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Contingent Preannounced Pricing Policies with Strategic Consumers

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Companies in diverse industries must decide the pricing policy of their inventories over time. This decision becomes particularly complex when customers are forward looking and may defer a purchase in the hope of future discounts and promotions. With such uncertainty, many customers may end up not buying or buying at a significantly lower price, reducing the firm's profitability. Recent studies show that a way to mitigate this negative effect caused by strategic consumers is to use a posted or preannounced pricing policy. With that policy, firms commit to a price path that consumers use to evaluate their purchase timing decision. In this paper, we propose a class of preannounced pricing policies in which the price path corresponds to a price menu contingent on the available inventory. We present a two-period model, with a monopolist selling a fixed inventory of a good. Given a menu of prices specified by the firm and beliefs regarding the number of units to be sold, customers decide whether to buy upon arrival, buy at the clearance price, or not to buy. The firm determines the set of prices that maximizes revenues. The solution to this problem requires the concept of equilibrium between the seller and the buyers that we analyze using a novel approach based on ordinary differential equations. We show existence of equilibrium and uniqueness when only one unit is on sale. However, if multiple units are offered, we show that multiple equilibria may arise. We develop a gradient-based method to solve the firm's optimization problem and conduct a computational study of different pricing schemes. The results show that under certain conditions the proposed contingent preannounced policy outperforms previously proposed pricing schemes. The source of the improvement comes from the use of the proposed pricing policy as a barrier to discourage strategic waiting and as a discrimination tool for those customers waiting.

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1. Introduction

Companies in diverse industries such as retailing, transportation, and the performing arts face the problem of pricing perishable goods in the presence of strategic consumers. In the high-fashion industry, for instance, a correct pricing policy may be the determinant of the company's success. This decision is complex and needs an adequate understanding of consumer preferences, particularly when consumers are forward-looking and incorporate expected future discounts into their purchasing behavior. These consumers may turn an otherwise successful pricing strategy into an ineffective or unprofitable one. Indeed, if consumers postpone purchasing decisions, firms may act accordingly and readily reduce their prices because of high observed inventories (reducing as a consequence their profitability). Interestingly, some consumers end up not buying either because they are hoping that prices decrease even further or because of the shortage due to the substantial number of customers waiting. In the first case, firms may end up with unsold items, whereas in the second, firms could have obtained higher revenues by keeping higher prices. For instance, to explain the low capacity utilization in the performing arts, Tereyagoglu et al. (2012, p. 11) state that "if the seats remain unsold until the last few weeks, then management decides to deeply discount tickets, which establishes in a vicious cycle because customers then postpone their purchases to receive the deep discounts in these weeks."

As a consequence of this growing problem, the topic of pricing with strategic consumers has recently been the focus of academic research in areas as diverse as economics, marketing, and operations management. There have been several studies analyzing conditions under which different pricing policies optimize the firm's profitability (Aviv and Pazgal 2008; Bansal and Maglaras 2009; Cachon and Feldman 2010; Elmaghraby et al. 2008, 2009; Jerath et al. 2010; Osadchiy and Vulcano 2010). In particular, two general pricing policies have been analyzed: dynamic and fixed preannounced pricing. In the first, the price depends on the number of remaining items at the end of each period. That is, the price path is unknown at the beginning of the horizon. In the second policy, the seller commits to a specific price path. This commitment implies that consumers know at each point exactly what the price will be in the future if inventory is still available. In this paper, we present a general preannounced policy that nests these two pricing policies as special cases. In the proposed *contingent preannounced pricing* policy, the seller announces, at the beginning of the season, a full menu of prices for each possible inventory level at the end of the selling period. This policy extends the fixed preannounced pricing policy, where the price is the same for all remaining units. The proposed policy has two appealing characteristics. First, and following the line of preannounced prices, one source of uncertainty is removed because customers know the price path for the entire selling season. This creates a barrier against strategic waiting, in particular against speculation about future prices. And second, customers have additional information about the price contingent on the number of available units. This information supports an effective discrimination tool to deal with those customers waiting in the second period. As we show later, the flexibility of the proposed contingent policy over the fixed policy provides an important tool to further discriminate customers and discourage strategic behavior. Interestingly, we show that in many cases, the optimal pricing policy will produce increasing unit prices for the discounted period. That is, the unit price could be higher when there are more remaining units to sell. Although this may seem counterintuitive, it serves as an effective mechanism to discourage strategic behavior (§§4.6 and 6.2).

Most of the studies mentioned above have demonstrated that if consumers are myopic (not forward looking), a dynamic pricing policy in which the seller does not commit and sets the price optimally given the leftover inventory is expected to produce better results. In contrast, when customers strategically adjust their purchase decision, this lack of commitment negatively affects the seller, making preannounced pricing policies more competitive. In this paper we concentrate on preannounced pricing policies, analyze their characteristics, and investigate the implications of adopting them. Examples of companies that have used such policies include Filene's Basement (Bell and Starr 1993), Lands' End overstocks,[1] and Dress for Less.

To investigate the properties of preannounced pricing policies, we set up a game-theoretic model and show that for a given set of contingent preannounced prices, an equilibrium for the buyers always exists. This equilibrium takes the form of a threshold function so that customers buy upon arrival if and only if their valuation is above their threshold. Next, we show that uniqueness of equilibrium is guaranteed when there is a single item to sell, independently of the other parameters of the problem, and we provide sufficient conditions for uniqueness in the multi-item case. Finally, we look at the seller's optimization problem. Despite its difficulty, we are able to explicitly compute the gradient of the objective function and therefore design an effective gradient method. We run extensive computational studies showing that the performance of the contingent preannounced pricing policy is only slightly higher than that of the fixed preannounced pricing for the basic situations studied in most of the literature. This difference, however, may become significant, for instance when the inventory is relatively small and the seller is uncertain about the product's success.

As mentioned by Aviv and Pazgal (2008, page 347) the complexity of these models makes it difficult to demonstrate the very existence of an equilibrium. Consequently, most of the analyses rely on numerical experiments. In this paper, we take a step forward and use a methodology based on ordinary differential equations to prove existence of equilibria. Our first methodological contribution is thus to model the problem of finding an equilibrium as that of finding a suitable threshold function and then via a transformation deduce that such an equilibrium can be seen as a solution of an appropriate differential equation, whose solution is guaranteed to exist using a fixed point argument. We note that the use of ordinary differential equations in dynamic pricing settings was pioneered by Kincaid and Darling (1963) (see also Talluri and van Ryzin 2005). However, here we use ordinary differential equations (ODEs) to model strategic behavior rather than for the pricing problem itself. Our second methodological contribution is our explicit computation of the gradient of the seller's optimization problem, which is critical for the proposed contingent policy. This computation leads to a method that extends previous research in preannounced pricing that relied on derivative free (and thus slower) methods.

In sum, the main contribution of this paper is threefold: It (i) presents a new pricing policy that generalizes two existing methods, (ii) provides formal proofs of existence of equilibrium and shows when uniqueness can be guaranteed by using a new modeling framework based on ordinary differential equations, and (iii) uses a gradient-based algorithm to solve the optimization problem and analyzes the performance of the proposed policy and its implied results.

The paper is organized as follows. We review the related literature in §2 and present the mathematical model in §3. In §4 we study the equilibrium of the contingent pricing policy. In §5 we design an optimization method for the seller's problem and present numerical simulations in §6. Concluding remarks are presented in §7.

2. Related Literature

Intertemporal price discrimination has been studied since the seminal work of Coase (1972), who postulates that a monopolist selling a durable good would have to sell the product at its marginal cost unless it commits to a pricing policy. The idea behind what is known as the *Coase conjecture* is that rational consumers know that the monopolist will reduce the price and therefore would wait until the price reaches the marginal cost. However, when the supply is finite or the good is perishable, this result does not hold because consumers may not have the incentive to wait for the lower price.

Revenue management is a more recent research area of intertemporal price discrimination with perishable goods (see, e.g., Talluri and van Ryzin 2005 for an in-depth treatment). In this context it is usually assumed that companies sell a finite number of units of a good during a finite time horizon, after which the unsold units have no value. The goal is to determine the markdown policy that maximizes profitability for the selling period. Hotels, airlines, high-fashion retailers, and movie theaters are just a few examples of the kind of companies applying this selling strategy. One of the initial assumptions of revenue management is that customers are not able (or willing) to anticipate price changes and cannot negotiate with the company because they have no effect on the pricing policy. This assumption is violated when customers have knowledge of (or form beliefs about) the pricing policy and incorporate it in their purchase decision, influencing as a consequence the firm's pricing decision. The evidence that consumers may act strategically (Sun et al. 2003, Li et al. 2014) has opened a new line of research in recent years, probably starting with the work of Lazear (1986). In what follows we discuss some of the operations management literature relevant to our work. It is worth mentioning that in other fields, particularly in economics, there have been recent efforts to understand these revenue management issues when the seller has full flexibility and can design any type of mechanism rather than just posted prices mechanisms more common in the pricing literature. Most notably, the work of Board and Skrzypacz (2015) applies a Myersonian approach to design the optimal mechanism to sell multiple identical items to customers arriving over discrete time periods, whereas that of Gershkov et al. (2014) extends the results to a continuos time setting.

There are two works closely related to our paper. First, Aviv and Pazgal (2008) investigate the problem of a monopolist selling a finite inventory of a product during two periods to a population of rational consumers. These customers arrive sequentially and have a decreasing valuation toward the good. Each consumer may decide to buy at the arrival time or wait for the clearance price. The authors compare the dynamic-without-commitment and fixed-preannounced pricing strategies. They conclude that the preannounced pricing scheme yields higher profits when the initial inventory and valuation heterogeneity are high, the discounts are offered at the end of the season, and there is a moderate decline in valuations. Second, Osadchiy and Vulcano (2010) investigate the case of selling with binding reservation. In this case, upon arrival strategic consumers decide whether to buy the product at the full price or place a reservation to have a chance of receiving the product in the clearance period if there are still units of the product available. They focus on the fixed preannounced pricing strategy and analyze the random and first-in-first-out (FIFO) allocation strategies for the clearance period. They conclude that, in general, the FIFO allocation policy yields better results for the seller than the random allocation policy.

Other papers relax some assumptions of the basic model presented in these two papers. For instance, Elmaghraby et al. (2009) investigate the problem of rational customers buying multiple units, in the context of finitely many customer classes. Similar to the Osadchiy and Vulcano (2010) paper, customers may place a bid with the number of units to purchase in a fixed preannounced pricing scheme. They investigate the effect of information about the clearance price on consumers' biding strategies. Additionally, they study the effect of information about consumer valuations (complete versus incomplete) on the optimal markdown strategy. They show that under different conditions, it is optimal for the buyer to submit all-or-nothing bids. They also show that the incomplete and complete information cases induce similar markdown strategies. Yin et al. (2009) analyze the effect of inventory display formats when facing strategic waiting. In particular, they analyze the display all (DA) and display one (DO) unit formats. They conclude that DO can increase consumer uncertainty about the inventory levels and therefore by creating a sense of shortage risk can reduce (but not eliminate) the detrimental impact of customers' strategic behavior. The implications of this research are fundamental to our proposed contingent preannounced pricing policy. In this paper, the display of the inventory at the end of the regular selling season is the condition that allows the firm to preannounce the full menu of prices.

The relationship between supply and demand is key when facing strategic consumers. If supply is scarce, then the cost of waiting is high because the likelihood of getting the product in the clearance period is low. Dasu and Tong (2010) show that there is no dominant strategy when the ratio supply-demand is lower than one, whereas when this ratio is higher than one, the preannounced pricing policy is better. Levin et al. (2010) show that it is more profitable for the seller to have a lower supply level when facing strategic consumers. To prevent strategic consumers from exerting their power, some studies suggest that the firm can strategically adjust its offering in such a way that consumers face higher uncertainty in the clearance period. Liu and van Ryzin (2008) investigate whether it is optimal for the firm to strategically understock some products. They show that the firm can successfully induce early purchases of risk-averse customers; however, this optimal rationing can be supported only in a market with few firms. Cachon and Swinney (2009) find that contingent pricing is generally better when the firm can dynamically adjust its capacity. Additionally, they show that there is a substantial benefit of a quick capacity response when facing strategic consumers, which can partially explain the success of fast-fashion retailers such as the Spanish retailer Zara. Cachon and Feldman (2010) limit consumers to have only one opportunity to buy (by incorporating visiting costs) and find that contingent pricing works better when the cost of visiting the store or the demand uncertainty is high.

We note that most of the studies assume that customers have rational expectations in that they can perfectly anticipate capacity (see Liu and van Ryzin 2011 for an exception) and rely on the notion of an equilibrium between buyers and the seller to find the optimal selling and purchasing decisions. Many of these papers assume the existence and uniqueness of the equilibrium without a formal demonstration of the implied consumer valuation thresholds. There are, however, some exceptions. For instance, Osadchiy and Vulcano (2010) prove existence and give conditions for the uniqueness of equilibrium. Interestingly, as we also show in our context, uniqueness of equilibrium is found only under some specific conditions.[2] Indeed, Cachon and Swinney (2009) find conditions under which uniqueness of equilibrium cannot be guaranteed. This finding, therefore, warns us of the limitations for the seller to find and implement an optimal pricing strategy.

The methodology proposed in the current paper allows us to first analytically study equilibria and then, by conducting a series of numerical studies, show the extent of the implied results across different instances. To derive the corresponding equilibrium we rely on the use of ordinary differential equations. Applications of ODEs can be seen in areas as diverse as biology, engineering, physics, economics, marketing, and operations research. In the latter fields, much of this work is dedicated to characterizing differential games. However, unlike applications in other fields, where numerical methods have been already introduced, the applications in management science and economics rely mostly on analytic approaches (Jorgensen and Zaccour 2007). In management science, most of this work is devoted to diffusion models starting from the work of Bass (1969) (see, e.g., Bass 2004). Other applications include the modeling of advertising dynamics (Naik et al. 2005) and marketing channels (Chintagunta and Jain 1992). Recently Crapis et al. (2015) analyze the problem of social learning about the quality of a new product and approximate the learning dynamics by using a system of ODEs. They present a fluid model asymptotic approximation (similar to Maglaras and Meissner 2006 and Osadchiy and Vulcano 2010). Rahmandad and Sterman (2008) compare the use of differential equations to agent-based models to analyze diffusion patterns in networks.

3. The Model

Consider a monopolistic seller that needs to determine the optimal pricing policy for a perishable good with finite inventory Q serving strategic consumers. The initial inventory Q is known to both the seller and the consumers. Also assume that the seller and the buyers know the distribution of consumer's valuations and the distribution of the arrival times. The selling season consists of two periods, the first (regular) period $[0, T]$ and the second (clearance) period that happens right after T. The seller commits to prices p_1 and p_2 for each period; customers take these prices as given and make purchasing decisions accordingly. In the fixed preannounced policy, p_2 is a scalar, whereas in our proposed contingent preannounced policy p_2 is a vector. Specifically,

FPP (*Fixed preannounced pricing*): At time 0 the seller announces both fixed prices p_1 and p_2.

CPP (*Contingent preannounced pricing*): At time 0 the seller announces p_1 and a set of prices $p_2(1), \ldots, p_2(Q)$, where $p_2(Q_T)$ is the price the seller charges for the second period if $Q_T = 1, \ldots, Q$ units are available at time T.

Upon arrival a customer decides whether to buy at price p_1 or wait until T for a chance of getting the item at price p_2, where the remaining items (if any) are allocated randomly to the interested consumers.

When consumers are rational these strategies influence the consumers' initial purchase decisions, and therefore the implications are not straightforward. Note that for $Q = 1$, *FPP* and *CPP* coincide. Moreover, *FPP* corresponds to a special case of *CPP* when the seller sets $p_2(Q_T)$ as a constant independent of Q_T. Similar to *CPP* is what we previously called *dynamic-without-commitment* pricing in which at time 0 the seller only announces p_1 and waits until time T to announce $p_2(Q_T)$, computed optimally for the second period, depending on the number of units left at the end of the first period. This latter strategy is optimal if consumers are myopic; however, it fails when strategic consumers anticipate the price at the second stage as a function of the remaining items since this influences their purchase decision upon arrival. With strategic customers the seller may simply announce prices in advance without further influencing customers' behavior since consumers anticipate p_2 as a function of Q_T. In conclusion, it is important to note that the best *CPP* may set prices for the second period that do not lead to a subgame perfect equilibria; i.e., the second period prices may be suboptimal for the remainder of the selling season.

3.1. Game Setting Specification

Consider a seller having Q units of an item to be sold in two periods. The first period is $[0, T]$ and items are sold at price p_1. The second period occurs immediately after T when the remaining items are offered at price $p_2 \leqslant p_1$, which may depend on the leftover inventory. The seller behaves as a Stackelberg leader setting the prices for the two periods in advance, with knowledge of the consumers' parameters and anticipating their behavior. In the second stage, customers observe p_1 and $\{p_2(k)\}_{k=1}^{Q}$ and make their purchase decision.[3]

Customers arrive according to a Poisson process of rate $\lambda(t) \in [0, \bar{\lambda}]$ in the interval $[0, T]$, and a customer arriving at time t has a valuation for the item equal to $v(t) = v_t$, where v_t is a random variable distributed according to F_t, a bounded density distribution. We assume that for any fixed x, $F_t(x)$ varies continuously in t. Moreover, a customer arriving at time t with valuation v_t who waits to buy the item at time $\tau \geqslant t$ will value the item at that time as $v(\tau) = v_t e^{-\alpha(\tau - t)}$, where $\alpha \geqslant 0$ is a discount factor that represents customers' willingness to wait. Throughout we assume that customers are strategic in that, depending of their valuation, they decide to buy upon arrival or to wait until time T for a chance to get the item at a lower price. As in Aviv and Pazgal (2008) and Osadchiy and Vulcano (2010), here we assume that at time T the remaining items are randomly distributed among those customers that decided to wait.[4]

The final piece of the model concerns the information customers have when they arrive at the store. We assume that customers know if there is stock available upon arrival (i.e., whether there is at least one unit remaining) and the initial inventory Q. Furthermore, customers know the arrival process, the valuation distribution of all other customers, and the fact that other customers behave strategically. Therefore a customer buys the item immediately if and only if

$$v(t) - p_1 \geqslant (v(T) - p_2)\,\mathbb{P}[G \mid A_t], \quad \text{and} \quad v(t) \geqslant p_1,$$

where, given the consumers' beliefs, G is the event of getting the item at time T and A_t is the event that at time t there is stock available. Alternatively, we may assume that customers do not have any information about the stock when they arrive. In that case a customer buys immediately if and only if

$$(v(t) - p_1)\,\mathbb{P}[A_t] \geqslant (v(T) - p_2)\,\mathbb{P}[G], \quad \text{and} \quad v(t) \geqslant p_1.$$

Interestingly, both models lead to the same condition because G can only happen when A_t happens, i.e., $G \subset A_t$, so that $\mathbb{P}[G \wedge A_t] = \mathbb{P}[G]$. Therefore,

$$\mathbb{P}[G \mid A_t] = \frac{\mathbb{P}[G \wedge A_t]}{\mathbb{P}[A_t]} = \frac{\mathbb{P}[G]}{\mathbb{P}[A_t]}.$$

The model of Aviv and Pazgal (2008) establishes the purchasing condition as $v(t) - p_1 \geqslant (v(T) - p_2)\,\mathbb{P}[G]$, which does not consider the information customers have at their arrival time. The purchasing behavior we use here was also considered by Osadchiy and Vulcano (2010),[5] although their work focuses on studying a different type of allocations (random versus reservations) whereas in this paper we focus on contingent pricing policies.

4. Contingent Preannounced Pricing

To analyze contingent pricing policies, we first study the equilibrium for a fixed menu of prices. Thus we turn our attention to the second stage equilibrium under a continent pricing policy, where the seller decides one price p_1 for the regular season and another set of prices for the clearance period $\{p_2(k)\}_{k=1}^{Q} \leqslant p_1$, depending on the unsold items at time T. Recall our previous discussion that this pricing scheme requires that the seller commits to markdown prices that may fail to be optimal for the second period alone because setting prices that are optimal for the second period only will induce customers to behave strategically, affecting the revenue obtained in the first period. Accordingly, it is in the seller's best interest to commit to these prices since failing to do so will quickly undermine the seller's credibility, thus inducing strategic behavior that will result in lower revenues. Similar commitment issues are common in electronic commerce where airlines such as LAN publish the remaining inventory at a certain price.

We model the equilibrium as a fixed point involving an ordinary differential equation. Then we prove existence and show that for the single item case ($Q = 1$) the equilibrium is unique, whereas for the multiple items case there may exist multiple equilibria. It is important to mention that existence of equilibria for the more restricted *FPP* was first established by Osadchiy and Vulcano (2010) using a sophisticated functional analysis approach. Here we present a different proof in a more general setting, and additionally we do not require the customer's valuation distribution be continuously differentiable and of bounded support.

To fix ideas, consider the *FPP* policy. In this case the seller announces at the beginning of the selling horizon both prices p_1 and p_2, which are fixed and independent on the posterior inventory level. When a customer arrives to the store at time $t \in [0, T]$ with valuation $v(t)$, she has the intention to buy if and only if $(v(t) - p_1)\,\mathbb{P}(A_t) \geqslant (v(T) - p_2)\,\mathbb{P}(G)$ and $v(t) \geqslant p_1$. That is, a customer buys when her valuation is over a threshold $\psi(t)$; i.e.,

$$v(t) \geqslant \psi(t) \doteq \max\left\{\frac{p_1\,\mathbb{P}(A_t) - p_2\,\mathbb{P}(G)}{\mathbb{P}(A_t) - e^{-\alpha(T-t)}\,\mathbb{P}(G)}, p_1\right\}. \tag{1}$$

Note that this threshold function $\psi(t)$ depends on the decision of other customers through the terms $\mathbb{P}(A_t)$ and $\mathbb{P}(G)$, and therefore an equilibrium corresponds to a function consistent with the above inequality for all customers and at all $t \in [0, T]$. It is, however, not clear in principle whether such a function always exists.

More generally, consider now *CPP*. Here customers face the uncertainty of the discounted price, which depends on the unsold items at time T. Thus, a customer arriving at time t decides to buy immediately if and only if $(v(t) - p_1)\,\mathbb{P}(A_t) \geqslant \mathbb{E}[(v(T) - p_2)\,\mathbb{P}(G)]$ and $v(t) \geqslant p_1$. Then the threshold function is defined as the following:

$$\psi(t) \doteq \max\left\{\frac{p_1\,\mathbb{P}(A_t) - \mathbb{E}[p_2\,\mathbb{P}(G)]}{\mathbb{P}(A_t) - e^{-\alpha(T-t)}\,\mathbb{P}(G)}, p_1\right\} = \begin{cases} p_1 & 0 \leqslant t < t^* \\ \dfrac{p_1\,\mathbb{P}(A_t) - \mathbb{E}[p_2\,\mathbb{P}(G)]}{\mathbb{P}(A_t) - e^{-\alpha(T-t)}\,\mathbb{P}(G)} & t^* \leqslant t \leqslant T, \end{cases} \tag{2}$$

where

$$t^* \doteq \max\left\{T - \frac{1}{\alpha}\ln\left(\frac{p_1\,\mathbb{P}(G)}{\mathbb{E}[p_2\,\mathbb{P}(G)]}\right), 0\right\}.$$

Let $q(t) \doteq \mathbb{P}(v(t) \geqslant \psi(t)) = 1 - F_t(\psi(t))$, $t \in [0, T]$ be the probability that a customer has the intention to buy the item upon her arrival at time t. In addition, define the expected demand by time t as $x(t) \doteq \int_0^t \lambda(u)q(u)\,du$ and the expected demand in the entire regular selling period as $\mu_0 \doteq \int_0^T \lambda(u)q(u)\,du$ to characterize $\mathbb{P}(A_t)$ and $\mathbb{P}(G)$ as follows. Conditioning on the number of items sold by time t we get that

$$\mathbb{P}(A_t) = \sum_{k=0}^{Q-1} \frac{(x(t))^k e^{-x(t)}}{k!}.$$

In our random allocation model, given that there are k items remaining and i customers willing to buy them at the clearance price, the probability of getting the product is $\min\{1, k/(i+1)\}$. By conditioning on the number of items remaining and on the number of customers that arrive and wait for a chance of getting the item, we can express the terms $\mathbb{P}(G)$ and $\mathbb{E}[p_2\mathbb{P}(G)]$ as

$$\mathbb{P}(G)=\sum_{i=0}^{\infty}\sum_{k=1}^{Q}\min\left\{1,\frac{k}{i+1}\right\}\mathbb{P}(N^I=Q-k)\mathbb{P}(N^{II}(k)=i),$$

$$\mathbb{E}[p_2\mathbb{P}(G)]=\sum_{i=0}^{\infty}\sum_{k=1}^{Q}p_2(k)\min\left\{1,\frac{k}{i+1}\right\}\cdot\mathbb{P}(N^I=j-k)\mathbb{P}(N^{II}(k)=i).$$

Here N^I, the number of items bought by time T, is distributed Poisson with mean μ_0. Also $N^{II}(k)$, the number of customers waiting to buy at price $p_2(k)$, is distributed Poisson with mean

$$\int_0^T \lambda(t)\,dt-\mu_0-\int_0^{t^*(k)}\lambda(t)F_t(p_1)\,dt - \int_{t^*(k)}^{T}\lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\,dt-\mu_k,$$

where $\mu_k \doteq \int_0^{t^*(k)}\min\{\lambda(t)-\dot{x}(t), \lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\}-\lambda(t)F_t(p_1)\,dt$, and $t^*(k)=\max\{T-(1/\alpha)\ln(p_1/p_2(k)),0\}$ for $k\in\{1,\ldots,Q\}$. Here, the first term represents all arriving customers, the second subtracts those who buy immediately, the third and forth term subtract customers who buy neither upon arrival nor at the second period, and the fifth subtracts customers who wait to buy in the second period although they will not buy if the clearance price is $p_2(k)$. The important fact is that the previous quantities depend on $x(t)$ (or $\psi(t)$) only through the quantities $\mu_0,\ldots,\mu_Q$.

Next, letting $\vec{\mu}=(\mu_0,\ldots,\mu_Q)$ we can call $C_{\vec{\mu}}\equiv\mathbb{P}(G)$ and $D_{\vec{\mu}}(\vec{p}_2)\equiv\mathbb{E}[p_2\mathbb{P}(G)]$ and treating $\vec{\mu}$ as a vector of parameters we consider the following differential equation, where $\phi(x)=\sum_{j=0}^{Q-1}(x^j e^{-x}/j!)$:

$$\dot{x}(t)=\begin{cases}\lambda(t)(1-F_t(p_1)) & 0\leqslant t<t^*\\ \lambda(t)\left(1-F_t\left(\dfrac{p_1\phi(x(t))-D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t))-e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\right) & t^*\leqslant t\leqslant T\end{cases}\tag{3}$$
$$x(0)=0.$$

This equation has a unique solution in the domain of $(\mu_0,\ldots,\mu_Q)$, namely,

$$\mathcal{D}\doteq\left\{\vec{\mu}\in\mathbb{R}^{Q+1}\colon \mu_i\geqslant 0\ \forall\, i\in\{0,\ldots,Q\}, \mu_0+\mu_i \leqslant\int_0^T\lambda(t)(1-F_t(p_1))\,dt\ \forall\, i\in\{1,\ldots,Q\}\right\}.$$

Proposition 1. *Consider a fixed $\vec{\mu}=(\mu_0,\ldots,\mu_Q)$. Then (3) has a unique solution in $\mathcal{D}$.*

Proof. See Appendix A.

Remark 1. Note that the differential equation with initial value (3) is nonautonomous since the right-hand side depends on t and $x(t)$. Although this means that we cannot apply the rich existing machinery for autonomous systems, we may still establish existence and uniqueness. Furthermore, the equation has good stability properties. Indeed, since the right-hand side depends continuously on the parameters μ_i and it does not vanish in $x=0$, we can apply Kuchment (2013, Theorem 19) to conclude that the solution varies continuously in both the initial condition and the parameters μ_i. This result may also be derived as follows. Consider $\Phi(t,x,\vec{\mu})$ the right-hand side of Equation (3) and let $\Psi(x,\vec{\mu})=\int_0^t\Phi(s,x,\vec{\mu})\,ds$ so that (3) is equivalent to the functional fixed point equation $x=\Psi(x,\vec{\mu})$. By Proposition 1 we know that this equation has a unique solution and since Ψ is well behaved, the solution $x(t)$ is continuous with respect to small perturbation in $\vec{\mu}$.

Let us call $x_{\vec{\mu}}(t)$ as the unique solution to (3), for a given $\vec{\mu}$, and consider the function $h\colon\mathcal{D}\to\mathbb{R}^{Q+1}$ defined by

$$h_0(\vec{\mu})=x_{\vec{\mu}}(T),$$
$$h_k(\vec{\mu})=\int_0^{t^*(k)}\min\{\lambda(t)-\dot{x}_{\vec{\mu}}(t),\lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\}-\lambda(t)F_t(p_1)\,dt,\quad \text{for } k=1,\ldots,Q.$$

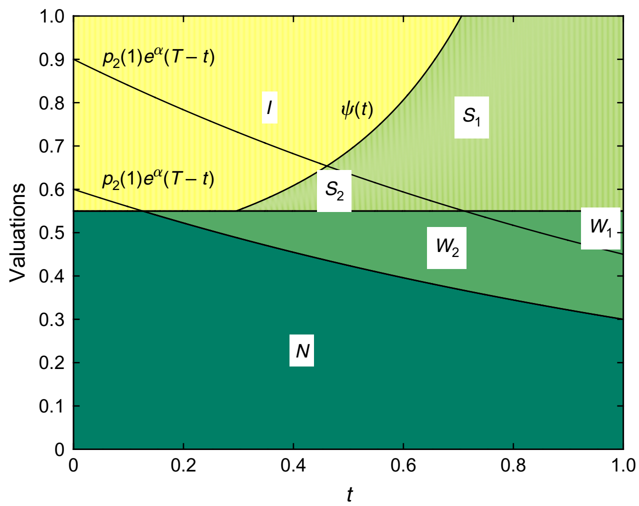
Therefore, we can finally write the following ODE with initial value and coupled with a system of equations to define equilibrium:

$$\dot{x}(t)=\begin{cases}\lambda(t)(1-F_t(p_1)) & 0\leqslant t<t^*\\ \lambda(t)\left(1-F_t\left(\dfrac{p_1\phi(x(t))-D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t))-e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\right) & t^*\leqslant t\leqslant T\end{cases}\tag{4}$$
$$x(0)=0,$$
$$h(\vec{\mu})=\vec{\mu}.$$

Definition 1. Given the parameters of the game, namely, the arrival process rate $\lambda(t)$, the discount rate α, the valuation distributions F_t, the initial inventory Q, and the prices p_1, $\{p_2(k)\}_{k=1}^{Q}$, we define an *equilibrium* as a function $x\colon[0,T]\to\mathbb{R}$ satisfying (4). Furthermore, we define the corresponding *threshold* function as

$$\psi(t)=F_t^{-1}\left(1-\frac{\dot{x}(t)}{\lambda(t)}\right)\quad\text{for all } t\in[0,T]$$

Figure 1. (Color online) Customer types for the instance $(F, T, Q, \lambda, \alpha, p_1, p_2(1), p_2(2)) = (U[0,1], 1, 2, 1.5, -\ln(0.5), 0.55, 0.45, 0.3)$.



Note. *I*: customers that buy *I*mmediately, *S*: wait *S*trategically, *W*: *W*ait nonstrategically, and *N*: do *N*ot buy.

4.1. Illustrative Example

We present an instance of the *CPP* strategy for uniform valuation distributions and two units of initial inventory. Figure 1 depicts the four customer types for the instance. Customers in zone (I) *buy immediately*, i.e., decide to buy upon arrival. The expected demand from these customers up to time t is exactly $x(t)$, whereas the expected demand up to time T is μ_0. Some customers *wait strategically* (S) and decide to wait even though their valuation is higher than the price in the first period. In Figure 1 these customers are represented by S_1 and S_2. S_1 denotes the customers that try to obtain the item at time T (as long as there are units available), whereas S_2 denotes those customers that intend to buy only when there are two items available at time T. The third type of customers *wait nonstrategically* (W) for the second period price because their valuation is lower than the price of the first period. In Figure 1 these customers are represented by W_1 and W_2. Similar to the case of strategic waiters, W_1 customers intent to buy at time T regardless of the number of remaining units, whereas W_2 are only interested on buying if there are two units available. Finally, customers (N) do *Not buy*.

4.2. Existence of Equilibria

We now present our main result establishing existence of equilibria for strategic consumers. Our approach is based on the modeling given by Equation (4). With this, and under a mild technical condition on the probability distribution of customers' valuation, the proof of existence of equilibrium becomes quite simple. This condition is that $\lim_{y\to\infty} F'(y)y^2 = 0$, which is very natural and easy to achieve; for instance, any finite mean distribution F whose density f is decreasing starting from some point τ satisfies such property (see Appendix B).

Theorem 1. *Assume that the distributions F_t are such that $\lim_{y\to\infty} F_t'(y)y^2 = 0$ for all $t \in [0, T]$; then there exists a solution to (4). Moreover, the solution $x(\cdot)$ is nondecreasing.*

Proof. See Appendix C. □

4.3. Uniqueness of Equilibria

In this section we first establish that when there is only one unit on sale, we have a unique equilibrium. Interestingly, this scenario is exactly the same as in the fixed preannounced pricing, and consequently we have uniqueness of equilibria there as well. We also give a sufficient condition for uniqueness of equilibria for the general setting, although this may not be easy to evaluate. This condition is similar in spirit to that of Osadchiy and Vulcano (2010) and it states that a certain mapping is a contraction. In general, this type of condition is actually very strong since the contraction requirement applies to the whole space rather than just locally. For instance, the sufficient condition of Osadchiy and Vulcano (2010) is only satisfied in cases where the initial inventory is high with respect to arrival rate, the normal period price is large with respect to the markdown, and the discount rate is close to zero. In other words, uniqueness is only guaranteed in very limited situations. Our condition for $Q > 1$ suffers the same weakness. This situation makes our general uniqueness result for $Q = 1$ fairly interesting. We close this section by showing that when $Q > 1$, multiplicity of equilibria can actually occur even in *FPP*.

Theorem 2. *If there is a single item on sale, there exists a unique solution to (4).*

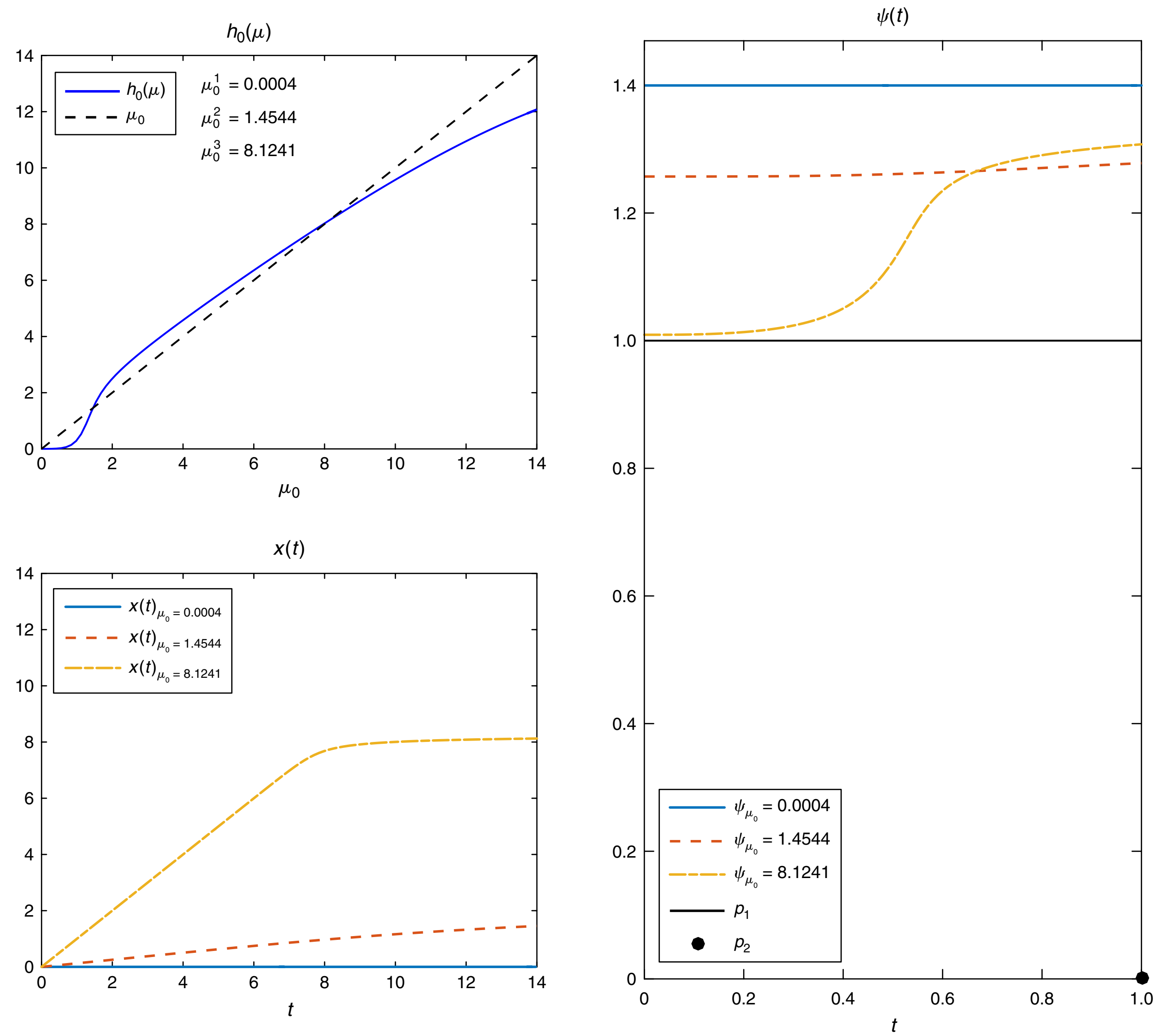
Proof. See Appendix D. □

In Appendix E, we also establish a standard condition guaranteeing uniqueness in the multi-item case. The condition reduces to noting that (4) has a unique solution if the mapping h is a contraction. Unfortunately, checking the condition is computationally expensive.

To complement the previous results, we exhibit (numerically computed) instances showing that when $Q > 1$, although equilibrium is guaranteed to exist, it may not be unique. Indeed, this is the case even with $Q = 2$ and with very general distributions for customers' valuations such as normal or uniform. Furthermore, we construct these examples for *FPP*, i.e., when $p_2 = p_2(k)$ for $k = 1, \ldots, Q$. In this case $\mu_1, \ldots = \mu_Q = 0$, and thus the only parameter in ODE (4) is μ_0. Therefore we consider the function $h_0(\mu_0) = x_{\mu_0}(T)$ defined as the unique solution to (4) only considering the initial value $x(0) = 0$, evaluated at time T (so that μ_0 is treated as a variable parameter). Let us point out that equilibria correspond to the fixed points of $h_0(\mu_0)$; see Appendix A for details.

We next describe one of these instances. Consider that consumers' valuation is distributed according to a normal

Figure 2. (Color online) Multiplicity of equilibria for the instance $(F, T, Q, \lambda, \alpha, p_1, p_2) = (\mathcal{N}(1.2, 0.05^2), 1, 4, 14, 0, 1, 0)$.



Note. Note that the equilibrium threshold functions cross.

distribution with mean 1.2 and standard deviation 0.05 ($\mathcal{N}(1.2, 0.05^2)$). Also suppose that there are four items $Q = 4$ to be sold in a selling season of length $T = 1$ and that customers arrive according to a Poisson of rate $\lambda = 14$. Finally, for simplicity assume that the firm set $p_1 = 1$ and $p_2 = 0$. In the top left panel of Figure 2, we plot the function $h_0(\mu_0)$ and observe that there are three fixed points. Each fixed point is derived from a specific threshold function $\psi(t)$ that are plotted on the right panel. Observe that these threshold functions actually cross at $t = 0.7$. Note that the derived equilibria are qualitatively very different. For example, in the equilibrium corresponding to μ_0^1, (E_1) basically everyone waits until time $t = 1$ for a chance of getting the item at price $p_2 = 0$, whereas in the equilibrium corresponding to μ_0^3 (E_3), more than 58% of the customers buy immediately, and most of these customers correspond to those that arrive early.

From the seller's perspective this situation is problematic. How can it expect to optimize over p_1 and p_2 if it cannot a priori identify the equilibrium that will arise? In terms of the collected revenues, this example is quite dramatic. In the equilibrium E_1 the expected revenue for the seller is very close to zero because with high probability all items are sold at price $p_2 = 0$. However, in the equilibrium E_3 the expected revenue is close to 4, the largest one could expect with such a $p_1 = 1$. Therefore, the assumptions that the seller could make regarding the expected equilibrium have important consequences on the optimal pricing policy and its corresponding expected revenue.

4.4. Equilibrium Selection

In the case of multiple equilibria, we may think that equilibria where most customers wait to buy at the discounted price deliver higher surplus for these customers (which is certainly true on average). As a consequence, the more customers postpone their buying, the lower the seller's profits are. This intuition implies that it is best for all buyers to play according to the equilibrium with the largest expected number of customers waiting (smallest μ_0). The following proposition captures this intuition for *FPP*; it shows that all consumers are better off in the equilibrium giving the least profit to the seller, making it reasonable to assume this behavior. We believe that a more general version of this fact also holds, but we have been unable to show it.

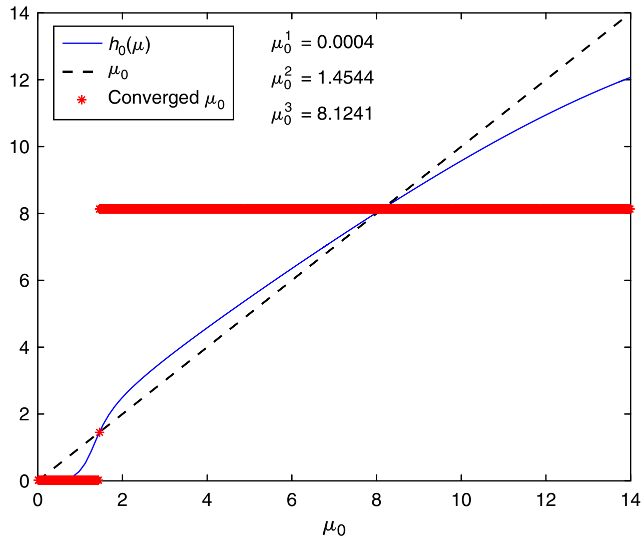
Proposition 2. *Consider the FPP policy with the initial inventory Q. Let $\psi^1(t)$ and $\psi^2(t)$ be two different equilibria of the instance where $\mu_0^1 > \mu_0^2$. Then the surplus for each customer is higher in the second equilibrium, whereas the surplus for the seller is higher in the first equilibrium.*

Proof. See Appendix F. □

Proposition 2 may also be seen as an extension of a result of Osadchiy and Vulcano (2010). They consider the (deterministic) asymptotic regime of the game, taking $\lambda, Q \to \infty$ but keeping the ratio λ/Q constant. This leads to a situation in which a *rate* of consumers arrive at the store to buy infinitely divisible goods. In such setting, Osadchiy and Vulcano (2010) observe that multiple equilibria may arise but the equilibrium in which more customers wait to buy in the second period Pareto dominates other equilibria.

On the other hand, for the general *CPP* policy when the discount rate is zero ($\alpha = 0$), one can observe that $\mu_1 = \cdots = \mu_Q = 0$ and that the only parameter is μ_0. Again, equilibria correspond to fixed points of the real function $h_0(\mu_0)$. Interestingly, the equilibrium with smallest μ_0 is stable with respect to the best response mapping. Indeed, at these equilibria the derivative of h_0 is less than 1 since it is the smallest fixed point. Consequently, the best response mapping is locally a contraction. (Observe that iterating the function h_0 is equivalent to iterating the best response mapping for the consumers.) This does not hold for equilibria in which the slope of h_0 is greater than one. Figure 3 depicts the situation for the instance where we show multiplicity of equilibria.

Figure 3. (Color online) Equilibrium convergence depending on the initial μ_0.



4.5. Pricing Policy Implementation

Because *CPP* involves a menu of prices, its implementability requires two basic considerations regarding the inventory level and committed prices.

Verifiability. In order to offer such a menu of prices, ideally there should be some way of verifying the level of inventory at time T, for either the customers or a regulator. Currently, the technological conditions are such that they allow online tracking of the inventory level. Indeed, there are many companies such as online retailers (e.g., Amazon.com) that reveal such information when the inventory level is low—*"only x in stock—order soon."* Other companies such as airlines show the exact seats available not only to travel agencies but also to final customers. More recently, when selling tickets to sport events and performing shows, companies are revealing not only the number of seats available but also their exact location in the venue.

Commitment. From an equilibrium perspective, strategic customers take the preannounced prices as given and decide accordingly. Sellers therefore have to commit to those prices, which may, however, be suboptimal when exclusively considering the second period. Arguably, committing to those preannounced prices is in the seller's best interest not only because its credibility and reputation are put on trust but also because if a seller fails to do so, its action would rather likely be discovered, which could induce customer strategic behavior that would result in lower revenues in the long run. In a way, this commitment issue is related to a result by Su and Zhang (2009) stating that a seller may be better off by committing to a certain availability guarantee.

4.6. Insights on the Potential Gains of *CPP*

To wrap up this section, we provide two simplified situations that illustrate how and why *CPP* can outperform *FPP*. In the first *CPP* is used as a *barrier* in revenue management terms. The intuition is that by setting (and announcing) a high second period price if the inventory is high, the seller induces high valuation customers to purchase the item upon arrival (discouraging strategic waiting). Additionally, when the inventory is low enough, the seller understands that high value customers have already purchased and thus it sets a low price so that low valuation customers have incentives to buy. In the second situation *CPP* is simply used as a price discrimination tool over those customers waiting in the second period. These two phenomena can also be observed in the computational results of §6, although here we present them in stylized situations that do not fit the specifics of our model.

Consider first a simple example in which there are two customers: one with valuation 1 and the other with valuation 2. Both customers arrive uniformly in the interval [0, 1] and there are two items on sale. Here, the optimal contingent pricing is to set $p_1 = 2-\varepsilon$, $p_2(1) = 1-\varepsilon$, and $p_2(2) = 2-\varepsilon$. With these prices the high valuation customer buys upon arrival (actually she is indifferent between buying and waiting, but adding an arbitrarily small discount rate would make her strictly prefer buying) since if she does not buy, the two units would be left at time 1 and thus the price will not decrease. Also, the customer with low valuation just waits and buys at price $p_2(1) = 1-\varepsilon$. The revenue for the seller is then arbitrarily close to 3. On the other hand, the best policy in which $p_2(1) = p_2(2)$, i.e., the best *FPP* policy, satisfies $p_1 = 3/2-\varepsilon$, $p_2(1) = p_2(2) = 1-\varepsilon$. Here again the high valuation customer buys upon arrival and the low valuation waits until time 1, for a revenue of almost 5/2. Thus, the revenue under *CPP* is 20% higher and, by varying the parameters of this example, we can obtain instances in which *CPP* collects twice as much revenue as the optimal *FPP*. Note that this last example shows that under *CPP* the prices $\{p_2(k)\}_{k=1}^{Q}$ may increase in k. This is rather unforeseen since in order to maximize the seller's revenue, one may expect that the price increases as units are more scarce.

To illustrate the price discrimination effect, we consider a discrete time situation in which the regular selling season is condensed to a single point. Customers arrive in the regular season and wait until the markdown, which occurs in the second period: there are two items and many customers. Two of the customers have valuation 2 and arrive with probability 1/2. Another customer has valuation 1 and arrives with probability 1. Finally, there is a large number customers with very small valuation $v > 0$ who arrive with probability 1. In this case the optimal *FPP* is to set $p_1 = 2-\varepsilon$ and $p_2 = v$. For these prices the equilibrium is that the valuation 2 customers buy at price p_1, whereas all others wait for price v leading to a revenue of $(1/4)\times(4-2\varepsilon)+(1/2)\times(2-\varepsilon+v)+(1/4)\times(2v)$, which is arbitrarily close to $2+v$.[6] On the other hand, in *CPP* we may set $p_1 = 1.75$, $p_2(1) = 1$, and $p_2(2) = v$, which leads to the equilibrium in which customers with valuation 2 buy at price p_1, the customer with valuation 1 buys at price p_2 independent of the leftover inventory, and customers of valuation v only buy if two units are left. Conditioning on the number of customers of valuation 2 that arrive, the revenue can be expressed as $(1/4)\times(3.5)+(1/2)(1.75+1)+(1/4)\times(2v) = 2.25+v/2$. For a small v, this *CPP* policy is 12.5% higher than the optimal *FPP*.

Another situation in which *CPP* may significantly outperform *FPP* occurs when the seller is uncertain about how the market will react to the product on sale. Say there are two possible scenarios, each occurring with some probability known by the seller: the *success* scenario in which the arrival rate and/or the customers' valuation are relatively high and the *failure* scenario in which the arrival rate and/or the customers' valuation are relatively low. In this context, as explored in §6.4, *CPP* provides significantly more flexibility to respond to the market conditions.

5. Optimal Pricing

In this section we formulate the seller's optimization problem for contingent preannounced pricing schemes and propose a gradient ascending method to obtain the corresponding solution.

Given the results in the previous sections, we can explicitly write the seller's problem. The problem can be considered as a Stackelberg game where the seller first announces a menu of prices p_1 and $\{p_2(k)\}_{k=1}^{Q}$, and then customers respond based on their threshold function ($\psi(t)$), which we know exists although it may not be unique. In the case of nonuniqueness, we consider the worst case for the seller as argued in §4.4. This corresponds to the lowest value of μ_0. Accordingly, the seller's revenue function can be written as

$$\begin{aligned}\pi_{CPP}(p_1,\vec{p}_2) &= p_1(Q-\mathbb{E}(Q_T))+\mathbb{E}(p_2(Q_T)\min\{Q_T, S+W\})\\ &= p_1\sum_{k=0}^{\infty}\min\{k,Q\}\mathbb{P}(N^I=k)\\ &\quad+\sum_{k=1}^{Q}\sum_{i=0}^{\infty}p_2(k)\min\{i,k\}\mathbb{P}(N^I=Q-k)\\ &\quad\cdot\mathbb{P}(N^{II}(k)=i).\end{aligned}$$

Here S and W denote respectively the number of customers waiting strategically and not strategically to buy in the second period (§4.1). Also N^I and $N^{II}(k)$ denote respectively the number of customers who buy the item at their arrival and those who wait to buy at time T at price $p_2(k)$ (§4). Then the problem for the seller is to solve $\pi^*_{CPP} = \max_{0\leqslant p_2(k)\leqslant p_1,\,\forall k\in\{1,\dots,Q\}} \pi_{CPP}(p_1,\vec{p}_2)$.

To illustrate the complexity of the problem, we consider a seller offering a single item during a unit horizon to patient consumers (i.e., $\alpha = 0$), whose valuation is deterministic and equal to 1. Assume also that the clearance price is fixed to $p_2 = 0$ (so the seller only decides p_1). The seller's objective function simplifies to the expression $\pi = p(1-e^{-\mu})$, where p is the price to charge in the first period. Note that μ, the expected demand up to time T, depends on p. To find the value of μ, we can obtain an implicit solution of Equation (4). Thus, the seller's optimization is

$$\max_{p\geqslant 0}\left\{p(1-e^{-\mu})\text{: s.t. } 0=\ln\frac{(1-p)(\lambda-\mu)}{1-e^{-\lambda+\mu}}\right\}.$$

Obtaining p from the constraint and replacing it in the objective function leads to the one dimensional problem $\max_{0\leqslant\mu\leqslant\lambda}(1-(1-e^{-\lambda+\mu})/(\lambda-\mu))(1-e^{-\mu})$. Unfortunately, even for this very simple instance a closed form solution cannot be obtained and thus we need to rely on numerical optimization tools.

In general, the customers' response can be incorporated into the first stage problem as a fixed point constraint. Thus the seller's optimization problem is

$$\max\{\pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q}): \text{ s.t. } h(\vec{\mu}) = \vec{\mu}\}. \tag{5}$$

Note that the objective function π_{CPP} also depends on $\vec{\mu}$ and that h also depends on p_1 and $\{p_2(k)\}_{k=1}^{Q}$. Thus, $\vec{\mu}$ depends on p_1 and $\{p_2(k)\}_{k=1}^{Q}$ through the fixed point equation $h(\vec{\mu}) = \vec{\mu}$. We now present a gradient method for the problem, involving an explicit computation of the gradient. This is new to the pricing literature.

Algorithm 5.1 (Optimization algorithm for CPP)

1 Start from an initial point $x^{(0)} = (p_1, \{p_2(k)\}_{k=1}^{Q})$ such that $p_1 \geqslant p_2(k)$ for all k.
2 Calculate $\nabla \pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q})$. This requires not only computing the partial derivatives of π_{CPP} with respect to prices but also those with respect to μ_i and then, by the chain rule and using the fixed point constraint in (5), the derivatives of μ_i with respect to prices.
3 Move in the gradient direction $x^{(m+1)} = x^{(m)} + \beta \cdot \nabla \pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q})$, imposing $p_1 \geqslant p_2(k) \geqslant 0$ for all k.
4 Stop if convergence criterion is satisfied. Otherwise, go to 2.

In Appendix G we derive the analytical expression for $\nabla \pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q})$ of problem (5). As mentioned above, this is nonobvious since the objective $\pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q})$ depends also on $\vec{\mu}$, which in turn depends on $p_1, \{p_2(k)\}_{k=1}^{Q}$ through the fixed point constraint. Therefore, to compute the total derivative of the objective with respect to some price, say, p_1, we compute $\partial \pi_{CPP}/\partial p_1 + \sum_{i=0}^{Q} (\partial \pi_{CPP}/\partial \mu_k)(\partial \mu_k/\partial p_1)$. Interestingly, we can obtain an explicit expression for this where $\partial \mu_k/\partial p_1$ is obtained by taking derivative in the fixed point constraint. The resulting gradient depends only on the prices of the iteration and on the corresponding $\vec{\mu}$ solving $h(\vec{\mu}) = \vec{\mu}$.

A consequence of the previous comment is that in step 2, for a given price vector, it may happen that there are multiple equilibria, and therefore the gradient will depend on which one we choose. If this is the case we select the equilibrium with the smallest value of μ_0, as discussed in §4.4. To this end we use the following heuristic that performs quite well in our experiments. Given a set of prices, the idea is to search over values of μ_0 while adjusting the remaining $\mu_1, \ldots, \mu_Q$ so that an equilibrium is attained. Specifically, we start by letting $\mu_k^{(0)} = 0$ for all $k = 1, \ldots, Q$ and explore the values of $\mu_0 = \mu_0^{(0)}$ in the interval $[0, \bar{\mu}_0]$ from left to right until a value such that $h_1(\vec{\mu}^{(0)}) = \mu_0^{(0)}$ is found. Then we update $\vec{\mu}^{(i+1)} = h(\vec{\mu}^{(i)})$ and if $\|\vec{\mu}^{(i+1)} - \vec{\mu}^{(i)}\|_2 < \epsilon$ we stop. Otherwise, we use these fixed values $\mu_k^{(i+1)}$ for $k = 1, \ldots, Q$ and again search from left to right the value of $\mu_0^{(i+1)}$ such that $h_1(\vec{\mu}^{(i+1)}) = \mu_0^{(i+1)}$.

Note that Algorithm 5.1 can be used to obtain a local optimal solution starting from an arbitrary initial solution. Thus, to solve a particular instance we use it for different starting points and select the best solution obtained among those found. The gain of this gradient method compared to a derivative free method in which the objective is evaluated multiple times around a given point in order to estimate the gradient is quite significant. Indeed, for instances with medium initial inventory (say $Q = 8$) our method already appears to be 100 times faster.

6. Numerical Experiments

In this section we numerically solve the seller's optimization problem for contingent preannounced pricing schemes using the gradient method. We examine the solution and equilibrium properties via a numerical experiment where we vary the number of items to be sold Q, the arrival rate λ, and the discount time valuation α in a factorial design. Since we are interested on the cases when there is some scarcity, we consider instances in which the expected arrival of customers is at least the initial inventory. The time horizon is set to $T = 1$ and customers' valuations distributed $U[0, 1]$.[7] Note that α determines the relative valuation of a customer between $t = 0$ and $t = T$, so if $\alpha = -\ln(0.25)$, customers value the item at $t = T$ at 25% of what they value it at $t = 0$.

6.1. Detailed Instance

For illustration purposes we show one instance in full detail. Customers arrive to the store following a Poisson process of rate 8 and, discount the future moderately ($\alpha = -\ln(0.75)$); there are $Q = 4$ items to be sold. Table 1 shows the results for *CPP*, *FPP*, and a single price **SP** policy, which sets a single price for the whole season. This specific instance is useful to clarify the main characteristics of the proposed policy. Note first that, as expected, in this example $\pi_{CPP} > \pi_{FPP} > \pi_{SP}$. Second, although the preannounced policies decrease the percentage of customers buying upon arrival, the total demand increases because the percentage of nonbuyers decreases. Third, the contingent policy reduces the percentage of customers strategically waiting; by setting a higher price in the first period, more customers wait nonstrategically.

6.2. *CPP* vs. *FPP*

We start the study by comparing the performance of *CPP* and *FPP* under mild conditions. The instances run for this case are the combinations of $Q \in \{1, 2, \ldots, 10\}$, $\lambda \in \{1, 2, \ldots, 10\}$, and $\alpha \in \{0, -\ln(0.75), -\ln(0.50), -\ln(0.25)\}$, considering only the cases were $\lambda \geqslant Q$. Because *FPP* is a special case of *CPP*, the latter yields higher revenues than does the former. Despite this fact, it is interesting to investigate the factors that mediate this improvement.

(i) *Main effects*. On average *CPP* yields 0.92% higher revenues than *FPP* does. This difference can reach up to 4.4%. Greater differences were found on instances with medium Q/λ ratio (between one and two) and medium discount factor (moderate impatience). Intuitively, if Q/λ is too large a scarcity effect pops in, eliminating the discrimination tool *CPP* has at the clearance season. On the other hand,

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Table 1. Results for the instance $(F, T, Q, \lambda, \alpha) = (U[0, 1], 1, 4, 8, -\ln(0.75))$.

	CPP	*FPP*	SP
π^*	1.729	1.696	1.684
p_1^*	0.603	0.594	0.595
p_2^*	0.603 ($p_2(1)$)	0.490	
	0.603 ($p_2(2)$)		
	0.418 ($p_2(3)$)		
	0.408 ($p_2(4)$)		
Average I (%)	30.6	29.2	40.5
Average S (%)	9.1	11.4	0.0
Average W (%)	13.0	3.5	0.0
Average N (%)	47.3	55.8	59.5
π^* I (%)	79.9	77.1	100.0
π^* S (%)	8.3	17.5	0.0
π^* W (%)	11.8	5.4	0.0
π^* N (%)	0.0	0.0	0.0
μ	2.451 (μ_0)	2.336 (μ)	
	0.111 (μ_1)		
	0.111 (μ_2)		
	0.000 (μ_3)		
	0.000 (μ_4)		

Note. I: customers that buy *I*mmediately, S: wait *S*trategically, W: *W*ait nonstrategically, and N: do *N*ot buy.

when customers are very impatient (high discount factor α), the advantage of the *CPP* strategy vanishes as customers became more myopic.

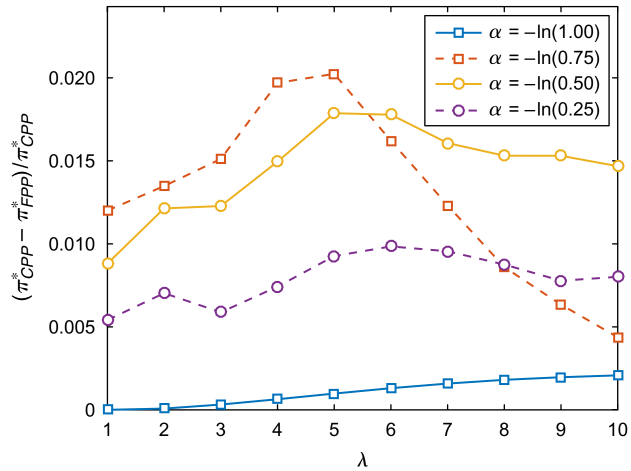
(ii) *Increasing prices with remaining inventory.* The prices in the second period may increase with Q for the *CPP* strategy. Indeed, we observe this condition on 87% of the instances with $Q > 1$. This arises in cases where customers have some degree of impatience and there is a relatively high number of units to sell. To understand why the seller would offer such prices, consider the discount price to be equal to the first period price ($p_2(Q) = p_1$) when no units are sold in the first period, whereas if some units are sold in the first period, say k, the discount price could be such that $p_2(k) < p_2(Q)$ $(0 < k < Q)$. Then impatient customers realize that if they wait, they could get the product for the same price as in the first period, but with a lower surplus ($\alpha > 0$). Table 2 depicts instances in which discount price equals the regular price if no units are sold in the first period. However, as demand increases the seller changes its strategy and stop adopting nondecreasing discount prices.

(iii) *Effect of customers' patience.* We observe that greater differences in profits between *CPP* and *FPP* are obtained when customers have medium impatience ($\alpha \sim \ln(0.75)$; see, e.g., Figure 4). When customers have medium impatience, the seller's *CPP* strategy consists of further discriminating the customers who wait. Interestingly, if customers are very impatient, customers' valuation in the second period will be low independently of how perfect the information is about Q; as a consequence, the advantage of *CPP* versus *FPP* diminishes.

6.3. *CPP* vs. *FPP* for Large Initial Inventory

Now we study how our methodology can be applied to bigger initial inventories. For a high initial inventory (Q),

Figure 4. (Color online) Revenue comparison for *CPP* vs. *FPP*, $v \sim U[0, 1]$, $Q = 3$.



the optimization problem in *CPP* becomes more challenging since the dimension of the space grows at the same rate as the number of available units. We solved this optimization problem applying the gradient ascend method starting from a number of different price menus. These price menus were generated by grouping prices $p_2(k)$ for similar values of k and then considering all combinations in a multidimensional grid (taking prices to be multiples of 0.1). The instances run for this case are the combinations of $T = 1$, $Q \in \{10, 20, 30, 40, 50\}$, $\lambda \in \{10, 20, 30, 40, 50, 100\}$, $\alpha \in \{0, -\ln(0.75), -\ln(0.50), -\ln(0.25)\}$, and customers' valuation $U[0, 1]$. As before, we consider the cases where $\lambda \geqslant Q$. As it is somewhat expected, we observed smaller relative differences between the preannounced policies than in the case of smaller inventories. Specifically, *CPP* delivers on average 0.70% higher revenues than does *FPP*. The intuition behind this fact is that with larger arrival rates the process becomes more deterministic, making it easier for the seller to anticipate the number of units left at time T. Naturally this implies that the advantage of *CPP* as a discrimination tool vanishes. Table 3 exhibits the situation and also shows that, as before, the differences are higher with moderately impatient consumers.

Table 3. Percentage difference between *CPP* and *FPP* for $(F, T) = (U[0, 1], 1)$.

α	Q \ λ	10	20	30	40	50	100
0	10	0	0.17	0.36	0.39	0.35	0.19
	20		0.02	0.01	0.11	0.17	0.34
	30			0	0.11	0.01	0.39
	40				0	0.01	0.2
	50					0	0.05
$-\ln(0.75)$	10	1.32	1.59	0.79	0.35	0.25	0.06
	20		1.29	1.43	1.27	1.35	0.18
	30			1.24	1.19	1.6	0.39
	40				1.23	1.24	0.7
	50					1.18	1.32

Table 2. Optimal prices in *CPP* for instances $(F, T, Q, \alpha) = (U[0, 1], 1, 2, -\ln(0.75))$.

λ	1	2	3	4	5	6	7	8	9	10
p_1^*	**0.527**	**0.555**	**0.579**	**0.613**	**0.644**	**0.672**	**0.695**	0.698	0.718	0.736
$p_2(1)^*$	0.360	0.392	**0.579**	**0.613**	**0.644**	**0.672**	**0.695**	0.505	0.519	0.531
$p_2(2)^*$	**0.527**	**0.555**	0.424	0.429	0.435	0.442	0.450	0.487	0.497	0.508

6.4. Uncertain Market

We explore how *CPP* behaves when there is a finite number of customer segments that can be attracted by the new product. Each segment differs in the valuation toward the product and the arrival rate. For simplicity we consider two segments and that only one of them is finally attracted by the product.

Consider that there is a finite set of possible scenarios S, where each scenario $s \in S$ corresponds to a customer arrival rate λ_s and a valuation distribution function F_s for them. The probability of each scenario is $r_s \geqslant 0$ $(\sum_{s\in S} r_s = 1)$. Customers know which scenario is realized (i.e., what type of customers they are); however, the seller only knows the probability of each scenario. This information structure may be natural when new products are introduced in the market and in particular was studied by Kanoria and Nazerzadeh (2014). Note that Theorems 1 and 2 hold because they are applied to each scenario. To solve the optimization problem, we apply Algorithm 5.1 with a slight change in step 2, in which the gradient should be computed as the weighted sum of the gradients of each scenario.

We ran the set of instances described as follows. There are two possible scenarios, each occurring with probability 1/2. In one scenario the valuation distribution is $U[1.99, 2.01]$, whereas in the other it is $U[0.99, 1.01]$. That is, it is equally likely that the product is a *success* or a *failure*. For both scenarios we consider the same values of Q, λ, and α, where $Q \in \{1, 2, 3, 4, 5\}$, $\lambda \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $\alpha \in \{0, -\ln(0.75), -\ln(0.5), -\ln(0.25)\}$, and $T = 1$.

We observed that over the studied instances, *CPP* outperforms *FPP* by 4.7% on average, with differences of up to 23%. Figure 5 shows these percentage differences among the policies for different levels of impatience and arrival rates. It can be seen that for each level of impatience there is an arrival rate level that delivers maximum difference between the strategies. Unlike the case with one customer type, the greater differences are observed when customers are more impatient. Indeed, we found that *CPP* benefits from the arrival of low valuation customers when customers are patient. In contrast, if customers are impatient, *CPP* benefits from the arrival of high valuation customers.

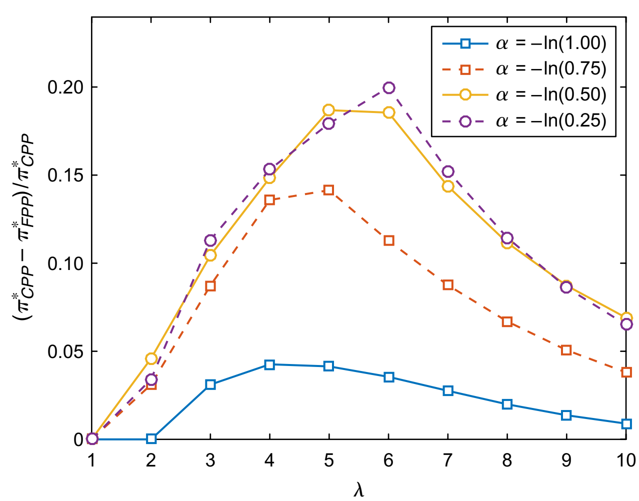
To understand the reason behind these results, we analyze in more detail the following instance: $T = 1$, $Q = 2$, $\alpha = 0$, $\lambda = 4$. In this case, *CPP* charges prices $p_1 = 1.53$, $p_2(1) = 1.53$, and $p_2(2) = 0.99$, whereas *FPP* charges $p_1 = 1.46$ and $p_2 = 0.99$. Thus if arriving customers are of low type, customers wait for the discount in the second period and both pricing strategies get the same revenues. However, if arriving customers are of high type, *CPP* only discounts the price in the second period when there are no sales in the first period. This allows the seller to increase the price in the first period since now buying at time T is not as attractive as in *FPP*.

Additionally, when customers are impatient, *FPP* charges high prices without taking into consideration low valuation customers, whereas *CPP* will leave some discounted prices low in order to extract surplus from them. For example, in the instance $T = 1$, $Q = 2$, $\alpha = -\ln(0.75)$, and $\lambda = 4$, the optimal prices are $p_1 = 1.99$, $p_2(1) = 1.99$, and $p_2(2) = 0.56$ for *CPP* and $p_1 = 1.99$ and $p_2 = 1.99$ for *FPP*.

7. Conclusion, Implications, and Future Work

The research investigating customers' strategic behavior and best pricing strategies to implement has driven a great deal of attention in the management science community. In this paper, we explore preannounced pricing schemes and their equilibrium solution for a seller facing strategic consumers who have uncertainty about the initial inventory level. We prove that existence of equilibrium is guaranteed but not its uniqueness and obtain interesting insights by solving numerical instances of the problem. Along the way we propose a new modeling approach based on differential equations, defining equilibria as the fixed point of a function. With this novel approach we are able to prove existence of equilibria and its uniqueness in the case of a single unit. In addition, we warn that uniqueness cannot be guaranteed in

Figure 5. (Color online) Revenue comparison for *CPP* and *FPP* where $v \sim U[0.99, 1.01]$ with probability 1/2 and $v \sim U[1.99, 2.01]$ with probability 1/2.



the multiunit case. Because even for very simple instances it was not possible to obtain closed-form solutions, we develop a gradient optimization method and solve a set of numerical instances to better understand the characteristics of the problem. In particular, we investigate how the performance of different pricing policies depends on customer willingness to wait, customer arrival rate, the initial number of items offered, and consumer uncertainty about the initial inventory level. Despite the theoretical multiplicity of equilibria, we find that in the vast majority of instances there is only one equilibrium. Our numerical results confirm the advantage of using the contingent preannounced pricing policies when facing strategic consumers. These advantages come from having two discrimination tools instead of one in the case *FPP*. The menu of prices in the second period can help as a barrier to discourage strategic waiting and simultaneously a discrimination mechanism for the customers waiting. In contrast, *FPP* cannot do that. We also explore a situation in which the seller is uncertain about how the market will perceive the product on sale. Naturally this uncertainty turns out to amplify the advantage of using *CPP* over *FPP* since its higher flexibility permits to better adapt to the different scenarios.

We conclude by noting some additional extensions of the present research. First, the assumption that all customers have the same willingness (or ability) to wait can be relaxed by allowing for heterogeneity in the parameter α. This may complicate the analysis but may enrich the results by offering the seller an additional source of potential discrimination. Second, it would be interesting to extend the results presented here to more pricing periods and ultimately to a continuous time price path. This latter problem may have more structure and provide deeper insights on how an optimal pricing strategy should look like. Also, by taking advantage of the ODE approach, one could explore the design of more effective algorithms for solving the price optimization problem. Finally, an interesting direction is to consider further pricing methods for the clearance period, such as running an appropriate auction.

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Appendix A: Proof of Proposition 1

Let us call $\Phi(t,x)$ the right-hand side of the ODE (3).

$$\dot{x}(t)=\begin{cases}\lambda(t)(1-F_t(p_1)) & 0\leqslant t<t^*\\ \lambda(t)\left(1-F_t\left(\dfrac{p_1\phi(x(t))-D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t))-e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\right) & t^*\leqslant t\leqslant T\end{cases}$$

$$x(0)=0.$$

We first prove that the solution is unique and then prove existence of a solution.

(i) *Uniqueness*: Note that $\Phi(t,x)$ is nonincreasing in x since for $t\in[t^*,T]$,

$$\frac{\partial\Phi(t,x)}{\partial x}=-\lambda(t)F_t'\left(\frac{p_1\phi(x(t))-D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t))-e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\cdot\frac{p_1e^{-\alpha(T-t)}-D_{\vec{\mu}}(\vec{p}_2)}{(\phi(x)-C_{\vec{\mu}}e^{-\alpha(T-t)})^2}\phi'(x)\leqslant 0.$$

The last inequality follows since $D_{\vec{\mu}}$ is the expected discount price, which is always below $p_1e^{-\alpha(T-t)}$, for $t\geqslant t^*$. Then for each t, we can write $\Phi(t,x)$ as minus the derivative (with respect to x) of a convex function $\varphi(t,x)$; i.e., $\dot{x}(t)=-\partial_x\varphi(t,x(t))$. To prove uniqueness, let us suppose that there exist two solutions $x(t)$ and $y(t)$ satisfying $\dot{x}(t)=-\partial_x\varphi(t,x(t))$ and $\dot{y}(t)=-\partial_x\varphi(t,y(t))$. Then

$$\begin{aligned}\frac{d(x(t)-y(t))^2}{dt}&=2(x(t)-y(t))(\dot{x}(t)-\dot{y}(t))\\&=-2(x(t)-y(t))(\partial_x\varphi(t,x(t))-\partial_x\varphi(t,y(t)))\\&\leqslant 0,\end{aligned}$$

where the last inequality follows because any nondecreasing real function g satisfies that $(x-y)(g(x)-g(y))\geqslant 0$, and $\partial_x\varphi(t,\cdot)$ is nondecreasing. Then $(x(t)-y(t))^2$ is nonincreasing. Adding that $x(0)-y(0)=0$ and $(x(t)-y(t))^2\geqslant 0$, we conclude that $x(t)=y(t)$.

(ii) *Existence*: Let us redefine the initial value ODE equivalently as

$$\dot{x}(t)=\begin{cases}\lambda(t)(1-F_t(p_1)) \quad \text{if } 0\leqslant t<t^*,\ x\in[0,+\infty)\\ \lambda(t)\left(1-F_t\left(\dfrac{p_1\phi(x(t))-D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t))-e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\right)\\ \qquad \text{if } t^*\leqslant t\leqslant T,\ 0\leqslant x<\phi^{-1}(C_{\vec{\mu}}e^{-\alpha(T-t)})\\ 0 \quad \text{if } t^*\leqslant t\leqslant T,\ \phi^{-1}(C_{\vec{\mu}}e^{-\alpha(T-t)})\leqslant x,\end{cases}\tag{A1}$$

$$x(0)=0.$$

In this differential equation we have simply extended the value of $\dot{x}(t)$ as zero whenever the denominator $\phi(x(t))-e^{-\alpha(T-t)}C_\mu$ becomes zero.

Clearly $\Phi(t,x)$ is continuous in t and x; then by Peano's Theorem (Hale 1980, Theorem 1.1), there exists a solution to (A1) near the initial value. Furthermore, if $\bar{\lambda}$ denotes the maximum possible value of $\lambda(t)$, we have that $0\leqslant\Phi(t,x)\leqslant\bar{\lambda}$, so that $0\leqslant x(t)\leqslant\bar{\lambda}T$. Then by extending the solution to the maximal interval (Hale 1980, Theorem 2.1), there exists a solution to (A1) for all $t\in[0,T]$.

Thus we have proved that (3), or equivalently (A1), has a unique solution. □

Appendix B

PROPOSITION 3. *Let F be a finite mean distribution whose density f is decreasing starting from some point τ; then* $\lim_{y\to\infty} F'(y)y^2 = 0$.

PROOF. Assume for a contradiction that $\limsup_{y\to\infty} f(y)y^2 = 2\varepsilon > 0$; then for all $M > 0$ there exists $x > 2M$ such that $f(x)x^2 > \varepsilon$. Consider a sequence $\{x_i\}_{i=1}^{\infty}$ where $x_1 = \tau$, x_2 is some $x > 2x_1$ so that $x^2 f(x) > \varepsilon$, x_3 is some $x > 2x_2$ so that $x^2 f(x) > \varepsilon$, and so on. Note that for all $x \in [x_i, x_{i+1}]$ we have that $f(x) \geqslant \varepsilon/x_{i+1}^2$ because $x^2 f(x) \geqslant x^2 f(x_{i+1}) = x_{i+1}^2 f(x_{i+1})x^2/x_{i+1}^2 > \varepsilon x^2/x_{i+1}^2$. Then

$$\begin{aligned}\int_\tau^\infty xf(x)\,dx &= \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} xf(x)\,dx\\ &> \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} \frac{\varepsilon x}{x_{i+1}^2}\,dx\\ &= \sum_{i=1}^\infty \frac{\varepsilon}{2}\left(1-\left(\frac{x_i}{x_{i+1}}\right)^2\right)\\ &> \frac{\varepsilon}{2}\sum_{i=1}^\infty \frac{3}{4} = \infty,\end{aligned}$$

which contradicts the finite mean assumption. □

Appendix C: Proof of Theorem 1

By the Brouwer fixed point Theorem, we need to prove that $h(\cdot)$ is continuous and that it transforms a convex compact set into itself. Clearly $\mathcal{D}$ is bounded and closed, then compact. Also $\mathcal{D}$ is convex because it is polyhedral because it can be specified by finite number of linear inequalities.

Let us now verify that $h(\mathcal{D}) \subseteq \mathcal{D}$. Clearly, the solution $x_{\vec{\mu}}(t)$ of (3) is nonnegative for all $t \in [0, T]$, so $x_{\vec{\mu}}(T) \geqslant 0$. Because of the form of the right-hand side of (3) and the definition of t^*, we have that $\lambda(t) - \dot{x}_{\vec{\mu}}(t) = \lambda(t) \cdot F_t(\max\{(p_1\phi(x(t)) - D_{\vec{\mu}}(\vec{p}_2))/(\phi(x(t)) - e^{-\alpha(T-t)}C_{\vec{\mu}}), p1\}) \geqslant F_t(p_1)$ for all $t \in [0, T]$, in particular for $t \in [0, t^*(k)]$ with $1 \leqslant k \leqslant Q$. Also, from the definition of $t^*(k)$, we have $F_t(p_2(k)e^{\alpha(T-t)}) \geqslant F_t(p_1)$ for $t \leqslant t^*(k)$. Therefore we have that $\min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), \lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\} - \lambda(t)F_t(p_1) \geqslant 0$ and $\int_0^{t^*(k)} \min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), \lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\} - \lambda(t)F_t(p_1)\,dt \geqslant 0$, concluding that h is nonnegative.

To finish the proof of $h(\mathcal{D}) \subseteq \mathcal{D}$, we need to observe that

$$\begin{aligned}x_{\vec{\mu}}(T) + \int_0^{t^*(k)} &\min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), \lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\}\\ &- \lambda(t)F_t(p_1)\,dt\\ \leqslant \int_0^T &\lambda(t)(1 - F_t(p_1))\,dt.\end{aligned}$$

Indeed,

$$\begin{aligned}x_{\vec{\mu}}(T) &+ \int_0^{t^*(k)} \min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), F_t(p_2(k)e^{\alpha(T-t)})\}\\ &- \lambda(t)F_t(p_1)\,dt\\ &= \int_0^T \dot{x}_{\vec{\mu}}(t)\,dt\\ &\quad + \int_0^{t^*(k)} \min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), \lambda(t)F_t(p_2(k)e^{\alpha(T-t)})\}\\ &\quad - \lambda(t)F_t(p_1)\,dt\\ &\leqslant \int_0^T \dot{x}_{\vec{\mu}}(t)\,dt + \int_0^{t^*(k)} \lambda(t) - \dot{x}_{\vec{\mu}}(t) - \lambda(t)F_t(p_1)\,dt\\ &\leqslant \int_0^T \dot{x}_{\vec{\mu}}(t)\,dt + \int_0^T \lambda(t) - \dot{x}_{\vec{\mu}}(t) - \lambda(t)F_t(p_1)\,dt\\ &= \int_0^T \lambda(t)(1 - F_t(p_1))\,dt.\end{aligned}$$

Finally, we observe that $h(\cdot)$ is a continuous function in $\mathcal{D}$. For this to hold it is sufficient to prove that the right-hand side of (A1) is continuous in t and μ and locally Lipschitz in x (Hale 1980, Theorem 3.2) . The continuity in t and μ is straightforward, whereas the locally Lipschitz property in x follows since

$$\begin{aligned}\frac{\partial\Phi(t,x)}{\partial x} &= -\lambda(t)F_t'\left(\frac{p_1\phi(x(t)) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t)) - e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\\ &\qquad \cdot \frac{p_1 e^{-\alpha(T-t)} - D_{\vec{\mu}}(\vec{p}_2)}{(\phi(x(t)) - C_{\vec{\mu}}e^{-\alpha(T-t)})^2}\phi'(x)\\ &= -\lambda(t)F_t'\left(\frac{p_1\phi(x(t)) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t)) - e^{-\alpha(T-t)}C_{\vec{\mu}}}\right)\\ &\qquad \cdot \left(\frac{p_1\phi(x(t)) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x(t)) - C_{\vec{\mu}}e^{-\alpha(T-t)}}\right)^2\\ &\qquad \cdot \frac{p_1 e^{-\alpha(T-t)} - D_{\vec{\mu}}(\vec{p}_2)}{(p_1\phi(x(t)) - D_{\vec{\mu}}(\vec{p}_2))^2}\phi'(x),\end{aligned}$$

and thus the conditions $\lim_{y\to\infty} F_t'(y)y^2 = 0$ and F_t' bounded for every $t \in [0, T]$ imply that $\partial\Phi(t,x)/\partial x$ is bounded. Then by the Brouwer fixed point theorem, we conclude that there is at least one fixed point of h in $\mathcal{D}$. □

Appendix D: Proof of Theorem 2

Because in we are in the case in which $Q = 1$, we note immediately that $\mu_1 = 0$, and for simplicity we denote $\mu = \mu_0$. Observe that when $t \in [0, t^*]$ the solution is clearly unique and given by $x(t) = \int_0^t \lambda(t)(1 - F_t(p_1))\,dt$, and thus $\psi(t) = p_1$. Moreover, it is clear that in this interval $x(\cdot)$ is nondecreasing.

We can then concentrate in the case $t \in [t^*, T]$. The idea of the proof is to show that the real function $h_0(\mu) = x_\mu(T)$, defined in Equation (4), has derivative at most one for μ between 0 and $\bar{\mu} = \int_0^T \lambda(t)(1 - F_t(p_1))\,dt$, which is an upper bound on μ. For this we consider the quantity $a_\mu = -\mu + \int_0^T \lambda(t)\,dt - \int_0^{t^*} \lambda(t)F_t(p_1)\,dt - \int_{t^*}^T \lambda(t)F_t(p_2 e^{\alpha(T-t)})\,dt$ and note that C_μ can be computed in terms of a_μ. Thus consider the following change of variables in the right-hand side of (4):

$$K(\mu) \doteq -\ln(C_\mu) = \mu - \ln \frac{1-e^{-a_\mu}}{a_\mu}.$$

Note that since $C_\mu \in (0,1)$, then $K(\mu) \in (0,+\infty)$. So for $t \in [t^*, T]$, we can rewrite the differential equation (4) as

$$\dot{x}(t) = \lambda(t)\left(1 - F_t\left(\frac{p_1 - p_2 e^{x(t)-K}}{1-e^{-\alpha(T-t)}e^{x(t)-K}}\right)\right) \quad t \in [t^*, T],$$
$$x(t^*) = \int_0^{t^*} \lambda(t)(1-F_t(p_1))\,dt \tag{D1}$$
$$x_\mu(T) = \mu.$$

In what follows we prove that h_0 is Lipschitz of constant equal to 1 with respect to K (for $K > 0$) and that $dK(\mu)/d\mu \leqslant 1$.

Claim 1. *$x_\mu(T)$ is Lipschitz of constant 1 as a function of K. Recall that $x(t)$ is nondecreasing and consider $K_1 < K_2$; let $x_1(t)$ and $x_2(t)$ be the corresponding solutions to (D1). Equivalently to prove that $(x_2(T)-x_1(T))/(K_2-K_1) \leqslant 1$, we will show that $x_1(T)-K_1 \geqslant x_2(T)-K_2$. To show this we see that the inequality holds for any $t \in [0,T]$. Indeed, the assertion is obvious for $t \in [0,t^*]$ since x_1 and x_2 coincide in that interval, so $x_1(t^*)-K_1 > x_2(t^*)-K_2$. Recall that both x_1, and x_2 are increasing and continuous functions. Consider then the point $u \in [t^*, T]$ satisfying $x_1(u)-K_1 = x_2(u)-K_2$ (if no such u exists, then we have finished the proof). From the form of (D1) it is clear that $\dot{x}_1(u) = \dot{x}_2(u)$ and $x_1(u)-K_1 = x_2(u)-K_2$, so then both curves $x_1(t)-K_1$ and $x_2(t)-K_2$ will be exactly the same up to $t=T$. Hence $x_1(t)-K_1 \geqslant x_2(t)-K_2$ for every $t\in[0,T]$. In particular $x_1(T)-K_1 \geqslant x_2(T)-K_2$.*

To verify that $(x_2(T)-x_1(T))/(K_2-K_1) > -1$, let us call $\Phi(t,x,K)$ the right part of the ODE (D1) and observe

$$\frac{\partial \Phi(t,x,K)}{\partial K} = \lambda(t) F_t'\left(\frac{p_1 - p_2 e^{x-K}}{1-e^{-\alpha(T-t)}e^{x-K}}\right) \cdot e^{x-K}\frac{p_1 e^{-\alpha(T-t)} - p_2}{(1-e^{-\alpha(T-t)}e^{x-K})^2} \geqslant 0.$$

Consider $K_1 < K_2$ and the respective solutions to the ODE (D1) $x_1(t)$ y $x_2(t)$. Clearly these are equal up to $t=t^*$, and because of the last inequality, $\dot{x}_2(t^*) \geqslant \dot{x}_1(t^*)$, which means that the curve $x_2(t)$ goes over $x_1(t)$ just after t^*.

Thus, if these curves do not intersect in $[t^*, T]$ the claim is proved. Otherwise, suppose that both curves intersect in $u \in [t^*, T]$. Then because $x_1(u) = x_2(u)$ and using the inequality just above, it implies that $\dot{x}_1(t) \leqslant \dot{x}_2(t)$. With the same argument, $x_1(t)$ can never cross $x_2(t)$. However, if at any t they touch, the second curve will continue with a steeper slope. Consequently, $x_2(T) \geqslant x_1(T)$, which means that $(x_2(T)-x_1(T))/(K_2-K_1) \geqslant 0 > -1$, concluding the proof of the first claim.

Claim 2. *$dK(\mu)/d\mu < 1$. This comes from some algebraic manipulations that we outline here.*

$$\begin{aligned}\frac{dK(\mu)}{d\mu} &= 1 - \frac{d}{d\mu}\left(\ln\frac{1-e^{-a_\mu}}{a_\mu}\right)\\ &= 1 - \frac{d}{da_\mu}\left(\ln\frac{1-e^{-a_\mu}}{a_\mu}\right)\frac{da_\mu}{d\mu}\\ &= 1 - \left(\frac{1-e^{-a_\mu}}{a_\mu}\right)^{-1}\frac{d}{da_\mu}\left(\frac{1-e^{-a_\mu}}{a_\mu}\right)(-1)\\ &= 1 + \left(\frac{1-e^{-a_\mu}}{a_\mu}\right)^{-1} e^{-a_\mu}\frac{1+a_\mu-e^{a_\mu}}{a_\mu^2}.\end{aligned}$$

Therefore $((1-e^{-a_\mu})/a_\mu)^{-1} > 0$, $e^{-a_\mu} > 0$ and $1+a_\mu - e^{a_\mu} \leqslant 0$, i.e., for a single value of μ. Then $dK(\mu)/d\mu \leqslant 1$, and this can only hold with equality if $a_\mu = 0$. We conclude the proof of the claim by noting that $dK(\mu)/d\mu = 1 + (1/(1-e^{-a_\mu}))e^{-a_\mu}((1+a_\mu-e^{a_\mu})/a_\mu) = (e^{-a_\mu}+a_\mu-1)/(a_\mu(1-e^{-a_\mu})) > 0$.

In this way we have shown that $dh_0(\mu)/d\mu = dh_0(K(\mu))/d\mu = (dh_0(K(\mu))/dK)(dK(\mu)/d\mu) \leqslant 1$ and the inequality is strict except for a single value of μ. Therefore we can conclude that there is only one fixed point of h_0. This in turn implies that we have a unique function $x(t)$ in each interval $[0,t^*]$ and $[t^*,T]$. This concludes the uniqueness of the threshold when $Q=1$. □

Appendix E: Condition for Uniqueness in the Multi-Item Case

Theorem 3. *For $\vec{\mu} \in \mathcal{D}$, consider the function $\Phi(t,x,\mu)$ corresponding to the right-hand side of ODE (3) and x its corresponding solution; i.e., $x = x_{\vec{\mu}}(t)$. Let $t^\diamond(k)$ be the intersection of $F_t^{-1}(1-\dot{x}(t)/\lambda(t))$ and $p_2(k)e^{\alpha(T-t)}$ in case it exists, and $t^\diamond(k)=0$ otherwise.*

Then if for all $\vec{\mu}\in\mathcal{D}$ we have that

$$\begin{aligned}&\max_{j=0,\dots,Q}\sum_{k=0}^{Q}\left|\frac{\partial h_k}{\partial \mu_j}\right|\\ &= \max_{j=0,\dots,Q}\int_0^T \exp\left(\int_s^T \frac{\partial\Phi(u,x,\mu_j)}{\partial x}du\right)\frac{\partial\Phi(s,x,\mu_j)}{\partial\mu_j}ds\\ &\quad + \sum_{k=1}^{Q}\int_0^{t^\diamond(k)}\exp\left(\int_s^{t^\diamond(k)}\frac{\partial\Phi(u,x,\mu_j)}{\partial x}du\right)\frac{\partial\Phi(s,x,\mu_j)}{\partial\mu_j}ds\\ &< 1\end{aligned}$$

There exists a unique solution to (4).

Proof. Since $h\colon \mathcal{D}\to\mathcal{D}$, where $\mathcal{D}\subset\mathbb{R}^{Q+1}$, a sufficient condition to show that h has a unique fixed point (and thus that (4) has a unique equilibrium) is that h is a contraction, i.e.,

$$\|J_h\|_\infty = \max_{j=0,\dots,Q}\sum_{k=0}^{Q}\left|\frac{\partial h_k}{\partial\mu_j}\right| < 1,$$

where the elements of the Jacobian J_h are

$$\frac{\partial h_0}{\partial \mu_j} = \frac{\partial x_\mu(T)}{\partial \mu_0}, \quad \frac{\partial h_k}{\partial \mu_j} = -\frac{\partial x_\mu(t^{\diamond(k)})}{\partial \mu_j}.$$

It turns out that we can explicitly compute these terms as a function of the solution to (3). To this end we need to find the derivative of the solution of a differential equation with respect to a parameter evaluated at some time t. Consider a parameter $\mu \in \mathbb{R}$, the parameter (which takes the form of μ_j) with respect to which we want to take the derivative, and consider the differential equation for x

$$\frac{dx(t)}{dt} = \Phi(t, x, \mu) \qquad \text{(E1)}$$
$$x(0) = 0.$$

Taking derivative respect to t of $dx(t)/d\mu$, we have that

$$\frac{d}{dt}\frac{dx(t)}{d\mu} = \frac{d}{d\mu}\frac{dx(t)}{dt} = \frac{d\Phi}{d\mu} = \frac{\partial\Phi}{\partial x}\frac{\partial x}{\partial \mu} + \frac{\partial \Phi}{\partial \mu}$$
$$= \frac{\partial\Phi}{\partial x}\frac{dx}{d\mu} + \frac{\partial \Phi}{\partial \mu}.$$

Then

$$\frac{d}{dt}\frac{dx(t)}{d\mu} - \frac{\partial \Phi(t,x,\mu)}{\partial x}\frac{dx}{d\mu} = \frac{\partial \Phi(t,x,\mu)}{\partial \mu}. \qquad \text{(E2)}$$

Thus we obtain another ODE where the unknown is $dx(t)/d\mu$. Note that we are taking derivative of the solution $x(t)$ of (E1) with respect to μ; thus the second argument of $\Phi(t, x, \mu)$ corresponds more precisely to that solution $x_\mu(t)$. To simplify notation, we express $\Phi(t, x_\mu(t), \mu)$ simply as $\Phi(t, x, \mu)$. Interestingly we can solve ODE (E2) using the integrating factor method, obtaining that its solution is given by

$$\frac{dx(t)}{d\mu} = \int_0^t \exp\left(\int_s^t \frac{\partial \Phi(u,x,\mu)}{\partial x}\,du\right)\frac{\partial \Phi(s,x,\mu)}{\partial \mu}\,ds,$$

proving the result. □

Note that in the previous result the partial derivatives of Φ can be explicitly computed as

$$\frac{\partial \Phi(t,x,\mu)}{\partial x} = 0\cdot \mathbf{1}(t<t^*) - \left[\lambda(t)F_t'\left(\frac{p_1\phi(x) - D_{\vec\mu}(\vec p_2)}{\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)}}\right) \cdot \frac{p_1 C_{\vec\mu}e^{-\alpha(T-t)} - D_{\vec\mu}(\vec p_2)}{(\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)})^2}\frac{x^{Q-1}e^{-x}}{(Q-1)!}\right]\cdot \mathbf{1}(t \geqslant t^*)$$

$$\frac{\partial \Phi(t,x,\mu_0)}{\partial \mu_0} = 0\cdot \mathbf{1}(t<t^*) - \left[\lambda(t)F_t'\left(\frac{p_1\phi(x) - D_{\vec\mu}(\vec p_2)}{\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)}}\right) \cdot\left(\left(-\frac{\partial D_{\vec\mu}(\vec p_2)}{\partial \mu_0}\right)(\phi(x) - e^{-\alpha(T-t)}C_{\vec\mu}) + (p_1\phi(x) - D_{\vec\mu}(\vec p_2))e^{-\alpha(T-t)}\frac{\partial C_{\vec\mu}}{\partial \mu_0}\right) \cdot((\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)})^2)^{-1}\right]\cdot \mathbf{1}(t\geqslant t^*)$$

$$\frac{\partial \Phi(t,x,\mu_k)}{\partial \mu_k} = 0\cdot \mathbf{1}(t<t^*) - \left[\lambda(t)F_t'\left(\frac{p_1\phi(x) - D_{\vec\mu}(\vec p_2)}{\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)}}\right) \cdot\left(\left(-\frac{\partial D_{\vec\mu}(\vec p_2)}{\partial \mu_k}\right)(\phi(x) - e^{-\alpha(T-t)}C_{\vec\mu}) + (p_1\phi(x) - D_{\vec\mu}(\vec p_2))e^{-\alpha(T-t)}\frac{\partial C_{\vec\mu}}{\partial \mu_k}\right) \cdot((\phi(x) - C_{\vec\mu}e^{-\alpha(T-t)})^2)^{-1}\right]\cdot \mathbf{1}(t\geqslant t^*)$$

where $\mathbf{1}(A)$ is the indicator function $=1$ if A is true and $=0$ if A is false.

Appendix F: Proof of Proposition 2

For ease of notation we use $\mu = \mu_0$ throughout the proof, which requires three technical lemmas.

Lemma 1. $dC_\mu/d\mu \leqslant 0$.

Proof. It is intuitively clear that C_μ must be decreasing on μ, since if the expected number of customers who buy immediately at their arrival increases, then there will be fewer units left by the end of the season, and the chances of getting the item for a waiting customers decrease. More formally,

$$C_\mu = \sum_{i=0}^{\infty}\sum_{k=1}^{Q}\min\left\{1, \frac{k}{i+1}\right\}\frac{\mu^{Q-k}e^{-\mu}}{(Q-k)!}\frac{(a_\mu)^i e^{-a_\mu}}{i!}$$
$$= e^{-\mu-a_\mu}\sum_{i=0}^{\infty}\sum_{k=1}^{Q}\min\left\{1, \frac{k}{i+1}\right\}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^i}{i!},$$

where $a_\mu = \int_0^T \lambda(t)dt - \mu - \int_0^{t^*}\lambda(t)F_t(p_1)\,dt - \int_{t^*}^T \lambda(t)\cdot F_t(p_2e^{\alpha(T-t)})\,dt$, and thus $da_\mu/d\mu = -1$. Then $e^{-\mu-a_\mu} > 0$ is independent of μ. Instead of $dC_\mu/d\mu < 0$ we are going to show that $e^{\mu+a_\mu}(dC_\mu/d\mu) < 0$.

$$e^{\mu+a_\mu}\frac{dC_\mu}{d\mu}$$
$$= \sum_{k=1}^{Q-1}\sum_{i=1}^{\infty}\min\left\{1, \frac{k}{i+1}\right\} \cdot\left[\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!} - \frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!}\right] + \sum_{k=1}^{Q-1}\frac{\mu^{Q-k-1}}{(Q-k-1)!} - \sum_{i=1}^{Q-1}\frac{(a_\mu)^{i-1}}{(i-1)!} - \sum_{i=Q}^{\infty}\frac{Q}{i+1}\frac{(a_\mu)^{i-1}}{(i-1)!}$$
$$= \sum_{k=1}^{Q-1}\sum_{i=0}^{k-1}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q}\sum_{i=1}^{k-1}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!} + \sum_{k=1}^{Q-1}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!}$$

$$= \sum_{k=1}^{Q-1}\sum_{i=0}^{k-1}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!} - \sum_{k=2}^{Q}\sum_{i=1}^{k-1}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!}$$
$$+ \sum_{k=1}^{Q-1}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!}$$
$$- \sum_{k=1}^{Q}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!}$$
$$= \sum_{k=1}^{Q-1}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!}$$
$$- \sum_{k=1}^{Q}\sum_{i=k}^{\infty}\frac{k}{i+1}\frac{\mu^{Q-k}}{(Q-k)!}\frac{(a_\mu)^{i-1}}{(i-1)!}$$
$$= -\sum_{k=1}^{Q-1}\sum_{i=k}^{\infty}\frac{i+1-k}{(i+1)(i+2)}\frac{\mu^{Q-k-1}}{(Q-k-1)!}\frac{(a_\mu)^i}{i!}$$
$$- \frac{\mu^{Q-1}}{(Q-1)!}\sum_{i=1}^{\infty}\frac{1}{i+1}\frac{(a_\mu)^{i-1}}{(i-1)!} \leqslant 0. \quad \square$$

Lemma 2. *Consider $\psi^1(t)$, $\psi^2(t)$ two different equilibria of an instance were $\mu^1 > \mu^2$; then it holds that $x_{\mu^1}(t) \geqslant x_{\mu^2}(t)$ $\forall t \in [0, T]$.*

Proof. Assume $\mu^1 > \mu^2$ and recall the ODE that defines an equilibria

$$\dot{x}(t) = \begin{cases} \lambda(t)(1-F_t(p_1)) & 0 \leqslant t < t^* \\ \lambda(t)\left(1-F_t\left(\dfrac{p_1\sum_{i=0}^{Q-1}((x(t))^i e^{-x(t)}/i!) - p_2 C_\mu}{\sum_{i=0}^{Q-1}((x(t))^i e^{-x(t)})/i! - e^{-\alpha(T-t)}C_\mu}\right)\right) & t^* \leqslant t \leqslant T, \end{cases}$$
$$x(0)=0, \quad x(T)=\mu. \tag{F1}$$

Calling $\Phi(t, x, \mu)$ the right-hand side of the differential equation of (F1), we need to show $\partial\Phi(t, x, \mu)/\partial\mu \geqslant 0$. First we calculate $\partial\Phi(t, x, \mu)/\partial C_\mu$. In $[0, t^*]$ this is equal 0, whereas in $[t^*, T]$

$$\frac{\partial\Phi(t, x, \mu)}{\partial C_\mu} = -\lambda(t)F_t'\left(\frac{p_1\phi(x) - p_2 C_\mu}{\phi(x) - C_\mu e^{-\alpha(T-t)}}\right) \cdot \frac{p_1 e^{-\alpha(T-t)} - p_2}{(\phi(x) - C_\mu e^{-\alpha(T-t)})^2}\phi(x) \leqslant 0. \tag{F2}$$

By Lemma 1 we have $dC_\mu/d\mu \leqslant 0$, so we have the result. $\square$

The following is a standard result for Poisson random variables.

Lemma 3. *If N is a Poisson random variable with mean $x > 0$, then $(d/dx)P(N(x) \leqslant n) = -\mathbb{P}(N(x) = n) < 0$ for all $n \in \mathbb{Z}^+$.*

Proof of Proposition 2 Recall that $\mathbb{P}(A_t) = \sum_{k=0}^{Q-1}(x(t))^k e^{-x(t)}/k!$; then by Lemma 3, $(d/dx)\mathbb{P}(A_t) \leqslant 0$. By Lemma 2, if $\mu_1 > \mu_2$, then $x_{\mu_1}(t) \geqslant x_{\mu_2}(t)$ $\forall t \in [0, T]$; thus

$$\mathbb{P}(A_t \mid \psi^1) \leqslant \mathbb{P}(A_t \mid \psi^2). \tag{F3}$$

Also, because of Lemma 1 we have that

$$\mathbb{P}(G \mid \psi^1) \leqslant \mathbb{P}(G \mid \psi^2). \tag{F4}$$

We will prove the desired result by distinguishing six cases. Recall that threshold functions may cross each other; hence we will consider all possible cases for the customers' valuations. Consider a customer arriving at time $t \in [0, T]$ with valuation v_t.

(i) $\psi^2(t) \leqslant \psi^1(t) \leqslant v_t$. From Equation (F3) in this case it is clear that $(v_t - p_1)\mathbb{P}(A_t \mid \psi^1)$ is at most $(v_t - p_1)\cdot\mathbb{P}(A_t \mid \psi^2)$.

(ii) $\psi^2(t) \leqslant v_t < \psi^1(t)$. Since in the first equilibria customers prefer to buy immediately rather than waiting and from Equation (F3), we have that

$$(v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^1) \leqslant (v_t - p_1)\mathbb{P}(A_t \mid \psi^1) < (v_t - p_1)\mathbb{P}(A_t \mid \psi^2).$$

We conclude that the surplus in the first equilibria is less than that under ψ^2; i.e.,

$$(v_t - p_1)\mathbb{P}(A_t \mid \psi^1) < (v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^2).$$

(iii) $\min\{p_1, p_2 e^{\alpha(T-t)}\} \leqslant v_t < \psi^2(t) \leqslant \psi^1(t)$. Because of Equation (F4), customers surplus is higher in the second equilibria

$$(v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^1) \leqslant (v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^2).$$

(iv) $\psi^1(t) < \psi^2(t) \leqslant v_t$. This case is exactly as case (i); customers' surplus in the second equilibria is $(v_t - p_1)\mathbb{P}(A_t \mid \psi^1)$, which at most can be equal to customers' surplus in the first equilibria $(v_t - p_1)\mathbb{P}(A_t \mid \psi^2)$.

(v) $\psi^1(t) \leqslant v_t < \psi^2(t)$. From Equation (F3) and because in the first equilibria customers prefer to wait rather than buying immediately, we have

$$(v_t - p_1)\mathbb{P}(A_t \mid \psi^2) < (v_t - p_1)\mathbb{P}(A_t \mid \psi^1) \leqslant (v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^1).$$

We conclude that the surplus in the first equilibria is less than that under the second; i.e.,

$$(v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^1) < (v_t - p_1)\mathbb{P}(A_t \mid \psi^2).$$

(vi) $\min\{p_1, p_2 e^{\alpha(T-t)}\}p_1 \leqslant v_t < \psi^1(t)\psi^2(t)$. This case is exactly as case (iii); indeed, by Equation (F4) customer's surplus in the first equilibria $(v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^1)$ is at most customers' surplus on the second equilibria $(v_t e^{-\alpha(T-t)} - p_2)\mathbb{P}(G \mid \psi^2)$. The latter inequality is true because of Equation (F4).

We excluded the cases in which $v_t < \min\{p_1, p_2 e^{\alpha(T-t)}\}$ since the surplus is zero in both equilibria. It follows that equilibria where more consumers decide to wait (lower μ) is preferred by all customers.

Now we check that the seller's utility is larger under the first equilibrium. This utility is

$$\pi_{FPP} = p_1 \underbrace{\sum_{k=1}^{\infty} \frac{\mu^k e^{-\mu}}{k!} \min\{Q, k\}}_{A} + p_2 \underbrace{\sum_{k=1}^{Q} \sum_{i=0}^{\infty} \frac{\mu^{Q-k} e^{-\mu}}{(Q-k)!} \frac{(a_\mu)^i e^{-a_\mu}}{i!} \min\{k, i\}}_{B}.$$

$$\begin{aligned}
\frac{dA}{d\mu} &= e^{-\mu} \sum_{k=1}^{\infty} \left[\frac{\mu^{k-1}}{(k-1)!} - \frac{\mu^k}{k!}\right] \min\{Q, k\} \\
&= e^{-\mu} \sum_{k=1}^{Q-1} k \left[\frac{\mu^{k-1}}{(k-1)!} - \frac{\mu^k}{k!}\right] + e^{-\mu} \sum_{k=Q}^{\infty} Q \left[\frac{\mu^{k-1}}{(k-1)!} - \frac{\mu^k}{k!}\right] \\
&= e^{-\mu} \left[1 - (Q-1) \frac{\mu^{Q-1}}{(Q-1)!} + \sum_{k=1}^{Q-2} \frac{\mu^k}{k!}\right] + e^{-\mu} Q \frac{\mu^{Q-1}}{(Q-1)!} \\
&= \sum_{k=0}^{Q-1} \frac{\mu^k e^{-\mu}}{k!}.
\end{aligned}$$

$$\begin{aligned}
&e^{-\mu - a_\mu} \frac{dB}{d\mu} \\
&= \sum_{k=1}^{Q-1} \sum_{i=0}^{\infty} \min\{k, i\} \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q} \sum_{i=1}^{\infty} \min\{k, i\} \frac{\mu^{Q-k}}{(Q-k)!} \frac{(a_\mu)^{i-1}}{(i-1)!} \\
&= \sum_{k=1}^{Q-1} \sum_{i=0}^{k} i \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q} \sum_{i=1}^{k} i \frac{\mu^{Q-k}}{(Q-k)!} \frac{(a_\mu)^{i-1}}{(i-1)!} \\
&\quad + \sum_{k=1}^{Q-1} \sum_{i=k+1}^{\infty} k \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q} \sum_{i=k+1}^{\infty} k \frac{\mu^{Q-k}}{(Q-k)!} \frac{(a_\mu)^{i-1}}{(i-1)!} \\
&= -\frac{\mu^{Q-1}}{(Q-1)!} - \sum_{k=1}^{Q-1} \sum_{i=0}^{k} \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} \\
&\quad - \frac{\mu^{Q-1}}{(Q-1)!} \sum_{i=1}^{\infty} \frac{(a_\mu)^i}{i!} - \sum_{k=1}^{Q-1} \sum_{i=k+1}^{\infty} \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} \\
&= -\sum_{k=1}^{Q-1} \sum_{i=0}^{\infty} \frac{\mu^{Q-k-1}}{(Q-k-1)!} \frac{(a_\mu)^i}{i!} - \frac{\mu^{Q-1}}{(Q-1)!} \sum_{i=0}^{\infty} \frac{(a_\mu)^i}{i!}.
\end{aligned}$$

Then

$$\frac{dB}{d\mu} = -\sum_{k=1}^{Q-1} \frac{\mu^{Q-k-1} e^{-\mu}}{(Q-k-1)!} - \frac{\mu^{Q-1} e^{-\mu}}{(Q-1)!} = -\sum_{k=0}^{Q-1} \frac{\mu^k e^{-\mu}}{k!}$$

so that

$$\frac{d\pi_{FPP}}{d\mu} = (p_1 - p_2) \sum_{k=0}^{Q-1} \frac{\mu^k e^{-\mu}}{k!} \geqslant 0,$$

concluding that the seller's surplus is higher in equilibria number 1. ☐

Appendix G: Optimization

We now obtain an analytical expression for $\nabla \pi_{CPP}(p_1, \{p_2(k)\}_{k=1}^{Q})$. Let us call a_k the Poisson rate of the customers who may intend to buy in the second period when there are k items available at time T; more precisely,

$$a_k = \int_0^T \lambda(t)\, dt - \mu_0 - \int_0^{t^*(k)} \lambda(t) F_t(p_1)\, dt - \int_{t^*(k)}^{T} \lambda(t) F_t(p_2(k) e^{\alpha(T-t)})\, dt - \mu_k.$$

The seller problem including the consumers' response can be written as

$$\max \quad p_1 \sum_{k=0}^{\infty} \min\{k, Q\} \frac{\mu_0^k e^{-\mu_0}}{k!} + \sum_{k=1}^{Q} \sum_{i=0}^{\infty} p_2(k) \min\{i, k\} \frac{\mu_0^{Q-k} e^{-\mu_0}}{(Q-k)!} \frac{(a_k)^i e^{-a_k}}{i!} \tag{G1}$$

$$\text{s.t.} \quad x_{\vec{\mu}}(T) = \mu_0 \tag{G2}$$

$$\int_0^{t^*(k)} \min\{\lambda(t) - \dot{x}_{\vec{\mu}}(t), \lambda(t) F_t(p_2(k) e^{\alpha(T-t)})\} - \lambda(t) F_t(p_1)\, dt = \mu_k, \quad \text{for } k = 1, \dots, Q, \tag{G3}$$

where $x_{\vec{\mu}}(T)$ is the solution of Equation (3). Note that we need to compute the total derivatives of the objective function π_{CPP} with respect to prices after replacing constraints (G2) and (G3); then by the chain-rule we obtain that

$$\frac{d\pi_{CPP}}{dp_1} = \frac{\partial \pi_{CPP}}{\partial p_1} + \frac{\partial \pi_{CPP}}{\partial \mu_0} \frac{\partial \mu_0}{\partial p_1} + \sum_{k=1}^{Q} \frac{\partial \pi_{CPP}}{\partial \mu_k} \frac{\partial \mu_k}{\partial p_1}$$

$$\frac{d\pi_{CPP}}{dp_2(j)} = \frac{\partial \pi_{CPP}}{\partial p_2(j)} + \frac{\partial \pi_{CPP}}{\partial \mu_0} \frac{\partial \mu_0}{\partial p_2(j)} + \sum_{k=1}^{Q} \frac{\partial \pi_{CPP}}{\partial \mu_k} \frac{\partial \mu_k}{\partial p_2(j)}, \quad \text{for } j = 1, \dots, Q$$

To obtain explicit expressions for $\partial \mu_0 / \partial p_1$ and $\partial \mu_0 / \partial p_2(k)$ for $k = 1, \dots, Q$, we take derivative of constraint (G2) with respect to p_1 and $p_2(k)$ and identify the terms in both sides of the resulting equation.

$$\frac{\partial x_\mu(T)}{\partial \mu_0} \frac{\partial \mu_0}{\partial p_1} + \sum_{j=1}^{Q} \frac{\partial x_\mu(T)}{\partial \mu_j} \frac{\partial \mu_j}{\partial p_1} + \frac{\partial x_\mu(T)}{\partial p_1} = \frac{\partial \mu_0}{\partial p_1}$$

$$\frac{\partial x_\mu(T)}{\partial \mu_0} \frac{\partial \mu_0}{\partial p_2(k)} + \sum_{j=1}^{Q} \frac{\partial x_\mu(T)}{\partial \mu_j} \frac{\partial \mu_j}{\partial p_2(k)} + \frac{\partial x_\mu(T)}{\partial p_2(k)} = \frac{\partial \mu_0}{\partial p_2(k)}, \quad \text{for } k = 1, \dots, Q$$

Rearranging terms we obtain

$$\frac{\partial \mu_0}{\partial p_1} = \frac{\partial x_\mu(T)/\partial p_1 + \sum_{j=1}^{Q} (\partial x_\mu(T)/\partial \mu_j)(\partial \mu_j/\partial p_1)}{1 - \partial x_\mu(T)/\partial \mu_0},$$

$$\frac{\partial \mu_0}{\partial p_2(k)} = \frac{\partial x_\mu(T)/\partial p_2(k) + \sum_{j=1}^{Q} (\partial x_\mu(T)/\partial \mu_j)(\partial \mu_j/\partial p_2(k))}{1 - \partial x_\mu(T)/\partial \mu_0}.$$

Similarly, $\partial \mu_k/\partial p_1$ and $\partial \mu_k/\partial p_2(j)$, are obtained by deriving constraints (G3) with respect to p_1 and $p_2(j)$:

$$\frac{\partial \mu_k}{\partial p_1} = -\frac{\partial x_\mu(t^\diamond(k))}{\partial p_1} - \int_0^{t^*(k)} \lambda(t) F_t'(p_1)\, dt$$

$$\frac{\partial \mu_k}{\partial p_2(j)} = \left[-\frac{\partial x_\mu(t^\diamond(k))}{\partial p_2(j)} - \int_{t^\diamond(k)}^{t^*(k)} \lambda(t) F_t'(p_2(k) e^{\alpha(T-t)}) \cdot e^{\alpha(T-t)}\, dt \right] \cdot \mathbf{1}(j=k) - \frac{\partial x_\mu(t^\diamond(k))}{\partial p_2(j)} \cdot \mathbf{1}(j \neq k)$$

where $t^\diamond(k)$ is defined in the proof of Theorem 3. In case there is an intersection of $\psi(t)$ and $p_2(k)e^{\alpha(T-t)}$, this is unique since $\psi(t)$ is nondecreasing whereas$p_2(k)e^{\alpha(T-t)}$ is strictly decreasing when $\alpha > 0$. Note that for $\alpha = 0$ we impose $t^\diamond(k) = 0$ since all μ_k for $k = 1, \ldots, Q$ will be zero.

To make the gradient explicit we still need a number of expressions thatare detailed below. Note that for some we need again the integrating factor method used in Theorem 3.

$$\frac{\partial \pi_{CPP}}{\partial p_1} = \sum_{k=1}^{\infty} \frac{\mu_0^k e^{-\mu_0}}{k!} \min\{Q, k\} + \sum_{k=1}^{Q} \sum_{i=1}^{\infty} p_2(k) \min\{i, k\} \cdot \frac{\mu_0^{Q-k} e^{-\mu_0}}{(Q-k)!} \frac{(a_k)^{i-1} e^{-a_k}}{(i-1)!} \left(1 - \frac{a_k}{i}\right) \frac{da_k}{dp_1}$$

$$\frac{\partial \pi_{CPP}}{\partial p_2(k)} = \sum_{i=1}^{\infty} \min\{i, k\} \frac{\mu_0^{Q-k} e^{-\mu_0}}{(Q-k)!} \frac{(a_k)^{i-1} e^{-a_k}}{(i-1)!} \cdot \left[\frac{a_k}{i} + p_2(k)\left(1 - \frac{a_k}{i}\right) \frac{da_k}{dp_2(k)} \right]$$

$$\frac{\partial \pi_{CPP}}{\partial \mu_0} = p_1 \sum_{k=0}^{Q-1} \frac{\mu_0^k e^{-\mu_0}}{k!} + \sum_{k=1}^{Q-1} \sum_{i=1}^{\infty} p_2(k) \min\{i, k\} \cdot \frac{\mu_0^{Q-k-1} e^{-\mu_0}}{(Q-k-1)!} \frac{(a_k)^{i-1} e^{-a_k}}{(i-1)!} \left(\frac{a_k}{i} - \frac{\mu_0}{Q-k} \right) - \sum_{i=1}^{\infty} p_2(Q) \min\{i, Q\} e^{-\mu_0} \frac{(a_k)^{i-1} e^{-a_k}}{(i-1)!}$$

$$\frac{\partial \pi_{CPP}}{\partial \mu_k} = \sum_{i=1}^{\infty} p_2(k) \min\{i, k\} \frac{\mu_0^{Q-k} e^{-\mu_0}}{(Q-k)!} \frac{(a_k)^{i-1} e^{-a_k}}{(i-1)!} \cdot \left(1 - \frac{a_k}{i}\right)(-1)$$

$$\frac{dx_\mu(T)}{dp_1} = \int_0^T \exp\left(\int_s^T \frac{\partial \Phi(u, x, p_1)}{\partial x} du \right) \frac{\partial \Phi(s, x, p_1)}{\partial p_1}\, ds$$

$$\frac{dx_\mu(T)}{dp_2(k)} = \int_0^T \exp\left(\int_s^T \frac{\partial \Phi(u, x, p_2(k))}{\partial x} du \right) \cdot \frac{\partial \Phi(s, x, p_2(k))}{\partial p_2(k)}\, ds$$

$$\frac{dx_\mu(T)}{d\mu_k} = \int_0^T \exp\left(\int_s^T \frac{\partial \Phi(u, x, \mu_k)}{\partial x} du \right) \frac{\partial \Phi(s, x, \mu_k)}{\partial \mu_k}\, ds$$

$$\frac{\partial \Phi(t, x, p_1)}{\partial p_1} = -\lambda(t) F_t'(p_1) \cdot \mathbf{1}(t < t^*) - \left[\lambda(t) F_t'\left(\frac{p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)}} \right) \cdot \left(\left(\phi(x) - \frac{\partial D_{\vec{\mu}}(\vec{p}_2)}{\partial p_1} \right)(\phi(x) - e^{-\alpha(T-t)} C_{\vec{\mu}}) + (p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)) e^{-\alpha(T-t)} \frac{\partial C_{\vec{\mu}}}{\partial p_1} \right) \cdot ((\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)})^2)^{-1} \right] \cdot \mathbf{1}(t \geqslant t^*)$$

$$\frac{\partial \Phi(t, x, p_2(k))}{\partial p_2(k)} = 0 \cdot \mathbf{1}(t < t^*) - \left[\lambda(t) F_t'\left(\frac{p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)}} \right) \cdot \left(\left(-\frac{\partial D_{\vec{\mu}}(\vec{p}_2)}{\partial p_2(k)} \right)(\phi(x) - e^{-\alpha(T-t)} C_{\vec{\mu}}) + (p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)) e^{-\alpha(T-t)} \frac{\partial C_{\vec{\mu}}}{\partial p_2(k)} \right) \cdot ((\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)})^2)^{-1} \right] \cdot \mathbf{1}(t \geqslant t^*)$$

$$\frac{\partial \Phi(t, x, \vec{\mu})}{\partial x} = 0 \cdot \mathbf{1}(t < t^*) - \left[\lambda(t) F_t'\left(\frac{p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)}} \right) \cdot \frac{p_1 C_{\vec{\mu}} e^{-\alpha(T-t)} - D_{\vec{\mu}}(\vec{p}_2)}{(\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)})^2} \frac{x^{Q-1} e^{-x}}{(Q-1)!} \right] \cdot \mathbf{1}(t \geqslant t^*)$$

$$\frac{\partial \Phi(t, x, \mu_0)}{\partial \mu_0} = 0 \cdot \mathbf{1}(t < t^*) - \left[\lambda(t) F_t'\left(\frac{p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)}} \right) \cdot \left(\left(-\frac{\partial D_{\vec{\mu}}(\vec{p}_2)}{\partial \mu_0} \right)(\phi(x) - e^{-\alpha(T-t)} C_{\vec{\mu}}) + (p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)) e^{-\alpha(T-t)} \frac{\partial C_{\vec{\mu}}}{\partial \mu_0} \right) \cdot ((\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)})^2)^{-1} \right] \cdot \mathbf{1}(t \geqslant t^*)$$

$$\frac{\partial \Phi(t, x, \mu_k)}{\partial \mu_k} = 0 \cdot \mathbf{1}(t < t^*) - \left[\lambda(t) F_t'\left(\frac{p_1 \phi(x) - D_{\vec{\mu}}(\vec{p}_2)}{\phi(x) - C_{\vec{\mu}} e^{-\alpha(T-t)}} \right) \right.$$

$$\cdot\left(\left(-\frac{\partial D_{\vec{\mu}}(\vec{p}_2)}{\partial \mu_k}\right)(\phi(x)-e^{-\alpha(T-t)}C_{\vec{\mu}}) + (p_1\phi(x)-D_{\vec{\mu}}(\vec{p}_2))e^{-\alpha(T-t)}\frac{\partial C_{\vec{\mu}}}{\partial \mu_k}\right) \cdot ((\phi(x)-C_\mu e^{-\alpha(T-t)})^2)^{-1}\Bigg]\cdot \mathbf{1}(t\geqslant t^*)$$

$$\frac{da_k}{dp_1}=-\int_0^{t^*}\lambda(t)F_t'(p_1)\,dt$$

$$\frac{da_k}{dp_2(j)}=\begin{cases}-\int_{t^*}^{T}\lambda(t)F_t'(p_2(j)e^{\alpha(T-t)})e^{\alpha(T-t)}\,dt & j=k\\ 0 & j\neq k\end{cases}$$

$$C_{\vec{\mu}}=\sum_{i=0}^{\infty}\sum_{k=1}^{Q}\min\left\{1,\frac{k}{i+1}\right\}\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\frac{(a_k)^i e^{-a_k}}{i!}$$

$$\frac{dC_{\vec{\mu}}}{dp_1}=\sum_{k=1}^{Q}\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(1-\frac{a_k}{i}\right)\frac{da_k}{dp_1}-\sum_{k=1}^{Q}\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}e^{-a_k}\frac{da_k}{dp_1}$$

$$\frac{dC_{\vec{\mu}}}{dp_2(k)}=\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(1-\frac{a_k}{i}\right)\frac{da_k}{dp_2(k)}-\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}e^{-a_k}\frac{da_k}{dp_2(k)}$$

$$\frac{dC_{\vec{\mu}}}{d\mu_0}=\sum_{k=1}^{Q-1}\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{\mu_0^{Q-k-1}e^{-\mu_0}}{(Q-k-1)!}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(\frac{a_k}{i}-\frac{\mu_0}{Q-k}\right)+\sum_{k=1}^{Q-1}\frac{\mu_0^{Q-k-1}e^{-\mu_0}}{(Q-k-1)!}e^{-a_k}-\sum_{i=1}^{\infty}\min\left\{1,\frac{Q}{i+1}\right\}\frac{(a_Q)^{i-1}e^{-a_Q}}{(i-1)!}e^{-\mu_0}$$

$$\frac{dC_{\vec{\mu}}}{d\mu_k}=\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\cdot\left[e^{-a_k}+\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\left(\frac{a_k}{i}-1\right)\right]$$

$$D_\mu=\sum_{i=0}^{\infty}\sum_{k=1}^{Q}p_2(k)\min\left\{1,\frac{k}{i+1}\right\}\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\frac{(a_k)^i e^{-a_k}}{i!}$$

$$\frac{dD_{\vec{\mu}}}{dp_1}=\sum_{k=1}^{Q}p_2(k)\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(1-\frac{a_k}{i}\right)\frac{da_k}{dp_1}-\sum_{k=1}^{Q}p_2(k)\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}e^{-a_k}\frac{da_k}{dp_1}$$

$$\frac{dD_{\vec{\mu}}}{dp_2(k)}=\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\left[\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(\frac{a_k}{i}+p_2(k)\left(1-\frac{a_k}{i}\right)\frac{da_k}{dp_2(k)}\right)+e^{-a_k}\left(1-p_2(k)\frac{da_k}{dp_2(k)}\right)\right]$$

$$\frac{dD_{\vec{\mu}}}{d\mu_0}=\sum_{k=1}^{Q-1}\sum_{i=1}^{\infty}p_2(k)\min\left\{1,\frac{k}{i+1}\right\}\frac{\mu_0^{Q-k-1}e^{-\mu_0}}{(Q-k-1)!}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\cdot\left(\frac{a_k}{i}-\frac{\mu_0}{Q-k}\right)+\sum_{k=1}^{Q-1}p_2(k)\frac{\mu_0^{Q-k-1}e^{-\mu_0}}{(Q-k-1)!}e^{-a_k}-\sum_{i=1}^{\infty}p_2(Q)\min\left\{1,\frac{Q}{i+1}\right\}\frac{(a_Q)^{i-1}e^{-a_Q}}{(i-1)!}e^{-\mu_0}$$

$$\frac{dD_{\vec{\mu}}}{d\mu_k}=p_2(k)\frac{\mu_0^{Q-k}e^{-\mu_0}}{(Q-k)!}\cdot\left[e^{-a_k}+\sum_{i=1}^{\infty}\min\left\{1,\frac{k}{i+1}\right\}\frac{(a_k)^{i-1}e^{-a_k}}{(i-1)!}\left(\frac{a_k}{i}-1\right)\right]$$

Endnotes

1. Through their *On the Counter* website http://www.landsend.com/otc/index.html.
2. The conditions for equilibrium specified for Osadchiy and Vulcano (2010) can be tough to satisfy. Indeed, for the example in their paper uniqueness can be guaranteed for only about 10% of the cases.
3. Note that from now on we use $\{p_2(k)\}_{k=1}^{Q}$ in addition to $p_2(Q_T)$ to emphasize the dependency of p_2 on the number of items remaining. Where not needed we just use p_2.
4. More precisely, in the model of Aviv and Pazgal (2008) there is a second period of positive length starting at T.
5. The only difference between our modeling approach for *FPP* and Osadchiy and Vulcano's is that for a buyer arriving at time t, with valuation v_t, and buying at time τ at price p, they consider the utility function $(v_t-p)e^{-\alpha(\tau-t)}$, whereas we consider it as $v_t e^{-\alpha(\tau-t)}-p$. However, this slight difference does not affect the implied results.
6. It is easy to observe that if we want $p_2=1$ we need to set $p_1=1.25$ to induce high valuation customers to buy upon arrival, which leads to a revenue of 2.
7. The results are robust under different distributions of customers' valuations.

References

Aviv Y, Pazgal A (2008) Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing Service Oper. Management* 10(3):339–359.
Bansal M, Maglaras C (2009) Dynamic pricing when customers strategically time their purchase: Asymptotic optimality of a two-price policy. *J. Revenue Pricing Management* 8(1):42–66.
Bass F (1969) A new product growth for model consumer durables. *Management Sci.* 15(5):215–227.
Bass F (2004) Comments on "A new product growth for model consumer durables the bass model." *Management Sci.* 50(12 supplement): 1833–1840.
Bell DE, Starr D (1993) (Revised 1998) Filene's Basement. Harvard Business School Case 594-018, Cambridge, MA.
Board S, Skrzypacz A (2015) Revenue management with forward-looking buyers. *J. Political Econom.* Forthcoming.
Cachon G, Feldman P (2010) Dynamic versus static pricing in the precense of strategic consumers. Working paper, The Wharton School, University of Pennsylvania, Philadelphia.

Cachon G, Swinney R (2009) Purchasing, pricing, and quick response in the presence of strategic consumers. *Management Sci.* 55(3):497–511.

Chintagunta P, Jain D (1992) A dynamic model of channel member strategies for marketing expenditures. *Marketing Sci.* 11(2):168–188.

Coase R (1972) Durability and monopoly. *J. Law Econom.* 15(1):143–49.

Crapis D, Ifrach B, Maglaras C, Scarsini M (2015) Monopoly pricing in the presence of social learning. Working paper, Columbia Business School, Columbia University, New York.

Dasu S, Tong C (2010) Dynamic pricing when consumers are strategic: Analysis of posted and contingent pricing schemes. *Eur. J. Oper. Res.* 204(3):662–671.

Elmaghraby W, Gülcü A, Keskinocak P (2008) Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *Manufacturing Service Oper. Management* 10(1):126–148.

Elmaghraby W, Lippman S, Tang C, Yin R (2009) Pre-announced pricing strategies with reservations. *Production Oper. Management* 18(4): 381–401.

Gershkov A, Moldovanu B, Strack P (2014) Revenue maximizing mechanisms with strategic customers and unknown demand: Name-your-own-price. Working paper, Hebrew University of Jerusalem, Israel.

Jerath K, Netessine S, Veeraraghavan SK (2010) Revenue management with strategic customers: Last-minute selling and opaque selling. *Management Sci.* 56(3):430–448.

Jorgensen S, Zaccour G (2007) Developments in differential game theory and numerical methods: Economics and management applications. *Computational Management Sci.* 4(2):159–181.

Hale JK (1980) *Ordinary Differential Equations* (Robert E. Krieger Publishing, Huntington, NY), 46.

Kanoria Y, Nazerzadeh H (2014) Dynamic reserve prices for repeated auctions: Learning from bids. Working paper, Columbia Business School, Columbia University, New York.

Kincaid WM, Darling DA (1963) An inventory pricing problem. *J. Math. Anal. Appl.* 7(2):183–208.

Kuchment P (2013) A survey of main theorems on ODEs. Lecture notes http://www.math.tamu.edu/~kuchment/ode.pdf.

Lazear EP (1986) Retail pricing and clearance sales. *Amer. Econom. Rev.* 76(1):14–32.

Levin Y, McGill J, Nediak M (2010) Optimal dynamic pricing of perishable items by a monopolist facing strategic consumers. *Production Oper. Management* 19(1):40–60.

Li J, Granados N, Netessine S (2014) Are consumers strategic? Structural estimation from the air travel industry. *Management Sci.* 60(9): 2114–2137.

Liu Q, van Ryzin G (2008) Strategic capacity rationing to induce early purchases. *Management Sci.* 54(6):1115–1131.

Liu Q, van Ryzin G (2011) Strategic capacity rationing when customers learn. *Manufacturing Service Oper. Management* 13(1):89–107.

Maglaras C, Meissner J (2006) Dynamic pricing strategies for multiproduct revenue management problems. *Manufacturing Service Oper. Management* 8(2):136–148.

Naik P, Raman K, Winer R (2005) Planning marketing-mix strategies in the presence of interaction effects. *Marketing Sci.* 24(1):25–34.

Osadchiy N, Vulcano G (2010) Selling with binding reservations in the presence of strategic consumers. *Management Sci.* 56(12):2173–2190.

Rahmandad H, Sterman J (2008) Heterogeneity and network structure in the dynamics of diffusion: Comparing agent-based and differential equation models. *Management Sci.* 54(5):998–1014.

Su X, Zhang F (2009) On the value of commitment and availability guarantees when selling to strategic consumers. *Management Sci.* 55(5):713–726.

Sun B, Neslin S, Srinivasan K (2003) Measuring the impact of promotions on brand switching under rational consumer behavior. *J. Marketing Res.* 40(4):389–405.

Talluri K, van Ryzin G (2005) *The Theory and Practice of Revenue Management* (Springer, New York).

Tereyagoglu N, Fader P, Veeraraghavan S (2012) Filling seats at a theater: Estimating the impact of posted prices and dynamic discounts. Working paper, The Wharton School, University of Pennsylvania, Philadelphia.

Yin R, Aviv Y, Pazgal A, Tang CS (2009) Optimal markdown pricing: Implications of inventory display formats in the presence of strategic customers. *Management Sci.* 55(8):1391–1408.

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