic, symmetrically informed liquidity traders dynamically choose between limit and market orders. Even though this approach does not focus on the transparency issue, it provides a simple and natural way to analyse a model of two markets, necessary for the fragmentation analysis and it is very helpful to quickly discover the difficulties of this kind of theoretical approaches.

In his work, Rosu (2009) starts with a basic scenario where there are only limit sell orders and market buy orders. These assumptions are not realistic but allow the problem to be very tractable and also with closed-form solution. He characterizes the mean and standard deviation of the bid-ask spread and proves that there exists a minimum spread. Also, characterizes the price distance between two limit sell orders and the maximum possible number of sellers in the book.

We begin by presenting the description of the model. Then, we describe the equilibrium of the basic scenario. Finally, we study an extension for two markets.

1.1 Model description

The market. Consider a market for an asset that pays no dividends. Assume there is a constant range $A > B$, where A represents an infinite supply when price is A, and B represents and infinite demand for the asset when price is B. Prices can take any value in this range, i.e., the tick size is zero.

Trading. The time horizon is infinite, and agents trade in continuous in time. The only orders allowed are limit and market orders, and the limit orders are subject to the usual time and price priorities. If several market orders are submitted at the same time, only one of them, randomly chosen, is executed, while other orders are cancelled. An important assumption is that limit orders can be cancelled for no cost at any time. Trading is based on a publicly observable limit order book (i.e. a lit market), and has no delay, which means that orders are processed immediately.

Agents. Traders arrive randomly due to liquidity needs that are not included in the model. The arrival process is described below. Once traders arrive, they choose strategically between market and limit orders. The traders are either buyers or sellers, and can trade at most one unit, after which they exit the model forever. Traders are risk neutral, and lose utility proportionally to their expected waiting time. If τ is the random execution time and P_τ is the price obtained at τ, the expected utility of a seller with patience coefficient $\tilde{r}$ is $f_t = \mathbb{E}\{P_\tau - \tilde{r}(\tau - t)\}$, and as opposed the expected utility of a buyer is $-g_t = \mathbb{E}\{P_\tau - \tilde{r}(\tau - t)\}$.

The discount coefficient $\tilde{r}$ is a constant that can take only two values r or r'. If $\tilde{r} = r$, the trader is called *patient*, and if $\tilde{r} = r'$, the trader is called *impatient*. For simplicity, it is assumed that $r' \gg r$, so impatient traders always submit market orders.

Arrivals. All traders arrive in the market according to independent Poisson processes with constant, exogenous intensity rates $\lambda_{PB}, \lambda_{PS}, \lambda_{IB}, \lambda_{IS}$ respectively to patient buyers, patient sellers, impatient buyers and impatient sellers. The basic scenario mentioned at the beginning considers $\lambda_{PB} = \lambda_{IS} = 0$.

Strategies. The game is setted in continuous time, hence agents can respond immediately. For instance, one can use strategies that specify: "Keep the limit order at a_1 as long as the other agent stays at a_2 or below. If at some time t the other agent places an order above a_2, then *immediately after* t undercut at a_2."

Setting the game in continuous time requires extra care, for more details see Rosu (2006). The type of equilibrium used are subgame perfect equilibrium, and Markov perfect equilibrium. Finally, in this approach, all information, including agents' strategies and beliefs, is common knowledge.

1.2. Equilibrium

In this section we describe the equilibrium for the basic scenario which assumes $\lambda_{PB} = \lambda_{IS} = 0$. Consider a limit order book with upper bound A, lower bound B, the sellers' patient coefficient r, the patient seller arrival rate λ_1 and the impatient buyer arrival rate λ_2.

The main intuition of the equilibrium is that all patient sellers have the same expected utility. To see this, suppose the book is empty, and a patient seller, namely T_1, arrives first to the market. This trader maximizes his expected payoff by submitting a limit sell at the maximum level which is $a_1 = A$. Suppose that a second patient seller arrives, say T_2. Now, both traders are competing for the market buy order that will sometime arrive. Hence, trader T_2 will put his limit order slightly lower than T_1, say $a_2 = a_1 - \delta$, because he gains price priority for the upcoming market buy order and has a marginal advantage, but trader T_1 can modify his order at a_1 too and undercut a_2 in the same manner, and so on. The output of this game is that trader T_2 put his order sufficiently far from T_1, but how far? Such that the expected utility of both a_2 and a_1 are the same, hence neither T_1 nor T_2 have the incentive to undercut the other. Therefore, traders compensate price and expected execution time of the orders to get the same expected payoff. In other words, if the price is low (high) then the waiting time is lower (higher) too.

Denote the number of sellers by m, and their expected utility f_m. The number m evolves according to a Markov process (with m being a state variable), and so the sellers' utility f_m satisfies a system of equations, called *recursive system*.

In order to define the system an important fact to take into account is that the number of states must be finite: otherwise, the expected execution time of the top-limit seller would be infinite, hence his utility would be negative infinity, but then he would rather submit a market order at the lower bound B, where there is always sufficient demand. Hence, denote M the largest number of limit orders the book can accommodate.

The system considers four aspect that are quite natural in the model and derive in four equations. First, when the book is empty, define that $f_0 = A$, which is the starting case. Second, when the book is full, all traders have expected gain $f_M = B$, otherwise an incoming patient seller would want to join in to get more than B and there would be $M+1$ sellers, but M is the maximum. Third, consider that from a book with m sellers ($m = 1, ..., M-1$) the market can go either to state $m+1$ due to a patient seller arrival, or to state $m-1$ due to the arrival of an impatient buyer. Considering that these arrivals follow Poisson processes and with the properties of such processes, one obtains the formula

$$f_m = \frac{\lambda_1}{\lambda_1+\lambda_2} f_{m+1} + \frac{\lambda_2}{\lambda_1+\lambda_2} f_{m-1} - r \cdot \frac{1}{\lambda_1+\lambda_2}$$

Finally, when the state have M sellers, the system can go only to $M-1$ either if an impatient buyer arrives or if the seller at the Ask price, i.e. at the bottom, places a market sell at B and exits, which happens with a Poisson process with intensity ν. Then, similarly to the previous, one obtains the formula

$$f_M = f_{M-1} - r \cdot \frac{1}{\lambda_2+\nu}$$

Definición 0.1 *The* recursive system *is a collection* (f_m, M, ν) *of the sellers' expected utility* f_m*, the maximum number of sellers in the book* M*, and the mixed strategy Poisson rate for the bottom seller* $\nu \geq 0$*, which satisfy*

$$f_0 = A$$

$$f_m = \tfrac{\lambda_1}{\lambda_1+\lambda_2} f_{m+1} + \tfrac{\lambda_2}{\lambda_1+\lambda_2} f_{m-1} - r \cdot \tfrac{1}{\lambda_1+\lambda_2}, \quad m = 1, ..., M-1$$

$$f_M = f_{M-1} - r \cdot \tfrac{1}{\lambda_2+\nu},$$

$$f_M = B.$$

Teorema 0.2 *There exists a Markov perfect equilibrium of the recursive system, with maximum number* M *of sellers, such that in the state with* $m \leq M$ *patient sellers, the sellers' expected utility* f_m *is given by*

$$f_m = A + C\left(\left(\frac{\lambda_2}{\lambda_1}\right)^m - 1\right) + \frac{r}{\lambda_1-\lambda_2} m, \quad \textit{if } \lambda_1 \neq \lambda_2$$

$$f_m = A - bm + \frac{r}{\lambda_1+\lambda_2} m^2, \quad if\ \lambda_1 = \lambda_2,$$

where $C > 0$, $b > 0$ are given by

$$C = \frac{r}{\lambda_1 - \lambda_2} \frac{\frac{\lambda_1+\nu}{\lambda_2+\nu}}{(\frac{\lambda_2}{\lambda_1})^{M-1} - (\frac{\lambda_2}{\lambda_1})^{M}},$$

$$b = \frac{2r}{\lambda_1+\lambda_2}\left(M - \frac{\nu-\lambda_1}{2(\nu+\lambda_1)}\right),$$

and $M > 0$ is the unique positive integer that for some $\nu \geq 0$ satisfies

$$\frac{A-B}{\frac{r}{\lambda_1-\lambda_2}} = \frac{\lambda_1+\nu}{\lambda_2+\nu} \frac{(\frac{\lambda_1}{\lambda_2})^M - 1}{(\frac{\lambda_2}{\lambda_1}) - 1} - M, \quad \boldsymbol{if}\ \lambda_1 \neq \lambda_2,$$

$$M = \frac{\nu-\lambda_1}{2(\nu+\lambda_1)} + \sqrt{\left(\frac{\nu-\lambda_1}{2(\nu+\lambda_1)}\right)^2 + \frac{A-B}{\frac{r}{\lambda_1+\lambda_2}}}, \quad \boldsymbol{if}\ \lambda_1 = \lambda_2$$

The ask price in the state with m sellers is given by

$$a_m = f_{m-1}, \quad if\ m < M,$$

$$a_M = B + \frac{r}{\lambda_2}.$$

The strategy of each agent in the state with m sellers is the following: If $m = 1$, then place a limit order at $a_1 = A$. If $m = 2, ..., M-1$, place a limit order at any level above a_m, as long as someone has stayed at a_m or below; otherwise place an order at a_m. If $m = M$, the strategy is the same as for $m = 2, ..., M-1$, except for the bottom seller at a_M, who exits at the first arrival in Poisson process with rate $\nu \geq 0$ by placing a market order at B (Agregar footnote). If $m > M$, then immediately place a market order at B.

The equilibrium described above is Markov, with state variables: the number of existing sellers and the ask price. The equilibrium is unique in the calss of rigid equilibria (insertar footnote), in the sense that any other rigid equilibrium leads to the same evolution of the state variables.

The proof of the theorem is also presented in Rosu (2009) along with the variety of implications, but so far we do not focus on that. Instead, our purpose is to study an extension of this setup, which is presented in the next section.

1.3. Extension for two markets

Consider two markets, $\mathcal{M}_1$ and $\mathcal{M}_2$, each one with its own constants A_1, B_1, A_2 and B_2, that relate to the price rank $B_1 < A_1$ and $B_2 < A_2$. As well as the one market case, there is a sufficient demand at B_1 and B_2, and a sufficient supply at A_1 and A_2. It is straightforward to take $A = \text{mín}\{A_1, A_2\}$ and $B = \text{máx}\{B_1, B_2\}$, since prices higher than A or lower than B are suboptimal. Each market is similar to the one market case, that is, the tick size is zero in both venues.

Trading rules are the same, but now traders must choose the market in which they will place their orders. Also they can modify their orders, which may include changing to the other market, without any cost. Agents characteristics and arrivals are the same as well, in particular, we still consider that $\lambda_{PB} = \lambda_{IS} = 0$.

Let (m_1, m_2) be the number of sellers in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, and let $n = m_1 + m_2$ be the total number of sellers in the state. Following the same argument as in the single market case, there is a maximum number N of sellers spread over both markets. Also, as discussed before, all patient sellers must have the same expected utility, denote f_{m_1,m_2} the expected utility of m_1 in $\mathcal{M}_1$ and m_2 in $\mathcal{M}_2$.

Suppose the book is empty and a patient seller T_1 arrives. As in the single market case, T_1 submit an order at the maximum level $a_1^i = A$, but he has to decide in which market $\mathcal{M}_{i\in\{1,2\}}$ place it. In this situation, both a_1^1 and a_1^2 have the same expected payoff since both markets are equal, so T_1 has no further preferences. Consider w.l.o.g. that T_1 submit a_1^1 and then a second patient seller arrive. Now, there are two quite different options. First, if T_2 places his order in $\mathcal{M}_1$, then a similar game than single market will arise, and the output is known: $a_2^1 < a_1^1$, such that both orders have the same expected payoff $f_{2,0}$. It holds that $f_{2,0} = f_2$, with f_2 being the value obtained in the single market scenario for a state with two patient sellers. Second, if T_2 places his order in $\mathcal{M}_2$ he also puts his order in $a_2^2 = A = a_1^1$, at the same price that T_1, because maximizes the price available in $\mathcal{M}_2$. In this situation, both traders compete for the upcoming market buy at the exact same level. In the buyer's point of view, both orders are equally valued, so, both T_1 and T_2 have one half probability of being chosen and their orders have the same expected payoff, denoted by $f_{1,1}$. However, $f_{1,1} = \frac{1}{2}f_1$, with f_1 equal the expected payoff of one patient seller in the single market scenario, because of the probability issue. Therefore, the main problem is exposed: what configuration does T_2 prefer? $f_{2,0}$ or $f_{1,1}$? Using Theorem 0.2 we can deepen in this values. First, consider the case when $\lambda_1 \neq \lambda_2$, then:

$$\frac{1}{2}f_1 = \frac{1}{2}A + \frac{1}{2}C\left(\left(\frac{\lambda_1}{\lambda_2}\right) - 1\right) + \frac{1}{2}\cdot\frac{r}{\lambda_1 - \lambda_2}$$

$$f_2 = A + C\left(\left(\frac{\lambda_1}{\lambda_2}\right)^2 - 1\right) + 2\cdot\frac{r}{\lambda_1 - \lambda_2}$$

$$\Rightarrow f_2 - \frac{1}{2}f_1 = \frac{1}{2}A + C\left(\frac{\lambda_2}{\lambda_1} - 1\right)\left(\frac{\lambda_2}{\lambda_1} + \frac{1}{2}\right) + \frac{3}{2}\frac{r}{\lambda_2 - \lambda_1} \tag{5}$$

On the other hand, when $\lambda_1 = \lambda_2$, then:

$$\frac{1}{2}f_1 = \frac{1}{2}A - \frac{1}{2}b + \frac{1}{4}\frac{r}{\lambda}$$

$$f_2 = A - 2b + 2\frac{r}{\lambda}$$

$$\Rightarrow f_2 - \frac{1}{2}f_1 = \frac{1}{2}A - \frac{3}{2}b + \frac{7}{4}\frac{r}{\lambda} \tag{6}$$

These expressions provide equations in terms of the parameters of the model, that could derive in preferences for $f_{1,1}$ or $f_{2,0}$. Thus, if $f_2 - \frac{1}{2}f_1 > 0$, configuration $f_{2,0}$ is preferred over $f_{1,1}$, and the trading would turn over one venue because any further similar configuration would be solved in a similar manner. On the other hand, if $f_2 - \frac{1}{2}f_1 < 0$, configuration $f_{1,1}$ is preferred over $f_{2,0}$, and trading would occur equated in both markets. Finally, if $f_2 - \frac{1}{2}f_1 = 0$, there are no preferences for any of the configurations, so how would trading occur?.

Denote $p_{m_1,m_2 \to m_1',m_2'}$ the probability to jump from state (m_1, m_2) to state (m_1', m_2'), given by the random arrivals. Notice that in a state with n sellers, there are $n+1$ feasible combinations of book configurations: $(0, n), (1, n-1), ..., (n-1, 1), (n, 0)$. Recall that from a state with n patient sellers, the model can go either to $n+1$ (in $n+2$ ways) if a patient seller arrives, or to state $n-1$ (in n ways) if an impatient buyer arrives. Similarly to the single market scenario, one can obtain the formula

$$f_{m_1,m_2} = \sum_{i+j=n+1} p_{m_1,m_2 \to i,j} \cdot f_{i,j} + \sum_{i+j=n-1} p_{m_1,m_2 \to i,j} \cdot f_{i,j} - r \cdot \frac{1}{\lambda_1 + \lambda_2}$$

where $n = m_1 + m_2$.

The same can be done when the book is full and, in a similar fashion to the single book case, a recursive system can be properly defined, but in terms of the probabilities $p_{\cdot,\cdot}$ and the conditions derived from expressions (1) and (2). It is clear that the resulting expression is not trivial and requires some work that exceeds the purpose of this thesis, but exposes the framework and the difficulties. Yet, it is an interesting approach and is let proposed for future work.

2. Details of the numerical algorithm

In this section, we describe several assumption to make our simulations computationally tractable. First, the fundamental value evolves across time, thus the set of prices feasible for trade is huge (although finite since our simulations are finite). However, traders only care about the difference between the price and the fundamental value v_t, so we can write prices relatively to v_t. Furthermore, historical prices can be also expressed in terms of v_t which significantly reduces the state space in our numerical simulations. Ideally, we would like to condition agents strategies across all information available in the order books, but this is computationally intractable. Instead, we restrict the state space for a trader as follows. Let s_t denote the state observed by an agent at the time t, then:

$$s_t = \{L_{1,t}, L_{2,t}, \delta_t, v_t, \alpha, \rho, \tilde{a}_t\} \quad (7)$$

where $L_{m,t}, m \in \{1, 2\}$ is a set of variables of the book m at time t that depends on the transparency of each book. For our simulation we consider the same transparency for both books. Accordingly the variables in $L_{m,t}$ includes:

- Bid and Ask prices $(B_{m,t}, A_{m,t})$
- Depths at these prices $(l^B_{m,t}, l^A_{m,t})$

- Depth on buy and sell side of the market $\sum\{l^i_{m,t} > 0\}$ and $\sum\{l^i_{m,t} < 0\}$.

Besides, δ_t indicates the most recent transaction which includes the book, the price and if the last transaction was a buy or sell, v_t is the fundamental value, α and ρ are respectively agents' private value and delay rate, and $\tilde{a} = (\tilde{b}, \tilde{p}, \tilde{x}, \tilde{q})$ represent the status of his previous order. In order to summarize, we consider 3 exogenous events:

- Changes in the fundamental value: Every time the fundamental value v changes, it increases or decreases its value with probability $\frac{1}{2}$ and appropriately shift all orders in the books relative to the new value of v
- New trader arrivals: Every time a new trader arrives to the market, he observes the current state s and takes the optimal action a^*. If a^* is a market order, he executes and leaves the market forever, if he takes any other order we draw a random time for his re-entry.
- Re-entry of 'old' traders who have not yet executed: Every time a trader returns to the market, he observes the current state s, which includes the status of his previous order. Then, he takes an optimal action (which could include retaining his previous order). If the order he selects is a market order, he executes and leaves the market forever, if he takes any other order we draw a new return time for his re-entry.

3. Learning Processes

In the algorithm, at the time t, each action[22] a in the state s has an associated expected payoff $U_t(a|s)$, which is a real number and represent current belief about the payoff from this action at state s. Consequently, a trader selects the action a^* in the state s at time t iff $a^* \in \underset{a \in \Gamma(s)}{\operatorname{argmax}} U_t(a|s)$. Then, the optimal value of state s from Bellman equation for the the traders' dynamic maximization problem is determined as $V(s) = U_t(a^*|s)$.

Since agents are uninformed about the fundamental value v_t, they have to estimate its value from a lagged one $v_{t-\Delta t}$. They use the estimated fundamental value to determine their action set. Denote this estimation as $\hat{v}_t = \mathcal{E}(v_t|m_t)$, with m_t being the market conditions observed in t. Let be $\delta_t(m_t) = \hat{v}_t - v_{t-\Delta t}$, denote the extent by which agents at time t revises his belief about v_t, given lagged value $v_{t-\Delta t}$.

We define the learning process for the estimation of the fundamental value as follows: Start with an initial belief δ_0 for each market condition m. Let $c(m)$ be an integer denoting the number of times market conditions m are encountered in the simulation. Each time an agent observes market conditions m, we increment c by one and set:

$$\delta_c(m) = \frac{c-1}{c}\delta_{c-1}(m) + \frac{1}{c}(v_t - v_{t-\Delta t}) \tag{8}$$

Hence, the estimate is $\hat{v} = v_{t-\Delta t} + \delta_{r-1}(m)$, since the market condition m has been observed $c-1$ times prior to his entry. With this estimate $\hat{v}_t$, the action set for each agent is properly defined.

[22] Recall that given a state s, each action a has a finite action set feasible, denoted by $\Gamma(s)$

Now, let's consider the learning process for actions. Each pair (a, s) has a initial belief $U_0(a|s)$. Note that any $U_0(\cdot)$ can lead to an equilibrium, so we choice it for reaching the equilibrium quickly, similar to Goettler et al. (2009). These initial beliefs are updated with the progress of the algorithm. Suppose the optimal action a^* is not a market order, that is, it is either a limit order or no order. Further that, suppose at some future time t' the trader re-enters the market and observes a state s' with an optimal value $V(s')$. Then, the belief $U_t(a^*|s)$ is updated as:

$$U_{t'}(a^*|s) = \frac{n}{n+1}U_t(a^*|s) + \frac{1}{n+1}e^{-\rho(t'-t)}V(s') \tag{9}$$

where $n = n(a^*, s)$ is a positive integer that is incremented by one each time action a^* is chosen in state s. Similar to Goettler et al. (2009), periodically during the simulation, we restart n to n_0 to obtain quicker convergence.

Similarly, suppose a trader submit a limit sell (denoted by a^*) when he faces the state s at time t, and this order executes against a market order submitted by another trader at time t'. In this case we update the belief $U_t(a^*|s)$ with the realized payoff associated with the transaction[23] as:

$$U_{t'}(a^*|s) = \frac{n}{n+1}U_t(a^*|s) + \frac{1}{n+1}e^{-\rho(t'-t)}x(\alpha + v_{t'} - p) \tag{10}$$

In the simulations, if all traders traders take the optimal action given their current beliefs, there is a possibility that the algorithm would be in a sub-optimal equilibrium, maybe because traders have not learned payoffs of other actions. To ensure that beliefs are updated for all actions in every state, with a small probability ε a trader trembles over all suboptimal limits orders with same probability, which allows the trader to visit states that he never would have been able to visit without trembles.

4. Convergence Criteria

The simulation has three main phases:

(i) As a first phase, we update the beliefs of agents for sufficient time to ensure equilibrium is reached (two billion of new arrivals for single market scenarios and five billion for multi market).

(ii) After first phase, we check that the algorithm converges properly.

(iii) If convergence criteria is satisfied means the algorithm reaches the equilibrium. We fix the belief of agents (i.e., there is not more learning process), disallow trembles (i.e., $\varepsilon = 0$) and simulate the model another 300 million of new arrivals to obtain our results in equilibrium.

Related to (ii), we use a similar convergence criteria than Goettler et al. (2009) Goettler et al. (2009). We take a snapshot of beliefs at the end of (i) and simulate several millions of

[23]We always know the instantaneous payoff associated to a market order, which is $x(\alpha + v - p)$

additional new arrivals. Then, we take another snapshot of beliefs. The intuition suggest if both snapshots are similar enough, then the algorithm has converged.

Formally, we evaluate every 600 million new trader arrivals, the weighted relative difference among the beliefs at the end of (i), $U_{t_1}(\cdot)$ and the belief at the end of the current 600 million new trader arrivals, $U_{t_2}(\cdot)$. We require the weighted relative difference not to exceed 1 %, which can be expressed as:

$$\sum_{(a,s)\in\mathcal{X}} \frac{U_{t_1}^{k_1}(a|s) - U_{t_2}^{k_2}(a|s)}{(k_2 - k_1)U_{t_1}^{k_1}(a|s)} < 0.01 \tag{11}$$

where $\mathcal{X}$ is the set of all actions a selected in the state s during the first phase, k_1 is the number of times the action a has been taken in state s at the end of (i) and $k_2 \geq k_1$ the number of times it has been chosen at the end of current 600 millions of new traders arrivals.