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RESIDENTIAL SEGREGATION: A PERVASIVE CONSEQUENCE OF SPATIAL EXTERNALITIES

TESIS PARA OPTAR AL GRADO DE MAGÍSTER EN ECONOMÍA APLICADA

MEMORIA PARA OPTAR AL TÍTULO DE INGENIERO CIVIL MATEMÁTICO

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El presente trabajo tiene por objetivo proveer una herramienta simple, pero rigurosa, que permita estudiar el fenómeno de segregación residencial. Para ello se desarrolla un modelo de equilibrio general en el cual un continuo de hogares escogen óptimamente un nivel de consumo privado y un lugar para vivir. La particularidad del marco de trabajo es que la ciudad es continua y los agentes, al estar divididos en distintos tipos, se ven afectados por externalidades espaciales generadas por quién compone su vecindario. Es decir, los individuos se involucran en relaciones que dependen de la identidad y la distancia de sus vecinos, que afectan el precio del mercado inmobiliario, pero que son externas a este. El principal resultado del trabajo muestra la íntima conexión existente entre externalidades espaciales y segregación residencial. La menor discrepancia en las preferencias por estas externalidades, independientemente de su naturaleza, trae como consecuencia segregación entre los grupos. Luego, al aplicar el modelo se obtienen varias lecciones tanto sobre segregación socioeconómica como racial. En cuanto a la primera, se demuestra que la excesiva concentración de inversión en bienes públicos locales desencadena una gentrificación del vecindario. Y en cuanto a la segunda, el mayor aporte es que se logra extender los resultados clásicos de Thomas Schelling a un modelo con mercados y precios. Finalmente, se analiza cómo el comportamiento de precios deja pistas, en el borde de los vecindarios segregados, sobre las preferencias de los individuos.

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The main objective of our work is to provide a simple,but rigorous tool to analyze the phenomenon of residential segregation. In order to fo so, we develop a general equilibrium model in which a continuum of agents optimally choose a level of private consumption and a place to live. The relevant feature of our framework is that the city is continuous and the agents, who are divided into different types, face spatial externalities generated by who surrounds them. In other words, individuals get involved in relations that depend on the distance and the identity of their neighbors, that affect the housing market prices, but that are external to it. The main result of our work shows the oervasive connection between spatial externalities and residential segregation. Just a tiny discrepancy in preferences for these externalities, regardless of its nature, results in segregation between groups. Then, we apply the model to different scenarios, obtaining valuable lessons about socioeconomic and racial segregation. For the first, we show that an excesive concentration of placed-based public investment triggers a gentrification of the neighborhood. And for the second, the main contribution is that we extend the classic results of Thomas Schelling to a model with markets and prices. Finally, we analyze how the behavior of prices provides clued, spacially at the border of neighborhoods, about individuals' preferences.

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Chapter 1

Introduction

It has been extensively studied how residential segregation negatively influences minorities. For example, individuals who live in poor segregated areas show lower educational attainments and have lower earnings expectations. As first exposed by Kain [18], and then revisited by Boustan & Margob [9], there is evidence that black neighborhoods tend to be located far away from employment opportunities. Housing segregation also has been recognized as a fundamental cause of racial disparities in health[1]. For these and many other of its consequences, this issue is one of the main topics in urban economics[2]. However, its causes are still a matter of debate, because there is a plenty of different forces that may influence segregation, but it is very difficult to point out the most important one. Historically, white collective discrimination towards black population have been remarked as a crucial factor for segregation; and despite the institutional changes in favor of racial equality, Ondrich, Ross & Yinger [25] found evidence that black households still face discrimination form real state agents. But, on the other hand, Thernstrom & Thernstrom [34] showed data supporting the hypothesis that segregation emerges simply because blacks prefer to self-segregate. But to these two opposing theories it should be also added the postulate, developed by Schelling [31, 32] in his famous tipping models, that says that segregation may be caused by uncoordinated individual actions that trigger massive movements away from a neighborhood. And even more explanations could be mentioned. The phenomena of socioeconomic segregation, treated in several papers like in Guerrieri, Hartley & Hurst [13], is another important factor because it shows that segregation could be driven simply by income inequality rather than racial attitudes.

In this context, with the objective of providing a rigorous framework to analyze the causes of residential segregation, we develop a general equilibrium model in which a continuum of households populate a continuous city, choosing optimally between their location and a level of private consumption. The particularity of our environment is that we leave aside urban variables, such as local amenities or commutation costs, and we concentrate in the housing market and how preferences for neighborhood composition influence it. In our model, this market is supplied by absentee landlords, who are collapsed in a unique competitive firm that

[1]See, for example, Williams & Collins [36].

[2]We highlight Boustan [8], who wrote a great essay analyzing social and economic consequences of segregation, and suits perfectly as an introduction to this matter.

faces increasing costs related to the housing density they provide. In addition, we incorporate differences between households by dividing them into disjoint groups, each of which has its own preferences for the type of neighbors surrounding each location. We study then if those groups tend to separate and agents of similar characteristics allocate together in equilibrium.

The most important feature of our model is that we conceive the city as a connected entity with rather smooth neighborhoods and rich interactions at the borders. This goes in the opposite direction of other wokrs, like Banzhaf & Walsh [4] and Somanathan & Sethi [33], who think the city as divided into isolated jurisdictions that do not interact with each other. And it separates from the urban model of Yinger [37], who does not take advantage of the continuity of the city. This approach let us incorporate spatial externalities that depend on the identity and the distance between neighbors, and that influence individuals' decisions. Also it let us analyze how agents and prices behave at the boundary between neighborhoods, which is relevant because it provides variability to neighborhood composition, that would be constant within each neighborhood if the city was composed by independent jurisdictions. Taking seriously the continuity of the city is an important novelty of our framework because there is recent evidence, found by Rossi-Hansberg, Sarte & Owens III [28], that there are non market interactions related to the location of a house that affect its price. And as far as we know, there are no other papers that treat the problem of residential segregation as a consequence of housing externalities generated by the characteristics of different groups of households, with the generality that we do.

And the relevance of a connected city becomes clear in Theorem 1 that states that if groups have different preferences for the spatial externalities generated by their neighbors, in equilibrium they segregate or they coexist in perfect integration, regardless of the nature of the difference in preferences. In other words, it eliminates the possibility of different mixed nieghborhoods in equilibrium when households do not coincide in their preferences. Despite its apparent technicality this result is driven by a basic economic principle. In equilibrium, a house is assigned to the type of agent that values it the most. So integrated areas are composed by houses that all groups value equally. But in a continuous city, the type of neighbors changes as one moves trough the city, so that every house has distinct characteristics. Therefore, any discrepancy in preferences makes impossible to sustain integrated neighborhoods because one of the groups is willing to pay more for that area, and they segregate there. The only exception to this rule is when every house has the same characteristics, and that case corresponds to a perfectly integrated city where every location has the same proportion of agents. It might be the main contribution of our work because it sheds lights on how pervasive the problem of residential segregation is when housing externalities are incorporated into the analysis. The divergence in preferences could be a discrepancy in the sign, as in the case of racial segregation where agents prefer areas with alike households. Or in the quantity, as in the case of socioeconomic segregation, where both groups prefer to live with richer households, but one has more wealth to spend on this item than the other. But it could also be a combination of both, and many other effects.

Even though Theorem 1 is a strong result, it does not discard the possibility of perfect integration, nor it says which of the infinitely many distinct patterns of segregation is more likely to emerge in equilibrium. For that reason, we develop the notion of equilibria stability, as a complementary tool to determine, within the set of equilibrium allocations, which is more robust. To do so we adapt the classic theory of *tâtonnement stability*, that is based on the principle of supply and demand, to our particular set-up. The main difficulty is that the housing market is actually represented by infinite markets, for which each agent consumes exactly one unit of only one good. Therefore the standard method, first taken by Samuelson [30], of expressing the dynamics of the economy as a system of differential equations for the excess of demand function is not possible, because in our model the excess of demand cannot be simply expressed as a function. Instead of that, we based our extension of stability on the neighborhood tipping models of Schelling [31, 32] who describes dynamics by introducing exogenous migrations of agents into a neighborhood and analyzing how it affects equilibrium. The main result of this section is Theorem 2 that gives a method to determine whether an equilibrium is stable based on the derivatives of preferences for neighborhood composition at mixed and frontier locations. Interestingly, this result is closely related to the well known characterization of stable equilibria in terms of the derivative of the excess of demand function, tightening the link between tâtonnement stability theory and our framework. And as corollaries we obtain a method to check if the perfect integrated equilibrium is stable, and we conclude that a completely segregated allocation is always stable if it part of equilibria.

For the second part of this work we apply our model to different scenarios in which residential segregation may arise, considering specifically two main themes: socioeconomic and racial segregation. For the first one we assume that one type of households produces positive spatial externalities to all agents, so we call them the *leading group*. Besides predicting segregation, we are able to analyze how placed based investments may cause gentrification. Our main result is that spreading the public expenditure, instead of concentrating it on a point, is a possible way to avoid gentrification. For example, investing in a too expensive hospital may encourage rich households to move nearby, whereas two separated smaller ones does not, because the distance between them disperse the benefits.

For the second theme, which is governed by homophily, the first contribution is that we recover the classical results of Schelling, but in a more rigorous and economically founded framework. In his models, there are no markets nor prices, so households behave according to a predetermined algorithm. The main drawback of his set up, is that agents cannot outbid others for more appealing locations, but instead they are limited to the available empty slots. However, his contributions had been very influential due to their significance and simplicity. In Schelling's spatial proximity model [31], he proves that segregation may arise even when agents prefer integrated neighborhoods, simply because they do not want to be minority. In Proposition 4.3.3 we extend this result and reveal two crucial dimensions of households' minority aversion that cannot be captured in the former framework: how it is compared to the utility of private consumption, and how it is compared to the others' willingness to pay. And in Schelling's tipping model [31], he shows how individual movements of intolerant agents may trigger uncoordinated massive movements of more tolerant households away from a neighborhood, due to the change in neighborhood composition that the first action causes. This theory have drawn attention recently because of empirical evidence that whites households show tipping-like behavior, found by Card, Mas & Rothstein [11]. In Section 4.4 we reconcile this finding with our model and argue that it could be a consequence of indifferent agents who have a threshold tolerance level, beyond which they prefer to abandon their neighborhood.

Also our model let us clarify two topics directly related to racial segregation. First we obtain a couple of results that suggest a change in the standard approach for testing the black self-segregation hypothesis. This theory says that blacks' tendency to self segregate plays a major role in explaining the whole problem of racial segregation. It is an important issue because it have had evidence supporting both sides. Specifically, we show that whites' preferences should be taken into account when testing this hypothesis, because they drastically change the impact of black' preferences. And then we prove that how black households rank neighborhood composition does not matter. Instead, we demonstrate that how unwilling to be minority they are influences equilibrium. Disregarding any of these two findings may generate biased estimations. Secondly, we study the counterintuitive negative correlation that have been found between racial inequality and segregation. We demonstrate that whites' desire to integrate could be an alternative explanation for this puzzle. When whites are tolerant and significantly richer, a segregated distribution cannot exist in equilibrium. Frontier locations are more valued by white households because those locations are more heterogeneous; and then whites outbid blacks for they side of the border, which contradicts equilibrium. This integrated scenario disappears as racial inequality diminishes because, when wealthier, blacks can afford locations near the boundary.

Finally, the continuity of our set up let us exploit price behavior at the boundary of segregated areas to make inferences about individuals' preferences for neighborhood composition. Inside segregated neighborhoods, the composition is almost constant and predominately of only one type; but near the frontier it has variation. So if households care about the characteristics of their neighbors, these changes should be reflected in housing market prices. When agents prefer integrated neighborhoods, prices should be higher at the frontier; when they prefer the opposite, prices should behave contrariwise. And when there is a leading group generating positive spatial externalities, prices should strictly increase to the direction of their neighborhood. This argument stresses the relevance of agents having a composite and smooth neighborhood perception. When a 'next-door' type of neighborhood is considered, such as in Yinger model, there are no transition effects, but a discontinuity at the frontier. And when a model of isolated jurisdiction is utilized, as is common in the literature, it is impossible to study prices at the frontier. Therefore, our work may be a contribution to hedonic regressions, which are important for identifying the relevant sources of housing segregation.

The rest of this work organizes as follows. In Chapter 2 we present the theoretical model. In Section 2.1 we give its general set-up, in Section 2.2 we introduce definitions related to neighborhood composition, in Section 2.3 we state the equilibrium conditions and in Section 2.4 we derive some general properties of equilibria. In Chapter 3 we present our main theoretical results. In Section 3.1 we show that segregation may be driven by subtle differences in individuals' preferences, and in Section 3.2 we develop the theory of equilibria stability. In Chapter 4 we apply our model to different scenarios. In Section 4.1 we explain the methodology, in Section 4.2 we study what happens when one group generates positive spatial externalities, in Section 4.3 when there is homophily, in Section 4.4 we discuss the importance of indifference in preferences and in Section 4.5 we analyze the behavior of prices in equilibrium and how it could be used to make inferences about households' preferences. Finally in Chapter 5 we present some conclusions and extensions. All proofs are placed in the Appendix.

Chapter 2

The Model

2.1 Set Up

In order to study analytically the phenomenon of residential segregation, we set up a general equilibrium model in which a continuum of households H optimally choose a unique place to live in a continuous city X. The city is represented by the unit segment $X = [0, 1]$ with its standard topology. To incorporate differences between agents, the households space is a disjoint union of n subspaces $H = \bigsqcup \Theta_i$, doted with a measure function ν. Each of these sets represent different types of households, each of which has a mass β_i, but in total they have unit measure; i.e. $\nu(\Theta_i) = \beta_i$ and $\nu(H) = \sum \beta_i = 1$. For example, one can take $\Theta_i = [0, \beta_i]$ with its standard measure, as a model for the households space. Each type of agents may have distinct properties such as diverse preferences. This is a key feature of our model, since it makes it possible to include prejudice or externalities from one type of household, which may be causes of residential segregation, as demonstrated later. Although a more general case could be studied, we stick to the case of only two types of agents throughout this work.

Each household has an exogenously determined income y, the same for every group. That wealth can be spent in two different markets: private consumption and housing. The first one consists of a composite budget, represented by a unique numeraire good c, which is provided by a perfectly elastic supply. This can be thought as an imported good in an open economy. To avoid carrying a constant, the price in this market is normalized to $P_c = 1$. The second one is more complex and relevant for the analysis as it determines how households allocate throughout the city.

On the one hand, the demand side of the housing market can be described by *sorting functions* $x_i : \Theta_i \longrightarrow X$ $i = 1, 2$, for which $x_i(h) \in X$ represents where household $h \in \Theta_i$ chooses to live. If x_i is measurable, then it can be seen as a random variable from Θ_i to X, and therefore it induces a law over the city described by the push-forward measure $\nu \circ x_i^{-1}$. This measure in turn can be represented as usual by a distribution function

$$\begin{aligned} D_i : X &\longrightarrow [0, \beta_i] \\ t &\longmapsto D_i(t) = \nu\left(\{h \in \Theta_i \mid x_i(h) \leq t\}\right) \end{aligned} \tag{2.1}$$

It represents the mass of Θ_i over X, so we call it the *distribution* of type i households throughout the city. Notice that in general for a given distribution there might be infinitely many a.e-different allocation functions that induce the same law over the city. But in our model agents are indistinguishable one from the other, so we only focus in their distribution and forget about the underlying allocation function. Also, we point out in advance that in equilibrium D_i will be absolutely continuous with respect to the Lebesgue measure, hence it is possible to take a function d_i defined almost everywhere such that $d_i = D_i'$, and define it as the *density* of type i households. The existence of this function is crucial in the next section, when the neighborhood composition is defined.

It is worth noting that, by the definition of the allocation function, nothing prevents two or more households to live at the same place, furthermore, it will be the case of some equilibria. We allow this to happen because otherwise there is no natural way to define integration in the continuum case. This last statement comes from the Lebesgue's density theorem. It is a special case of the Lebesgue's differentiation theorem[1] and basically states that for every subset A of a $\mathbb{R}^n$ the metric density of A exists and is equal to the indicator function of A almost everywhere. More precisely, let dx be the Lebesgue measure, $A \subseteq \mathbb{R}$ any measurable set, then for almost every point $x \in \mathbb{R}$

$$\mathbb{1}_A(x) = \lim_{\epsilon \to 0^+} \frac{dx\,(A \cap (x-\epsilon, x+\epsilon))}{2\epsilon}$$

where $\mathbb{1}_A$ represents the characteristic function of the set A. This result implies that if one let households of type 1 and 2 occupy disjoint areas, then for a sufficiently small neighborhood one would see segregation almost everywhere. Hence $x \in X$ should be interpreted as part of a continuous neighborhood in which there are densities of both types of agents, rather than a separated building occupied by a single household.

On the other hand, the supply side of the market is assumed to be provided by absentee landlords who face increasing and convex costs related to how dense is the land they offer for rent. In a more general set up the city could be divided into disjoint segments each owned by a single proprietor, but because in this model agents behave competitively we treat them as a single firm that provides a density $d^s(x)$ of housing at each location $x \in X$. To do so, it must incur in a cost $\int_X C(d^s(x))\ dx$. This means that when the *rent price function* is described by $R : X \longrightarrow \mathbb{R}_+$, landlords' utilities for building housing density d^s in the city is

$$\pi(d^s, R) = \int_X R(x)d^s(x) - C(d^s(x))\ dx \tag{2.2}$$

where $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is assumed to be increasing, convex and $C(0) = 0$. We anticipate that in equilibrium landowners may earn positive profits, and because they are not considered part of the city this welfare is taken away form households.

For the set up of this model we base on the classical urban model of Alonso [2], but with the crucial difference that we are not particularly interested in the costs related to commutation or other source of urban externalities related to the producers. Instead, we focus on neighbors interactions and the effect that the composition of a neighborhood might have on individuals' preferences. Consequently, we leave aside some common features of

[1] For a generalization and a proof see for example Theorem 7.7 of Rudin (1987) [29].

urban models, such as the central business district; so we define preferences in terms only of the allocation of households. In general, they can be represented by a utility function $U_i(c, x, D)$, different between groups. We suppose that U satisfies the Inada conditions in the first variable[2]. However, for the purpose of incorporating different situations U may take any form in the second and third variable. Particularly, we are concerned about positive or negative effects in sharing neighborhood with other type of people.

2.2 Neighborhood Composition

Our key assumption is that the distribution of households generates spatial externalities so it determines how agents perceive their neighborhood, then it is crucial to establish the way by which D affects households' utility. When the city is analyzed separately point by point, the proportion of type i households at each location $x \in X$ is given by $\gamma_i(x) = {}^{d_i(x)}/_{d_1+d_2(x)}$. Even though it is not obvious that d_i exists, recall we already said in advance that it is indeed well defined in equilibrium, and thus we can use it safely to define other quantities. This function γ_i was used by Yinger [37] to derive racial segregation in an urban model, and it represents the infinitesimal neighborhood composition at each point, or in Yinger's words: the type of their 'next-door neighbors'. The problem with this quantity is that it is not a good measure if one wants to take advantage of the continuity of the city. When choosing neighborhoods, agents should care about the composition of their vicinity, not only about the point at where they live. Even though Yinger explained how a smoother measure of the proportion of the population around a point could be obtained, he discarded this possibility arguing intractability for the problem. However, we are not interested in building a precise urban model, nor in obtaining explicit expression for allocations or prices, but rather in describing qualitative properties. For that reason, to define the neighborhood composition function Γ_i we are going to take a weighted average of γ_i around x.

Informally, we want to define a function Γ_i as the convolution of γ_i and a weight function w. Rossi-Hansberg, Sarte & Owens III [28] found empirical evidence that housing externalities decline exponentially with distance; and therefore they take $w(y) = e^{\delta|y|}$ as the weight function for their model, where δ is the exponential rate of decline. Their choice shows two natural properties that any weight should satisfy. First, it has to be a pair function because there are no reasons to believe that households would put more weight on one side than the other. Secondly, $w(y)$ should be increasing for $y < 0$, and decreasing for $y > 0$, so that agents are more interested in the proportion of households at locations near them. But we also want households to be able to distinguish perfectly the local proportion of households at each point just by looking at the composition function. More precisely, we ask Γ_i to be an injective transformation from $L^\infty(a, b)$ to $L^\infty(0, 1)$ for any interval $[a, b]$.

The last one is a rather strong assumption, but it is important to capture the idea that agents do not perceive different neighborhoods equally. Lets formally define

$$\Gamma_i(x) = K(x) \int_X w(x-y)\gamma_i(y)\, dy \tag{2.3}$$

[2] That is $U_c > 0$, $U_{cc} < 0$, $\lim_{c\to 0} U_c = \infty$ and $\lim_{c\to\infty} U_c = 0$.

where $K(x) = (\int_X w(x-y)\, dy)^{-1}$ is a normalizing constant. To visualize the importance of the injectiveness of Γ_i, lets take the function considered by Rossi-Hansberg, Sarte & Owens III, namely $w(y) = e^{|y|}$, as the weight function and choose any points $0 < a < b < 1$. Then take r_1, r_2 such that $r_1 = \ln\left(\frac{1+e^a}{2}\right)$ and $r_2 = -\ln\left(\frac{e^{-b}+e^{-1}}{2}\right)$. And finally consider the allocation where type 1 households are segregated in the intervals $(0, r_1)$ and (b, r_2), type 2 in (r_1, a) and $(r_2, 1)$, and (a, b) is mixed in proportion 1:1. Notice that for this allocation, $\gamma_1 = \mathbb{1}_{(0,r_1)\cup(b,r_2)} + \frac{1}{2}\mathbb{1}_{(a,b)}$, where $\mathbb{1}_A$ represents the characteristic function of the set A. A direct computation shows that $\forall x \in (a, b)$ $\Gamma_1(x) = 1/2$, which says that agents perceive exactly the same neighborhood composition in every location of (a, b) even though they are qualitatively very different. We want to avoid these pathological situations, so we focus only on weight functions that make Γ injective. However, we would also like to recover the empirical evidence of exponential decline in housing externalities reported in the literature.

Fortunately, a similar problem arises in the field of statistics in both estimation and testing theory. In this literature, a family of measures $\mathcal{P}_\Omega = \{P_\omega \mid \omega \in \Omega\}$ on a measurable space is said to be *complete* if for any measurable function ϕ, $\int \phi\, dP_\omega = 0$ $\forall \omega \in \Omega$ implies $\phi = 0$ almost everywhere, and *boundedly complete* if the same is true but only for bounded functions. Observe that our weight function w can be interpreted as the density function of a measure M_w, and then one can consider the location family $\mathcal{M}_{[a,b]}$ generated by the density functions $\{w(\,\cdot\, - x)\}_{x\in[a,b]}$. Therefore, the problem of injectivity of Γ is reduced to the bounded completeness of $\mathcal{M}_{[a,b]}$. We borrow this term from the literature and say that Γ is *complete* if this property is satisfied for every interval $[a, b]$.

The seminal papers of Lehmann & Scheffé [20, 21] provide a useful theorem to find exponential families of complete measures. As a corollary of this theorem, the normal distributions are known to be complete in the sense discussed above. Supported by their results we take our weight function w to be normal, and define Γ_i as in equation (2.3). This approach let us be partially consistent with the empirical evidence, but also it let us tackle the problem of injectivity of neighborhood perception.

Definition 2.2.1. Given $\sigma > 0$, we define the *neighborhood composition function* $\Gamma_i : X \longrightarrow [0, 1]$ as

$$\Gamma_i(x) = K(x) \cdot \int_0^1 e^{\frac{-(y-x)^2}{2\sigma^2}} \gamma_i(y)\, dy \quad i = 1, 2$$

where $K(x) = \left(\int_0^1 e^{\frac{-(y-x)^2}{2\sigma^2}}\, dy\right)^{-1}$ is a normalizing constant.

Using such a small class of weight functions might seem too restrictive, but the problem of deciding whether a given family of measures is boundedly complete turns out to be quite difficult and the amount of known distributions satisfying this property is scarce[3]. And despite the apparent generality of Lehmann-Schaffé theorem, the Dynkin-Ferguson theorem[4] shows that every exponential location family is, apart from a scale parameter, normal or logarithm of gamma distributions. However, it is important to highlight that, notwithstanding we do not know of any boundedly complete family that fits our framework, bounded completeness

[3]See for example Mattner [23] and Andrews [3].
[4]See Mattner [22] and Lehmann & Casella [19].

is a strictly weaker property than completeness[5]. Consequently we use the normal family because it satisfies the assumptions necessaries for our model. Any other weight function should be studied separately as we are not aware of any general characterization.

Going back to the definition of the neighborhood composition function, it is easy to see that $\Gamma_i \in [0,1]$ and that it is differentiable because it is a convolution multiplied by a smooth factor. Parameter σ^2 is exogenously determined, corresponds to the variance of the normal distribution, and represents how much more important are nearby locations relatively to distant ones. But it can also be interpreted as how large is the city; the larger σ^2 is, the more concentrated is the weight, so the larger the city is in terms of households' perception. Finally, notice that Γ depends on how households are distributed, so we are going to write Γ^D to specify that the neighborhood composition function is given by distribution D, and simply Γ when there is no confusion. From now on, given a level of private consumption $c > 0$, a place in the city $x \in X$ and a distribution function D, we take the utility function of type i households as

$$U_i(c, x, D) = u(c) + v_i\left(\Gamma^D(x)\right) \tag{2.4}$$

where u satisfies the Inada conditions and v_i is any 'well behaved' function, i.e with finite discontinuities, that needs to be described depending on the situation. In fact, Chapter 4 is dedicated to study how different assumptions on v_i influence equilibria. Finally, observe that we are leaving aside other interesting features that can be easily included, such as commutation costs and other externalities of common study in urban economics. We do this to isolate the effect of neighborhood composition on households' behavior. To sum up, in our further analysis, agents only care about their private consumption and how their neighborhood is composed.

But besides determining how the distribution of households is related to their preferences, we also need to establish criteria for measuring the segregation level of different allocations. Precisely because that phenomenon and its relation to preferences and prices is our main motivation. For that purpose, we proceed as before classifying first locations, that is points $x \in X$, by its level of heterogeneity, to then give a global definition.

Definition 2.2.2. Given a distribution of households D, we say that a location $x \in X$ is

a. *Mixed* if $\exists \epsilon > 0\ \forall x' \in (x-\epsilon, x+\epsilon)\ \gamma_1(x'), \gamma_2(x') \in (0,1)$ a.e.
b. *Segregated* if $\exists \epsilon > 0\ \exists i \in 1,2\ \forall x' \in (x-\epsilon, x+\epsilon)\ \gamma_i(x') = 1$ a.e.
c. *Frontier* if it is neither of the above.

The classification given by Definition 2.2.2 is exhaustive and exclusive, that is every location must satisfy exactly one proposition. These concepts are represented in Figure 2.1, and we use them to define if the whole allocation is integrated, segregated or none. These last definitions are represented in Figure 2.2.

[5] In regard to this issue there is also a famous result first exposed by Ghosh & Singh [12], which uses the Wiener's tauberian theorem to characterize the bounded completeness of a translation family $\{w(\,\cdot\, - x)\}_{x \in \mathbb{R}}$ by the zero-freeness of the Fourier transform of $w(\,\cdot\,)$. However, this theorem cannot be applied to our framework, since we ask for a stronger concept that requires bounded completeness on every interval of $[0,1]$ and not in the entire line. For example, if one takes the location family generated by $e^{|\,\cdot\,|}$, $\{e^{|\,\cdot\,-x|}\}_{x \in \mathbb{R}}$ is boundedly complete, as proven by Oosterhoff & Schriever [26], but $\{e^{|\,\cdot\,-x|}\}_{x \in [a,b]}$ is not as demonstrated above in this section.

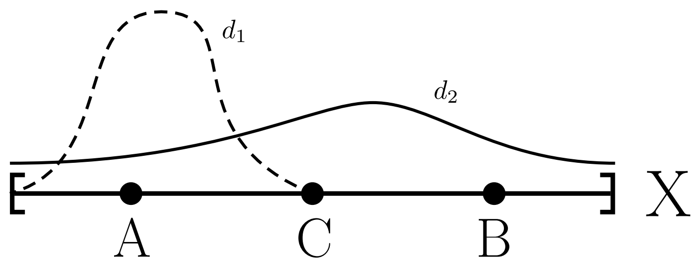
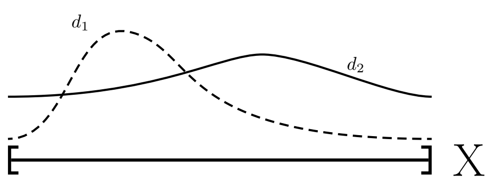


Figure 2.1: Representation of the three types of locations: A Mixed , B Segregated and C Frontier.

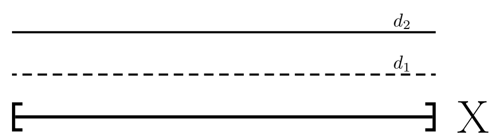
Definition 2.2.3. We say that an allocation is

a. *Integrated* if every location is mixed. And *perfectly integrated* if in addition Γ is a constant function.

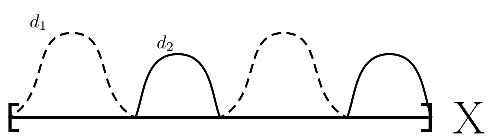
b. *Segregated* if there are no mixed locations. And *completely segregated* if in addition there is only one frontier location.



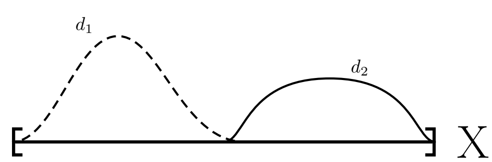
(a) Integrated



(b) Perfectly Integrated



(c) Segregated



(d) Completely Segregated

Figure 2.2: Representation of different allocations in Definition 2.2.3.

As seen in Figure 2.2a, there are some integrated allocations for which the neighborhood composition might change, for example if d_1 increases at the same time that d_2 decreases, but they are both positive. Those allocations are integrated, but not perfectly. Observe that, thanks to the completeness of Γ, in a perfectly integrated allocation γ is also constant; that is, the proportion of type i households at each point of the city remains constant. For the other classification, as shown in Figure 2.2d, if a segregated allocation has exactly one frontier location, then the city is divided into two connected areas each of which is occupied by a different type of household. If there were more frontier locations, then there would be two non connected areas occupied by the same type of household, as in Figure 2.2c.

2.3 Equilibrium Conditions

In this section we establish which allocations are going to be considered as part of equilibria. To do so we rely on the standard general equilibrium conditions: market clearing, agents utility maximization and firms profit maximization; in a competitive attitude, that is taking as given market prices. Consequently, we say that a sorting function of households throughout the city $(x_i : \Theta_i \longrightarrow X)_{i=1,2}$, with its distribution $D = (D_i : X \longrightarrow [0, \beta_i])_{i=1,2}$, a supply density function $d^s : X \longrightarrow \mathbb{R}_+$, and a rent price function $R : X \longrightarrow \mathbb{R}_+$ constitute an equilibrium iff

E1. Housing market clears

$$D_1(x) + D_2(x) = \int_0^x d^s(y)\, dy$$

E2. Households maximize utility

$$\forall h \in \Theta_i \subseteq H \quad x_i(h) \in \underset{x \in X}{\operatorname{argmax}}\, U_i(y - R(x), x, D)$$

E3. Landlords maximize profit

$$d^s \in \underset{d: X \longrightarrow \mathbb{R}_+}{\operatorname{argmax}} \int_X R(x)d(x) - C(d(x))\, dx$$

Condition E1 imposes that the demand for housing must be equal to the supply. Notice that this also implies that D_1 and D_2 are absolutely continuous. In fact, let $A \subseteq [0, 1]$ be a null set, if E1 holds then

$$\nu \circ x_1^{-1}(A) + \nu \circ x_2^{-1}(A) = \int_A d^s(x)\, dx = 0 \Rightarrow \nu \circ x_1^{-1}(A) = \nu \circ x_2^{-1}(A) = 0 \tag{2.5}$$

By the Radon-Nikodym theorem, equation (2.5) let us take d_1 and d_2 measurable functions such that $D_i(x) = \int_0^x d_i(x)\, dx$, $i = 1, 2$; as in Section 2.1, we call them density functions. The constrains set by housing market clearing that x_i, and consequently D_i, must satisfy are described by Lebesgue's decomposition theorem: $\nu \circ x_i^{-1}$ should not concentrate on a set of measure zero. This in turn implies that D_i should not increase on null sets. Firstly, it cannot have jump discontinuities because it would mean that the image of a point under x_i has positive measure; and secondly, it cannot fluctuate on uncountable null sets, such as the Cantor set for example. Finally, note that equation (2.5) also justifies our definition of γ_i in Section 2.2, because no matter how Γ_i is defined when d_i does not exist, such an allocation cannot be part of equilibria.

Interpreting condition E2 and E3 is straightforward. The first one requests that households reside in their preferred location, given their budget constraint. Observe that private consumption does not appear because there are only two markets, and by the Walras law it can be deduced that $c = y_i - R(x)$. And the second one imposes that landlords chose to provide the density d^s that maximizes their profits, given prices and costs. However, there is a subtle point regarding E2 that should be mentioned. Jara, Jofré & Martínez [16] present a general equilibrium urban model distinguishing two approaches for equilibrium conditions.

The first is based on auction theory, and the second on utility maximization theory. As can be noticed, we take the latter, and we do that safely because they demonstrate that under weak assumptions both definitions are equivalent.

It is important to notice that the density function d_i characterizes how households are distributed in equilibrium, since D_i can be obtained integrating the density. But the underlying sorting function x_i cannot be recovered completely from D_i, there may be infinitely many measurable functions inducing the same distribution over X. However, we are not interested in the sorting function itself but in the law it defines, because households are indistinguishable in our model. Accordingly, throughout this work we only specify the density functions and forget about the allocation function, as it is of no interest itself. More precisely, we denote an equilibrium as a vector (d_1, d_2, d^s, R) satisfying E1, E2 and E3, where d_i is the density of type i households, d^s is the housing supply density and R is the rent price function.

2.4 Equilibrium Properties

There are some results related to equilibria that are driven by rather weak assumptions over households' preferences, and that might help understanding better the model because they show general behavior of agents, firms and prices. In favor of exposition, in this section we present some propositions that hold true regardless of the particular selection of v_i. By doing so, we think that it is easier for the reader to identify the forces that determine agents' behavior in equilibrium.

The first result is about the price function. Because Γ is a continuous function, there should be small variations in the neighborhood composition between nearby locations. Consequently, if preferences for neighborhood composition are also continuous, then prices behave in a similar manner in equilibrium. Since households only care about private consumption and neighborhood composition, in equilibrium, a small increase (decrease) in the second must be accompanied by a small decrease (increase) in private consumption, which is equivalent to a small decrease (increase) in prices; otherwise households may be willing to change their location.

Proposition 2.4.1. *If v_i is continuous, then in equilibrium the rent price function R is continuous.*

But apart form its continuity, condition E2 gives us a direct way to compute the the rent price function. Utility maximization of agents impose that households should be indifferent between locations occupied by individuals of the same type. Other way of saying the same is that the utility level attained in equilibrium should be constant within each group of agents. This implies that its derivative with respect to x must be equal to zero for every interval occupied by the same type of households. Suppose that v_i is differentiable and that $d_i(x) > 0$ a.e. in (a, b), then applying E2, in equilibrium $u(y - R(x)) + v_i(\Gamma(x)) = \bar{U}_i \ \forall x \in (a, b)$ for some constant value $\bar{U}_i$. Deriving with respect to x and then rearranging one obtains

$$R'(x)u'(y - R(x)) = v_i'(\Gamma(x))\frac{d\Gamma}{dx}(x) \tag{2.6}$$

which provides an equation for characterizing prices in equilibrium.

After establishing the behavior of prices, we make a remark about the behavior of landlords in equilibrium. Profit maximization condition E3 gives an expression for the integral problem of profit maximization, however this integral optimization is equivalent to optimizing the integrand pointwise almost everywhere. Therefore prices are equal to marginal costs, and because cost function is convex, profits are positive in equilibrium. Throughout the following sections, we do not analyze the total density $d^s = d_1 + d_2$ in depth, and we focus only in the proportion between them γ_i. This is mainly because equation (2.7) characterize its behavior in equilibrium: locations with high prices are more dense than those of lower prices.

Proposition 2.4.2. *In equilibrium for almost every $x \in X$*

$$R(x) = C'(d^s(x)) \tag{2.7}$$

And if there are no fixed costs $C(0) = 0$, then profits are strictly positive $\pi(d^s, R) > 0$.

Finally, to conclude this section we present a type of equilibrium that always exists, regardless of the particular choice of v_i. Observe that if Γ and prices are constant, then no household can improve her utility by changing location, so the utility maximization condition E2 is met. And if the sum of demand densities are constant and equal to β_i and the supplied density d^s is constant and equal to 1, then the market clearing condition E1 is also satisfied. So a perfectly integrated allocation with constant density and constant prices $R = C'(1)$ is an equilibrium of the model.

Proposition 2.4.3. *Let $v_i : [0,1] \longrightarrow \mathbb{R}$ for $i = 1, 2$ be any preferences for neighborhood composition and $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ any continuous, increasing and convex cost function. Then, the perfectly integrated allocation with constant density $d_i(x) = \beta_i$, and constant prices $R(x) = C'(1)$ is an equilibrium.*

Notice that, even tough there are many perfectly integrated allocations, this is the only one that can be part of equilibria due to the constant prices faced by landlords; for that reason we call it *the perfectly integrated equilibrium.* Despite this equilibrium always exists, it cannot be concluded that it is a robust outcome. In many other models, for example Schelling [31, 32], integration is a steady state because agents fail to find a better place to live given the distribution of households. But when dynamics are taken into account, this type of equilibrium can be categorized as infrequent because there might be only a few initial conditions that lead to it. Perfect integration may also be vulnerable to perturbations, for example a small exogenous shock can change the composition of an area in such a way that it triggers a massive movement of households. Therefore Proposition 2.4.3 simply states that integration is always possible, but does not establish if it is a plausible outcome. In the next chapter we deduce results in the opposite direction, demonstrating that perfect integration is the only integrated distribution that can arise in equilibrium.

Chapter 3

Segregation in Equilibrium

3.1 Divergent Preferences

To begin this chapter it is important to spend some time understanding how agents sort in equilibrium. The first observation is that, thanks to utility maximization condition E2, in equilibrium households should be indifferent between their location and other occupied by the same type of agents. That is because every household of type i must attain the same utility level, say $\bar{U}_i$, otherwise someone would not be maximizing her utility. An important remark about this issue is that, given a particular equilibrium, the number $\bar{U}_i$ is arbitrary because preferences do not change by adding a constant value to the utility function. Therefore, without loss of generality, we can assume that $\bar{U}_1 = \bar{U}_2$ so households of different types share the same utility level. We adopt this convention because for every frontier or mixed location $x \in X$, that is a location occupied by both types of agents, one has that

$$\begin{aligned} \bar{U}_1 = \bar{U}_2 &\Rightarrow u(y - R(x)) + v_1(\Gamma(x)) = u(y - R(x)) + v_2(\Gamma(x)) \\ &\Rightarrow v_1(\Gamma(x)) = v_2(\Gamma(x)) \end{aligned} \tag{3.1}$$

Hence in equilibrium, every possible mixed or frontier location can be interpreted as an intersection between v_1 and v_2. Particularly, the chain of implications in equation (3.1) shows that the amount of different mixed locations is bounded by the number of possible intersections of the functions. Actually, this last conclusion will be later the main explanation for Theorem 1. However, it should be highlighted that if this convention is adopted, then the exact value of the functions v_i will change depending on the equilibrium, so computations should be made carefully.

As is economically intuitive, locations are taken by those who value them the most. But which group is willing to pay more for a house with a particular neighborhood composition depends in turn on the whole distribution of households. When the convention of equal utility level is adopted, this problem disappears because it is easy to identify which areas are occupied by which type of agents. When a location $x \in X$ provides a higher utility for neighborhood composition to type 1 agents they outbid the others and hence live in that point, and vice versa. This principle can be restated formally as: if $\bar{U}_1 = \bar{U}_2$ then

$$v_i(\Gamma(x)) > v_j(\Gamma(x)) \Rightarrow \gamma_i(x) = 1 \text{ a.e.} \tag{3.2}$$

So if both groups attain the same utility level in equilibrium, agents who are willing to pay more for a particular location are simply those whose utility function for neighborhood composition v_i at that point is greater.

Relying just on equations (3.1) and (3.2) we are ready to obtain the main result of our work. When trying to find integrated equilibria, the first equation tells us that in order to have mixed locations both type of agents must value neighborhoods equally. One simple way of ensuring that, is by eliminating all differences between neighborhoods; that is, having constant neighborhood composition. This case corresponds to the perfectly integrated equilibrium exposed in Proposition 2.4.3. But if one tries to find more integrated allocations that may also be equilibriums, it is easy to see that they are somewhat rare, if not inexistent. The equal value condition imposed by equation (3.1) is very difficult to fulfill when there is variation on the neighborhood composition function.

Suppose we have an equilibrium in which there is an interval of mixed locations (a, b), because Γ_i is a continuous function the image set $\Gamma_i(a, b)$ must also be an interval (possibly degenerated and not necessarily open). Lets call p_1 and p_2 the frontier points of such image interval, having variation on the neighborhood composition is equivalent to ask for $p_1 \neq p_2$, lets say $p_1 < p_2$. We know that for every $p \in (p_1, p_2)$ $v_1(p) = v_2(p)$, which in turn implies that $v_1'(p) = v_2'(p)$. Or in other words, both types of households are willing to give up (or gain) exactly the same amount of private consumption for a marginal increase in type i neighborhood proportion. Notice that this conclusion is not related to the convention discussed above, but it is rather a condition directly imposed on preferences. The difficulty of having convergent preferences is definitely the main source of residential segregation in our model, we devote Chapter 4 to study different utility functions that are of general interest, but for the moment we take the divergence of preferences as an assumption and then present a quite strong result.

Definition 3.1.1. We say that preferences for neighborhood composition are *divergent* if for every $p_1, p_2 \in [0, 1]$ $v_1|_{(p_1,p_2)} - v_2|_{(p_1,p_2)}$ is not constant.

Theorem 1. *Suppose that preferences for neighborhood composition are divergent then, besides the perfectly integrated equilibrium, the only allocations that occur in equilibrium are segregated.*

There is a simple economic principle behind this theorem that might be overshadowed by the technical definition of divergent preferences. The set of available neighborhood compositions in the city can be interpreted as the set of goods that are to be assigned in the economy. Under this interpretation, divergence in preferences is equivalent to say that individuals have different valuations for those goods. In equilibrium, locations are occupied by the agents who value them the most; consequently, mixed areas are places where all households are willing to pay exactly the same price for each house in the neighborhood. But if the two groups of households value these goods differently, then it is impossible to assign all of them to both at the same time because for each location there would be a type that outbids the other. Thus different mixed locations cannot coexist in equilibrium, or equivalently the only possible outcomes in equilibrium are perfect integration or some degree of segregation.

This is a crucial result because it sheds lights on how extensive the problem of residential segregation is, and its tight connection with spatial externalities in the city. Households may segregate by type because of many different kinds of housing externalities, when these effects depend continuously on distance. To obtain segregation it is only necessary to introduce some discrepancy in individual's preferences. For example, preferences could go on opposite directions, when agents prefer to live in neighborhood with a higher share of similar households. Or it could be the case that one function is strictly steeper than the other because one group is more sensitive to neighborhood changes than the other. Segregation may also arise simply because only one group cares about the identity of neighbors. Even if the other group is absolutely indifferent, preferences diverge if the former rank neighborhoods in some arbitrary strict order.

However, it is important to observe that nothing is said in Theorem 1 about the existence of segregated equilibria. In fact it cannot be ensured in the general case, as demonstrated for example in Proposition 4.2.2, where there are no completely segregated equilibrium. Neither it says which pattern of residential segregation is more likely to be observed, nor if some of them is more plausible than perfect integration. In the following section we establish some criterion to classify equilibria by its robustness.

3.2 Equilibria Stability

In general, it is not possible to characterize equilibria, even for simple and explicit preferences for neighborhood composition. Moreover, there might be several different distributions of households that, for some rent price function, are part of equilibria. Proposition 2.4.3 and Theorem 1 say that the only possible outcomes are perfect integration and some degree of segregation, we would like to extend these results by providing tools to distinguish, in some sense, which predictions are more likely to be observed. To do so we take inspiration from Schelling's models, which are based on the best response dynamics of individuals. For example, in his linear spatial proximity model[1] a perfectly integrated distribution is a steady state, because taken that allocation as given no individual wants to change his neighborhood. However, when a random initial distribution is chosen it is highly unlikely that perfect integration will come up as a resulting state for the best response dynamics. We want to include these ideas by saying that equilibrium allocations are steady states, but there are some of them which we can label as unstable because they are vulnerable to small perturbations.

The issue of incorporating dynamics into an economic model is rather a hard one. Mainly because economist have struggled to describe precisely how an economy evolves from a disequilibrium state into equilibrium. Hahn [14] does a general review of the topic and concludes there is no general axiomatic foundation of equilibrium stability, but instead a vast variety

[1]This model considers that agents are divided into two types and distributed randomly along a line. Then agents who do not have at least 50% of alike neighbors take turns, following an arbitrary rule, to change to another location in the line. See Schelling [31].

of models built for particular examples. The classical way to tackle this problem is with the theory of *tâtonnement stability*[2], in which the excess of demand for a good z_k is studied as a function of prices p, and the dynamics of the economy is described by $\dot{p}_k = z_k(p)$. Under these definitions equilibria is characterized by $z_k(p) = 0$ for every good k; and in the case of only two goods stable equilibria can easily be defined as those equilibrium prices p for which $\partial_{p_1} z_1(p) < 0$, because around those points an increase (decrease) in prices causes a decrease (increase) in demand that in turn makes prices fall (raise) and return to the original state. Throughout this section we attempt to reconcile these principles with our model, facing two major difficulties: there are infinitely many goods, one for each location in the city; and there are externalities generated by the distribution of agents, perceived by households through the neighborhood composition function. As a consequence of that, we do not describe the entire process by which the hosing market attains an equilibrium, because such an ambitious goal would be very complex and based on flimsy assumptions. Instead of that, we simply want to determine which states are steady when small perturbations are introduced.

The first idea to analyze stability would be to define an excess of demand function, yet there is not a natural way to do that because agents consume only a unit of only one of the housing goods, so an increase (decrease) in prices would make everybody unwilling (willing) to rent in that place. The only way that there could be a non degenerated difference between supply and demand is when households are indifferent between locations, that is if they attain the same utility level. Hence instead of quantifying an excess of demand when prices are distorted, our approach is studying qualitatively individuals' behavior and prices when a perturbation in neighborhood composition is introduced. This concept comes from Schelling model of neighborhood tipping[3]. In this model an exogenous replacement of a small number of type i by type $-i$ agents changes neighborhood composition, which may trigger a massive migration of the former because the proportion of the latter after the perturbation exceeds their tolerance level. This threshold migration, at which the distribution of agents rapidly changes, is called the *tipping point*; and it is closely related to stability because the smaller it is, the easier the equilibrium is dissolved. To incorporate it to our model we need to establish what we mean by a small change in the distribution of agents. We choose two points x_1 and x_2 of X and then make a mass ε of type 1 agents around x_1 to 'migrate' into x_2, and an equivalent mass of type 2 agents around x_2 make the inverse movement, maintaining the housing supply unchanged.

Definition 3.2.1. Given an equilibrium $\left(\vec{d}, d^s, R\right)$ we say that a density vector $\vec{d^\varepsilon}$ is an *ε-migration* between $x_1, x_2 \in X$ if there exists $\mathcal{V}_i$ open neighborhood of x_i ($i = 1, 2$) such that

i. $\forall x \in X \setminus (\mathcal{V}_1 \cup \mathcal{V}_2) \quad \vec{d^\varepsilon}(x) = \vec{d}(x) \quad \text{a.e.}$
ii. $\forall x \in \mathcal{V}_i \quad d_i^\varepsilon(x) = d^s(x) \ \wedge\ d_{-i}^\varepsilon(x) = 0 \quad \text{a.e.} \quad i = 1, 2$
iii. $\int_{\mathcal{V}_1} d_1 \, dx = \int_{\mathcal{V}_2} d_2 \, dx = \varepsilon$

Item i of Definition 3.2.1 states that outside the neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ the distribution

[2] The foundations of this theory come from the law of demand and supply discussed by Walras in the 19th century, which was then translated into differential equations by Samuelson [30] to precisely describe the dynamic process by which an economy converges to equilibrium.

[3] See for example Schelling [31].

of agents remains unchanged. Condition *ii* says that households of type i occupy all estate available in $\mathcal{V}_i$. And finally statement *iii* ensures that the mass of agents does not change in the process. The size of ε determines how much mass of agents is involved in the migration, so small ε means small changes in neighborhood composition.

The next step is to determine how agents would react to such a distortion. We already discussed the difficulties of quantifying the excess of demand generated, nevertheless it is possible to describe it qualitatively. The idea is that if an invasion of type i agents into a neighborhood is more valued by the same type of agents, then this change in the distribution of households is 'persistent' in the sense that it would attract more alike neighbors. On the contrary, if the new neighborhood composition yields more utility for the other group, then they would outbid the invaders reverting the perturbation. This principle is analogous to the supply and demand law behind the system of equations described by Samuelson [30]. And it is also present in Schelling model, that says that the perturbation is amplified if it surpass the tipping point. Therefore, we use it to give the definition of a persistent migration.

Definition 3.2.2. Given an equilibrium $(\vec{d}, d^s, R)$, where households attain utility level $\bar{U}_i$ for $i = 1, 2$. We say that an ε-migration $\vec{d^\varepsilon}$ between $x_1, x_2 \in X$ is *φ-persistent* if there exists a rent price function R^ε at uniform distance less than φ of R, that is $\|R - R^\varepsilon\|_\infty < \varphi$, such that for every $x \in \mathcal{V}_i$

$$U_i\left(y - R^\varepsilon(x), x, D^\varepsilon\right) - \bar{U}_i > 0 > U_{-i}\left(y - R^\varepsilon(x), x, D^\varepsilon\right) - \bar{U}_{-i} \tag{3.3}$$

where $\mathcal{V}_i$ denotes the open neighborhood around x_i into which type i agents migrated.

Here the price function acts as a separator between agents. If the migration is persistent, then R^ε is less than what the invaders are willing to pay, and more than what the other group does. We do not know exactly how prices are set, but there is no need to clarify it either: in order to maintain the perturbation prices must lie between the willingness to pay of agents. Now, if for a given $\phi, \varepsilon > 0$ an equilibrium has a φ-persistent ε-migration, then for a price function at a distance φ form the original a mass ε of households are willing to exchange locations. But if the new prices are too different from the former, or the mass of households is too large, then the shock might be too big to test the stability of the equilibrium. And because we are dealing with a continuous model there is not an obvious way to distinguish between small and large quantities. Therefore we say that an equilibrium is *unstable* when there are arbitrarily small persistent migrations sustained by prices arbitrarily close to the ones of equilibrium.

Definition 3.2.3. We say that an equilibrium $\left(\vec{d}, d^s, R\right)$ is *unstable* if there exists $x_1, x_2 \in X$ such that for every $\varphi > 0$ there is a φ-persistent ε-migration between x_1 and x_2 and $\lim_{\varphi \to 0} \varepsilon = 0$. Otherwise we say the equilibrium is *stable*.

In other words, an equilibrium is unstable if there are two different locations at which the tipping point is arbitrarily small. Any perturbation in the neighborhood composition at those locations triggers a collective movement of agents. One type of households completely leave the neighborhood, while the others take possession of it. Conversely, if the tipping point is strictly above the current neighborhood composition at some point x of the city, regardless of how close those quantities are, we say that this allocation is stable, in the sense that it cannot be part of an arbitrarily small persistent migration.

The main result of this section is a characterization of equilibria stability in terms of the slope of preferences for neighborhood composition at frontier and mixed locations. The mentioned property of stable equilibriums for a two goods economy that $\partial_{p_1} z_1(p) < 0$ is analogous to Theorem 2. In the former, a sufficiently small price perturbation causes a reverse effect in excess of demand, an then the economy returns to its original state; in the latter, a sufficiently small invasion of type i agents is valued more by type $-i$ agents, causing them to return to their original location. The fact that segregated locations cannot get involved in migrations is because households occupying those point value strictly more the composition of their neighborhood, hence a sufficiently small change in Γ is not enough to make the invaders outbid them.

Theorem 2. *Suppose v_1 and v_2 are continuously differentiable functions, i.e. v_1, v_2 are of class $\mathcal{C}^1$. Then an equilibrium $\left(\vec{d}, d^s, R\right)$ is unstable if there are $x_1, x_2 \in X$ mixed or frontier locations such that for $i = 1, 2$*

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x_i)) \tag{3.4}$$

Conversely, if $\left(\vec{d}, d^s, R\right)$ is unstable, then there are $x_1, x_2 \in X$ mixed or frontier locations such that for $i = 1, 2$

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) \geq \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x_i)) \tag{3.5}$$

Recall that if preferences are divergent, then perfect integration or some degree of segregation are the only possible outcomes in equilibrium. Thus we can apply the last theorem to characterize when these equlibriums are also stable. Classifying equilibria in such a way let us determine which of the steady distributions are more likely to be observed, and which pattern of segregation is more plausible to emerge.

Corollary 2.1. *The perfectly integrated equilibrium is unstable if $\frac{\partial v_1}{\partial \Gamma_1}(\beta) > \frac{\partial v_2}{\partial \Gamma_1}(\beta)$, and stable if $\frac{\partial v_1}{\partial \Gamma_1}(\beta) < \frac{\partial v_2}{\partial \Gamma_1}(\beta)$.*

This result is explained by the fact that in a perfectly integrated equilibrium, the neighborhood composition is constant, an thus it coincides at each point with the population share of each group. Then its stability depends on the relative value of an exogenous perturbation; if a migration of one type of households is valued more by the others, then it cannot be persistent because the latter outbid the former, countering the perturbation.

Corollary 2.2. *Let $\left(\vec{d}, d^s, R\right)$ be a segregated equilibrium. Then $\left(\vec{d}, d^s, R\right)$ is unstable if there are at least two frontier location $x_1, x_2 \in X$, such that $\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x_i))$. Particularly, if it is a completely segregated allocation it is always stable.*

As a first remark, a completely segregated equilibrium is always stable because, by definition, it has only one frontier location. This point analyzed separately is unstable, in the sense that it could be part of an arbitrarily small persistent migration. But it is the only location with this property, and then there is no other source of invaders that could feed a perturbation that would make the equilibrium unstable.

Secondly, it is worth understanding what is necessary in order to have a frontier location that satisfies equation (3.4) in a segregated distribution. On the one hand, preferences should be such that at the neighborhood composition of the frontier location, agents are willing to pay more than the other type for an increase in the proportion of unlike households. This puts some restrictions over the utility function of agents, but it is completely possible and we present some examples in Section 4.3. However, on the other hand, it constrains considerably segregation patterns. Equation (3.4) imposes that, around the frontier, locations with higher proportion of one type of agent are valued more by the other type; and then the interval with grater Γ_i is populated by $-i$ households. The existence of this kind of pathological allocations depends strongly on the choice of the weight function w used to define the neighborhood composition function Γ_i, thus it contains a mathematical complexity that is out of the scope of this work.

Following these two observations, we interpret the existence of a completely segregated equilibrium as a sign that strong segregation is the most plausible outcome. The stability of the perfect integration as a sign that integration is more likely to be found. And if none of them are part of stable equilibria, then we can conclude that mild segregation should be observed.

Chapter 4

Preferences for Neighborhood Composition

4.1 Methodology

In this chapter we consider specific classes of preferences for neighborhood composition in order to analyze different situations where segregation occurs. The first scenario is when there is a 'good' type of agents that every household wants to live near to, we call them the *leading group*. We intend to represent the problem of socioeconomic segregation, where one type of agents generate a positive externality over the rest, so they act as a leading group increasing the utility function of those who live in a neighborhood populated by them. We take the ideas of Rossi-Hansberg, Sarte & Owens III [28], who found evidence that agents get involved in non market interactions that are determined by the location of their houses, and that are reflected in the rent price. We test if these interactions could generate residential segregation, having in mind that they are strongly correlated with households' income. The second case is when households prefer to live among similar people, and we call it *homophily*. It can be interpreted as a model for racial or religious segregation, where agents of the same group share some characteristics that make them prefer to live together. As discussed in the introduction, this issue has had a lot of attention and there are a lot of analytical works about it. We specifically take the approach of Pancs & Vriend [27], who explicitly describe Schelling's ideas through utility functions, and try to incorporate it into our model. In both cases we give an explicit expression for the utility function, and then describe how parameters determine the existence and stability of equilibria, discussing results and contrasting them with previous works and what should be observed in reality.

As presented in Section 3.1, the main segregating force is divergence in preferences. To understand how restrictive this condition is suppose that utility functions are described by polynomials, with possibly different degrees, that is $v_i(x) = p_i(x)$ with p_i a polynomial. And suppose in addition that preferences are not divergent, then there must be a non degenerated open interval I in which they coincide up to a constant, more precisely $q_1(x) - q_1(0) = q_2(x) - q_2(0)$ $\forall x \in I$, so the polynomial $q_1(x) - q_2(x)$ is constant in I and therefore it is a constant function in the whole segment $[0, 1]$. In conclusion, if two polynomial are not divergent, they are the same function up to a constant; that is, they represent the same preferences for neighborhood composition. So for a considerably big family of utility functions it is impossible to have mixed neighborhoods in equilibria, other than in the perfectly integrated case.

Although this result weakens if, for example, piecewise polynomial functions are considered, there is another cause of divergence in preferences arising from the relative importance that households put into neighborhood composition compared to private consumption. Hence, even if the shape of utility functions coincides in an interval, any discrepancy in the value of neighborhood improvement may trigger divergence. For that reason we introduce the following notation for the utility function

$$v_i(\Gamma) = \lambda_i \cdot \tilde{v}_i(\Gamma) \tag{4.1}$$

where we define $\lambda_i > 0$ as the *sensitiveness* for neighborhood composition changes, and $\tilde{v}_i$ as the *shape* of preferences. The first parameter plays a central role in the analysis as it has a clear economic interpretation: it represents the rate at which agents are willing to resign utility of private consumption for an increase in the quality of their neighborhood. As remarked before, preferences not only must have the same shape, i.e. monotonicity, concavity and properties of higher order derivatives, but also must maintain the same relationship with the utility of consumption. And this last condition seems almost impossible to satisfy, especially in this model because λ_i is exogenously determined. We cannot think of any justification to believe that different types of households should have the same sensitiveness for neighborhood composition; actually, there are reasons to think that it should never coincide, if, for example, it changes over time.

Another important feature of λ_i is that, because it determines the rate at which agents are willing to trade private consumption for neighborhood improvement, it can be used to introduce income inequality in the analysis without actually varying the parameter y. It can be easily noticed that if a group is richer than the other they should be more interested in improving their neighborhoods, or equivalently, they are more sensitive. This approach may not be sufficiently rigorous, but it has been used in the literature, for example by Bayer, Fang & McMillan [5]; and we simply use it to make qualitative inferences about equilibria.

In the following sections we try to derive conditions for which integration may occur, but we anticipate that indifference might be the only robust way of achieving integration. We already discussed how difficult it is to have preferences with the same shape, and how arbitrary is to make them coincide also in sensitiveness. Notwithstanding, a plausible way to elude this problem is when agents are indifferent between a range of compositions. Incorporating indifference into the analysis is also relevant because it is present in the heart of Schelling model, where individuals prefer to be majority rather than minority, but they do not care about the exact proportion. We devote a section to this topic, in which we extend Schelling utility function and describe two different classes of mixed equilibria.

Finally, we emphasize the importance of prices around the frontier to make inferences about individuals' preferences. Throughout this chapter we derive segregation from different assumptions, so we propose a way to determine which of the possible causes better explains a given allocation of households. Particularly, we take advantage of the continuity of our model to describe prices in the transition between the interior of segregated areas to the boundary of the neighborhoods; they give clues about preferences because their behavior changes dramatically depending on how v_i is defined.

4.2 Leading Group

We first look at the case where both types of households prefer a neighborhood composed by individuals of one particular type, which we call the *leading group*. There are spatial externalities that can be interpreted as nonmarket interactions between agents, such as in Rossi-Hansberg, Sarte & Owens III [28] or in Guerrieri, Hartley & Hurts [13]. These effects could be positive and generated by the 'good' type of households, like more and better public goods. Or they could be negative and generated by the 'bad' ones, like higher rates of crime. For example, Somanathan & Sethi [33] discuss how individuals care about the level of affluence in their community, as it provides better public schools or entry into social networks which can be lucrative. But it can also represent prejudice towards low income classes, that is class discrimination or *classism*. This approach has been infrequent in the literature, but it plays an important role in some Latin American societies. For example, Núñez & Gutiérrez [24] describe how prejudice towards lower-class individuals considerably affects earning expectations in Chile[1]. But regardless of the underlying reasons, we simply suppose that agents prefer neighborhoods with a higher proportion of one type of agents, and we model this situation by assuming that preferences of both groups are increasing in Γ_i, where i is the leading group.

Without loss of generality we adopt the convention that type 1 agents are the ones providing positive externalities. Therefore we let v_1 and v_2 be any increasing functions of Γ_1. We represent v_i like in equation (4.1)

$$v_i(\Gamma_1) = \lambda_i \cdot \tilde{v}(\Gamma_1) \tag{4.2}$$

where $\tilde{v}$ is an increasing function. Consequently, sensitiveness for neighborhood composition is the only parameter studied in this section. As discussed before, changing λ_i let us partially recover wealth inequality without considering different incomes between groups. The more affluence of one group translates into stronger incentives to improve neighborhood composition, because the marginal utility of private consumption is lower, which is exactly what happens when the sensitiveness is higher. Throughout this section we assume v_1 and v_2 satisfy equation (4.2).

The first step is to determine whether preferences are divergent. This problem is easily solved by noticing that, by definition, v_1 is the multiplication of v_2 by a scalar factor, and then if this factor is different from 1 they can coincide at most in one point. This is because $\tilde{v}$ is assumed to be increasing, and hence it cannot be zero in an open interval. Thus it can be concluded that:

[1]Strikingly, they found that there is an earning gap in the Chilean labor market, explained only by socioeconomic background, that is larger than other reported salaries discrepancies, such as those generated by gender, race or physical appearance.

Proposition 4.2.1. *If $\lambda_1 \neq \lambda_2$ preferences are divergent.*

By applying Theorem 1 to the last result one can directly obtain information about the allocation of agents in equilibrium:

Corollary 4.2.1.1. *If $\lambda_1 \neq \lambda_2$ then, besides the perfectly integrated equilibrium, the only allocations that occur in equilibrium are segregated.*

Hence perfect integration or some degree of segregation are the only possible outcomes. Even when agents rank neighborhood composition equally, but they differ in their willingness to give up private consumption for an improvement in the quality of their vicinity. Although from the point of view of equilibrium conditions all equilibriums are equal, we show that there are significant differences between them when stability is incorporated into the analysis. Like in the previous section, we study how parameters, in this case only λ_i, make one prediction more robust than the other.

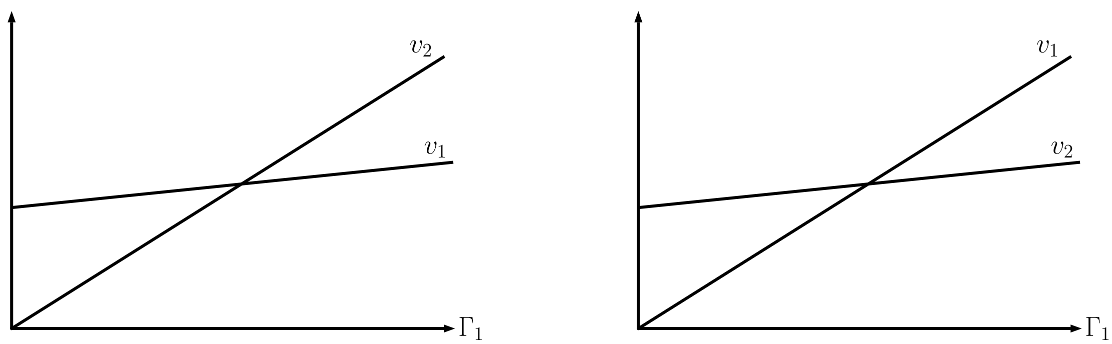
As a remark, notice that we are assuming that preferences have the same shape and so they only differ in their sensitiveness. Thus we leave aside any convexity effect that might influence equilibria. For example, it could be the case that type i agents have a critical level of composition at which a marginal improvement is highly valued, like preferences in the Schelling model, while the other group have a linearly increasing utility function. In this scenario equilibria might be different from the cases we study, particularly because type i agents would value more locations around the threshold level, and value less more homogeneous locations. In other words, preferences might have more than only one intersection, which is impossible when preferences are described by equation (4.2). Therefore we forget about any other factor that might be interesting to consider in this model, and isolate the effect of differences in sensitiveness, that is λ_i. We demonstrate that a crucial determinant of segregation is whether the leading group is more sensitive or not, producing completely opposite results in each case.

Proposition 4.2.2. *If $\lambda_1 < \lambda_2$, then the perfectly integrated equilibrium is stable, and for sufficiently small σ there is no completely segregated equilibrium.*

Intuitively this result is explained by the fact that, regardless of the neighborhood composition they face, type 2 households value more than the others an increase in the proportion of type 1 agents. Therefore no perturbation would be persistent, making any equilibrium stable. And the lack of a completely segregated equilibrium comes from the same observation. In a segregated allocation, type 2 agents outbid type 1 agents at locations with a high share of 'good' households. This makes impossible to sustain complete segregation in equilibrium because agents from the leading group are not willing to pay the amount of money necessary to keep possession of their area. And exactly the opposite occurs when 'bad' households are less sensitive. Every migration between frontier or mixed location, in any equilibrium, would be persistent. And then no equilibrium can be stable, other than a completely segregated because it has exactly one frontier and no mixed locations.

Proposition 4.2.3. *If $\lambda_1 > \lambda_2$, then for every $\sigma > 0$ the only stable equilibrium is completely segregated.*

In Figure 4.1 the same argument can be seen graphically. When $\lambda_1 > \lambda_2$, v_2 lies below v_1 after the interception, which implies that type 1 households value more than the others locations with Γ_1 close to 1, that is compositions at the right side of Figure 4.1b. Hence this case is compatible with complete segregation, both groups of households value more than the others areas with high proportion of alike households.



(a) Preferences when $\lambda_1 < \lambda_2$

(b) Preferences when $\lambda_1 > \lambda_2$

Figure 4.1: Graphic explanation for Propositions 4.2.2 and 4.2.3.

Bringing these two propositions together it can be concluded that, in the context of spatial externalities from a leading group, differences in sensitiveness for neighborhood composition induce segregation, but only when the 'good' group is more willing to pay. In the opposite case it produces perfect integration. Notice that this is indeed the case of socio-economic segregation, rich households can spend more money on improving their neighborhood, accordingly they are more sensitive. In our model, complete segregation is the only robust outcome when there is income inequality.

Residential segregation based on households income is also present in the context of racial segregation, but the evidence is quite counterintuitive. Bayer, McMillan & Rueben [5] show that when racial inequality diminishes, housing segregation increases because there emerge middle-class black neighborhoods. This finding is consistent with what we have demonstrated in this section, because when black population is divided into two different classes, they also segregate into disjoint areas because of their difference in sensitiveness for neighborhood composition. However, in the next section we study specifically the problem of segregation in the presence of homophily, and we come up with an alternative explanation for the increase of segregation when the racial wealth gap narrows.

Public Goods and Gentrification

A particular problem of socioeconomic segregation is the phenomenon of *gentrification*. This term refers to a process in which a poor neighborhood receives an incoming flux of richer households, that over the time transforms the area into an upper-class neighborhood. It has been widely observed, and specifically studied many times. Recently, Guerrieri, Hartley & Hurts [13] predict gentrification endogenously as a consequence of housing demand shocks. It has also been identified as an outcome of regulatory policies, for example by Kahn, Vaughn & Zasloff [17], and placed based investments, like by Zheng & Kahn [38] and Rossi-Hansberg, Sarte & Owens III [28]. But not every government intervention causes a significant change in neighborhood composition, as shown by Busso, Gregory & Kline [10]. Thus we use our model to analyze how public investment may change equilibria.

Specifically we address the problem of how the provision of a local public good would affect the distribution of households when there are spatial externalities generated by a leading group. So we maintain the assumption that type 1 are the 'good' type of agents, but we now simplify equation (4.2) by assuming that $\tilde{v}$ is the identity function. In addition, we place N public goods of size $P_k > 0$ at $x_k \in X$, that provide positive externalities to every households. So that type i utility function of living at $x \in X$, given the distribution D of households and the public goods $(P_k, x_k)_{k=1}^N$, is

$$v_i\left(\Gamma(x), (P_k, x_k)_{k=1}^N\right) = \lambda_i \left(\Gamma_1(x) + \sum_{k=1}^{N} P_k \cdot \psi(|x - x_k|)\right) \tag{4.3}$$

Where $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a decreasing and convex function, and $\psi(0) = 1$. Thus the benefits of public goods decline with distance at a decreasing rate.

In Proposition 4.2.3 we show that when there are no public goods, if $\lambda_1 > \lambda_2$, then complete segregation is the only stable equilibrium. We can extend this proposition to a more general case. Notice that when there are fixed sources of spatial externalities, so they do not depend on the allocation of households, the effect of a marginal change in the neighborhood composition is the same as in a plain city. More precisely,

$$\frac{\partial v_i}{\partial \Gamma_1}\left(\Gamma(x), (P_k, x_k)_{k=1}^N\right) = \frac{\partial v_i}{\partial \Gamma_1}\left(\Gamma(x)\right)$$

for every x, D and $(P_k, x_k)_{k=1}^N$. So Theorem 2 can also be applied in this context, and then the next proposition follows.

Proposition 4.2.4. *Let v_i be defined by equation (4.3) and $\lambda_1 > \lambda_2$. Then for every $\sigma > 0$ the only stable equilibrium is completely segregated.*

Therefore, any effort to make the city more integrated would be pointless, at least by providing public goods. As a consequence, we proceed by starting from a segregated initial condition, and then checking if the provision of public goods change the segregation pattern. Hence we assume that the distribution of households D_0 is such that $\gamma_1(x) = \mathbb{1}_{[x, x_0]}$. So that type 1 households are segregated at the left side of the city, type 2 at the right, and x_0 represents the frontier of the segregated neighborhoods.

Given the segregated distribution D_0 and a collection of public goods $(P_k, x_k)_{k=1}^N$, lets define the *absolute value* of x as

$$V(x) := \Gamma_1(x) + \sum_{k=1}^{N} P_k \cdot \psi(|x - x_k|)$$

It does not depend on the type of agent, that is, locations give the same absolute utility to every household. And under this definition, $v_i = \lambda_1 \cdot V$. Observe that both types of households coexist at the frontier, so in equilibrium the relative value of that point must be the same for both groups. Now, for the remaining locations, they are occupied by who values them the most. More precisely, when $\lambda_1 > \lambda_2$, for almost every $x \in X$

$$d_1(x) > 0 \iff V(x) > V(x_0) \tag{4.4}$$

So in equilibrium those locations with grater absolute value than the frontier are assigned to type 1 agents.

Based on this equation we can provide a concrete example where the initial segregated distribution is not an equilibrium when a local public good is provided.

Example 4.2.1 (Gentrification). Suppose that there is only one public good of size P and located at the type 2 segregated area; that is, $x_1 > x_0$. A necessary and sufficient condition to maintain this allocation in equilibrium is given by equation (4.4): D_0 is an equilibrium iff $[0, x_0] = \{x \in X \,|V(x) \geq V(x_0)\}$ almost everywhere. In particular, lets define

$$\bar{P}(x) := \frac{\Gamma_1(x_0) - \Gamma_1(x)}{1 - \psi(|x - x_0|)} \tag{4.5}$$

Hence if $P > \bar{P}(x_1)$, then

$$V(x_1) = \Gamma_1(x_1) + P > \Gamma_1(x_0) + P\psi(|x_1 - x_0|) = V(x_0)$$

Thus for $x_1 > x_0$ and $P > \bar{P}(x_1)$, D_0 it is not an equilibrium. Figure 4.2 shows that when the public good is too large, it attracts households from the leading group. Because it is more appealing to live near the local amenity than near the frontier, despite how bad neighborhood composition is at the right side of the city. This effect causes a gentrification, type 2 agents close to x_1 are replaced by 'good' agents.

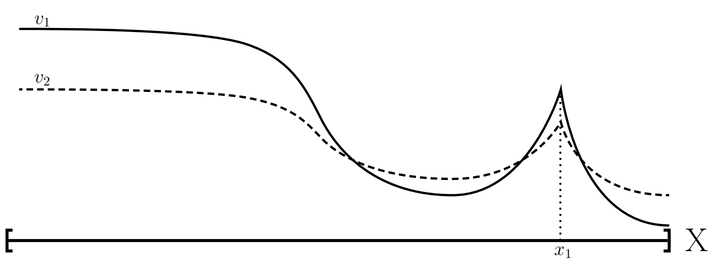


Figure 4.2: Example of gentrification when a large public good is provided at x_1.

Now suppose that there are two public goods at x_1 and x_2 of size

$$P^* = \frac{\Gamma_1(x_0) - \Gamma_1(x)}{1 - \psi(|x_1 - x_0|) + \psi(|x_1 - x_2|) - \psi(|x_2 - x_0|)}$$

Then $V(x_1) = V(x_2) = V(x_0)$, so that locations near the public good have less absolute value than the frontier. An important observation is that thanks to the convexity of ψ, $2P^* > \max\{\bar{P}(x_1), \bar{P}(x_2)\}$, so that the sum of the sizes of the public goods provided is greater than the upper bound found for a single good. This process can be iterated to disperse the benefits even more, and consequently to be able to expend a larger expenditure without altering the equilibrium. The explanation for this result is that, when separated, public goods do not provide enough local benefits for the 'good' households to be willing to change their neighborhood. Even if an equivalent amenity concentrated at a single point would encourage them to move, the convex decline with distance reduces the appealing of the most attractive neighborhood when the sources of spatial externalities are split apart.

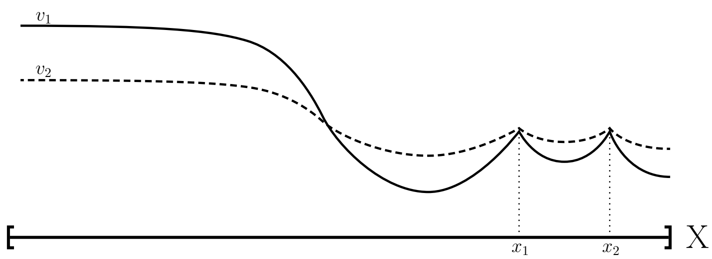


Figure 4.3: Example of how gentrification is avoided when smaller public good are placed at x_1 and x_2.

Gentrification might be an undesired effect from the point of view of public policies. The intended target group, which clearly agglomerates in a particular area at the beginning, is not benefited by the local amenity in the long term due to the movement of households. One alternative to deal with this problem is to divide the public good into smaller pieces separated by positive distance. Spreading public expenditure may avoid making some location so attractive that richer households prefer to live there than near the frontier, what would gentrify that neighborhood.

4.3 Homophily

We now look at the case where households prefer a neighborhood composed by individuals of the same type. In the classical literature, this is thought as direct effects of different characteristics of agents in individuals' preferences. It usually represents negative externalities generated by each group to the other, such as racial prejudice like in Yinger [37]. But in more recent works, it has been pointed out other indirect effects that may encourage households to interact with similar agents. In the seminar paper of Alesina & La Ferrara [1] it is shown that in more heterogeneous communities, social capital is lower; and therefore there is less participation in social activities and lower levels of trust. Following the same line, Banzhaf & Walsh [4] incorporate the effect of exogenous local public goods in a residential sorting model, deriving stronger segregation when there is grater investment. Also, Waldfogel [35] find evidence that local private goods are correlated with the composition of its neighborhood, providing benefits to those agents that live among households with similar preferences. All these three findings are examples of how neighborhood composition indirectly influences households' utility function. Consequently, we abstract from the underlying reason that agents might have for preferring to live in more homogeneous areas, simply assuming that they do and giving a parametrization for preferences that is flexible enough to capture any of these direct or indirect effects.

Given the set up that we are considering in this section, we have in mind the problem of racial segregation which has been largely studied, and we discuss different topics that are recurrent in the literature. We recover the results of Schelling's spatial proximity model, in which segregation is predicted even when individuals have strict preferences for integration; the only assumption being the reluctance of agents to be minority in their neighborhood. We draw on the extension made by Pancs & Vriend [27], especially their formalization of Schelling preferences, to give the explicit expression for the utility function v_i. We also address the theory of *black self-segregation*, which hypothesize that racial segregation arises because black households prefer to live in ghettos. And finally we discuss the counterintuitive finding that a decrease in racial inequality increases the level of housing segregation. Throughout this section we provide a parametrization for v_i based on Schelling's ideas, and then derive equilibrium properties related to these topics.

Following Pancs & Vriend, we incorporate some degree of preferences for integration, we assume that agents have a unique ideal neighborhood composition, that is $\text{argmax}_{\Gamma_i \in [0,1]} v_i(\Gamma_i) = \rho_i$; and that this favorite composition includes at least a half of own type agents, that is $\rho_i \geq 1/2$. We interpret ρ_i as the *intolerance* of type i agents, because greater values imply that agents prefer more homogeneous neighborhoods. In the extreme case that $\rho_i = 1/2$ we say that households have strict preferences for integration, and when $\rho_i = 1$ we say that they have strict preferences for segregation. And because we want to make the exposition as simple as possible, we choose a piecewise polynomial utility function with a unique local and global maximum at ρ_i. So v_i is described by:

$$v_i(\Gamma_i) = \begin{cases} -\lambda_i \cdot \mu_i \cdot (\rho_i - \Gamma_i)^\alpha & \text{if } \Gamma_i < \rho_i \\ -\lambda_i \cdot (\Gamma_i - \rho_i)^\alpha & \text{if } \Gamma_i \geq \rho_i \end{cases} \tag{4.6}$$

where $\lambda_i, \alpha > 0$, $\mu_i \geq 1$ and $\rho_i \in [1/2, 1]$. The parameter α simply represents the degree of the polynomial. For $\alpha > 1$ the function is concave, so agents value more a marginal improvement in the quality of their neighborhood when they are away from the optimal composition; and for $\alpha < 1$ the opposite happens. The case $\alpha = 1$ is piecewise linear and shows a little bit different behavior as seen in Proposition 4.3.1, but we include it in our analysis because it was explicitly studied by Pancs & Vriend. Finally, we say that μ_i is the *minority aversion* of type i households and it tries to captures the driving force of segregation in Schelling's spatial proximity model. It makes steeper the increasing portion of v_i by amplifying it by a constant for compositions below the intolerance level, as seen in Figure 4.4. For all the results presented in this section we assume v_i is described by equation (4.6).

Lemma 4.3.1 formalizes the effect of $\mu_i \geq 1$ in preferences by proving that households prefer an increase in the proportion of own type of agents than an equal amount of decrease in their share around the optimal composition. Also they are more sensitive to marginal improvements in the neighborhood composition to the left side of ρ_i.

Lemma 4.3.1. *For all* $\Delta \leq \min\{\rho_i, 1 - \rho_i\}$

i. $v_i'(\rho_i - \Delta) \geq -v_i'(\rho_i + \Delta)$

ii. $v_i(\rho_i + \Delta) \geq v_i(\rho_i - \Delta)$

and both inequalities hold with equality if and only if $\mu_i = 1$.

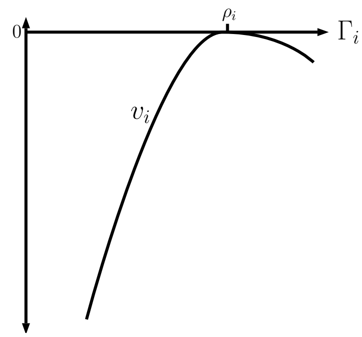


Figure 4.4: Preferences for neighborhood composition defined by equation (4.6).

Like in the previous section, the first step is to analyze divergence in preferences. Notice that for any neighborhood composition that is between the intolerance levels, agents prefer opposite changes in neighborhood composition, both would like to increase the proportion of own type households; so in that case it is obvious that preferences diverge. For more segregated compositions, both types would like a more integrated neighborhood, but only for a specific choice of parameters they value a marginal improvement equally. For example, suppose they have the same sensitiveness, that is $\lambda_1 = \lambda_2$, then the minority type is willing to pay more for an increase in the proportion of alike households, even if both types would pay a positive amount for a change in that direction. These arguments are resumed in proposition 4.3.1.

Proposition 4.3.1. *Suppose $\alpha \neq 1$, then preferences are non divergent iff $\rho_1 = \rho_2 = 1/2$ and $\lambda_1\mu_1 = \lambda_2$ or $\lambda_1 = \lambda_2\mu_2$. When $\alpha = 1$ preferences are non divergent iff $\lambda_1\mu_1 = \lambda_2$ or $\lambda_1 = \lambda_2\mu_2$.*

Combining this result with Theorem 1 one can straightforwardly conclude that, when agents differ in their sensitiveness for neighborhood composition changes, segregation is the only possible outcome in equilibrium, other than perfect integration. If in addition utility functions are not linear, strict preferences for integration are also needed in order to have different mixed locations in equilibrium.

Corollary 4.3.1.1. *If $\lambda_1\mu_1 \neq \lambda_2$ and $\lambda_1 \neq \lambda_2\mu_2$, then besides the perfectly integrated equilibrium the only allocations that occur in equilibrium are segregated. Additionally, if $\alpha \neq 1$ and $\rho_1 \neq 1/2$ or $\rho_2 \neq 1/2$ the same statement is deduced.*

Corollary 4.3.1.1 tells us that if preferences for neighborhood composition are described by equation (4.6), then there are only two possible outcomes: perfect integration or some degree of segregation; with the exception of a tiny set of parameters. Therefore, we focus on whether one equilibrium is more plausible than the others. In the previous chapter we developed the concept of stability, which we can use as an auxiliary tool to establish when segregation is more likely to emerge for a particular selection of parameters. From Corollary 2.2 we know that whenever an equilibrium is completely segregated, then it is also stable; hence, it is crucial to determine if such an allocation is part of equilibria. The second approach is to check if the perfectly integrated equilibrium is stable, we know form Proposition 2.4.3 that it always exists, but it might not be robust if a small shock causes it to fall apart. Answering these two questions for different parameters enables us to identify the different reasons for segregation; if complete segregation is part of equilibria then it is a probable outcome, if not then some mild degree of segregation should appear, and if perfectly integration is stable then segregation should not be observed.

Preferences for Segregation

As a benchmark, we first analyze the case when agents have strict preferences for segregation, that is $\rho_1 = \rho_2 = 1$. In this case they always strictly prefer a more homogeneous neighborhood, regardless of the other parameters. Observe that the derivative of v_1 and v_2 have opposite signs, hence the condition of Theorem 2 is satisfied for every mixed or frontier point; this is because type i agents value positively a marginal increase in type i proportion of the neighborhood, while type $-i$ value it negatively. Therefore, the only possible stable equilibrium is when there is a unique frontier location, namely a completely segregated distribution. And in fact it always exists, because for this allocation the city is divided into two large ghettos whose neighborhood composition is always more valued for the agents living in those locations, despite how sensitive or minority reluctant they are. In conclusion we can state the following proposition:

Proposition 4.3.2. *If $\rho_1 = \rho_2 = 1$, then for every $\sigma > 0$ the only stable equilibrium is completely segregated.*

Thus if households preferences are strictly increasing in the proportion of alike households in their neighborhood, complete segregation is the only robust outcome. This is a relevant result, because it tells that if agents are better off segregated, then there is nothing that can prevent segregation; neither changing the initial conditions, nor increasing (or decreasing) inequality, as suggested by Bayer, Fang & McMillan [5].

Preferences for Integration

Now we analyze the case when agents prefer integrated neighborhoods. Even with these preferences segregation may arise as a consequence of the unwillingness of agents to be minority. This principle was first exposed by Schelling in his famous spatial proximity model. In his set up, households are tokens on a chessboard or a line, they are divided into two different colors, and they take turns to change their neighborhood if there are less than a half of alike tokens surrounding them. He shows that it is very likely that starting from an arbitrary initial condition, best response dynamics leads to a segregated outcome. Observe that to predict segregation he did not need to introduce strong preferences for segregation, in his model households are indifferent between a completely segregated and heterogeneous neighborhoods with half share of each group. More precisely, the underlying preferences in Schelling's spatial proximity model can be described as

$$v_i(\Gamma_i) = \begin{cases} \lambda_i & \text{if } \Gamma_i \geq 1/2 \\ 0 & \text{if } \Gamma_i < 1/2 \end{cases} \tag{4.7}$$

where $\lambda_i > 0$. This utility function was made explicit by Pancs & Vriend [27], who revisited this model putting special attention on how Schelling's results are robust to other preferences.

Observe that the exact value of λ_i in equation (4.7) is irrelevant, because in the spatial proximity model there is no private consumption to trade with. We would like to obtain similar results in our general equilibrium set up, and with v_i described by equation (4.6). In other words, we want to study equilibria when agents have strict preferences for integration, that is $\rho_1 = \rho_2 = 1/2$. In this case, both types of households strictly prefer a more heterogeneous neighborhood, regardless of the other parameters, so the derivatives of v_1 and v_2 have the same sign. Thus segregation cannot be the only outcome in the general case. If for example, type 1 agents are insensitive compared to type 2 agents, namely $\lambda_1 \ll \lambda_2$, and they are not reluctant to be minority, $\mu_1, \mu_2 \approx 1$, segregation does not arise. This is explained because a segregated area of type 1 households would be valued more by type 2 agents. Individuals of the first type would be willing to pay very little to live in a segregated area because they strongly value heterogeneity; and on the contrary, the second type would be willing to pay similar prices for all locations because they are almost indifferent. Therefore the latter would outbid the former, so complete segregation does not emerge.

However, there are some other scenarios in which segregation is the most likely outcome. And in the next proposition we give a characterization for the existence of a completely segregated equilibrium in the case of strict preferences for integration:

Proposition 4.3.3. *Let $\rho_1 = \rho_2 = 1/2$. If*

$$\mu_i > \frac{\lambda_{-i}}{\lambda_i} \tag{4.8}$$

for $i = 1$ and $i = 2$, then for every $\sigma > 0$ the only stable equilibrium is completely segregated. Conversely, if the reverse strict inequality holds for $i = 1$ or $i = 2$, then for sufficiently small σ there is no completely segregated equilibrium.

This proposition says that when μ_i is large enough, that is when households are sufficiently uncomfortable being minority, then complete segregation is the only plausible equilibirum. Observe that if $\lambda_1 \neq \lambda_2$, then μ_1 or μ_2 must be greater than one in order to have complete segregation. In other words, households must have some degree of minority aversion in order to maintain homogeneous neighborhoods.

It extends the results of Schelling to a model in which there are market and prices. This extension is important, because in our set up, agents can take away attractive locations from other individuals simply by paying a higher price. On the contrary, in the former model agents may get stuck in unappealing neighborhoods just because there are no better options available. The lack of a housing market omits the economic behavior of agents. And when these factors are considered, minority aversion acquires new dimensions. First it can be compared to the utility of private consumption; agents might not want to be minority, but how much private consumption are they willing to resign to avoid it. And also it can be compared to the others' preferences; are households minority averse enough to outbid the other type. Proposition 4.3.3 relates these two effects and provides an equation that characterizes segregated equilibrium when agents prefer segregation in a more rigorous framework.

Now we study two topics directly related to racial segregation: black self-segregation and the impact of racial inequality in segregation. These two issues have had a lot of empirical and theoretical attention, especially when the subjects are North American metro areas.

Black Self-Segregation

There is a controversial discussion about the importance of blacks' preferences for self-segregation in explaining residential segregation. It was first exposed by Thernstrom & Thernstrom [34], who argued that whites' discriminatory actions have diminished over time, and that black households prefer to live with similar neighbors; and therefore it should be a relevant factor. Then his theory was belittled by empirical studies, such as Ihlanfeldt & Scafidi [15], who found that it played a minor role in explaining the whole phenomenon. But recently this empirical evidence has been challenged by the idea of incorporating the effect of local amenities in the analysis. When Ihlanfeldt & Scafidi measure preferences they mainly look at prejudice between groups, which was the standard approach. But Banzhaf & Walsh [4] and Waldfogel [35] introduced a new dimension, highlighting that local amenities encourage agents to self-segregate, regardless of any prejudice. In our model, homophily might be driven by any of the two components, so black self-self segregation could be analyzed from a general point of view.

To study this issue it is necessary first to determine what we understand by self-segregation. Intolerance, represented by parameter ρ_i, is a natural component because it directly ranks neighborhoods by its degree of homogeneity. But also μ_i and λ_i say something about preferences for integration. If black households are minority averse one can conclude that they are better off segregated than living in a predominantly white neighborhood. And if black individuals have a high level of sensitiveness, not explained by their wealth, it means that neighborhood composition is an important factor for them, and then their attitude for self-segregation is amplified. Consequently, we understand *self-segregation* as a three dimensional property characterized by high levels of intolerance ρ_i, minority aversion μ_i and sensitiveness for neighborhood composition λ_i. However, we consider the last component as a second order effect because, given the racial inequality observed in most American cities, blacks' low income should be its main explanation.

As this topic makes sense only in the context of racial segregation, we adopt the convention that type 1 agents are blacks and type 2 are whites. Thus it is natural to assume that $\beta_1 < \beta_2$ because blacks are in general a minority. And that $\lambda_1 < \lambda_2$ because whites tend to be wealthier, hence they are willing to pay larger amounts of money to live in a neighborhood with better composition. We divide the analysis in two parts, depending on whether whites are tolerant or not; as we prove below, it is a crucial matter.

When whites' utility function is increasing in the proportion of alike households in their neighborhood, that is when $\rho_2 = 1$, and their are more sensitive than blacks, segregation is the only outcome no matter how tolerant blacks are. This is because the intolerance and sensitiveness of whites overcome preferences for integration that blacks might have, thus segregated black neighborhoods are valued more by black households.

Proposition 4.3.4. *Suppose $\rho_2 = 1$. If*

$$\lambda_1 < \lambda_2 \tag{4.9}$$

then the only stable equilibrium is completely segregated.

Therefore if whites have strict preferences for segregation, black self-segregation plays almost no role in explaining the problem. Neither blacks' tolerance, nor their unwillingness to be minority influence the global outcome when sensitiveness levels are different enough. Recall that when racial inequality exists we have $\lambda_1 < \lambda_2$, and then it can be directly concluded that complete segregation is the only stable equilibrium when $\rho_2 = 1$ regardless of blacks' preferences for segregation.

In the case that whites and blacks like heterogeneous neighborhoods the converse part of Proposition 4.3.3 tells us that if blacks are sufficiently minority averse, then complete segregation arises; and the opposite occurs when they are not. In the following proposition we extend this result by eliminating its dependence to blacks' tolerance, and incorporating the stability of perfect integration.

Proposition 4.3.5. *Suppose $\rho_2 = 1/2$ and $\beta_2 > 1/2$. Then there exists a constant value M, function only of α and β_2, such that if*

$$\mu_1 < \frac{\lambda_2}{\lambda_1} \cdot M \tag{4.10}$$

then the perfectly integrated equilibrium is stable, and for sufficiently small σ there is no completely segregated equilibrium.

Again in this case, intolerance of black individuals does not affect equilibrium, it is rather the unwillingness to be minority the principal determinant of segregation. Last proposition shows that if the term $\mu_1 \cdot \lambda_1$ is not large enough then complete segregation does not sustain, but instead segregation is stable. So in this case blacks tendency to segregate is relevant, but only through these two dimensions. In conclusion, if self-segregation is understood simply as a rank for neighborhood composition, then according to our model it plays a minor role in explaining racial residential segregation. However if the other components are considered, especially minority aversion, then it affects equilibria only when whites are better off in heterogeneous neighborhoods.

From these results we obtain two general lessons that might help evaluating the impact of blacks' desire for segregation in the whole phenomenon of residential segregation. First, white preferences should be also considered in the analysis. Because the allocation of households depends not only on what they prefer, but also on how their preferences are related to others'. For example, if the theoretical model utilized for estimating assumes that preferences are increasing in the proportion of alike neighbors, then results might be biased against the black-self segregation hypothesis. Second, the most important component of blacks' preferences is not how they rank different neighborhood composition, but instead how unwilling to be minority they are. This is because the former, represented by parameter ρ_1 in our model, does not have major impact on equilibrium. And on the contrary, minority aversion, represented by μ_1, may change equilibria if white households are tolerant. Therefore, if the optimal neighborhood composition of black households, and not their minority aversion, is the variable used for testing, then again the importance of blacks' preferences might be underestimated.

Racial Segregation and Income Inequality

The final topic that we analyze is how racial inequality is related to housing segregation. It has been described many times in the literature that a decrease in inequality is followed by an increase in the level of segregation. For example, it was empirically confirmed by Bayer, Fang & McMillan [7] and Bayer, McMillan & Rueben [5], and theoretically explained by Somanathan & Sethi [33]. We analyze the effect of inequality in residential segregation by varying the relative sensitiveness for neighborhood composition between groups, which is represented by the quotient λ_2/λ_1. We assume that this fraction is grater than one because whites are wealthier and hence they should be willing to spend more money in improving the composition of their neighborhoods. Consequently, a decline in racial inequality can be modeled as λ_1 approaching to λ_2. Like in the previous topic we adopt the convention that type 1 agents are blacks and type 2 are whites, so $\beta_1 < 1/2 < \beta_2$ because blacks are in general a minority, and that $\lambda_1 < \lambda_2$ because whites are wealthier.

When both groups are intolerant this issue is solved by Proposition 4.3.2. They both say that when agents prefer to be segregated, then nothing can prevent it, so the effect of inequality in this case is null. But also in the previous topic, specifically in Proposition 4.3.4, we show that it suffices that white households are intolerant to generate housing segregation. On the contrary when white households have some degree of preferences for integration, racial wealth disparity becomes an important factor, as we show in the next proposition.

Proposition 4.3.6. *Let* $\bar{\rho}_i := \frac{2+\sqrt[\alpha]{\mu_i}}{2+2\sqrt[\alpha]{\mu_i}}$, *and* $\beta_2 > 1/2$. *Then for every* $\rho_2 < \min\{\bar{\rho}_2, \beta_2\}$ *there exists* $\eta > 0$ *such that if* $\lambda_1/\lambda_2 < \eta$, *then the perfectly integrated equilibrium is stable, and for sufficiently small* σ *there is no completely segregated equilibrium.*

In simple words, this proposition says that if whites households are tolerant, below the threshold level $\min\{\bar{\rho}_2, \beta_2\}$, then there is always a sufficiently large difference between relative sensitiveness for which integration emerges. The explanation for this result is that when whites are tolerant they highly value the frontier, because it has mixed neighborhood composition. But when in addition, blacks are much less sensitive they are not able to maintain possession of their side of the border, because whites outbid them.

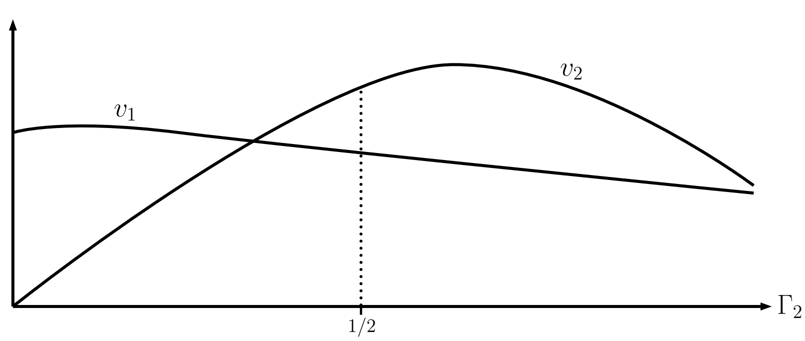


Figure 4.5: Graphic explanation for Proposition 4.3.6.

In Figure 4.5 Proposition 4.3.6 is explained. Curves v_1 and v_2 represent the respective willingness to pay for locations in terms of white neighborhood composition Γ_2. In a completely segregated equilibrium, highest values of Γ_2 are valued more by white households because they occupy those location. For that reason, in Figure 4.5 v_2 lies above v_1 at the right side of the graph. However, v_2 is still above at the left side of the frontier, which is incompatible with a segregated equilibrium. Notice that the composition of the frontier location is always around $1/2$, so when whites outbid blacks at the boundary, it does not simply move to the left, because wherever it is located its composition is the same.

Supported by the link between income and sensitiveness for neighborhood composition, we can use this proposition to provide an alternative explanation for the empirical puzzle of the negative correlation between racial wealth gap and residential segregation. Proposition 4.3.6 demonstrates that when whites are tolerant and significantly more sensitive than blacks, complete segregation does not emerge in equilibrium, and perfect integration is stable. If one accepts that when racial inequality is relevant whites care considerably more about neighborhood improvements; that is, $\lambda_1 \ll \lambda_2$. Then when whites are tolerant and blacks are poor, integration is a more plausible outcome. This statement is explained by the high prices that tolerant whites are willing to pay to live close to the border of a completely segregated allocation, that cannot be matched by poor black households. Notice that when blacks are relatively wealthier, we have shown in previous sections that segregation arises even if whites have strict preferences for integration. Thus when whites are tolerant a decrease in the racial inequality may be followed by a segregation process, because black households are now able to pay to live near the frontier.

4.4 Indifference and Neighborhood Tipping Point

In this section we try to reconcile mixed neighborhoods with our model. Theorem 1 makes a strong statement about how just a tiny discrepancy in preferences immediately eliminates the possibility of different mixed neighborhoods coexisting in equilibrium. And even if the utility functions were such that there is a mixed equilibrium that is not perfect, this allocation would have a fundamental weakness. Any change in the sensitiveness of households would cause a divergence in preferences that is incompatible with non-perfect integration, unless v_i is constant. Consequently, we study equilibria when agents are indifferent between a range of compositions, that is when v_i is constant, because in this case mixed distributions might be robust outcomes. For the moment we assume preferences are described by equation (4.1).

Proposition 4.4.1. *If there is an open interval in which $\tilde{v}_1$ and $\tilde{v}_2$ are constant, then preferences are non divergent* $\forall \lambda_1, \lambda_2 > 0$.

Recall that in general v_i is a function of the neighborhood composition, so it can be parametrized by a single variable Γ_1 or Γ_2. For example, in Section 4.3 we give an expression for v_i as a function of Γ_i, but in Section 4.2 we use Γ_1 as the argument of both functions. This may generate confusion in the last proposition when we ask for $\tilde{v}_1$ and $\tilde{v}_2$ to be constant in an open interval, because in principle they might be parametrized by different arguments. What we precisely mean is that both preferences, as functions of the same argument, are constant in an open interval[2].

[2] For example, preferences described by the utility function p50 considered by Pancs & Vriend [27] do not

If this interval of neighborhood composition, for which both groups are indifferent, contains the population proportion, then one can build integrated allocations that are part of equilibria, and such that Γ is not constant. Again, what we mean by this is that, regardless of the parameter used to express preferences, both type of agents are indifferent around $\Gamma_i = \beta_i$. This equilibrium is built upon indifference, hence prices are constant. And even more, households do not care about small changes in the composition of their neighborhoods, and then perturbations cannot be persistent; in other words, this equilibrium is stable. The following proposition formalizes this idea.

Proposition 4.4.2. *Suppose there is an open interval I in which $\tilde{v}_1$ and $\tilde{v}_2$ are constant. If this interval contains the population proportion, then there exists a stable non-perfectly integrated equilibrium.*

Indifference in preferences for neighborhood composition is important, not only because it makes possible to obtain mixed areas in our model, but also because it is present in Schelling utility function represented in equation (4.7). Preferences described by that equation are non divergent and, even more, they coincide in every interval not containing $1/2$. Thus by the last proposition there are stable mixed equilibria when households have these preferences. We would like to generalize these preferences by allowing the jump discontinuity to be at ρ_i, not necessarily equal to one half as in Schelling model. More precisely, we would like to consider

$$v_i(\Gamma_i) = \begin{cases} \lambda_i & \text{if } \Gamma_i \geq \rho_i \\ 0 & \text{if } \Gamma_i < \rho_i \end{cases} \tag{4.11}$$

where $\lambda_i > 0$ and $\rho_i \in [0, 1]$. With this utility function, households of type i are indifferent between any neighborhood composed of $\Gamma_i < \rho_i$, and also between neighborhoods composed of $\Gamma_i \geq \rho_i$. But they strictly prefer the second over the first, and they are willing to exchange λ_i units of utility of private consumption for moving from a neighborhood of the first type to another of the second.

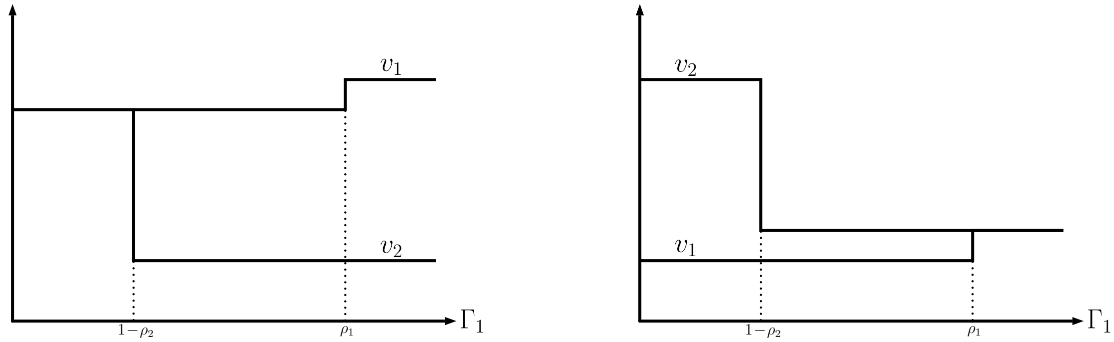
There are two different types of equilibria when preferences are represented by equation (4.11). Like in Section 4.3, we adopt the convention that type 1 agents are blacks; and consequently we assume that $\lambda_1 < \lambda_2$ due to the income disparities between the two groups. The different types of equilibria are shown in Figure 4.6. In the first one, mixed areas have a proportion of black households lesser than the minimum between ρ_1 and $1-\rho_2$. If Γ_1 exceeds any of the two values for some location, then it is segregated and occupied by black households. Prices are a little bit higher for areas with $\Gamma_1 \geq \rho_1$, because black agents are willing to pay more for that places. In the second type of equilibrium the roles are reversed, mixed areas have a black majority and prices are very high in segregated locations with $\Gamma_1 \leq 1-\rho_2$.

Observe that in Figure 4.6a, if we maintain the analogy with the case of racial segregation, $1-\rho_2$ is the maximum proportion of black households that a white individual is willing to

satisfy the hypothesis of Proposition 4.4.1. It can be expressed as

$$v_i(\Gamma_i) = \begin{cases} a + \lambda_i(1-\Gamma_i) & \text{if } \Gamma_i \geq 1/2 \\ 0 & \text{if } \Gamma_i < 1/2 \end{cases}$$

where $a > 0$. Preferences in this case are divergent because if we take Γ_1 as the argument of both functions, v_1 is constant only in the interval $(0, 1/2)$ and v_2 only in the interval $(1/2, 1)$.



(a) Majority white mixed neighborhoods. (b) Majority black mixed neighborhoods.

Figure 4.6: Two types of equilibria when preferences are described by equation (4.11).

accept in his neighborhood. After this point, any arbitrarily small migration of black agents into the area would be persistent, in the sense that it would provide incentives for whites to leave the neighborhood. This is pretty much a neighborhood tipping point, in which a given proportion of black residents triggers a massive movement of white residents away from a neighborhood that was predominately white. Therefore, we are able to explain the tipping-like behavior of white population, found empirically in Card, Mas & Rothstein [11], from another perspective. When agents are indifferent in a range of neighborhood compositions they can coexist in equilibrium, but when the threshold level of tolerance is exceeded they are prompted to collectively leave the neighborhood, rapidly changing the composition of the area.

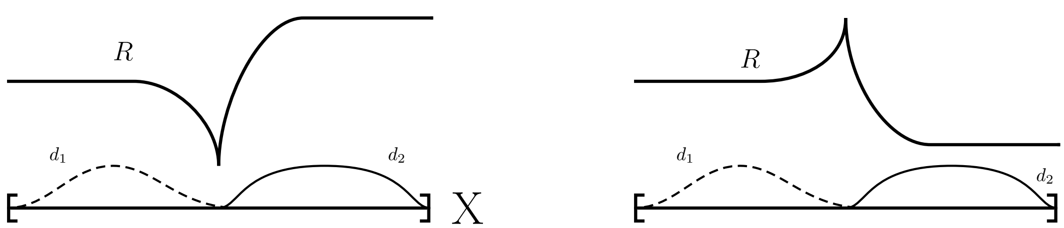
4.5 Price Behavior

If households value the composition of their neighborhood, it should be reflected in the housing market as price differences paid by households for improving some characteristics of their surrounding. And thus price rent function could be used to make inferences about individuals' preferences. However, there is a substantial difference between our model and what is observed in reality. In our framework, agents only care about the characteristics of their neighbors, but in reality there is a strong correlation between the identity of neighbors and unobserved neighborhood quality. Therefore, if, for example, one wants to isolate the impact of race in preferences for neighborhood composition, then data should be controlled by neighborhood quality. Regarding this issue, Bayer, Ferreira & McMillan [6] propose a general method to deal with the correlation problem that arises when preferences are measured. They take advantage of the discontinuity provided by schools attendance boundaries to deduce the value that agents put on neighborhood composition.

Throughout this section, we neglect the endogeneity that may emerge when measuring preferences for neighborhood composition, even though we are aware of its existence. We proceed this way because we want merely to expose how prices can give clues about how agents rank different neighborhoods according to the characteristics of their neighbors. As we have demonstrated in previous sections, residential segregation can appear in many different situations under rather weak assumptions. Hence prices can help determining which of the possible causes are in fact influencing households' decisions. But deciding and implementing a proper econometric method to do so is out of the scope of this work.

Our main finding is that prices around the frontier of segregated areas are the ones that convey the most information about individuals' preferences, because at these locations is where neighborhood composition has variation. Unlike inside the segregated areas, where the proportion of agents is almost entirely of one type, near the frontier Γ_i takes values from 1 to $1/2$. So prices should behave accordingly to preferences, increasing as approaching to the boundary if households prefer more heterogeneous locations, and decreasing if the opposite is preferred. At the same time, the slope of these price changes gives hints about how disposed are agents to trade private consumption for an improvement in the quality of their neighborhood; that is the derivative of R can be used to estimate the sensitiveness of agents, λ_i.

In Figure 4.7 we present how prices behave in the case studied in Section 4.3, namely racial segregation. On the left side of the figure it is shown the case where households strictly prefer to live in neighborhoods with a high share of alike neighbors, that is when their intolerance is $\rho_i = 1$. In this case, locations close to the frontier have a more heterogeneous composition, and therefore agents are willing to pay a lower price to live there. One can conclude that if segregation driven by strong preferences for segregation, which can be a possible cause as proved in Proposition 4.3.2, then prices attain their lower level at the frontier. On the right side exactly the opposite happens, segregation can also be occasioned by strict preferences for integration and a reluctance to be minority as shown Proposition 4.3.3, but in this case agents prefer to live near the frontier because their favorite composition is $\rho_i = 1/2$. Hence prices attains their maximum at the frontier and inside the segregated areas are lower.



(a) Households prefer segregation: $\rho_i = 1$. (b) Households prefer integration: $\rho_i = 1/2$.

Figure 4.7: Rent price function R in a completely segregated equilibrium in the case of homophily.

It is noteworthy that in both cases type 2 agents are more sensitive, that is $\lambda_1 < \lambda_2$. It can be determined in the figures because the rent price function is steeper at the type 2 side of the boundary, which in turn generates a larger difference between the price of the frontier location and the price of a location inside the segregated area of type 2 agents. This remark is important because it shows that complete racial segregation is consistent with differences in prices between white and black neighborhoods. Surprisingly, Boustan [8] argues exactly the opposite saying that if there is enough housing in each neighborhood to accommodate both groups, then there is no reason to expect higher prices in the white neighborhood, regardless of how much they are willing to pay to avoid living among the other group. This reasoning may sustain in a model where neighborhoods are independent and isolated one from the other;

but from the perspective of a continuous city, the transition from the interior of a segregated neighborhood to the boundary generates changes in neighborhood composition that forces prices to respond according to the different sensitiveness of agents. If white households are intolerant and sensitive, then prices should indeed be higher in white neighborhoods, because marginal changes are more important for them.

But also there is a third cause of segregation that we explicitly analyzed: socioeconomic segregation. This issue is represented in Figure 4.8, and it can be observed that prices are strictly increasing as the proportion of the 'good' households increases. But there is a significant spike around the frontier, where the neighborhood compositions changes from $\Gamma_1 \approx 0$ to $\Gamma_1 \approx 1$. Recall that, in order to have segregation caused by externalities generated by one group, the leading group should be more sensitive than the other. This translates into a change in the slope of prices at the frontier. More precisely, prices increases more at the 'good' side of the boundary, because those agents are willing to pay more for marginal improvements in their neighborhood.

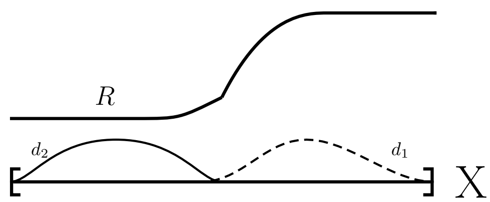


Figure 4.8: Rent price function R in a completely segregated equilibrium in the case of a leading group.

After analyzing prices in equilibrium, it can be seen that the most relevant hints about household preferences are given near the frontier. At least from a qualitative point of view, prices can be used to decide which of the multiple possible causes for housing segregation are more important when studying a given segregated city. This stresses the importance of considering a continuous set up, with smooth transitions between neighborhoods. Models in which neighborhood are thought as isolated jurisdictions do not capture phenomena caused by the connectedness of the city, where living at the boundary of a region is not the same as inside of it. For that reason, we think that it could be a good contribution to incorporate frontier price behavior to improve hedonic regressions.

Chapter 5

Concluding Remarks

It is not our intention to provide a realistic model of a city. But rather to expose, in a simple set-up, the different forces interacting in the phenomenon of residential segregation. Consequently, we do not build this model only to show that segregation can be predicted in an analytical framework, because it has been done successfully multiple times in the past. Instead, we expect to contribute with a rigorously founded basic tool that may help isolating the different causes of housing segregation, based on rational behavior of agents in an economic environment. It separates from the classical Schelling dynamic models as it incorporates market interactions between agents. But it is also different from the standard models where neighborhoods are understood as isolated jurisdictions, and others where agents only care about 'next-door neighbors'. Because in our set up there are spatial externalities generated by the identity and distance of neighbors at each location. At this point we deeply studied how this external effects are perceived by agents, unveiling its complexity and connecting it to a branch of statistics. Our approach let us capture the rich interactions between agents at the frontier of neighborhoods that would be impossible to analyze in a discontinuous model.

The main result of our paper is Theorem 1 that connects segregation to subtle discrepancies in preferences for spatial externalities. It says that if agents have different valuations for changes in neighborhood composition, then there are only two possible outcomes: perfect integration or some degree of segregation; but different mixed areas cannot coexist in equilibrium. It is an important result because it shows how pervasive is the problem of housing segregation in any city. It suffices that households get involve in non market interactions that depend on distance, and that they value differently. Besides, it can be explained by a simple economic reasoning. In a continuous city with spatial externalities, each place is distinct form the others. So if households of different types have dissimilar preferences, they cannot live in the same neighborhood, because to share a location they should be willing to pay exactly the same price for it, otherwise one would outbid other.

The development of a theory of equilibria stability, based on the canonical theory of tâtonnement stabilty and in the early neighborhood tipping models, is an important contribution because of its generality. It let us deal with the problem of multiple equilibria, common when modeling residential segregation, because it reduces the set of plausible outcomes and allows us to make more robust predictions. With Theorem 2 we give a characterization of stable equilibria in terms of the derivatives of preferences for neighborhood composition at segregated and frontier locations. It is relevant because its similarity to the well known characterization of stable equilibria in terms of the derivative of the excess of demand function serves as a proof of the link between the classical theory of dynamics and our modification. And also because it allows us to make explicit calculations to determine whether segregation or integration is more likely to be observed in each situation.

When we apply our model to particular utility functions, we obtain some theoretical results that are of general interest. We derive segregation when one group generates positive spatial externalities, but only when that group cares more about neighborhood composition, which is exactly the case of socioeconomic segregation. Extending our set up, we were able to analyze the phenomenon of gentrification as a consequence of local public goods. And the main consequence of this extension is that we show how the concentration of placed based investments generate gentrification. Avoiding this problem is important if governments want to benefit low-income households in the long term.

We also study segregation in the presence of homophily, and we recover the classical results of Schelling, but in a more rational framework. First, in Proposition 4.3.3 we demonstrate that households segregate even when they prefer an integrated city just because they are reluctant to be minority. The famous result of Schelling's spatial proximity model is analogous to this one, but we introduced new components to the minority aversion of agents, mainly relating it to private consumption. Second, we find that agents present tipping-like behavior when there is indifference in preferences. This is an alternative explanation to the actions described by Schelling's neighborhood tipping model, which is important considering the recent empirical evidence supporting this behavior among white population.

But we also deduce some lessons about two topics directly related to racial segregation. We first demonstrate that to test the black self-segregation hypothesis there are some theoretical corrections to the standard approach. On the one hand, whites' preferences should be also taken into account, because they affect equilibria, and consequently the role of blacks' preferences. And on the other hand, how blacks rank different neighborhoods should not be considered as the main component of blacks' preferences. But instead how unwilling to be minority they are, because, as opposed to their tolerance, it can change equilibria. Secondly, we analyze how income disparities between races affect the allocation of households, providing a new explanation for this empirical puzzle. If whites prefer integration, in an hypothetical segregated allocation, areas near the frontier are highly valued by them because those locations provide mixed neighborhood composition. So when blacks are poor, it is impossible for them to maintain possession of their side of the border, because whites pay higher prices. Thus segregation does not emerge in equilibrium. But when blacks are relatively richer they can afford to live near the boundary, regardless of whites' preferences, so segregation arises.

Finally, we demonstrate the importance of studying the rent price function at the frontier of segregated neighborhoods as a method to deduce the causes of segregation. This function presents different shapes that depend on households preferences. But at the boundary is where the clues are more evident, because in that area is where neighborhood composition has variation. We propose studying price behavior as a method to make inferences about households preferences and, consequently, to identify the most important factors that determine its emergence. This finding highlights again the importance of a continuous set up. Other models where neighborhoods are independent entities do not capture the rich interactions at the boundaries, because locations of the same neighborhood are indistinguishable and are not affected by the characteristics of the other areas. And in the same line, models where the city is continuous, but agents only look at the local composition of their place, do not capture the effects of surrounding locations either. Because in this case each location can be interpreted as an isolated point, regardless of the topology of the city.

There are two natural extensions that can be added to our work in future investigation. The first one is related to the geometry of the city, it would be interesting to extend it to the 2-dimensional case. In regard to this issue, it should be noted that equilibrium properties would be equivalent, particularly, any divergence in preferences would immediately eliminate the possibility of different mixed neighborhoods. And many of the applied results about racial and socioeconomic segregation are driven by principles that are not related to the geometry of the city. However, the stability section need to be readjusted to the new set-up, especially a new characterization of stable equilibria should be deduced. The second extension is incorporating different preferences within each group. For example, it is usual in the literature to assume that preferences distribute according to a given law, and it would be interesting to incorporate it in our set up.

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Appendix A

Proofs for Chapter 2

Proofs for Section 2.4

Proof of Proposition 2.4.1. Let v_i be a continuous function. Suppose by contradiction that the rent price function $R(x)$ has a discontinuity at $x \in X$. Notice then that $u(y - R(x))$ is discontinuous at x too, because u is continuous. Without loss of generality lets assume that $u(y-R(x)) > \liminf_{z\to x} u(y-R(z))$, the other case is analogous and explained below. Thus there exists $K > 0$ and $(z_n)_{n\in\mathbb{N}} \subseteq X$ such that $z_n \to x$ and

$$u(y - R(x)) - u(y - R(z_n)) > K \quad \forall n \in \mathbb{N} \tag{A.1}$$

But $\Gamma_i(x)$ is continuous, hence so is $v_i(\Gamma(x))$, for all $i = 1, 2$. Thus we can choose $N \in \mathbb{N}$ and $\eta < K$ satisfying (A.1) and

$$|v_i(\Gamma(x)) - v_i(\Gamma(z_N))| < \eta \quad i = 1, 2 \tag{A.2}$$

Equations (A.1) and (A.2) together imply that for $i = 1, 2$

$$\begin{aligned} U_i(x) &= u(c - R(x)) + v_i(\Gamma(x)) \\ &> u(c - R(z_N)) + K + v_i(\Gamma(z_N)) - \eta \\ &> u(c - R(z_N)) + v_i(\Gamma(z_N)) \\ &= U_i(z_N) \end{aligned}$$

so every household would strictly prefer to live in x over z_N, which is impossible in equilibrium.

In the case that $u(y - R(x)) \leq \liminf_{z\to x} u(y - R(z))$, then it must be true that $u(y - R(x)) < \limsup_{z\to x} u(y - R(z))$, otherwise $u(y - R(x))$ would be continuous. Proceeding in a similar manner one can choose $z \in X$ such that $U_i(x) < U_i(z)$ for $i = 1, 2$, which is again impossible in equilibrium.

$\square$

Proof of Proposition 2.4.2. Let $R : X \longrightarrow \mathbb{R}_+$ the rent price function in equilibrium. And for every $x \in X$, let $d^*(x)$ be the pointwise solution of E3, that is

$$d^*(x) \in \operatorname*{argmax}_{x} R(x)d - C(d) \tag{A.3}$$

The FOC of this problem is $C'(d^*(x)) = R(x)$. To prove equation (2.7) it suffices to prove that d^* is an integral solution of E3, and that it is unique almost everywhere.

Suppose d is not equal to d^* as a function. Then there exists $A \subseteq X$, with $dx(A) > 0$, such that $d(x) \neq d^*(x)$ for every $x \in A$. Because C is a convex function, for every $x \in X$ the only solution to (A.3) is $d^*(x)$. Hence $\forall x \in X$

$$R(x)d(x) - C(d(x)) \leq R(x)d^*(x) - C(d^*(x))$$

with strict inequality for $x \in A$. And then

$$\int_X R(x)d(x) - C(d(x))\, dx < \int_X R(x)d^*(x) - C(d^*(x))\, dx$$

Therefore d is not a solution of $E3$. Or in other words, if d is a solution of E3, then $d = d^*$ a.e.

Finally, because C is convex and $C(0) = 0$, $\forall t > 0\ C(t) < tC'(t)$, so

$$\begin{aligned}
\pi(d^s, R) &= \int_X R(x)d^s(x) - C(d^s(x))\, dx \\
&= \int_X C'(d^s(x))d^s(x) - C(d^s(x))\, dx \\
&> \int_X C(d^s(x)) - C(d^s(x))\, dx \\
&= 0
\end{aligned}$$

□

Proof of Proposition 2.4.3. Let $v_i : [0,1] \longrightarrow \mathbb{R}$ for $i = 1, 2$ be any preferences for neighborhood composition and $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ any continuous, increasing and convex cost function. Suppose that density and prices are constant: $R(x) = C'(1)$ and $d_i(x) = \beta_i$.

Notice $\gamma_i(x) = \frac{d_i(x)}{d_1(x)+d_2(x)} = \beta_i$ is also constant, and then

$$\Gamma_i(x) = K(x) \cdot \int_0^1 e^{\frac{-(y-x)^2}{2\sigma^2}} \beta_i\, dy = \beta_i$$

is constant too. So no location is strictly preferred over another by any household, because prices and neighborhood composition are the same for every place in the city. More precisely, $R(x)$ and $\Gamma(x)$ are constant then $U(y - R(x), x)$ is also constant, so E2 holds.

Now by E1, $\int_0^x dy = x = D_1(x) + D_2(x) = \int_0^x d^s(y)\, dy$, thus supplied density is equal to 1. And this is consistent with E3 because the first order condition of profit maximization is

$C'(1) = C'(d^s(x))$, which is in fact the optimal condition thanks to the convexity of the cost function, as seen in the proof of Proposition 2.4.2.

In conclusion, E1, E2, E3 are satisfied regardless of particular choices of v_i. Hence, in general, this perfectly integrated allocation is an equilibrium.

$\square$

Appendix B

Proofs for Chapter 3

Proofs for Section 3.1

Proof of Theorem 1. Let $\left(\vec{d}, d^s, R\right)$ be an equilibrium. That is, let d_1, d_2 be the demand densities, d^s the supply density and R the rent price function in equilibrium. Suppose that preferences for neighborhood composition are divergent. To prove the theorem we will assume that there is a non segregated neighborhood, and we will demonstrate that the equilibrium must be perfectly integrated.

More precisely, lets assume that there is an interval (a, b) in which $\gamma_i|_{(a,b)} \neq 1$ a.e. This implies that there is a dense subset A of (a, b), for which $d_1, d_2 > 0$. Then by E2

$$u(y - R(x)) + v_i(\Gamma(x)) = u(y - R(z)) + v_i(\Gamma(z)) \quad \forall x, z \in A \quad i = 1, 2 \tag{B.1}$$

Joining equation (B.1) for the case $i = 2$ and the case $i = 1$, it is obtained

$$v_1(\Gamma(x)) - v_1(\Gamma(z)) = v_2(\Gamma(x)) - v_2(\Gamma(z)) \quad \forall x, z \in A \tag{B.2}$$

Together with the assumption of finite discontinuities of v_i equation (B.2) implies that $v_1(\Gamma(x)) - v_2(\Gamma(x))$ is constant for some interval, lets say without loss of generality (a, b). This is equivalent to state that $v_1 - v_2$ is a constant function in the interval $\Gamma(a, b)$. But preferences are assumed to be divergent, so $\Gamma(a, b)$ must be a degenerated interval (it must be a singleton), otherwise the definition of divergent preferences would be contradicted. Therefore, it can be concluded that $\Gamma(x)$ is constant for all $x \in (a, b)$, because it is the only way that the image of (a, b) under Γ could be a single point.

Finally, observe that $\Gamma_i(x) = T$ for every $x \in (a, b)$ is equivalent to $\int \gamma_i - T \, dM = 0$ for all $M \in \mathcal{M}_{(a,b)}$. And we know form Lehmann & Scheffé [20, 21] that the family of measures

$$\mathcal{M}_{(a,b)} = \left\{ K(x) e^{\frac{-(y-x)^2}{2\sigma^2}} \, dy \right\}_{x \in (a,b)}$$

is complete. So by the completeness of $\mathcal{M}_{(a,b)}$, γ_i is constant in the whole city. Or in other words, the allocation is perfectly integrated.

$\square$

Proofs for Section 3.2

In this section we denote total density by $d^s = d_1 + d_2$ as before, and characterize equilibrium by giving its density vector and rent price function $\left(\vec{d}, d^s, R\right)$, where the density vector is given by by $\vec{d} = (d_1, d_2)$.

Proof of Theorem 2. Let $\left(\vec{d}, d^s, R\right)$ be an equilibrium, and v_1, v_2 be continuously differentiable. In addition, denote $\bar{U}_i$ the utility level that type i agents attain in equilibrium.

For the first part, let $x_1, x_2 \in X$ de two mixed or frontier locations such that that

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x_i)) \tag{B.3}$$

We will show that there are φ-persistent ε-migrations between x_1 and x_2 for arbitrarily small φ and ε. So let $\vec{d}^{\varepsilon}$ be a ε-migration between x_1 and x_2 for some $\varepsilon > 0$, and denote $\mathcal{V}_1$ and $\mathcal{V}_2$ the open neighborhoods around x_1 and x_2, respectively, that are mentioned in the definition of migration, Definition 3.2.1. Because $v_i \in \mathcal{C}^1$, equation (B.3) implies that there exists open neighborhoods $\mathcal{U}_1$ of $\Gamma(x_1)$ and $\mathcal{U}$ of $\Gamma_i(x_2)$, such that for every $\Gamma \in \mathcal{U}_1 \cup \mathcal{U}_2$

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma)$$

Notice that when ε is small, for every $x \in \mathcal{V}_i$, $\Gamma_i(x) < \Gamma_i^{\varepsilon}(x)$, because after the migration type i households occupy all locations in $\mathcal{V}_i$. And also for a sufficiently small ε, $\Gamma^{\varepsilon}(x) \in \mathcal{U}_i$ because neighborhood composition changes just a little. Therefore, for every $x \in \mathcal{V}_i$

$$\begin{aligned}
v_i(\Gamma^{\varepsilon}(x)) - v_i(\Gamma(x)) &= \int_{\Gamma(x)}^{\Gamma^{\varepsilon}(x)} \frac{\partial v_i}{\partial \Gamma_i}(\Gamma)\, d\Gamma \\
&> \int_{\Gamma(x)}^{\Gamma^{\varepsilon}(x)} \frac{\partial v_{-i}}{\partial \Gamma_i}(\Gamma)\, d\Gamma \\
&= v_{-i}(\Gamma^{\varepsilon}(x)) - v_{-i}(\Gamma(x))
\end{aligned}$$

Then, for a sufficiently small ε, for every $x \in \mathcal{V}_i$

$$v_i(\Gamma^{\varepsilon}(x)) - v_i(\Gamma(x_i)) > v_{-i}(\Gamma^{\varepsilon}(x)) - v_{-i}(\Gamma(x_i))$$

So we can choose $R(x)$ such that

$$v_i(\Gamma^{\varepsilon}(x)) - v_i(\Gamma(x)) > u(y - R(x)) - u(y - R^{\varepsilon}(x)) > v_{-i}(\Gamma^{\varepsilon}(x)) - v_{-i}(\Gamma(x)) \tag{B.4}$$

But because x_i is a mixed or frontier location, we know that households attain the equilibrium utility level at that point. That is, $u(y - R(x_i)) + v_1(\Gamma(x_i)) = \bar{U}_1$ and $u(y - R(x_i)) + v_2(\Gamma(x_i)) = \bar{U}_2$. So rearranging equation (B.4)

$$u(y - R^{\varepsilon}(x)) + v_1(\Gamma^{\varepsilon}(x)) - \bar{U}_1 > 0 > u(y - R^{\varepsilon}(x)) + v_2(\Gamma^{\varepsilon}(x)) - \bar{U}_2$$

which means that for a sufficiently small ε-migration between x_1 and x_2, there is a rent price function R^ε that makes that migration φ-persistent, where $\varphi > \|R^\varepsilon - R\|_\infty$.

Now we need to show that φ can be taken arbitrarily small. Notice that v_i is continuous and defined on the compact set $[0,1]$, then it is uniformly continuous. Hence, for every $\eta > 0$ there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$

$$v_i(\Gamma^\varepsilon(x)) - v_i(\Gamma(x)) < \eta \quad \forall x \in \mathcal{V}_i, \quad i = 1, 2$$

Now, if we take $\eta = u(y) - u(y - 1/n)$ we have that for sufficiently small ε

$$\begin{aligned} u(y) - u(y - 1/n) &> v_i(\Gamma^\varepsilon(x)) - v_i(\Gamma(x)) \\ &> u(y - R(x)) - u(y - R^\varepsilon(x)) \\ &\geq u(y) - u\left(y - (R^\varepsilon(x) - R(x))\right) \end{aligned}$$

where for the last step we have used the concavity of u. Now, because u is increasing

$$u(y - 1/n) < u\left(y - (R^\varepsilon(x) - R(x))\right) \Longrightarrow R^\varepsilon(x) - R(x) < 1/n$$

and therefore $\|R^\varepsilon - R\|_\infty < 1/n$ for a sufficiently small ε.

For the converse part we divide the proof in two parts. First let $x \in X$ be a segregated location, we will show that for sufficiently small ε and φ, x cannot be part of a φ-persistent ε-migration. Without loss of generality, lets say that x is occupied by type 1 households, so that $u(y - R(x)) + v_2(\Gamma(x)) < \bar{U}_2$. In this case the only possible invaders around x are of type 2, because type 1 already live there. Lets call $\Delta \doteq \bar{U}_2 - (u(y - R(x)) + v_2(\Gamma(x))) > 0$. By the continuity of v_2 and $\Gamma(\ \cdot 0;)$ there exists $\bar{\varepsilon} > 0$ such that if $\varepsilon < \bar{\varepsilon}$ then

$$|v_2(\Gamma(x)) - v_2(\Gamma^\varepsilon(x))| < \Delta/2$$

Also, by the continuity of u there is a $\bar{\varphi} > 0$ such that if $|R(x) - R^\varepsilon(x)| < \bar{\varphi}$ then

$$|u(y - R(x)) - u(y - R^\varepsilon(x))| < \Delta/2$$

This imply that for every $\varphi < \bar{\varphi}$ and $\varepsilon < \bar{\varepsilon}$

$$\begin{aligned} u(y - R(x)^\varepsilon) + v_2(\Gamma^\varepsilon(x)) &< u(y - R(x)) + \Delta/2 + v_2(\Gamma^\varepsilon(x)) \\ &< u(y - R(x)) + \Delta/2 + v_2(\Gamma(x)) + \Delta/2 \\ &= u(y - R(x)) + v_2(\Gamma(x)) + \Delta \\ &= \bar{U} \end{aligned}$$

Therefore for sufficiently small ε and φ, a ε-migration at x is not φ-persistent.

Finally, let $x \in X$ be such that for arbitrarily small ε and φ there is a φ-persistent ε-migration at x. By the arguments given above, we know that x must be a frontier or mixed location, which means that $u(y - R(x)) - v_i(\Gamma(x)) = \bar{U}_i$. Lets suppose by contradiction that $\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x)) < \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x))$. Without loss of generality, lets assume that type 1 households migrate around x, the other case is analogous. So for sufficiently small ε, $\Gamma_1^\varepsilon(x) > \Gamma_2^\varepsilon(x)$, and then

$$v_1(\Gamma^\varepsilon(x)) - v_1(\Gamma(x)) > v_2(\Gamma^\varepsilon(x)) - v_2(\Gamma(x))$$

so, subtracting $u(y - R(x))$ at both sides

$$v_1(\Gamma^\varepsilon(x)) \underbrace{-u(y - R(x)) - v_1(\Gamma(x))}_{-\bar{U}_1} > v_2(\Gamma^\varepsilon(x)) \underbrace{-u(y - R(x)) - v_2(\Gamma(x))}_{-\bar{U}_2}$$

And then, adding $u(y - R^\varepsilon(x))$ at both sides, for any rent price function R^ε

$$u(y - R^\varepsilon(x)) + v_1(\Gamma^\varepsilon(x)) - \bar{U}_1 > u(y - R^\varepsilon(x)) + v_2(\Gamma^\varepsilon(x)) - \bar{U}_2$$

contradicting the persistence of the migration. In conclusion, $\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x))$ must be greater or equal than $\frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x))$, otherwise x cannot be part of a persistent migration when ε is too small. $\square$

Proof of Corollary 2.1. Let $\left(\vec{d}, d^s, R\right)$ be the perfectly integrated equilibrium. Notice that under this distribution, every location $x \in X$ is mixed and has neighborhood composition $\Gamma_i(x) = \beta_i$, because it is the mass of type i agents. Then any of the equations of Theorem 2 can be directly applied. Therefore, $\left(\vec{d}, d^s, R\right)$ is unstable if $\frac{\partial v_1}{\partial \Gamma_1}(\beta) > \frac{\partial v_2}{\partial \Gamma_1}(\beta)$.

$\square$

Proof of Corollary 2.2. Let $\left(\vec{d}, d^s, R\right)$ be a segregated equilibrium, by Theorem 2 the only candidates for unstable locations are the frontier ones, because there are no mixed locations in a segregated distribution. Hence it is unstable if there are two frontier locations x_1, x_2 such that $\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(x_i))$, so that there could be persistent migration between x_1 and x_2. Particularly, a completely segregated equilibrium is stable because there is only one frontier location.

$\square$

Appendix C

Proofs for Chapter 4

In favor of exposition, before proving the results of the body part, we demonstrate two lemmas that stem from general principles of our model and that can be applied in different contexts. They depend only on preferences for neighborhood composition, so in order to state them as general as possible, we do not assume any shape of v_i a priori. Instead, we treat its argument Γ as an ordered pair $\Gamma = (\Gamma_1, \Gamma_2)$, and to avoid confusions we use the notation $v_i(\Gamma_1, \Gamma_2)$. Observe that we have treated these functions as one-dimensional, because $\Gamma_1 = 1 - \Gamma_2$ so v_i can be parametrized by a single parameter. We give a brief explanation before each result for a better comprehension.

In a completely segregated equilibrium, each group must value be willing to pay more than the other for locations inside their segregated area. When the city is large enough, or equivalently when σ is small enough, the neighborhood composition at the most segregated area of type i is approximately $\Gamma_i \approx 1$. To determine the exact willingness to pay of agents, the frontier serves as a references, because both groups coexists at that point. If $x_0 \in X$ is the boundary, then the value of another location $x \in X$ for an agent of type i is $v_i(\Gamma(x)) - v_i(\Gamma(x_0))$. Thus if $v_1(1,0) - v_1(\Gamma(x_0)) < v_2(1,0) - v_2(\Gamma(x_0))$, then type 2 agents are the highest for locations of occupied by type 1 households, which is impossible with equilibrium.

Lemma C.0.1. *If*

$$v_1(1,0) - v_1(1/2, 1/2) < v_2(1,0) - v_2(1/2, 1/2)$$

or

$$v_2(0,1) - v_2(1/2, 1/2) < v_1(0,1) - v_1(1/2, 1/2)$$

then for a sufficiently small σ there is no completely segregated equilibrium.

Proof of Lemma C.0.1. Let d_1 and d_2 be densities such that $\gamma_1(x) = \mathbb{1}_{[0,x_0]}(x)$, that is type 1 agents are segregated in the interval $[0, x_0]$ and type 2 in the interval $[x_0, 1]$. This is an arbitrary completely segregated allocation, the case in which households occupy the other side of the city is analogous.

First, because both agents live at the frontier x_0 (or at least arbitrarily close), in equilibrium they should attain the optimal utility level $\bar{U}_i$ at that point. So we can use it to determine who values more a given location. If for any $x \in X$ we have that $v_i(\Gamma(x)) - v_i(\Gamma(x_0)) > v_j(\Gamma(x)) - v_j(\Gamma(x_0))$, then for any price $R(x)$

$$\begin{aligned} U_i(x) - \bar{U}_i &= u(c - R(x)) + v_i(\Gamma(x)) - u(c - R(x_0)) - v_i(\Gamma(x_0)) \\ &> u(c - R(x)) + v_j(\Gamma(x)) - u(c - R(x_0)) - v_j(\Gamma(x_0)) \\ &= U_j(x) - \bar{U}_j \end{aligned} \tag{C.1}$$

This means that type i households live at x because they value that location more.

Secondly, notice that under this completely segregated distribution, by definition

$$\Gamma_1(0) = \frac{\int_0^{x_0} e^{\frac{-y^2}{2\sigma^2}}\, dy}{\int_0^{1} e^{\frac{-y^2}{2\sigma^2}}\, dy} \xrightarrow[\sigma \to 0]{} 1$$

So when the variance gets close to zero, that is $\sigma \to 0$, $\Gamma_1(0)$ gets arbitrarily close to 1. In other words, when the city is big enough, the most segregated neighborhood of type 1 agents, namely $x = 0$, is almost completely composed by type 1 individuals, because households do not perceive the other agents due to the long distance between neighborhoods. The same argument can be made for the frontier $x = x_0$, thus for small σ

$$\Gamma_1(x_0) \xrightarrow[\sigma \to 0]{} 1/2$$

.

Now, we can use these two approximations and the first observation to obtain a necessary condition for equilibrium. Namely, type 1 households should value more $x = 0$, because they live there by hypothesis. Therefore, if

$$v_1(1, 0) - v_1(1/2, 1/2) < v_2(1, 0) - v_2(1/2, 1/2)$$

for sufficiently small σ, type 2 agents would value more location $x = 0$. Which would be incompatible with equilibrium, so the completely segregated allocation cannot be an equilibrium.

An analogous argument demonstrate the same when $v_2(0, 1) - v_2(1/2, 1/2) < v_1(0, 1) - v_1(1/2, 1/2)$.

$\square$

On the other hand, when households pay more than the others for a marginal increment in the proportion of alike households, regardless of the neighborhood composition, then complete segregation always exists. This is because, in a segregated allocation, Γ_i increases to the side of type i segregated area. So if v_i increases more rapidly than v_j in the same direction, then type i agents will always value more these locations than those at the other side of the the frontier. But assuming that the derivative of v_i with respect to Γ_i is grater the derivative of v_j with respect to the same argument for every neighborhood composition, is the same than assuming that equation (3.4) of Theorem 2 is satisfied for every $\Gamma \in [0, 1]$. In general, this would mean that every equilibrium should be unstable, because for every two different mixed or frontier locations one could build a persistent migration. However, there is a unique exception. If there is a completely segregated equilibrium, then it has only one frontier and no mixed locations. Consequently, it is stable, no matter how the derivatives of v_1 and v_2 are. In particular, if inequality (3.4) is satisfied for every possible composition, then it is the only possible stable equilibrium

Lemma C.0.2. *Let v_a and v_2 be differentiable functions. If for every $\Gamma \in [0,1]$*

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma, 1-\Gamma) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma, 1-\Gamma)$$

then for every $\sigma > 0$ the only stable equilibrium is completely segregated.

Proof of Lemma C.0.2. Suppose for every $\Gamma \in [0,1]$

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma, 1-\Gamma) > \frac{\partial v_2}{\partial \Gamma_1}(\Gamma, 1-\Gamma)$$

And like in the proof of the previous lemma, let d_1 and d_2 be densities such that $\gamma_1(x) = \mathbb{1}_{[0,x_0]}(x)$, that is type 1 agents are segregated in the interval $[0, x_0]$ and type 2 in the interval $[x_0, 1]$.

First, to prove the existence of a completely segregated equilibrium we use equation (2.6) to determine, up to a constant, a rent price function R for which households are indifferent between locations occupied by the alike agents. This method can always be applied, and it implies that each point where $d_i > 0$ individuals of type i attain the same utility level $\bar{U}_i$. But to completely satisfy E2 we need to ensure that households do not prefer areas where the other group lives. Observe that because the city is completely segregated and type 1 agents are concentrated at the left side, so for any $x < x'$, we have that $\Gamma_1(x) > \Gamma_1(x')$. This means that the neighborhood composition of type 1 agents is decreasing. So for every $x < x_0$, applying the fundamental theorem of calculus, the chain rule and the hypothesis

$$\begin{aligned}
v_1(\Gamma(x)) - v_1(\Gamma(x_0)) &= -\int_x^{x_0} \frac{\partial v_1}{\partial \Gamma_1}(\Gamma(y)) \cdot \frac{d}{dx}\Gamma(y)\, dy \\
&= \int_x^{x_0} \frac{\partial v_1}{\partial \Gamma_1}(\Gamma(y)) \cdot \underbrace{\left(-\frac{d}{dx}\Gamma(y)\right)}_{>0}\, dy \\
&> \int_{x_0}^{x} \frac{\partial v_2}{\partial \Gamma_1}(\Gamma(y)) \cdot \left(-\frac{d}{dx}\Gamma(y)\right)\, dy \\
&= v_2(\Gamma(x)) - v_2(\Gamma(x_0))
\end{aligned}$$

Then by the same reasoning of equation (C.1) we can conclude that $U_2(x) < \bar{U}_2$. And proceeding in a similar way it can be concluded that for every $x > x_0$ $U_1(x) < \bar{U}_1$. In conclusion, for any x_0 and constant R_0, households maximize utility given the distribution defined by x_0 and the rent price function $R(x) + R_0$. By imposing E1 and E3 the exact value of x_0 and R_0 can be determined to characterize equilibrium.

To prove that it s the only stable equilibrum, we let $\left(\vec{d}, d^s, R\right)$ be any equilibrium different from the completely segregated. Notice that the allocation of this equilibrium must have at least two frontier or mixed locations. If it has one mixed location, then it has infinitely many, because by Definition 2.2.2 mixed locations are surrounded by an interval of mixed locations. And if it has not mixed locations, then it must have at least one frontier location, because with two types of agents, there cannot be only segregated locations. Finally, it cannot have only one frontier location, because we assumed that the allocation is not completely segregated (see Definition 2.2.3). Now, let x_1 and x_2 be two frontier or mixed locations. By hypothesis we know that

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma(x_i)) > \frac{\partial v_2}{\partial \Gamma_1}((\Gamma(x_i))$$

for $i = 1, 2$. So by Theorem 2 the equilibrium is unstable. As a remark, recall that a completely segregated equilibrium is always stable, when it exists.

□

Proofs for Section 4.2

For this section we assume that v_i satisfy equation (4.2). Notice that in this case the notation $v_i(\Gamma_1, \Gamma_2)$, used in Lemmas C.0.1and C.0.2, is equivalent to $v_i(\Gamma_1)$.

Proof of Proposition 4.2.1. Suppose preferences are not divergent, then there is a non degenerated interval I in which $\lambda_1 \tilde{v}(\Gamma_1) - \lambda_2 \tilde{v}(\Gamma_1) = (\lambda_1 - \lambda_2)\tilde{v}(\Gamma_1)$ is constant. But because $\tilde{v}$ is increasing, then $\lambda_1 = \lambda_2$.

□

Proof of Corollary 4.2.1.1. This is a direct conclusion from Theorem 1 and Proposition 4.2.1.

□

Proof of Proposition 4.2.2. Let $\lambda_1 < \lambda_2$.

First, observe that $\frac{\partial \tilde{v}}{\partial \Gamma_1}(\beta) > 0$ because, by definition, $\tilde{v}$ is an increasing function of Γ_1. So the perfectly integrated equilibrium is stable, because when $\lambda_1 < \lambda_2$

$$\frac{\partial v_1}{\partial \Gamma_1}(\beta) = \lambda_1 \frac{\partial \tilde{v}}{\partial \Gamma_1}(\beta) < \lambda_2 \frac{\partial \tilde{v}}{\partial \Gamma_1}(\beta) = \frac{\partial v_2}{\partial \Gamma_1}(\beta)$$

Then by Corollary 2.1 of Theorem 2 this allocation is stable.

Now, for the second part notice that because $\lambda_1 < \lambda_2$, then

$$v_1(1) - v_1(1/2) = \lambda_1 \underbrace{(\tilde{v}(1) - \tilde{v}(1/2))}_{>0} < \lambda_2(\tilde{v}(1) - \tilde{v}(1/2)) = v_2(0) - v_2(1/2)$$

So we can use Lemma C.0.1 to conclude that for sufficiently small σ there is no completely segregated equilibrium. □

Proof of Proposition 4.2.3. Let $\lambda_1 > \lambda_2$.

To prove this proposition we use Lemma C.0.2. Let $\Gamma \in [0,1]$

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma) = \lambda_1 \underbrace{\frac{\partial \tilde{v}}{\partial \Gamma_1}(\Gamma)}_{>0} > \lambda_2 \frac{\partial \tilde{v}}{\partial \Gamma_1}(\Gamma) = \frac{\partial v_2}{\partial \Gamma_1}(\Gamma)$$

then for every $\sigma > 0$ the only stable equilibrium is completely segregated.

□

Proof of Proposition 4.2.4. Let v_i be defined by equation (4.3), and $\lambda_1 > \lambda_2$.

Notice that for every location $x \in X$, distribution of households D and local public goods $(P_k, x_k)_{k=1}^N$

$$\frac{\partial v_i}{\partial \Gamma_1}\left(\Gamma(x), (P_k, x_k)_{k=1}^N\right) = \frac{\partial v_i}{\partial \Gamma_1}\left(\Gamma(x)\right)$$

So Theorem 2 can be applied as if there were no public goods. Then by using Lemma C.0.2 and proceeding like in the proof of Proposition 4.2.3, it follows that for every $\sigma > 0$ he only stable equilibrium is completely segregated.

□

Proofs for Section 4.3

For this section we assume that v_i satisfy equation (4.6). Notice that in this case the notation, used in Lemmas C.0.1 and C.0.2, is equivalent to $v_i(\Gamma_1, \Gamma_2) = v_i(\Gamma_i)$.

Proof of Lemma 4.3.1. Let $\Delta \leq \min\{\rho_i, 1-\rho_i\}$.

i.

$$\begin{aligned} v_i'(\rho_i - \Delta) &= \lambda_i \mu_i \alpha(\Delta)^{\alpha-1} \\ &\geq \lambda_i \alpha(\Delta)^{\alpha-1} \\ &= -v_i'(\rho_i + \Delta) \end{aligned}$$

ii.

$$\begin{aligned} v_i(\rho_i + \Delta) &= -\lambda_i(\Delta)^{\alpha} \\ &\geq -\lambda_i \mu_i (\Delta)^{\alpha} \\ &= v_i(\rho_i - \Delta) \end{aligned}$$

both inequalities bind iff $\mu_i = 1$. □

Proof of Proposition 4.3.1. Suppose preferences are non divergent. Then, by definition, there is an interval (p_1, p_2) for which they coincide up to a constant, which is equivalent to say that their derivatives are the same. Without loss of generality $\Gamma_1 < 1/2$ in this interval, then it must be true that

$$\lambda_1\mu_1\alpha(\rho_1 - \Gamma_1)^{\alpha-1} = \lambda_2\alpha(1 - \rho_2 - \Gamma_1)^{\alpha-1} \quad \forall\Gamma_1 \in (p_1, p_2) \tag{C.2}$$

Equation (C.2) directly implies that $\lambda_1\mu_1 = \lambda_2$, otherwise the shape of the two functions would not be the same. When in addition $\alpha \neq 1$ then it also implies that $\rho_1 = 1 - \rho_2$, which is only possible when $\rho_1 = \rho_2 = 1/2$. Finally notice that when $\Gamma_1 > 1/2$ in (p_1, p_2) is assumed, the other possibility $\lambda_1 = \lambda_2\mu_2$ is obtained.

□

Proof of Corollary 4.3.1.1. This is a direct conclusion from Theorem 1 and Proposition 4.3.1.

□

Proof of Proposition 4.3.2. Let $\rho_1 = \rho_2 = 1$.

To demonstrate this proposition we use Lemma C.0.2. Notice that under this assumption, preferences are increasing in the proportion of alike, and decreasing in the proportion of unlike households. Let $\Gamma \in [0, 1]$

$$\frac{\partial v_1}{\partial\Gamma_1}(\Gamma) > 0 > \frac{\partial v_2}{\partial\Gamma_1}(1 - \Gamma)$$

then for any $\sigma > 0$ the only stable equilibrium is a completely segregated.

□

Proof of Proposition 4.3.3. Let $\rho_1 = \rho_2 = 1/2$.

For the first part we use Lemma C.0.2. Suppose $\mu_i > \lambda_{-i}/\lambda_i$ for $i = 1, 2$. Let $\Gamma \in [0, 1]$ we divide the proof in two parts. If $\Gamma > 1/2$

$$\begin{aligned}
\frac{\partial v_1}{\partial\Gamma_1}(\Gamma) &= -\underbrace{\lambda_1}_{<\mu_2\lambda_2}\alpha\cdot(\Gamma - 1/2)^{\alpha-1} \\
&> -\lambda_2\mu_2\cdot\alpha\cdot(\Gamma - 1/2)^{\alpha-1} \\
&= \frac{\partial v_2}{\partial\Gamma_1}(1 - \Gamma)
\end{aligned}$$

And if $\Gamma < 1/2$

$$\begin{aligned}\frac{\partial v_1}{\partial \Gamma_1}(\Gamma) &= \underbrace{\lambda_1\mu_1}_{>\lambda_2}\alpha \cdot (1/2 - \Gamma)^{\alpha-1} \\ &> \lambda_2 \cdot \alpha \cdot (1/2 - \Gamma)^{\alpha-1} \\ &= \frac{\partial v_2}{\partial \Gamma_1}(1-\Gamma)\end{aligned}$$

Thus by Lemma C.0.2 for every $\sigma > 0$ the only stable equilibrium is completely segregated.

Conversely, suppose $\mu_2 < \lambda_1/\lambda_2$, we will show with Lemma C.0.1 that for sufficiently small σ there is no completely segregated equilibrium. The case $\mu_1 < \lambda_2/\lambda_1$ is analogous.

$$\begin{aligned}v_1(1) - v_1(1/2) &= -\underbrace{\lambda_1}_{>\mu_2\lambda_2}(1/2)^{\alpha} \\ &< -\lambda_2\mu_2 \cdot (1/2)^{\alpha} \\ &= v_2(0) - v_2(1/2)\end{aligned}$$

Then by Lemma C.0.1 for sufficiently small σ there is no completely segregated equilibrium.

□

Proof of Proposition 4.3.4. Let $\rho_2 = 1$. Suppose $\lambda_1 < \lambda_2$. Recall that $\mu_2 \geq 1$, so $\lambda_1 < \lambda_2\mu_2$.

We use Lemma C.0.2 to prove this proposition. Let $\Gamma \in [0, 1]$, we divide the proof in two parts. First, if $\Gamma < \rho_1$, then

$$\frac{\partial v_1}{\partial \Gamma_1}(\Gamma) > 0 > \frac{\partial v_2}{\partial \Gamma_1}(1-\Gamma)$$

And if $\Gamma > \rho_1$

$$\begin{aligned}\frac{\partial v_1}{\partial \Gamma_1}(\Gamma) &= -\underbrace{\lambda_1}_{<\lambda_2\mu_2}\alpha \cdot (\Gamma - \rho_1)^{\alpha-1} \\ &> -\lambda_2\mu_2 \cdot \alpha \cdot (\Gamma - \rho_1)^{\alpha-1} \\ &> -\lambda_2\mu_2 \cdot \alpha \cdot (\Gamma)^{\alpha-1} \\ &= \frac{\partial v_2}{\partial \Gamma_1}(1-\Gamma)\end{aligned}$$

Therefore, for any $\sigma > 0$ the only stable equilibrium is completely segregated. □

Proof of Proposition 4.3.5. Let $\rho_2 = 1/2$ and $\beta_1 < 1/2$. Recall that $\beta_2 = 1 - \beta_1 > 1/2$.

For the stability of the perfectly integrated equilibrium we use Corollary 2.1 of Theorem 2. Let $M_1 = \min\left\{\left(\frac{\beta_2 - 1/2}{1-\beta_1}\right)^{\alpha-1}, \left(\frac{\beta_2 - 1/2}{1/2 - \beta_1}\right)^{\alpha-1}\right\}$. Notice that $M_1 < \left(\frac{1-\beta_2}{\rho_1 - \beta_1}\right)^{\alpha-1}$. So if $\mu_1 <$

$M_1 \cdot {}^{\lambda_2}/_{\lambda_1}$, then

$$\begin{aligned}\frac{\partial v_1}{\partial \Gamma_1}(\beta) &= \lambda_1 \mu_1 \alpha (\rho_1 - \beta_1)^{\alpha-1} \\ &< \lambda_2 \cdot M_1 \cdot \alpha (\rho_1 - \beta_1)^{\alpha-1} \\ &< \lambda_2 \left(\frac{\beta_2 - 1/2}{\rho_1 - \beta_1} \right)^{\alpha-1} \cdot \alpha (\rho_1 - \beta_1)^{\alpha-1} \\ &= \lambda_2 \alpha (\beta_2 - 1/2)^{\alpha-1} \\ &= \frac{\partial v_2}{\partial \Gamma_1}(\beta)\end{aligned}$$

Thus by Corollary 2.1 the perfectly integrated equilibrium is stable.

For the second part we use Lemma C.0.1. Let $M_2 = (1/2)^\alpha$. In this case

$$v_1(0) - v_1(1/2) = -\lambda_1 \mu_1 \underbrace{(\rho_1^\alpha - (1 - \rho_1)^\alpha)}_{<1} > -\lambda_1 \mu_1$$

So if $\mu_1 < M_2 \cdot {}^{\lambda_2}/_{\lambda_1}$, then

$$\begin{aligned}v_2(1) - v_2(1/2) &= -\lambda_2 \cdot (1/2)^\alpha \\ &= -\lambda_2 \cdot M_2 \\ &< -\lambda_1 \mu_1 \\ &< v_2(0) - v_2(1/2)\end{aligned}$$

Hence for a sufficiently small σ there is no completely segregated equilibrium.

Finally, let $M = \min\{M_1, M_2\}$. M depends only on α and β. And if $\mu_1 < M \cdot {}^{\lambda_2}/_{\lambda_1}$, then for a sufficiently small σ there is no completely segregated equilibrium, and the perfectly integrated equilibrium is stable.

$\square$

Proof of Proposition 4.3.6. Let $\bar{\rho}_i := \frac{2 + \sqrt[\alpha]{\mu_i}}{2 + 2\sqrt[\alpha]{\mu_i}}$, and $\beta_2 > 1/2$.

First, define

$$\eta_1 := \frac{(1 - \rho_2)^\alpha - \mu_2 (\rho_2 - 1/2)^\alpha}{\rho_1^\alpha - (\rho_1 - 1/2)^\alpha} \cdot \frac{1}{\mu_1}$$

Observe that the term $\rho_1^\alpha - (\rho_1 - 1/2)^\alpha$ is always positive, so the denominator of η_1 is positive. Then, it is not difficult to see that $\rho_2 < \bar{\rho}_i$ iff the numerator is positive iff $\eta_1 > 0$. Now, if ${}^{\lambda_1}/_{\lambda_2} < \eta_1$, then

$$\begin{aligned}\frac{\lambda_1}{\lambda_2} < \eta_1 &\iff \lambda_1 \mu_1 \rho_1^\alpha - \lambda_1 \mu_1 (\rho_1 - 1/2)^\alpha < \lambda_2 (1 - \rho_2)^\alpha - \lambda_2 \mu_2 (\rho_2 - 1/2)^\alpha \\ &\iff -\lambda_1 \mu_1 \rho_1^\alpha + \lambda_1 \mu_1 (\rho_1 - 1/2)^\alpha > -\lambda_2 (1 - \rho_2)^\alpha + \lambda_2 \mu_2 (\rho_2 - 1/2)^\alpha \\ &\iff v_1(0) - v_1(1/2) > v_2(1) - v_2(1/2)\end{aligned}$$

So by Lemma C.0.1, if ${}^{\lambda_1}\!/_{\lambda_2} < \eta_1$ then for a sufficiently small σ there is no completely segregated equilibrium.

Secondly, define

$$\eta_2 := \left(\frac{\beta_2 - \rho_2}{\rho_1 - \beta_1}\right)^{\alpha-1} \cdot \frac{1}{\mu_1}$$

Observe that $\beta_1 < {}^1\!/_2 \leq \rho_1$, so the denominator is positive. Thus if $\rho_2 < \beta_2$, then $\eta_2 > 0$. Now, if ${}^{\lambda_1}\!/_{\lambda_2} < \eta_2$, then

$$\begin{aligned}
\frac{\lambda_1}{\lambda_2} < \eta_2 &\iff \lambda_1\mu_1(\rho_1 - \beta_1)^{\alpha-1} < \lambda_2(\beta_2 - \rho_2)^{\alpha-1} \\
&\iff \lambda_1\mu_1\alpha(\rho_1 - \beta_1)^{\alpha-1} < \lambda_2\alpha(\beta_2 - \rho_2)^{\alpha-1} \\
&\iff \frac{\partial v_1}{\partial \Gamma_1}(\beta_1) < \frac{\partial v_2}{\partial \Gamma_1}(\beta_2)
\end{aligned}$$

So by Corollary 2.1 of Theorem 2, if ${}^{\lambda_1}\!/_{\lambda_2} < \eta_2$ the perfectly integrated equilibrium is stable.

Finally, for every $\rho_2 < \min\{\bar{\rho}_2, \beta_2\}$, $\eta := \min\{\eta_1, \eta_2\} > 0$. Therefore, if ${}^{\lambda_1}\!/_{\lambda_2} < \eta$, then the perfectly integrated equilibrium is stable, and for sufficiently small σ there is no completely segregated equilibrium. □

Proofs for Section 4.4

Proof of Proposition 4.4.1. Suppose there is a non degenerated interval I in which $\tilde{v}_1$ and $\tilde{v}_2$ are constant, then for every $\lambda_1, \lambda_2 > 0$, $\lambda_1\tilde{v}_1 - \lambda_2\tilde{v}_2$ is also constant in I. Therefore preferences are not divergent.

□

Proof of Proposition 4.4.2. Suppose there is an interval I in which $\tilde{v}_1$ and $\tilde{v}_2$ are constant, and let the population proportion be in that interval. That means that small changes around the perfectly integrated equilibrium does not affect households utility function. Therefore there are infinitely many variations to change distributions such that, maintaining the housing supply d^s and the rent price function R, households still maximize utility.

This equilibrium is stable because for sufficiently small ϵ a migration would no change households utility, an therefore it cannot be persistent. □