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# CONDICIÓN DE BOSHERNITZAN PARA SISTEMAS MINIMALES DE CANTOR

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En 1992 M. Boshernitzan [6] presenta una condición suficiente para que los subshifts minimales sean únicamente ergódicos. Usando el concepto de factores simbólicos extendemos esta condición a sistemas minimales de Cantor. Decimos que un sistema minimal de Cantor satisface la condición de Boshernitzan si todos sus factores simbólicos satisfacen la condición de Boshernitzan. Esta extensión resulta natural en cuanto todo sistema minimal de Cantor es topologicamente conjugado al límite inverso de ciertas secuencias factorizantes de factores símbolicos. Demostramos que la condición de Boshernitzan implica única ergodicidad para sistemas minimales de Cantor. También mostramos que esta condición puede ser verificada analizando cualquier representación de Bratteli-Vershik de un sistema minimal de Cantor dado. Luego tiene sentido buscar condiciones sobre los diagramas de Bratteli asociados a un sistema minimal de Cantor que sean necesarias y/o suficientes para que tal sistema satisfaga la condición de Boshernitzan. Presentamos varias de estas condiciones. Las más generales están relacionadas con el comportamiento asintótico de los vectores de altura y los vectores de medida de las representaciones de Bratteli-Vershik. Estas condiciones son luego reducidas, sacrificando un poco de generalidad, a condiciones concernientes a la repetición de un bloque de matrices positivas dado en una cantidad infinita de niveles de los diagramas. En todos los casos se considera una hipótesis de estandarización sobre el orden de los diagramas. Se explora el alcance y las limitaciones de los criterios presentados a través del estudio de ejemplos específicos. Se observa que la combinatoria de los sistemas influye de gran manera en el cumplimiento de la condición de Boshernitzan.

In 1992 M. Boshernitzan [6] provided a sufficient condition for minimal subshifts to be uniquely ergodic. By using the concept of symbolic factors we extend this condition to Cantor minimal systems. We say a Cantor minimal systems satisfies Boshernitzan's condition if all of its symbolic factors satisfy Boshernitzan's condition. This extension seems natural given the fact that every Cantor minimal system is topologically conjugate to the inverse limit of certain factoring sequences of symbolic factors. We prove that Boshernitzan's condition implies unique ergodicity for Cantor minimal systems. We also show that this condition can be verified by analyzing any particular Bratteli-Vershik representation of a given a Cantor minimal system. It then makes sense to look for diagram related necessary and/or sufficient condition for Cantor minimal systems to satisfy Boshernitzan's condition. We provide several of these conditions. The more general ones relate to the asymptotic behaviour of the height vectors and measure vectors of the Bratteli-Vershik representations. These conditions are then reduced, sacrificing some generality, to conditions concerning the repetition of a given block of positive matrices at inifinitely many levels of the diagrams. In all cases a standardization hypothesis on the order of the diagrams is made. We explore the scope and limitations of the criteria provided by studying specific examples. The combinatorics of the systems is seen to greatly influence the achievement of Boshernitzan's condition.

# Contents

# Introduction

In 1972 O. Bratteli [7] introduced in the theory of operator algebras a special class of infinite directed graphs, which are now referred to as Bratteli diagrams, with the purpose of describing directed sequences of finite-dimensional algebras. Later in 1985 A. M. Vershik [26] associated dynamics to Bratteli diagrams via what is now called the Vershik map, an "adic-transformation" defined by introducing a lexicographical order on the set of infinite paths of the diagram. This class of systems are now referred to as Bratteli-Vershik systems. In 1992 R. H. Herman, I. F. Putnam, and C. F. Skau [22] refined Vershik's construction to show that every Cantor minimal system is topologically conjugate to a Bratteli-Vershik system. Since then the theory of Cantor minimal systems has seen numerous amazing developments. Even today it keeps garnering great interest, in large part due to the diversity of its applications in many areas of mathematics. For instance, interval exchange transformations and translation surfaces have natural representations as Bratteli-Vershik systems. The structure of these representations is intimately related to the Rauzy-Veech induction algorithm. See for example [19] and [10]. Furthermore, many examples of groups with specific properties which were historically hard to find were discovered for the first time as full groups of Cantor minimal systems. See for instance [17] and [23].

In 1985, M. Boshernitzan [5] introduced a condition for minimal interval exchange transformations to be uniquely ergodic. He also conjectured that the same result should hold for an apparently significant weakening of such condition. In 1987 W. A. Veech [25] provided a proof of this conjecture. Later in 1992 M. Boshernitzan [6] provided an homologous condition for minimal subshifts to be uniquely ergodic. This condition is referred in this thesis as Boshernitzan's condition. In recent years this condition has sparked significant interest due to its applications to different problems in the theory of dynamical systems. For instance, D. Damanik and D. Lenz [12] used this condition to study the uniform convergence in the multiplicative ergodic theorem on aperiodic subshifts. Furthermore, A. Avila and V. Delecroix [1] considered a regime partially replicating Boshernitzan's condition to study the eigenvalues of non-arithmetic Veech surfaces with linearly recurrent flows.

The main purpose of this thesis is to extend Boshernitzan's condition to the class of Cantor minimal systems and to develop techniques for checking such condition by analyzing the structure of the ordered Bratteli diagrams representing those systems. We look for a condition which not only extends the original definition but which also extends M. Boshernitzan's result regarding the unique ergodicity of the system. By doing so we will arrive at simple criteria for checking the unique ergodicity of Cantor minimal systems.

The main tool used in the construction of Boshernitzan's condition for Cantor minimal systems is the concept of symbolic factors. The importance of symbolic factors comes from the fact that Cantor minimal systems are topologically conjugate to the inverse limit of some specific factoring sequences of symbolic factors. With this idea in mind we will say that a Cantor minimal system satisfies Boshernitzan's condition if all of its symbolic factors satisfy Boshernitzan's condition. This definition turns out to have the desired characteristics described in the previous paragraph. We will also see how symbolic factors help in extending another simpler property of subshifts, sublinear complexity in a subsequence, to Cantor minimal systems. We will derive diagram related conditions for Cantor minimal systems to have such property and use this as inspiration for developing diagram related Boshernitzan's condition criteria for Cantor minimal systems.

Describing a procedure for checking Boshernitzan's condition via Bratteli-Vershik representations turns out to be rather simple, though its application to particular examples can become extremely non-trivial. Once a procedure has been established we provide a series of sufficient conditions for Boshernitzan's condition to hold, which, although less general, are significantly easier to check. We show several examples of applications of these results. The conditions provided all make use of a specific hypothesis on the order of the diagram which may seem artificial at first. It is important to warn the reader that this hypothesis works as a standardization of the diagrams considered rather than as a restriction. A thorough explanation of the nature of this hypothesis is provided.

Even though a solid framework for understanding Boshernitzan's condition in the context of Cantor minimal systems is established, many questions remain unanswered. A major point which remains unsolved is establishing simple diagram related conditions for a Bratteli-Vershik system to not satisfy Boshernitzan's condition. In the same line, the desire of obtaining explicit diagram related necessary and sufficient conditions for a Bratteli-Vershik system to satisfy Boshernitzan's condition remains unfulfilled. The great deal of combinatorial mechanisms which appear to be involved in Boshernitzan's condition, as seen in some of the examples provided, point towards the hardness of these problems. These questions are further discussed in the conclusions.

This thesis is organized in the following way. In Chapter 1 the basic theory of Cantor minimal systems is introduced. We provide a complete self-contained survey of the results necessary to understand the work developed in this thesis. In Chapter 2 we present and thorougly explain the definition of Boshernitzan's condition for Cantor minimal systems via symbolic factors. We also provide diagram related procedures and conditions for this condition to be satisfied. In Chapter 3 we provide several examples of applications of these results. In the conclusions we round up the main ideas of this thesis, make some interesting observations, and discuss a series of questions that remain open.

# Chapter 1

# Cantor Minimal Systems

In this chapter we introduce the basics of the theory of minimal homeomorphic dynamics on Cantor spaces. This class of dynamical systems will be referred to as Cantor minimal systems. We begin by providing some basic definitions and results in the theory of dynamical systems, with the purpose of introducing the vocabulary and notation to be used in this thesis. We then introduce ordered Bratteli diagrams and the associated natural dynamics on the phase space of infinite paths of such diagrams, the Vershik maps. Such dynamical systems will be referred to as Bratteli-Vershik systems. These systems play a key role in the theory here considered as every Cantor minimal system can be represented by, in the sense that it is topologically conjugate to, a Bratteli-Vershik system. We provide a proof of this fact. We then strengthen this result by showing that any Cantor minimal system can be represented by a Bratteli-Vershik system with certain nice properties. We end this chapter with a brief survey of some criteria for classifying Cantor minimal systems according to the properties of their Bratteli-Vershik representations

## 1.1 Preliminary Definitions and Results

In everything that follows the set of positive integers will be denoted by $\mathbb{N} := \{1, 2, \dots\}$ while the set of non-negative integers will be denoted by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will also denote the set of positive real numbers by $\mathbb{R}_{>0}$, while the set of non-negative real numbers will be denoted by $\mathbb{R}_{\geq 0} := \mathbb{R}_{>0} \cup \{0\}$. We will say a matrix $A \in M_{n\times m}(\mathbb{R})$ is positive, and denote it by $A > 0$, if all of its entries are positive. We will say a matrix $A \in M_{n\times m}(\mathbb{R})$ is non-negative, and denote it by $A \geq 0$, if all of its entries are non-negative.

Detailed explanations to most of the concepts and results discussed in this section can be found in any introductory level text on dynamical systems and ergodic theory. See for example [28] and [24].

We begin by introducing the definition of dynamical system to be used from now on.

**Definition 1.1** *A dynamical system is a pair $(X, T)$, where $X$ is a compact metric space* *and $T\colon X \to X$ is a continuous function. The set $X$ is referred to as the phase space of the system and the function $T$ is said to provide the dynamics of the system.*

In general one considers dynamical systems $(X, T)$ where either $X$ or $T$ or both have special characteristics. For instance, we will focus our attention on systems where $T\colon X \to X$ is a homeomorphism. Such systems will be referred to as invertible dynamical systems. In this context we introduce the following definition of orbit and positive orbit of a point.

**Definition 1.2** *Let $(X, T)$ be an invertible dynamical system and let $x \in X$ be a point in the phase space. The orbit of $x$ through $T$ is defined as the set:*

$$\mathcal{O}_T(x) = \{T^n x \colon n \in \mathbb{Z}\}$$

*The positive orbit of $x$ through $T$ is defined as the set:*

$$\mathcal{O}_T^+(x) = \{T^n x \colon n \in \mathbb{N}_0\}$$

We will deal almost exclusively with dynamical systems $(X, T)$ that have the following minimality properties:

**Definition 1.3** *Let $(X, T)$ be an invertible dynamical system. We say that the system, or that the function $T$, is minimal if the orbit of any point is dense in the phase space, i.e.:*

$$\forall x \in X \colon \overline{\mathcal{O}_T(x)} = X$$

*We say that the system, or that the function $T$, is positive minimal if the positive orbit of any point is dense in the phase space, i.e.:*

$$\forall x \in X \colon \overline{\mathcal{O}_T^+(x)} = X$$

The compactness of the phase space $X$ considered as part of our definition of dynamical system yields a nice equivalence between the previous minimality properties:

**Proposition 1.4** *Let $(X, T)$ be an invertible dynamical system. Recall that the phase space $X$ is compact. Then $(X, T)$ is minimal if and only if it is positive minimal.*

Proof. It is straightforward that positive minimality implies minimality. To prove the converse we consider an arbitrary open set $A$ and prove that the positive orbit of every $x \in X$ intersects such set. It is easy to see that this is equivalent to $X = \bigcup_{n \in \mathbb{N}_0} T^{-n} A$. As $A$ is open and $T$ is minimal we see that $X = \bigcup_{n \in \mathbb{Z}} T^{-n} A$. As $X$ is compact there exists an $m \in \mathbb{N}$ such that $X = \bigcup_{n=-m}^{m} T^{-n} A$. Taking $T^{-m}$ in this last equality yields $X = T^{-m}(X) = \bigcup_{n=0}^{2m} T^{-n} A$. Which in particular proves the desired property. $\square$

We will also consider special characteristics for the phase space. Specifically we will work with systems whose phase space is a Cantor space, i.e.:

**Definition 1.5** *Let $X$ be a topological space. We say that $X$ is a Cantor space if it is a non-empty compact Hausdorff space which is perfect, i.e. it has no isolated points, and 0-dimensional, i.e. it has a countable base consisting of clopen sets.*

Two basic examples of Cantor spaces are the classical Cantor set on the real line and the countably infinite topological product of the discrete two point space $\{0,1\}^{\mathbb{N}}$. It is easy to see that this two topological spaces are homeomorphic. This homeomorphism property is shared by all Cantor spaces as the following theorem by L. E. J. Brouwer [9] shows:

**Theorem 1.6** *Any two non-empty compact Hausdorff spaces without isolated points and having countable bases consisting of clopen sets are homeomorphic to each other.*

As the Cantor set on the real line, or alternatively the countably infinite topological product of the discrete two point space $\{0,1\}^{\mathbb{N}}$, is easily seen to be metrizable and as metrizability is a topological invariant, we see in particular that every Cantor space is metrizable.

We are now ready to introduce the class of systems which are the main focus of this thesis:

**Definition 1.7** *Let $(X,T)$ be a dynamical system such that $X$ is a Cantor space and the $T$ is a minimal homemorphism. Then $(X,T)$ is said to be a Cantor minimal system.*

We will also work with shift spaces. We will always consider finite alphabets. Let $\mathcal{A}$ be a finite alphabet. Given a point $x \in \mathcal{A}^{\mathbb{Z}}$ and an integer $\mathrm{i} \in \mathbb{Z}$ we will denote $x_{\mathrm{i}} := x(\mathrm{i}) \in \mathcal{A}$. Also, given a point $x \in \mathcal{A}^{\mathbb{Z}}$ and a pair of integers $\mathrm{i}, j \in \mathbb{Z}$ with $\mathrm{i} < j$ we will denote $x_{\mathrm{i}:j} := x_{\mathrm{i}} \dots x_j \in \mathcal{A}^{j-\mathrm{i}+1}$. In this context let us provide a complete definition of shift spaces:

**Definition 1.8** *Let $\mathcal{A}$ be a finite set, which we will refer to as alphabet. Consider $\mathcal{A}^{\mathbb{Z}}$ the countably infinite topological product of the discrete space $\mathcal{A}$. Notice that by Tychonoff's theorem the space $\mathcal{A}^{\mathbb{Z}}$ is compact. Define the left shift map $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ which maps every point $x := (x_{\mathrm{i}})_{\mathrm{i}\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ to $\sigma(x) = (x_{\mathrm{i}+1})_{\mathrm{i}\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. It is easy to see that $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is a homeomorphism. A set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is said to be a subshift if $X$ is closed and shift invariant, i.e. $\sigma(X) \subseteq X$. It follows that if $X$ is a subshift then $(X, \sigma|_X)$ is a dynamical system. We will usually just write $\sigma$ in place of $\sigma|_X$ when no confusion arises. Such dynamical systems will also be referred to as subshifts or alternatively as shift spaces. In particular the system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is a shift space called the fullshift over the alphabet $\mathcal{A}$.*

It is straightforward to see that for every subshift $(X,\sigma)$ the phase space $X$ is 0-dimensional, compact, and Hausdorff. Moreover, it is not hard to check that if $(X,\sigma)$ is minimal and $|X| = +\infty$ then $X$ is also perfect. It follows then that every minimal subshift $(X,\sigma)$ with $|X| = +\infty$ is a Cantor minimal system. Then the class of Cantor minimal systems contains all minimal subshifts safe for some trivial cases.

Let $\mathcal{A}$ be a finite alphabet. Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ be integers, and let $\omega \in \mathcal{A}^n$ be a finite word. Then the cylinder associated to these parameters is the subset of $\mathcal{A}^{\mathbb{Z}}$ given by:

$$_k[\omega] = \{x \in \mathcal{A}^{\mathbb{Z}} \colon x_{k+\mathrm{i}} = \omega_{\mathrm{i}},\ \forall \mathrm{i} = 0, \dots, n-1\}$$

If $\varepsilon$ is the empty word, we will assume that $_k[\varepsilon] = \mathcal{A}^{\mathbb{Z}}$ for all $k \in \mathbb{Z}$. The family of all cylinder sets is a base for the topology of $\mathcal{A}^{\mathbb{Z}}$. Moreover, it is a generating $\pi$-system for the borelians of $\mathcal{A}^{\mathbb{Z}}$. The same is true for arbitrary subshifts when considering the restrictions of the cylinders to such spaces. Let us now state the definition of language of a subshift:

**Definition 1.9** *Let $X$ be a subshift over a finite alphabet $\mathcal{A}$. For each $n \in \mathbb{N}_0$ we define the language of words of length $n$ of $X$ as the set:*

$$L_n(X) := \{\omega \in \mathcal{A}^n : {}_0[\omega] \cap X \neq \emptyset\}$$

*We also define the language of $X$ as the union:*

$$L(X) := \bigcup_{n \in \mathbb{N}_0} L_n(X)$$

With the concept of language at hand we can define the complexity function of a subshift:

**Definition 1.10** *Let $X$ be a subshift over a finite alphabet $\mathcal{A}$. We define the complexity function of $X$ as the map $p_X : \mathbb{N}_0 \to \mathbb{N}$ given by $p_X(n) = |L_n(X)|$ for each $n \in \mathbb{N}_0$.*

It will be useful for our purposes to have a simple characterization of continuous shift-conmuting functions between shift spaces. Such characterization is provided by the following classical result by M. L. Curtis, G. A. Hedlund, and R. Lyndon [21]:

**Theorem 1.11** *Let $(X, \sigma_X)$ and $(Y, \sigma_Y)$ be shift spaces over finite alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\phi : X \to Y$ be a continuous functions that conmutes with the shifts of each space, i.e. $\sigma_Y \circ \phi = \phi \circ \sigma_X$. Then there exists integers $m, a \in \mathbb{N}_0$ and a function $\Phi : L_{m+1+a}(X) \to \mathcal{B}$ such that for all $x \in X$ and all $\mathrm{i} \in \mathbb{Z}$ we have $\phi(x)_{\mathrm{i}} = \Phi(x_{\mathrm{i}-m:\mathrm{i}+a})$. We then say that $\phi$ is a sliding block code with memory $m$, anticipation $a$, and block code $\Phi$.*

We now recall the notions of topological factor and conjugacy of dynamical systems:

**Definition 1.12** *Let $(X, T)$ and $(Y, S)$ be two dynamical systems. Suppose there exists a continuous surjective map $\pi : (X, T) \to (Y, S)$ that conmutes with the dynamics of the systems, i.e. $\mathcal{S} \circ \pi = \pi \circ T$. In this case we say that $\pi$ is a factor map, and that $(Y, S)$ is a topological factor of $(X, T)$ through the map $\pi$. If the map $\pi$ is a homeomorphism, we say that $\pi$ is a conjugacy, and that $(X, T)$ and $(Y, S)$ are topologically conjugate.*

A property of dynamical systems is said to be a topological conjugacy invariant if whenever a given dynamical system has this property, every topologically conjugate dynamical system must also have such property. It is straightforward to check that the property of being a Cantor minimal system is a topological conjugacy invariant. Another example of a topological conjugacy invariant is the property of being uniquely ergodic, which we discuss shortly.

We now briefly recall some basic results regarding invariant and ergodic measures of dynamical systems. We will always refer to borelian measures when talking about measures. We begin with the definitions of invariant and ergodic measures:

**Definition 1.13** *Let $(X, T)$ be a dynamical system and let $\mathcal{B}(X)$ be the borelians of $X$. Let $\mu$ be a borelian measure on $X$. We say that $\mu$ is an invariant measure of $(X, T)$, or that $\mu$ is*

*$T$-invariant, if the following holds:*

$$\forall A \in \mathcal{B}(X) \colon \mu(A) = \mu(T^{-1}A)$$

*If $T$ is a homeomorphism, this is equivalent to:*

$$\forall A \in \mathcal{B}(X) \colon \mu(A) = \mu(TA)$$

*We say the measure $\mu$ is ergodic if it is invariant and the following condition holds:*

$$\forall A \in \mathcal{B}(X) \colon \mu(A \triangle T^{-1}A) = 0 \Rightarrow \mu(A) \in \{0, 1\}$$

*The set of all invariant measures of $(X, T)$ will be denoted by $\mathcal{M}^*(X, T)$, while the set of all invariant probability measures of $(X, T)$ will be denoted by $\mathcal{M}(X, T)$. The set of all ergodic measures of $(X, T)$ will be denoted by $\mathcal{E}^*(X, T)$, while the set of all ergodic probability measures of $(X, T)$ will be denoted by $\mathcal{E}(X, T)$.*

Let us state a classical result by N. Bogoliouboff and N. Kryloff [3] regarding the existence of invariant measures of dynamical systems with compact phase space:

**Theorem 1.14** *Let $(X, T)$ be a (compact) dynamical system. Then $\mathcal{M}(X, T) \neq \emptyset$.*

We also recall basic results regarding the geometry of the sets $\mathcal{M}(X, T)$ and $\mathcal{E}(X, T)$:

**Theorem 1.15** *Let $(X, T)$ be a dynamical system. Then $\mathcal{M}(X, T)$ is a convex weak-$\star$ compact subset of $\mathcal{C}(X)^\star$, the topological dual of the space of continuous complex valued functions defined on $X$. Moreover the set $\mathcal{E}(X, T)$ amounts to all the extreme points of $\mathcal{M}(X, T)$. By Krein-Milman theorem this implies that $\mathcal{M}(X, T)$ is the weak-$\star$ closed convex hull of $\mathcal{E}(X, T)$.*

Let us introduce two more definition regarding the cardinality of the set $\mathcal{E}(X, T)$:

**Definition 1.16** *Let $(X, T)$ be a dynamical system. We say that $(X, T)$ is finitely ergodic if $|\mathcal{E}(X, T)| < +\infty$. We say that $(X, T)$ is uniquely ergodic if $|\mathcal{E}(X, T)| = 1$. By Theorem 1.15 this last property is equivalent to $|\mathcal{M}(X, T)| = 1$.*

Let us also introduce the concept of pushforward measure:

**Definition 1.17** *Let $(X, \mathcal{A})$ and $(X, \mathcal{B})$ be measurable spaces and let $\mu$ be a measure on $(X, \mathcal{A})$. Consider a measurable function $\pi \colon (X, \mathcal{A}) \to (X, \mathcal{B})$. We define the pushforward measure of $\mu$ through $\pi$ as the measure $\pi\mu$ given by $\pi\mu(B) = \mu(\pi^{-1}B)$ for all $B \in \mathcal{B}$.*

The pushforward measure of an invariant probability measure through a factor map is also an invariant probability measure, as the following result shows:

**Proposition 1.18** *Let $(X, T)$ and $(Y, S)$ be dynamical systems and let $\pi \colon (X, T) \to (Y, S)$ be a factor map between them. Consider an invariant probabilty measure $\mu \in \mathcal{M}(X, T)$. Then the pushforward measure $\pi\mu$ is also an invariant probability measure, i.e. $\pi\mu \in \mathcal{M}(Y, S)$.*

We now recall the notion of partition refinement:

**Definition 1.19** *Let $X$ be a set. Let $\mathcal{P}$ and $\mathcal{P}'$ be two partitions of $X$. We say $\mathcal{P}'$ is finer than $\mathcal{P}$, or that $\mathcal{P}$ is coarser than $\mathcal{P}'$, if for every set $A' \in \mathcal{P}'$ there exists a (unique) set $A \in \mathcal{P}$ such that $A' \subseteq A$. In this case we will write $\mathcal{P} \preceq \mathcal{P}'$. The relation between the elements of $\mathcal{P}$ and $\mathcal{P}'$ can be naturally represented by a map $\mathrm{i}_{\mathcal{P}',\mathcal{P}} \colon \mathcal{P}' \to \mathcal{P}$.*

Finally we provide the definition of immediate successor and predecessor of an element in a partially ordered set:

**Definition 1.20** *Let $(X, \geq)$ be a partially ordered set. Let $x \in X$ be an arbitrary element. An immediate predecessor of $x$ is any maximal element of the set $\{y \in X \colon y \leq x, y \neq x\}$. An immediate successor of $x$ is any minimal element of the set $\{y \in X \colon y \geq x, y \neq x\}$.*

In the context of the definition above, immediate predecessors and successors of a certain element $x \in X$ may not be unique or even exist at all.

## 1.2 Bratteli-Vershik Systems

We now introduce a special class of directed graphs called Bratteli diagrams, which will be the main abstract object we will work with:

**Definition 1.21** *A Bratteli diagram is an infinite directed graph $B = (V, E)$ where $V$ is the vertex set and $E$ is the edge set. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup \ldots$ and $E = E_1 \cup E_2 \cup \ldots$ where $V_0 = \{v_0\}$ is a one point set. The directions of the edges are determined by the maps $s, r \colon E \to V$ where $s$ is the source map and $r$ is the range map. These two maps are such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$ for all $n \in \mathbb{N}$, and $s^{-1}(\{v\}) \neq \emptyset$, $r^{-1}(\{v'\}) \neq \emptyset$ for all $v \in V$ and $v' \in V \backslash V_0$.*

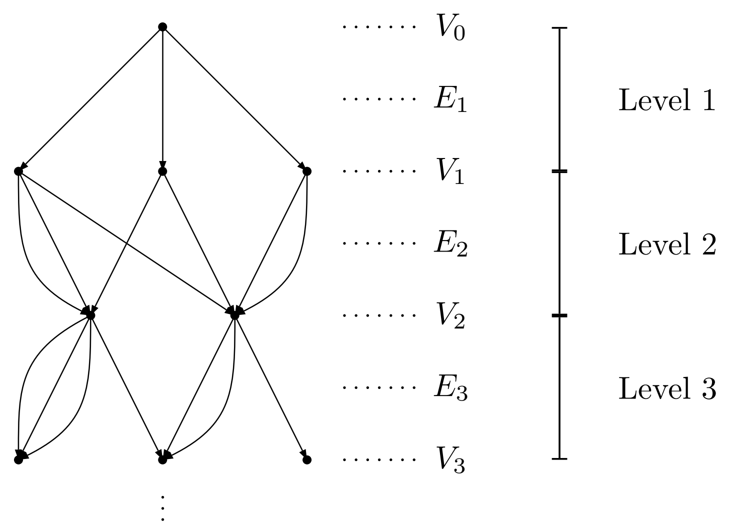


Figure 1.1: First three levels of a Bratteli diagram. In most cases lines are drawn instead of arrows and the orientation of these edges is assumed by the definition of Bratteli diagrams.

In the context of the definition above, for each $\mathrm{i} \in \mathbb{N}$ the pair $(V_\mathrm{i}, E_\mathrm{i})$ is called the i-th level of the diagram $B$. A finite sequence of edges $\overline{\mathrm{e}} = (\mathrm{e}_k, \ldots, \mathrm{e}_{k+m-1}) \in \prod_{\mathrm{i}=k}^{k+m-1} E_\mathrm{i}$ such that $r(\mathrm{e}_\mathrm{i}) = s(\mathrm{e}_{\mathrm{i}+1})$ for $\mathrm{i} = k, \ldots, k+m-2$ is called a path of length $m$ in $(V, E)$ starting at the vertex $s(\mathrm{e}_k) \in V_{k-1}$ and ending at the vertex $r(\mathrm{e}_{k+m-1}) \in V_{k+m-1}$. For $j, k \in \mathbb{N}_0$ with $j < k$, $v \in V_j$, and $w \in V_k$, we will denote by $E(v, w)$ the set of all finite paths starting at the vertex $v$ and ending at the vertex $w$. For $j, k \in \mathbb{N}_0$ with $j < k$ and $v \in V_j$, we will denote by $E(v, k)$ the set of all finite paths starting at the vertex $v$ and ending at some vertex of $V_k$. For $j, k \in \mathbb{N}_0$ with $j < k$ and $w \in V_k$, we will denote by $E(j, w)$ the set of all finite paths starting at the some vertex of $V_j$ and ending at vertex $w$. For $j, k \in \mathbb{N}_0$ with $j < k$ we will denote by $E(j, k)$ the set of all finite paths starting at vertices of $V_j$ and ending at vertices of $V_k$. In the same way we define infinite paths $\overline{\mathrm{e}} = (\mathrm{e}_k, \mathrm{e}_{k+1}, \ldots) \in \prod_{\mathrm{i}=k}^{\infty} E_\mathrm{i}$ for $k \in \mathbb{N}$ arbitrary. We will denote by $X_B$ the set of all infinite paths starting at the vertex $v_0$ and by $x \in X_B$ its elements, i.e.:

$$X_B = \left\{ x = (x_1, x_2, \ldots) \in \prod_{\mathrm{i} \in \mathbb{N}} E_\mathrm{i} \colon r(x_\mathrm{i}) = s(x_{\mathrm{i}+1}),\ \forall \mathrm{i} \in \mathbb{N} \right\}$$

For every element $x \in X_B$ and every $j, k \in \mathbb{N}$ with $j < k$ we will denote $x_{j:k} = (x_j, \ldots, x_k)$.

We endow $X_B$ with the topology induced by the countably infinite topological product $\prod_{\mathrm{i} \in \mathbb{N}} E_\mathrm{i}$ of the discrete spaces $E_\mathrm{i}$, analogous to the case of subshifts. Just as in the case of subshifts, this topology has as a base the collection of cylinder sets. Given a finite path $\overline{\mathrm{e}} = (\mathrm{e}_k, \ldots, \mathrm{e}_{k+m-1}) \in \prod_{\mathrm{i}=k}^{k+m-1} E_\mathrm{i}$, we will denote its associated cylinder set by:

$$[\mathrm{e}_k, \ldots, \mathrm{e}_{k+m-1}] = \{x \in X_B \colon x_\mathrm{i} = \mathrm{e}_\mathrm{i},\ \forall \mathrm{i} = 1, \ldots, k+m-1\}$$

It is straightforward to check that these cylinder sets are clopen sets in the topology of $X_B$. As the collection of all cylinder sets is countable, it follows that $X_B$ is 0-dimensional. Additionally, $X_B$ is clearly seen to be Hausdorff and compact. The only condition missing for $X_B$ to be a Cantor space is perfectness. For this condition to be true, more structure on the diagram will be required. Nonetheless, $X_B$ is easily seen to be metrizable by considering the metric $\mathrm{d} \colon X_B \times X_B \to \mathbb{R}_{\geq 0}$ defined for every $x, y \in X_B$ as:

$$\mathrm{d}(x, y) = 2^{-\Delta(x,y)}$$

where $\Delta \colon X_B \times X_B \to \mathbb{N}$ is defined for every $x, y \in X_B$ as:

$$\Delta(x, y) = \min\{n \in \mathbb{N} \colon x_n \neq y_n\}$$

Adjacency matrices are of great interest in the study of Bratteli diagrams:

**Definition 1.22** *Let $B$ be a Bratteli diagram. For each $n \in \mathbb{N}$ we define its adjacency matrix $A^{(n)}$ of level $n$ as the $|V_n| \times |V_{n-1}|$ matrix whose entry $A^{(n)}_{v,w}$ for each $v \in V_n$ and $w \in V_{n-1}$ is equal to the number of edges going from vertex $w$ to vertex $v$, i.e.:*

$$A^{(n)}_{v,w} = |\{\mathrm{e} \in E_n \colon s(\mathrm{e}) = w,\ r(\mathrm{e}) = v\}|$$

In the context of the definition above, notice that $A^{(1)}$ is always a vector. We will usually work with diagrams in which $A^{(1)} = (1, \ldots, 1)^T$.

Other objects of interest in the study of Bratteli diagrams are height vectors. This name will make sense once we introduce the notion of Kakutani-Rholin partition.

**Definition 1.23** *Let $B$ be a Bratteli diagram. Notice that for any vertex $v \in V$ the set $E(v_0, v)$ is finite. For any $n \in \mathbb{N}$ we define the height vector $h^{(n)}$ of level $n$ of the diagram as the $|V_n|$ vector with integer entries given by $h_v^{(n)} = |E(v_0, v)|$ for each $v \in V_n$. We will also define $h^{(0)} = (1)$.*

Height vectors of consecutive levels of a Bratteli diagram satisfy an important relation:

**Proposition 1.24** *Let $B$ be a Bratteli diagram. Then for each $n \in N_0$ the following holds:*

$$h^{(n+1)} = A^{(n+1)} h^{(n)}$$

Proof. For any $n \in \mathbb{N}_0$ and $v \in V_{n+1}$ we have:

$$h_v^{(n+1)} = |E(v_0, v)| = \sum_{w \in V_n} |E(v_0, w)| \cdot |E(w, v)| = \sum_{w \in V_n} h_w^{(n)} \cdot A_{v,w}^{(n+1)}$$

Writing these equalities in matricial form yields the desired relation. □

We now introduce order to Bratteli diagrams:

**Definition 1.25** *Let $B = (V, E)$ be a Bratteli diagram. For each $v \in V \backslash V_0$ we provide $r^{-1}(\{v\})$ with a linear ordering. This induces a partial order $\geq$ in $E$ in which two edges $\mathrm{e}, f \in E$ are comparable if and only if $r(\mathrm{e}) = r(f)$, and in such case they are compared through the linear order in $r^{-1}(\{r(\mathrm{e})\})$. We then say $(B, \geq)$ is an ordered Bratteli diagram.*

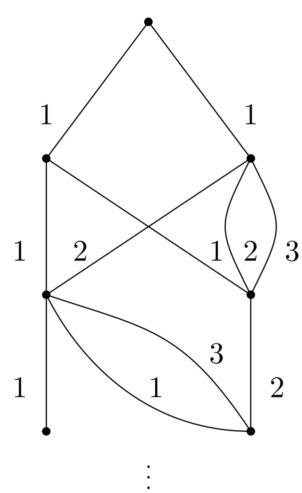


Figure 1.2: First three levels of an ordered Bratteli diagram. For each $v \in V \backslash V_0$ the order $\geq$ in $r^{-1}(\{v\})$ is represented by the numbers that label the edges in such set.

Given an ordered Bratteli diagram $(B, \geq)$ we can define for each $v \in V \backslash V_0$ a linear order in the set $E(v_0, v)$ which is in some sense a lexicographical order. Let $n \in \mathbb{N}$, $v \in V_n$, and

$\overline{\mathrm{e}} = (\mathrm{e}_1, \ldots, \mathrm{e}_n), \overline{f} = (f_1, \ldots, f_n) \in E(v_0, v)$. Then we say $\mathrm{e} \succeq f$ if there is a $k \in \{1, \ldots, n\}$ such that $\mathrm{e}_\mathrm{i} = f_\mathrm{i}$ for all $\mathrm{i} = k+1, \ldots, n$ and $\mathrm{e}_k \geq f_k$. These orders can then be used to define a partial order on the set of all infinite paths $X_B$. Two infinite paths $x, y \in X_B$ will be comparable if and only if they are cofinal, i.e.:

**Definition 1.26** *Let $B$ be a Bratteli diagram. Two infinite paths $x, y \in X_B$ are said to be cofinal, which we will denote by $x \sim y$, if $x_n = y_n$ for all sufficiently large $n \in \mathbb{N}$, i.e.:*

$$x \sim y \Leftrightarrow \exists N \in \mathbb{N}, \forall n \geq N \colon x_n = y_n$$

It is easy to check that cofinality is an equivalence relation in $X_B$. Now let $x, y \in X_B$ be infinite paths. We say that $x \succcurlyeq y$ if $x$ and $y$ are cofinal and if $N \in \mathbb{N}$ is such that $x_n = y_n$ for all $n \geq N$, then $(x_1, \ldots x_{N-1}) \succeq (y_1, \ldots, y_{N-1})$. Notice that this last property is true or false independent of the choice of $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$. It is easy to see that $\succeq$ is a partial order, so $(X_B, \succeq)$ becomes a partially ordered set.

Given an ordered Bratteli diagram $(B, \geq)$, we will denote by $E_{\min}$ and $E_{\max}$ the set of all minimal and maximal edges in the order $\geq$, respectively. An infinite path $x \in X_B$ is said to be minimal if all the edges making up the path are elements of $E_{\min}$. In the same manner we define infinite maximal paths. We will be particularly interested in ordered Bratteli diagrams that have unique minimal and maximal infinite paths, which we will denote by $x^-$ and $x^+$, respectively.

We now introduce dynamics to ordered Bratelli diagrams:

**Definition 1.27** *Let $(X, B)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths $x^-$ and $x^+$ respectively. We define the Vershik map $\phi \colon X_B \to X_B$ by:*

1. *If $x = (x_1, x_2, \ldots) \neq x^+$, then let $k = \min\{n \in \mathbb{N} \colon x_n \notin E_{\max}\}$, and define:*

$$\phi(x) = (x'_1, \ldots, x'_k, x_{k+1}, x_{k+2}, \ldots)$$

*where $x'_k$ is the immediate successor of $x_k$ in the linear ordering $\geq$ on $r^{-1}(r(x_k))$ and $x'_\mathrm{i}$ is the minimal element in the linear ordering $\geq$ on $r^{-1}(s(x_{\mathrm{i}+1}))$ for all $\mathrm{i} = k-1, \ldots, 1$.*

2. *For the unique infinite maximal path $x^+$ define $\phi(x^+) = x^-$.*

It is easy to check that the Vershik map $\phi \colon X_B \to X_B$ of any ordered Bratteli diagram $(B, \geq)$ with unique minimal and maximal infinite paths is biyective. Indeed, it is easy to show a two sided inverse of $\phi$ by defining a function in the same way but with the roles of minimal and maximal elements and paths interchanged. It is also easy to see that for $x \neq x^+$, the image $\phi(x)$ is the unique immediate successor of $x$ in the partial order $\succcurlyeq$ defined on $X_B$. In the same manner, for $x \neq x^-$, the inverse image $\phi^{-1}(x)$ is the unique immediate predecessor of $x$ in the partial order $\succcurlyeq$ defined on $X_B$. When denoting by $[x]$ the equivalence class of $x \in X_B$ under the cofinal equivalence relation $\sim$, one can easily see that for $x \notin [x^-] \cup [x^+]$ the orbit of $x$ under $\phi$ is its equivalence class under $\sim$, i.e $\mathcal{O}_\phi(x) = [x]$. For $x \in [x^+] \cup [x^-]$ one has $\mathcal{O}_\phi(x) = [x^-] \cup [x^+]$.

It will be convenient to introduce some notation which will also be later introduced in the more general context of Kakutani-Rholin partitions of Cantor minimal systems:

**Definition 1.28** *Let $(B, \geq)$ be an ordered Bratteli diagram. For any $n \in \mathbb{N}$ and $v \in V_n$ we will denote by $\tau_v^{(n)}$ the set $\tau_v^{(n)} := \bigsqcup_{\overline{\mathrm{e}} \in E(v_0,v)} [\overline{\mathrm{e}}]$ of all infinite paths in $X_B$ that start at the vertex $v_0$ and pass through the vertex $v$ and will refer to it as the tower $v$ of level $n$. The cylinder sets $[\overline{\mathrm{e}}]$ for finite paths $\overline{\mathrm{e}} \in E(v_0, v)$ will be referred to as the floors of tower $v$ of level $n$. These floors can be ordered according to the total order $\succeq$ defined on the finite set $E(v_0, v)$. This yields a partition of $\tau_v^{(n)}$ into floors $\mathcal{P}_v^{(n)} = \{\phi^k B_v^{(n)} \colon k = 0, \ldots, h_v^{(n)} - 1\}$, where $\phi^k B_v^{(n)}$ denotes the k-th floor of tower $v$ of level $n$. The 0-th floor of tower $v$ of level $n$ will be refered to as the base of tower $v$ of level $n$ and will be denoted by $B_v^{(n)}$. We also define the tower $v_0$ of level 0 as $\tau_{v_0}^{(0)} := X$. This tower has exactly one floor which corresponds to its base $B_{v_0}^{(0)} = X$.*[1]

Let $(B, \geq)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths. Consider an arbitrary integer $n \in \mathbb{N}$ and an arbitrary vertex $v \in V_n$. It is straightforward to see that if $\overline{\mathrm{e}}, \overline{f} \in E(v_0, v)$ and $\overline{f}$ is the unique immediate successor of $\overline{\mathrm{e}}$ in the total order $\succeq$ defined on the finite set $E(v_0, v)$, then $\phi([\overline{\mathrm{e}}]) = [\overline{f}]$. Thus when considering the partition $\mathcal{P}_v^{(n)} = \{\phi^k B_v^{(n)} \colon k = 0, \ldots, h_v^{(n)} - 1\}$ described in Definition 1.28, we can look at $\phi$ as the Vershik map on $X_B$ and at $B_v^{(n)}$ as the base of tower $v$ of level $n$. In what follows $\phi$ denotes de Vershik map on $X_B$ and $B_v^{(n)}$ denotes the base of tower $v$ of level $n$. We then can write $\tau_v^{(n)} = \bigsqcup_{k=0}^{h_v^{(n)}-1} \phi^k B_v^{(n)}$. This allows us to partition $X_B$ in the following way: $X_B = \bigsqcup_{v \in V_n} \bigsqcup_{k=0}^{h_v^{(n)}-1} \phi^k B_v^{(n)}$; i.e the family $\mathcal{P}^{(n)} = \{\phi^k B_v^{(n)} \colon v \in V_n, k = 0, \ldots, h_v^{(n)} - 1\}$ is a partition of $X_B$. This type of partitions, for which a precise definition will be provided in the next section, will be referred to as Kakutani-Rholin partititons. They play a crucial role in the construction of Bratteli-Vershik representations of Cantor minimal systems.

For any ordered Bratteli diagram $(B, \geq)$ with unique minimal and maximal infinite paths, the Vershik map $\phi \colon X_B \to X_B$ is more than just a biyection. Indeed, $\phi$ is also continuous and moreover a homeomorphism. This property calls for a more delicate proof we now provide. Notice that the proof relies heavily on the compactness of $X_B$ and the uniqueness of the minimal infinite path $x^-$.

**Theorem 1.29** *Let $(B, \geq)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths $x^-$ and $x^+$ respetively, and let $\phi \colon X_B \to X_B$ be its associated Vershik map. Then $\phi$ is a homeomorphism.*

Proof. Notice that we only need to prove that $\phi$ is continuous. Once that property has been established, as we already know that $\phi$ is biyective, and as $X_B$ is both compact and Hausdorff, we will automatically have that $\phi$ is a homeomorphism. The proof of the continuity of $\phi$ at points $x \neq x^+$ is just routinary work with the product topology on $X_B$. The continuity of $\phi$ at $x^+$ is more delicate and requires further discussion.

To check the continuity of $\phi$ at $x^+$ we will proceed by contradiction. Suppose that there is a sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ in $X_B$ such that $x^{(n)} \to x^+$ but $\phi(x^{(n)}) \not\to x^-$. By compactness of

[1] The reader should be warned that in the context of this definition $\phi$ is not a priori the Vershik map on $X_B$ (the diagram does not necessarily have unique minimal and maximal infinite paths) but just a notation. We will see shortly that $\phi$ con be considered as the Vershik map on $X_B$ in case the diagram has unique minimal and maximal infinite paths, which explains the use of this notation.

$X_B$ we can assume without loss of generality that $\phi(x^{(n)}) \to y$ for some point $y \in X_B$ with $y \neq x^-$. We will now show that $y$ is a minimal infinite path contradicting the uniqueness of such path. For this let $m \in \mathbb{N}$ be arbitrary. Because of the convergence $x^{(n)} \to x^+$, for big enough $n$, say $n \geq N$, we must have $x^{(n)}_{1:m} = x^+_{1:m}$, which is a finite maximal path. Thus from the definition of $\phi$ we see that $\phi(x^{(n)})_{1:m}$ is a finite minimal path for all $n \geq N$. Using the convergence $\phi(x^{(n)}) \to y$ we see that for big enough $n$, say $n > N'$, we must have $\phi(x^{(n)})_{1:m} = y_{1:m}$. Taking $N'' = \max\{N, N'\}$ we see that $y_{1:m} = \phi(x^{(N'')})_{1:m}$ which is a finite minimal path. As $m \in \mathbb{N}$ is arbitrary, we conclude as desired. □

This last theorem shows that given an ordered Bratteli diagram $(B, \geq)$ with unique minimal and maximal infinite paths, the pair $(X_B, \phi)$ is an invertible dynamical system. This class of systems will be referred to as Bratteli-Vershik systems. As mentioned previously, we will need to impose an additional condition on the diagrams to assure that the systems obtained are indeed Cantor minimal system, i.e. to assure that $X_B$ is Cantor and $\phi$ is minimal. We now provide a definition of this condition called simplicity:

**Definition 1.30** *A Bratteli diagram $B$ is said to be simple if for every $n \in \mathbb{N}_0$ there exists a big enough $m > n$ such that for every vertex $v \in V_n$ and every vertex $w \in V_m$ there exists a path from $v$ to $w$.*

Let us introduce one further definition to allow for shorter descriptions from now on:

**Definition 1.31** *An ordered Bratteli diagram $(B, \geq)$ is said to be properly ordered if $B$ is simple and $(B, \geq)$ has unique minimal and maximal infinite path.*

As was just mentioned, if a Bratteli diagram $B$ is simple, then $X_B$ will indeed be a Cantor space. Furthermore, given any order $\geq$ such that $(B, \geq)$ is properly ordered, the associated Vershik map $\phi\colon X_B \to X_B$ will be minimal. In this case $(X_B, \phi)$ will be a Cantor minimal system. Both of these facts are easily seen to be true from the definitions of simplicity and the Vershik map. For completeness sake we provide statements for them:

**Proposition 1.32** *Let $B$ be a simple Bratteli diagram. Then $X_B$ is a Cantor space.*

**Proposition 1.33** *Let $(B, \geq)$ be a properly ordered Bratteli diagram and let $(X_B, \phi)$ be the associated Bratteli-Vershik system. Then $X_B$ is a Cantor space and the Vershik map $\phi\colon X_B \to X_B$ is minimal. It follows that $(X_B, \phi)$ is a Cantor minimal system.*

We will see in the next section that every ordered Bratteli diagram $(B, \geq)$ with unique minimal and maximal infinite paths whose associated Bratteli-Vershik system $(X_B, \phi)$ is a Cantor minimal system must be properly ordered. Thus working exclusively with properly ordered diagrams will not represent a restriction to our ultimate goal of studying the whole class of Cantor minimal systems.

We now introduce a standard telescoping procedure for ordered Bratelli diagrams:

**Definition 1.34** *Let $(B, \geq)$ be an ordered Bratteli diagram. Let $\mathcal{N} = \{n_k\}_{k \in \mathbb{N}_0}$ be an increasing sequence of non-negative integers with $n_0 = 0$. We defined the $\mathcal{N}$-telescope of the diagram*

$(B,\geq)$ *as the ordered Bratteli diagram* $(B',\geq')$ *defined in the following way. The vertex set* $V'=V'_0\cup V'_1\cup\ldots$ *is defined by* $V'_k=V_{n_k}$ *for each* $k\in\mathbb{N}_0$*. The edge set* $E'=E'_1\cup E'_2\cup\ldots$ *is defined by* $E'_k=E(n_{k-1},n_k)$ *for each* $k\in\mathbb{N}$*. The source and range maps* $s',r'\colon E'\to V'$ *are defined for each* $k\in\mathbb{N}$ *and* $\overline{\mathrm{e}}=(\mathrm{e}_{n_{k-1}+1},\ldots,\mathrm{e}_{n_k})\in E'_k$ *by* $s'(\overline{\mathrm{e}})=s(\mathrm{e}_{n_{k-1}+1})$ *and* $r'(\overline{\mathrm{e}})=r(\mathrm{e}_{n_k})$*. Finally for each* $k\in\mathbb{N}$ *and each* $v\in V'_k$ *the linear order* $\geq'$ *is defined on the set* $r^{-1}(\{v\})$ *by extending each path in* $r^{-1}(\{v\})$ *up to* $v_0\in V_0$ *following minimal edges and then comparing them through the total order* $\succeq$ *defined on* $E(v_0,v)$*.*

We now define what it means for two ordered Bratteli diagrams to be equivalent:

**Definition 1.35** *We say two ordered Bratteli diagrams* $(B,\geq)$ *and* $(B',\geq')$ *are equivalent if there exists a sequence of ordered Bratteli diagrams* $\{(B_{\mathrm{i}},\geq_{\mathrm{i}})\}_{\mathrm{i}=1}^{n}$ *such that* $(B_1,\geq_1)=(B,\geq)$*,* $(B_n,\geq_n)=(B',\geq')$*, and for each* $\mathrm{i}=1,\ldots,n$ *one of the following three possibilities holds: either* $(B_{\mathrm{i}},\geq_{\mathrm{i}})$ *is a relabeling of* $(B_{\mathrm{i}+1},\geq_{\mathrm{i}+1})$*, or* $(B_{\mathrm{i}+1},\geq_{\mathrm{i}+1})$ *is a telescope of* $(B_{\mathrm{i}},\geq_{\mathrm{i}})$*, or* $(B_{\mathrm{i}},\geq_{\mathrm{i}})$ *is a telescope of* $(B_{\mathrm{i}+1},\geq_{\mathrm{i}+1})$*.*

The terminology used in the last definition is explained by the following result:

**Proposition 1.36** *Let* $(B,\geq)$ *and* $(B',\geq')$ *be properly ordered Bratteli diagrams and let* $(X_B,\phi)$ *and* $(X_{B'},\phi')$ *be the associated Bratteli-Vershik systems. Then the following properties are equivalent:*

1. *The diagrams* $(B,\geq)$ *and* $(B',\geq')$ *are equivalent.*
2. *The systems* $(X_B,\phi)$ *and* $(X_{B'},\phi')$ *are topologically conjugate.*

Finally let us state a result which shows how telescoping can be used to provide nice equivalent representations of simple ordered Bratteli diagrams:

**Proposition 1.37** *Let* $(B,\geq)$ *be a simple ordered Bratteli diagram. Then there exists* $\mathcal{N}=\{n_k\}_{k\in\mathbb{N}_0}$ *an increasing sequence of non-negative integers with* $n_0=0$ *such that the* $\mathcal{N}$*-telescope* $(B',\geq')$ *of* $(B,\geq)$ *has positive incidence matrices at all levels.*

We now study the invariant measures of Bratteli-Vershik systems. For this let us first define what we will refer to as the measure vectors of the diagram:

**Definition 1.38** *Let* $(B,\geq)$ *be a properly ordered Bratteli diagram,* $(X_B,\phi)$ *be the associated Bratteli-Vershik system, and* $\mu\in\mathcal{M}^*(X_B,\phi)$ *be a* $\phi$*-invariant measure. Then for any* $n\in\mathbb{N}$ *we define the* $\mu$*-measure vector of level* $n$ *of the diagram as the* $|V_n|$ *vector with non-negative entries given by* $\mu_v^{(n)}=\mu(B_v^{(n)})$ *for each* $v\in V_n$*, where we recall that* $B_v^{(n)}$ *is the base of tower* $v$ *of level* $n$*. Notice that by definition* $\mu_{v_0}^{(0)}=\mu(X_B)$*.*

In the context of the definition above, consider an arbitrary integer $n\in\mathbb{N}$ and an arbitrary vertex $v\in V_n$. Recall that if $\overline{\mathrm{e}},\overline{f}\in E(v_0,v)$ and $\overline{f}$ is the unique immediate succesor of $\overline{\mathrm{e}}$ in the order $\succeq$ defined on $E(v_0,v)$, then $\phi([\overline{\mathrm{e}}])=[\overline{f}]$. From the invariance of $\mu$ it follows that all cylinder sets $[\overline{\mathrm{e}}]$ for elements $\overline{\mathrm{e}}\in E(v_0,v)$ have the same measure $\mu_v^{(n)}$. Additionally, measure vectors satisfy the following important relations:

**Proposition 1.39** *Let $(B, \geq)$ be a properly ordered Bratteli diagram, $(X_B, \phi)$ be the associated Bratteli-Vershik system, and $\mu \in \mathcal{M}^*(X_B, \phi)$ be a $\phi$-invariant measure. Then the following condition holds:*

*(i)* $\forall n \in \mathbb{N}_0 \colon \mu^{(n)} = \left[A^{(n+1)}\right]^T \mu^{(n+1)}$

*Additionally, if $\mu$ is a probability measure, i.e. $\mu \in \mathcal{M}(X, T)$, then the following holds:*

*(ii)* $\forall n \in \mathbb{N}_0 \colon \langle h^{(n)}, \mu^{(n)} \rangle = 1$

Proof. We begin with a proof of the first condition. Let $\mu \in \mathcal{M}^*(X_B, \phi)$ be a $\phi$-invariant measure, $n \in \mathbb{N}$ be an arbitrary integer, and $v \in V_n$ be an arbitrary vertex of level $n$. Consider $\bar{\mathrm{e}} = (\mathrm{e}_1, \ldots, \mathrm{e}_n) \in E(v_0, v)$ an arbitrary path starting at $v_0$ and ending at $v$. We then do the following calculations:

$$\begin{aligned} \mu_v^{(n)} = \mu([\mathrm{e}_1, \ldots, \mathrm{e}_n]) &= \sum_{\mathrm{e}_{n+1} \in s^{-1}(v)} \mu([\mathrm{e}_1, \ldots, \mathrm{e}_n, \mathrm{e}_{n+1}]) \\ &= \sum_{\mathrm{e}_{n+1} \in s^{-1}(v)} \mu_{r(\mathrm{e}_{n+1})}^{(n+1)} = \sum_{w \in V_{n+1}} A_{w,v}^{(n+1)} \cdot \mu_w^{(n+1)} \end{aligned}$$

Where the last equality follows from the definition of the adjacency matrix $A^{(n+1)}$ of level $n+1$ of the diagram. Writing these equalities in matricial form yields the desired relation. The case $n = 0$ follows in the same way but with $X_B$ replacing the cylinder set $[\mathrm{e}_1, \ldots, \mathrm{e}_n]$.

We now prove the second condition. Let $\mu \in \mathcal{M}(X_B, \phi)$ be a $\phi$-invariant probability measure. For any integer $n \in \mathbb{N}$ we have:

$$\begin{aligned} 1 = \mu(X_B) &= \sum_{v \in V_n} \sum_{\bar{\mathrm{e}} \in E(v_0, v)} \mu([\bar{\mathrm{e}}]) \\ &= \sum_{v \in V_n} h_v^{(n)} \mu_v^{(n)} = \langle h^{(n)}, \mu^{(n)} \rangle \end{aligned}$$

Which proves the desired condition. The case $n = 0$ follows directly from the definitions. □

The conditions just stated turn out to be not only necessary but also sufficient, as the following theorem shows:

**Theorem 1.40** *Let $(B, \geq)$ be a properly ordered Bratteli diagram and let $(X_B, \phi)$ be the associated Bratteli-Vershik system. Consider a sequence of non-negative vectors $\{p^{(n)}\}_{n \in \mathbb{N}_0}$, with $p^{(n)} \in \mathbb{R}_{\geq 0}^{V_n}$ for all $n \in \mathbb{N}$, such that the following condition holds:*

*(i)* $\forall n \in \mathbb{N}_0 \colon p^{(n)} = \left[A^{(n+1)}\right]^T p^{(n+1)}$

*Then there exists a unique invariant measure $\mu \in \mathcal{M}^*(X_B, \phi)$ such that the for all $n \in \mathbb{N}_0$ the $\mu$-measure vector of level $n$ is given by $\mu^{(n)} = p^{(n)}$. Moreover, if the sequence of vectors $\{p^{(n)}\}_{n \in \mathbb{N}_0}$ satisfies the following condition:*

*(ii)* $\exists n \in \mathbb{N}_0 \colon \langle h^{(n)}, p^{(n)} \rangle = 1$

*Then it satisfies:*

*(iii)* $\forall n \in \mathbb{N}_0 \colon \langle h^{(n)}, p^{(n)} \rangle = 1$

*And the invariant measure* $\mu$ *is a probability measure, i.e.* $\mu \in \mathcal{M}(X,T)$.

Proof. We will first prove there exists a unique invariant measure $\mu \in \mathcal{M}^*(X_B, \phi)$ such that the for all $n \in \mathbb{N}_0$ the $\mu$-measure vector of level $n$ is given by $\mu^{(n)} = p^{(n)}$. The uniqueness of such measure is straightforward, as two invariant measures satisfying such property coincide on the generating $\pi$-system of cylinder sets and so will coincide on every mesurable set.

A natural way of constructing an invariant measure $\mu \in \mathcal{M}(X,T)$ satisfying the desired property is to use Kolmogorov's extension theorem to construct a measure $\mu$ on the space $\prod_{n\in\mathbb{N}} E_n$ such that its $\mu$-measure vectors are given by $\{p^{(n)}\}_{n\in\mathbb{N}_0}$ and which under further analysis is seen to be supported inside $X_B$, in the sense that $\mu(\prod_{n\in\mathbb{N}} E_n \backslash X_B) = 0$, and to be $\phi$-invariant when restricted to $X_B$. As all the sets $E_n$ are finite, we just need to provide a definition of the measure $\mu$ on the cylinder sets $[\mathrm{e}_1, \dots, \mathrm{e}_n]$ for all $n \in \mathbb{N}$ and all $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in \prod_{k=1}^n E_k$. The natural way of defining such quantities as to obtain the desired measure $\mu$ is to declare $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = p_v^{(n)}$ for all $v \in V_n$ and all $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in E(v_0, v)$, and $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = 0$ for all $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in \prod_{n=1}^k E_n \backslash E(0,n)$.

We need to check that the measure $\mu$ just defined satisfies Kolmogorov's extension criteria. For this we take any $n \in \mathbb{N}$ and any $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in \prod_{k=1}^n E_k$. If $(\mathrm{e}_1, \dots, \mathrm{e}_n) \notin E(0,n)$ the criteria is directly verified. Let us suppose then that $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in E(v_0, v)$ for some $v \in V_n$. We check the criteria by doing the following calculation in which we use condition (i):

$$\begin{aligned} \mu([\mathrm{e}_1, \dots, \mathrm{e}_n] \times E_{n+1}) &= \sum_{\mathrm{e}_{n+1} \in E_{n+1}} \mu([\mathrm{e}_1, \dots, \mathrm{e}_n, \mathrm{e}_{n+1}]) = \sum_{\mathrm{e}_{n+1} \in s^{-1}(v)} \mu([\mathrm{e}_1, \dots, \mathrm{e}_n, \mathrm{e}_{n+1}]) \\ &= \sum_{\mathrm{e}_{n+1} \in s^{-1}(v)} p^{(n+1)}_{r(\mathrm{e}_{n+1})} = \sum_{w \in V_{n+1}} A^{(n+1)}_{w,v} \cdot p^{(n+1)}_w \\ &= \left( \left[ A^{(n+1)} \right]^T p^{(n+1)} \right)_v = p_v^{(n)} = \mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) \end{aligned}$$

By Kolmogorov's extension theorem we obtain a measure $\mu$ defined on the product space $\prod_{n\in\mathbb{N}} E_n$ such that for any $n \in \mathbb{N}$ the measure of the cylinders $[\mathrm{e}_1, \dots, \mathrm{e}_n]$ for $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in \prod_{k=1}^n E_n$ is given by $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = p_v^{(n)}$ if $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in E(v_0, v)$ for a certain vertex $v \in V_n$, and $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = 0$ if $(\mathrm{e}_1, \dots, \mathrm{e}_n) \notin E(0,n)$.

We now study the support of the measure $\mu$. It is straightforward to check that:

$$\prod_{n\in\mathbb{N}} E_n \Big\backslash X_B = \bigcup_{n\in\mathbb{N}} \bigsqcup_{\substack{(\mathrm{e}_1, \dots, \mathrm{e}_n) \in \\ \prod_{i=1}^n E_n \backslash E(0,n)}} [(\mathrm{e}_1, \dots, \mathrm{e}_n)]$$

which is a countable union of sets of measure 0. We see then that $\mu(\prod_{n\in\mathbb{N}} E_n \backslash X_B) = 0$, i.e. $\mu$ is supported inside $X_B$. From now on, when writing $\mu$ we will refer to the restriction of the measure $\mu$ to the set $X_B$. This measure retains all the properties held by the original.

Before checking the $\phi$-invariance of $\mu$ let us do some simple calculations in the spirit of the proof of condition (ii) of Proposition 1.39. First we see that for any $n \in \mathbb{N}$ we have:

$$\mu(X_B) = \sum_{v \in V_n} \sum_{\overline{\mathrm{e}} \in E(v_0, v)} \mu([\overline{\mathrm{e}}]) = \sum_{v \in V_n} h_v^{(n)} p_v^{(n)} = \langle h^{(n)}, p^{(n)} \rangle$$

Now using condition (i) we see that for any $n \in \mathbb{N}_0$:

$$\langle h^{(n)}, p^{(n)} \rangle = \langle h^{(n)}, \left[A^{(n+1)}\right]^T p^{(n+1)} \rangle = \langle A^{(n+1)} h^{(n)}, p^{(n+1)} \rangle = \langle h^{(n+1)}, p^{(n+1)} \rangle$$

Using this for $n = 0$ we see that:

$$p_0^{(0)} = \langle h^{(0)}, p^{(0)} \rangle = \langle h^{(1)}, p^{(1)} \rangle = \mu(X_B)$$

We now check that $\mu$ is $\phi$-invariant. As $\phi$ is a homeomorphism, it will be equivalent to check the invariance of $\mu$ under images or preimages of $\phi$. Furthermore, as the cylinders sets of $X_B$ are a generating $\pi$-system, we just need to check the invariance of $\mu$ under images of $\phi$ for these sets. We first deal with cylinders described by non-maximal finite paths. Let $n \in \mathbb{N}$ and $v \in V_n$ be arbitrary. Let $\overline{\mathrm{e}} \in E(v_0, v)$ be any non-maximal path. Recall that $\phi([\overline{\mathrm{e}}]) = [\overline{f}]$, where $\overline{f} \in E(v_0, v)$ is the unique immediate succesor of $\overline{\mathrm{e}}$ according to the order $\succeq$. We then see that $\mu([\overline{\mathrm{e}}]) = p_v^{(n)} = \mu([\overline{f}]) = \mu(\phi([\overline{\mathrm{e}}]))$. Now we deal with maximal paths. Let $n \in \mathbb{N}$ and $v \in V_n$ be arbitrary. Let $(\mathrm{e}_1, \dots, \mathrm{e}_n) \in E(v_0, v)$ be a maximal path. For each $k \geq n + 1$, we will consider the set $\mathcal{R}_k = \{(\mathrm{e}_{n+1}, \dots, \mathrm{e}_k) \in E(r(\mathrm{e}_n, k)) \colon \mathrm{e}_{n+1}, \dots, \mathrm{e}_{k-1} \in E_{\max},\ \mathrm{e}_k \notin E_{\max}\}$. Using this sets we define $A = \bigsqcup_{k=n+1}^{\infty} \bigsqcup_{(\mathrm{e}_{n+1}, \dots, \mathrm{e}_k) \in \mathcal{R}_k} [\mathrm{e}_1, \dots, \mathrm{e}_k]$. The disjointness of the sets involved in these unions is straightforward to check. Now notice that $A \subseteq [\mathrm{e}_1, \dots, \mathrm{e}_n]$. It is also easy to check that $[\mathrm{e}_1, \dots, \mathrm{e}_n] \triangle A \subseteq \{x^+\}$. It follows that $\phi[\mathrm{e}_1, \dots, \mathrm{e}_n] \triangle \phi A \subseteq \{x^-\}$. Then if we could verify that $\mu(\{x^+\}) = \mu(\{x^-\}) = 0$ we would have $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = \mu(A)$ and $\mu(\phi[\mathrm{e}_1, \dots, \mathrm{e}_n]) = \mu(\phi A)$. To check that $\mu(\{x^+\}) = \mu(\{x^-\}) = 0$ we will use the fact that if $(B, \geq)$ is a properly ordered Bratteli diagram then $\min_{\mathrm{i} \in V_n} h_{\mathrm{i}}^{(n)} \to \infty$ when $n \to \infty$. We will provide a proof of this fact in the next section. Now recall that $p_0^{(0)} = \langle h^{(n)}, p^{(n)} \rangle$ for all $n \in \mathbb{N}$. As $p_0^{(0)} \in \mathbb{R}_{\geq 0}$ and as $\min_{\mathrm{i} \in V_n} h_{\mathrm{i}}^{(n)} \to \infty$ we must have that $\max_{\mathrm{i} \in V_n} \mu_{\mathrm{i}}^{(n)} \to 0$ when $n \to \infty$. This is easily seen to imply $\mu(\{x\}) = 0$ for all $x \in X_B$ and in particular $\mu(\{x^+\}) = \mu(\{x^-\}) = 0$. We then have $\mu([\mathrm{e}_1, \dots, \mathrm{e}_n]) = \mu(A)$ and $\mu(\phi[\mathrm{e}_1, \dots, \mathrm{e}_n]) = \mu(\phi A)$. With this we finally calculate:

$$\begin{aligned}\mu(\phi[\mathrm{e}_1, \dots, \mathrm{e}_n]) = \mu(\phi A) &= \sum_{k=n+1}^{\infty} \sum_{(\mathrm{e}_{n+1}, \dots, \mathrm{e}_k) \in \mathcal{R}_k} \mu(\phi[\mathrm{e}_1, \dots, \mathrm{e}_k]) \\ &= \sum_{k=n+1}^{\infty} \sum_{(\mathrm{e}_{n+1}, \dots, \mathrm{e}_k) \in \mathcal{R}_k} \mu([\mathrm{e}_1, \dots, \mathrm{e}_k]) = \mu(A) = \mu([\mathrm{e}_1, \dots, \mathrm{e}_n])\end{aligned}$$

The rest of the theorem follows directly from the calculations already done. □

Using Proposition 1.39 and Theorem 1.40 it is easy to provide a method for calculating all invariant probability measures of a given Bratteli-Vershik system. For this let us introduce the following useful notation for adjacency matrix products:

**Definition 1.41** *Let $B$ be a Bratteli diagram. For each $n \in \mathbb{N}$ denote by $A^{(n)}$ the adjacency matrix of level $n$ of $B$. For each pair of integers $n, m \in \mathbb{N}_0$ with $n < m$ define the $|V_m| \times |V_n|$ matrix $A^{(n,m)} := A^{(m)} \cdots A^{(n+1)}$. Notice that for each $v \in V_m$ and $w \in V_n$ the entry $(v, w)$ of $A^{(n,m)}$ is given by $A^{(n,m)}_{v,w} = |E(w, v)|$.*

It will also be useful to introduce the concept of measure cones of Bratteli diagrams:

**Definition 1.42** *Let $B$ be a Bratteli diagram. For each $n \in \mathbb{N}_0$ we define the measure cone of level $n$ of $B$ as the subset $C^{(n)} \subseteq \mathbb{R}^n_{\geq 0}$ given by:*

$$C^{(n)} = \bigcap_{m>n} \left[A^{(m,n)}\right]^T \mathbb{R}^m_{\geq 0}$$

*A simple proof shows that for all $n \in \mathbb{N}_0$ the following relation holds:*

$$C^{(n)} = \left[A^{(n+1)}\right]^T C^{(n+1)}$$

*Notice also that $C^{(0)} = \mathbb{R}_{\geq 0}$.*

We now describe a procedure for calculating all invariant measures of a given Bratteli-Vershik system. Let $(B, \geq)$ be a properly ordered Bratteli diagram and let $(X_B, \phi)$ be the associated Bratteli-Vershik system. For each $n \in \mathbb{N}_0$ calculate the measure cone $C^{(n)}$ of level $n$ of the diagram $B$. Choose a vector $p^{(0)} \in C^{(0)}$ with $\langle h^{(0)}, p^{(0)} \rangle = 1$, i.e. $p^{(0)} = (1)$. Then proceed inductively in the following manner: given a finite sequence of vectors $\{p^{(k)}\}_{k=0}^n$ choose a vector $p^{(n+1)} \in C^{(n+1)}$ such that $p^{(n)} = \left[A^{(n+1)}\right]^T p^{(n+1)}$. Such vector exists because $p^{(n)} \in C^{(n)} = \left[A^{(n+1)}\right]^T C^{(n+1)}$. Following this procedure we obtain a sequence of non-negative vectors $\{p^{(n)}\}_{n \in \mathbb{N}_0}$, with $p^{(n)} \in \mathbb{R}^{V_n}_{\geq 0}$ for all $n \in \mathbb{N}$, satisfying the conditions of Theorem 1.40. It follows that such sequence defines a unique invariant probability measure $\mu \in \mathcal{M}(X_B, \phi)$. Conversely, it is straightforward to see from Proposition 1.39 that every invariant probability measure $\mu \in \mathcal{M}(X_B, \phi)$ can be obtained by following the procedure described above; just choose $\{p^{(n)}\}_{n \in \mathbb{N}_0}$ as the sequence of $\mu$-measure vectors $\{\mu^{(n)}\}_{n \in \mathbb{N}_0}$. We summarize this discussion via the following proposition:

**Proposition 1.43** *Let $(B, \geq)$ be a properly ordered Bratteli diagram, $(X_B, \phi)$ be the associated Bratteli-Vershik system. Consider the convex subset $M \subseteq \prod_{n \in \mathbb{N}_0} C^{(n)}$ defined by:*

$$M = \left\{ \left(p^{(n)}\right)_{n \in \mathbb{N}_0} \in \prod_{n \in \mathbb{N}_0} C^{(n)} \colon p^{(0)} = 1 \wedge p^{(n)} = \left[A^{(n+1)}\right]^T p^{(n+1)},\ \forall n \in \mathbb{N}_0 \right\}$$

*The map $L \colon \mathcal{M}(X_B, \phi) \to M$ which maps each invariant probability measure $\mu \in \mathcal{M}(X_B, \phi)$ to its sequence of $\mu$-measure vectors $L(\mu) := \{\mu^{(n)}\}_{n \in \mathbb{N}_0}$ is a convex biyection.*

According to this last proposition, the study of the probability invariant measures of a given Bratteli-Vershik system can be carried out by studying the set $M$ described above. In particular the system will be uniquely ergodic if and only if $|M| = 1$. Notice also that the set $M$ does not depend on the order but just on the adjacency matrices of the diagram. We then see that the invariant probability measures of a Bratelli-Vershik system are completely determined by the adyacency matrices of the diagram, without influence by of the order.

## 1.3 Bratteli-Vershik Representations of Cantor Minimal Systems

Let $(X, T)$ be a Cantor minimal system. We now show how to construct a properly ordered Bratteli diagram $(B, \geq)$ such that the associated Bratteli-Vershik system $(X_B, \phi)$ is topologically conjugate to $(X, T)$. Such system is said to be a Bratteli-Vershik representation of $(X, T)$. This construction will require the introduction of some specific objects and the subsequent description of procedures for building such objects inside of $(X, T)$. We begin by introducing the concept of Kakutani-Rholin partitions:

**Definition 1.44** *Let $(X, T)$ be a Cantor minimal system. A Kakutani-Rholin partion or K-R partition of $(X, T)$ is a partition of $X$ of the form:*

$$\mathcal{P} = \{T^{\mathrm{i}} A_j \ : \ j = 1, \ldots, l, \ \mathrm{i} = 0, \ldots, h_j - 1\}$$

*where $l \in \mathbb{N}$, $h_j \in \mathbb{N}$ for all $j = 1, \ldots, l$, and the sets $A_j$ for $j = 1, \ldots, l$ are clopen. For each $j = 1, \ldots, l$, the subset $\tau_j = \bigsqcup_{\mathrm{i}=0,\ldots,h_j-1} T^{\mathrm{i}} A_j$ is called the $j$-th tower of $\mathcal{P}$, $h_j$ is called the height of the $j$-th tower of $\mathcal{P}$, $A_j$ is called the base of the $j$-th tower of $\mathcal{P}$, and the set $T^{\mathrm{i}} A_j$ is called the* i*-th floor of the $j$-th tower of $\mathcal{P}$, for all* $\mathrm{i} = 0, \ldots, h_j - 1$. *The set $\beta = \bigsqcup_{j=1}^{l} A_j$ is called the base of $\mathcal{P}$.*

An example of a Kakutani-Rholin partition was previously introduced when discussing this same notation in the context of Bratteli-Vershik systems.

We now describe a simple way of constructing K-R partitions of a given Cantor minimal system and later refine our construction as to control the diameter of the atoms that belong to such partitions. Let $(X, T)$ be a Cantor minimal system and let $A \neq \emptyset$ be any clopen subset of $X$. Let $r_A \colon A \to \mathbb{N} \cup \{\infty\}$ be the first return time map defined for each $x \in A$ as:

$$r_A(x) = \min\{n \geq 1 \ : \ T^n x \in A\}$$

As $T$ is minimal and $A$ is open, it follows that $r_A(x) < +\infty$ for all $x \in A$. From the continuity of $T$ and the fact that $A$ is clopen, it is straightforward to check that $r_A \colon A \to \mathbb{N}$ is continuous. Notice also that as $A$ is compact, the function $r_A$ takes finitely many values, say $h_1, \ldots, h_l$ for some $l \in \mathbb{N}$. Denote by $A_j = r_A^{-1}(\{h_j\})$ for each $j = 1, \ldots, l$. Then from the minimality of $T$ and the definition of the sets $A_j$ for $j = 1, \ldots, l$, it follows that the sets in $\mathcal{P} = \{T^{\mathrm{i}} A_j \ : \ j = 1, \ldots, l, \ \mathrm{i} = 0, \ldots, h_j - 1\}$ cover $X$ and are also pairwise disjoint. We have thus constructed a K-R partition whose base $\beta$ is precisely the set $A$.

Recalling that $X$ is a metric space for a certain metric $\mathrm{d} \colon X \times X \to \mathbb{R}_{\geq 0}$, we want to refine the partition just constructed as to have atoms of diameter less than a previously fixed $\varepsilon > 0$. For this we let $h = \max_{j=1,\ldots,l} h_j$ and using the uniform continuity of $T|_A^{\mathrm{i}}$ for all $\mathrm{i} = 0, \ldots, h-1$, we find $\delta > 0$ such that for any subset $C \subseteq A$ with $\mathrm{diam}(C) < \delta$ we have $\mathrm{diam}(T^{\mathrm{i}} C) < \varepsilon$ for all $\mathrm{i} = 0, \ldots, h-1$. Now for each $j = 1, \ldots, l$ we partition the set $A_j$ into a finite number of clopen sets of diameter less than $\delta$, say $A_j^k$ for $k = 1, \ldots, m_j$, where $m_j \in \mathbb{N}$. This is possible because $A$ is clopen, and in particular compact, and because $X$ is 0-dimensional. Then the desired K-R partition is:

$$\mathcal{P}' = \{T^{\mathrm{i}} A_j^k \ : \ j = 1, \ldots, l, \ k = 1, \ldots, m_j, \ \mathrm{i} = 0, \ldots, h_j - 1\}$$

We have thus proved the following result:

**Proposition 1.45** *Let $(X,T)$ be a Cantor minimal system, let $A \neq \emptyset$ be any clopen subset of $X$, and let $\varepsilon > 0$. Then there exists a K-R partition of $(X,T)$:*

$$\mathcal{P} = \{T^{\mathrm{i}} A_j \ : \ j = 1, \ldots, l, \ \mathrm{i} = 0, \ldots, h_j - 1\}$$

*such that the base $\beta$ of $\mathcal{P}$ is $A$ and such that all atoms of $\mathcal{P}$ have diameter less that $\varepsilon$.*

We now introduce another object of interest which is constructed from K-R partitions and which is the main tool for generating the desired Bratteli-Vershik representations of Cantor minimal systems:

**Definition 1.46** *Let $(X,T)$ be a Cantor minimal system. A topology generating filtered sequence of Kakutani-Rholin partitions of $(X,T)$ is a sequence of K-R partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$ of $(X,T)$ with bases $\{\beta^{(n)}\}_{n\in\mathbb{N}_0}$ for which the following conditions hold:*

1. *$\mathcal{P}^{(0)} = \{X\}$.*
2. *The atoms of $\bigcup_{n\in\mathbb{N}_0} \mathcal{P}^{(n)}$ generate the topology of $X$.*
3. *The partitions are filtered, i.e. $\mathcal{P}^{(n)} \preceq \mathcal{P}^{(n+1)}$ for all $n \in \mathbb{N}_0$.*
4. *The bases are filtered, i.e. $\beta^{(n+1)} \subseteq \beta^{(n)}$ for all $n \in \mathbb{N}_0$.*
5. *The bases converge to a point, i.e. $\bigcap_{n\in\mathbb{N}_0} \beta^{(n)} = \{x_0\}$ for some $x_0 \in X$.*

We now describe a method for constructing a topology generating filtered sequence of Kakutani-Rholin partitions of a Cantor minimal system $(X,T)$ whose topology is determined by a metric $\mathrm{d}\colon X \times X \to \mathbb{R}_{\geq 0}$. We start by taking any point $x_0 \in X$. We set $\mathcal{P}^{(0)} = \{X\}$ and notice that $\beta^{(0)} = X$. We then proceed inductively to define the rest of the partitions in the sequence. Given the K-R partition $\mathcal{P}^{(n)}$ with base $\beta^{(n)}$ and Lebesgue number $\delta^{(n)}$, we find a clopen set $\beta^{(n+1)}$ such that $x_0 \in \beta^{(n+1)}$, $\beta^{(n+1)} \subseteq \beta^{(n)}$, and $\mathrm{diam}(\beta^{(n+1)}) \leq 1/(n+1)$. Using Proposition 1.45 we then find a K-R partition $\mathcal{P}^{(n+1)}$ of $(X,T)$ whose base is $\beta^{(n+1)}$ and whose atoms are of diameter less than $\min\{\delta^{(n)}, 1/(n+1)\}$. It is straightforward to check that the sequence of K-R partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$ obtained through this procedure has the desired properties. We have thus proved the following theorem:

**Theorem 1.47** *Let $(X,T)$ be a Cantor minimal system and let $x_0 \in X$ be any point in $X$. Then there exists a topology generating filtered sequence of Kakutani-Rholin partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$ of $(X,T)$ with bases $\{\beta^{(n)}\}_{n\in\mathbb{N}_0}$ such that $\bigcap_{n\in\mathbb{N}_0} \beta^{(n)} = \{x_0\}$.*

As previously stated, topology generating filtered sequences of K-R partitions will be the main tool for representing Cantor minimal systems as Bratteli-Vershik systems. Of fundamental importance will be the fact that towers of K-R partitions traverse the towers of coarser K-R partitions in an exact manner. A precise statement of this fact, which follows directly from the definition of K-R partitions and the biyectivity of the dynamics considered, is provided in the following lemma:

**Lemma 1.48** *Let $(X,T)$ be a Cantor minimal system. Let $\mathcal{P} = \{T^{\mathrm{i}} A_j \ : \ j = 1, \ldots, l, \ \mathrm{i} = 0, \ldots, h_j - 1\}$ and $\mathcal{P}' = \{T^{\mathrm{i}} A'_j \ : \ j = 1, \ldots, l', \ \mathrm{i} = 0, \ldots, h'_j - 1\}$ be two K-R partitions with* *bases $\beta$ and $\beta'$ respectively, such that $\mathcal{P} \preceq \mathcal{P}'$ and $\beta' \subseteq \beta$. Then, for each $j = 1, \ldots, l'$, there exists a finite sequence $(j_1, \ldots, j_N)$ of integers in $\{1, \ldots, l\}$ such that:*

$$\forall k = 0, \ldots, N-1;\ \forall \mathrm{i} = 0, \ldots, h_{j_k} - 1 \colon T^{(h_{j_1} + \cdots + h_{j_k}) + \mathrm{i}} A'_j \subseteq T^{\mathrm{i}} A_{j_{k+1}}$$

*The vector $(j_1, \ldots, j_N)$ will be referred to as the $\mathcal{P}$-traversal of the $j$-th tower of $\mathcal{P}'$.*

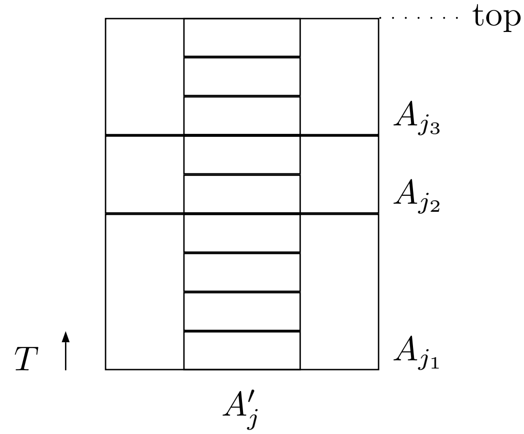


Figure 1.3: Example of tower traversal between K-R partitions.

Using this lemma we can finally construct the desired Bratteli-Vershik representations of Cantor minimal systems. The procedure is described in the following theorem:

**Theorem 1.49** *Let $(X, T)$ be a Cantor minimal system. Then there exists an ordered Bratteli diagram $(B, \geq)$ with unique minimal and maximal infinite paths such that the associated Bratteli-Vershik system $(X_B, \phi)$ is topologically conjugate to $(X, T)$.*

Proof. We first describe the construction of the ordered Bratteli diagram $(B, \geq)$ representing $(X, T)$. When defining vertices of different levels of the diagram we will assume they are distinct even though we will sometimes use the same labels for them. Take $x_0 \in X$ any point in $X$. Using Theorem 1.47 find a topology generating filtered sequence of Kakutani-Rholin partitions $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}_0}$ of $(X, T)$ with bases $\{\beta^{(n)}\}_{n \in \mathbb{N}_0}$ such that $\bigcap_{n \in \mathbb{N}_0} \beta^{(n)} = \{x_0\}$. We will denote $\mathcal{P}^{(n)} = \{T^{\mathrm{i}} A_j^{(n)} : j = 1, \ldots, l^{(n)},\ \mathrm{i} = 1, \ldots, h_j^{(n)} - 1\}$ for each $n \in \mathbb{N}_0$. For each $n \in \mathbb{N}_0$ let $V_n = \{1, \ldots, l^{(n)}\}$, i.e. there is one vertex in $V_n$ per tower in $\mathcal{P}^{(n)}$. For each $n \in \mathbb{N}$ the edges $E_n$, always directed from $V_{n-1}$ to $V_n$, are defined in the following way. For each $j = 1, \ldots, l^{(n)}$ use Lemma 1.48 to find the $\mathcal{P}^{(n-1)}$-traversal $(j_1, \ldots, j_N)$ of the $j$-th tower of $\mathcal{P}^{(n)}$. The vertex $j \in V_n$ will have an edge to each of the vertices $j_1, \ldots, j_N \in V_{n-1}$, allowing repetitions, and the order considered among these edges will be given by the order in the sequence $(j_1, \ldots, j_N)$. This completes the construction of $(B, \geq)$. Notice that for each $n \in \mathbb{N}$ and $j = 1, \ldots, l^{(n)}$ the $j$-th coordinate of the height vector of level $n$ of $(B, \geq)$ is $h_j^{(n)}$.

We now construct a homeomorphism $\varphi \colon X \to X_B$ which for all $n \in \mathbb{N}_0$ maps towers, floors, and bases of $\mathcal{P}^{(n)}$ to towers, floors, and bases of level $n$ of $X_B$, i.e.:

$$\forall n \in \mathbb{N},\ \forall j = 1, \ldots, l^{(n)},\ \forall \mathrm{i} = 1, \ldots, h_j^{(n)} - 1 \colon \varphi\, T^{\mathrm{i}} A_j^{(n)} = \phi^{\mathrm{i}} B_j^{(n)}$$

Given $x \in X$, define $T(x) = (\mathrm{e}_1, \mathrm{e}_2, \dots) \in X_B$ in the following way. For each $n \in \mathbb{N}$, let $j \in V_n$ be the tower of $\mathcal{P}^{(n)}$ to which $x$ belongs. Let $(j_1, \dots, j_N)$ be the $\mathcal{P}^{(n-1)}$-traversal of the $j$-th tower of $\mathcal{P}^{(n)}$. Then there exists a unique $k \in \{1, \dots, N\}$ such that $x$ belongs to the $k$-th traversed tower of $\mathcal{P}^{(n-1)}$, i.e. $x \in \bigsqcup_{\mathrm{i}=0}^{h_{j_k}-1} T^{(h_{j_1}+\cdots+h_{j_{k-1}})+\mathrm{i}} A_j^{(n)}$. Let $\mathrm{e}_n$ be the $k$-th edge according to the local order in $r^{-1}(j)$. This completes the definition of the map $\varphi\colon X \to \prod_{n\in\mathbb{N}} E_n$. It is straightforward to check that for each $x \in X$ the image $\varphi(x) = (\mathrm{e}_1, \mathrm{e}_2, \dots)$ in fact belongs to $X_B$, so the map $\varphi\colon X \to X_B$ is indeed well defined. It is also easy to check that $\varphi$ maps towers, floors, and bases of $\mathcal{P}^{(n)}$ to towers, floors, and bases of level $n$ of $X_B$. A simple argument using this property and Cantor's intersection theorem for compact metric spaces shows that $\varphi$ is a sujective map. The same property and the fact that the sequence of partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ is topology generating implies that $\varphi$ is inyective. Finally the continuity of $\varphi$ and its inverse $\varphi^{-1}$ follows directly from the same property, as the family of floors of towers of all levels generate the topology of both $X$ and $X_B$.

We now check that $(B, \geq)$ has unique minimal and maximal infinite paths. For this we notice that an infinite path $x \in X_B$ is minimal if and only if $\varphi^{-1}(x)$ belongs to the base $\beta^{(n)}$ of partition $\mathcal{P}^{(n)}$ for all $n \in \mathbb{N}$. Recall that $\bigcap_{n\in\mathbb{N}_0} \beta^{(n)} = \{x_0\}$. This means that for all infinite minimal paths $x \in X_B$ one has $\varphi^{-1}(x) = x_0$, i.e. there is a unique infinite minimal path $x^- = \varphi(x_0)$. Similarly, an infinite path $x \in X_B$ is maximal if and only if $\varphi^{-1}(x)$ belongs to the top floor of some tower of partition $\mathcal{P}^{(n)}$ for all $n \in \mathbb{N}$. This happens if and only if $T(\varphi^{-1}(x))$ belongs to the base $\beta^{(n)}$ of partition $\mathcal{P}^{(n)}$ for all $n \in \mathbb{N}$, i.e. if and only if $T(\varphi^{-1}(x)) = x_0$. It follows that there is a unique inifinite maximal path $x^+ = \varphi(T^{-1}(x_0))$. As $(B, \geq)$ has unique minimal and maximal infinite paths it has an associated Vershik map $\phi\colon X_B \to X_B$.

Finally we check that the homeomorphism $\varphi\colon (X,T) \to (X_B, \phi)$ is a topological conjugacy of dynamical systems, i.e. $\varphi \circ T(x) = \phi \circ \varphi(x)$ for all $x \in X$. For all points $x \neq T^{-1}(x_0)$ this property follows by carefully using the definition of $\varphi$ and $\phi$ to calculate both sides of the equality. The case $x = T^{-1}(x_0)$ follows directly from the fact that $x^+ = \varphi(T^{-1}(x_0))$, $x^- = \varphi(x_0)$, and $\phi(x^+) = x^-$. $\square$

Recall that given a properly ordered Bratteli diagram, the associated Bratteli-Vershik system is a Cantor minimal system. We will now show that every ordered Bratteli diagram with unique minimal and maximal infinite paths whose associated Bratteli-Vershik system is a Cantor minimal system must be properly ordered. As being a Cantor minimal system is a topological conjugacy invariant, this will in particular show that the ordered Bratteli diagram constructed in Theorem 1.49 is properly ordered.

We begin by proving a basic property of the height vectors of ordered Bratteli diagrams with unique minimal and maximal infinite paths whose associated Bratteli-Vershik system is a Cantor minimal system. This property was already used in the proof of in Theorem 1.40.

**Proposition 1.50** *Let $(B, \geq)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths. Suppose the associated Bratteli-Vershik system $(X,T)$ is a Cantor minimal system. Then the height vectors of the diagram satisfy:*

$$\min_{\mathrm{i}\in V_n} h_{\mathrm{i}}^{(n)} \nearrow +\infty, \text{ when } n \to \infty$$

Proof. Recall that each vertex of the diagram must have an edge coming in and an edge coming out. It is easy then to see that $\min_{\mathrm{i}\in V_n} h_{\mathrm{i}}^{(n)}$ is an increasing sequence. To prove that $\min_{\mathrm{i}\in V_n} h_{\mathrm{i}}^{(n)} \nearrow +\infty$ we proceed by contradiction. Suppose that this does not happen. Taking into account that $\min_{\mathrm{i}\in V_n} h_{\mathrm{i}}^{(n)}$ is an increasing sequence we have:

$$\exists H \in \mathbb{N}, \exists N \in \mathbb{N}, \forall n \geq N \colon \min_{\mathrm{i}\in V_n} h_{\mathrm{i}}^{(n)} = H$$

Notice then that any vertex $\mathrm{i} \in V_n$ with $n \geq N+1$ whose height is $h_{\mathrm{i}}^{(n)} = H$ must have exactly one edge coming in and this edge must come from some vertex $j \in V_{n-1}$ of height $h_j^{(n-1)} = H$. For each $n \geq N+1$ let $\mathrm{i}_n \in V_n$ be a level $n$ height minimizer, i.e. $h_{\mathrm{i}_n}^{(n)} = H$, and let $x^{(n)}$ be a point in the tower $\mathrm{i}_n$ of level $n$. Then for each integer $k \in \mathbb{N}$ with $N+1 \leq k \leq n$ the edge $x_k^{(n)}$ must be the unique edge coming into the vertex $r(x_k^{(n)})$. By compactness of $X$ there exists a sequence $(n_k)_{k\in\mathbb{N}} \nearrow \infty$ such that $x^{(n_k)} \to x$ for some $x \in X$ as $k \to \infty$. It is easy to see that for all $n \geq N+1$ the edge $x_n$ is the unique edge coming into the vertex $r(x_n)$. Indeed let us prove this for an arbitrary integer $n \geq N+1$. By convergence there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have $x_{1:n}^{(n_k)} = x_{1:n}$ and in particular $x_n^{(n_k)} = x_n$. Let us consider $k \geq K$ big enough such that $n_k \geq n$. The desired property then follows from the description provided above for the edges of $x^{(n_k)}$. The fact that for all $n \geq N+1$ the edge $x_n$ is the unique edge coming into the vertex $r(x_n)$ forces the orbit of $x$ to be finite. Due to the minimality of $(X,T)$ the phase space $X$ must then be finite. In particular, all points in $X$ are the isolated. This contradicts the property of $X$ being a Cantor space.

□

The following result, which we will use just in a moment, can be proved by the same arguments exposed in the discussion of first return time maps following Definition 1.44:

**Proposition 1.51** *Let $(X,T)$ be a Cantor minimal system. Let $A \subseteq X$ be a clopen set. Consider the first visit time map $r'_A \colon X \to \mathbb{N}_0 \cup \{\infty\}$ defined for each $x \in X$ as:*

$$r'_A(x) = \min\{n \geq 0 \colon T^n x \in A\}$$

*Then $r_A$ is a finite valued, continuous, and bounded map.*

We now show that every ordered Bratteli diagram with unique minimal and maximal infinite paths whose associated Bratteli-Vershik system is a Cantor minimal system must be properly ordered:

**Proposition 1.52** *Let $(B,\geq)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths. Suppose the associated Bratteli-Vershik system $(X,T)$ is a Cantor minimal system. Then $(B,\geq)$ is simple. As a consequence $(B,\geq)$ is properly ordered.*

Proof. For each $n \in \mathbb{N}_0$ we will denote by $\mathcal{P}^{(n)} = \{T^k A_{\mathrm{i}}^{(n)} \colon \mathrm{i} \in V_n, k = 0,\ldots,h_{\mathrm{i}}^{(n)}-1\}$ the natural Kakutani-Rholin partition of level $n$ of $(X,T)$ as described after Definition 1.28. Notice then that $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$ is a topology generating filtered family of K-R partitions. Now let $n \in \mathbb{N}_0$ be a fixed arbitrary integer. We need to find a big enough $m > n$ such that for every pair of vertices $v \in V_n$ and $w \in V_m$ there exists a path from $v$ to $w$. To find such $m$ we consider $H = \max_{\mathrm{i} \in V_n} \max_{x \in X} r'_{A_\mathrm{i}}(x) < +\infty$. Notice that $H$ is finite because of Proposition 1.52. Now take $m > n$ big enough so that $\min_{j \in V_m} h_j^{(m)} \geq H$. Such $m$ exists because of Proposition 1.50. Then for each $\mathrm{i} \in V_n$ and $j \in V_m$ the $\mathcal{P}^{(n)}$-traversal $(j_1, \ldots, j_N)$ of tower $j$ of level $m$ must include the tower i of level $n$. It is straightforward to check that this implies the desired property. □

Combining this last result with Theorem 1.49 we obtain the following important corollary:

**Corollary 1.53** *Let $(X, T)$ be a Cantor minimal system. Then there exists a properly ordered Bratteli diagram $(B, \geq)$ such that the associated Bratteli-Vershik system $(X_B, \phi)$ is topologically conjugate to $(X, T)$.*

Taking this result into consideration, from now on we will exclusively work with Bratteli-Vershik systems that originate from properly ordered Bratteli diagrams. Keep in mind that such systems are always Cantor minimal systems.

Many of the results of this thesis make use of specific assumptions on the structure of the order of the Bratteli diagrams considered. At first these assumptions may seem somewhat artificial. The following proposition shows that such assumptions are natural in the sense that they work as a standardization of the diagrams and not as a real restriction:

**Proposition 1.54** *Let $(B, \geq)$ be a propertly ordered Bratteli diagram. Then for each $n \in \mathbb{N}_0$ there exists a vertex $1 \in V_n$ and a big enough $m > n$ such that all finite minimal paths in $E(n, m)$ start from the same vertex $1 \in V_n$. Equivalently, for each $n \in \mathbb{N}_0$ there exists a vertex $1 \in V_n$ and a big enough $m > n$ such that when considering the sequence $\mathcal{N} = \{0, \ldots, n, m, m+1, \ldots\}$ the $\mathcal{N}$-telescope $(B', \geq')$ of $(B, \geq)$ has the property that all minimal edges in $E'_{n+1}$ start from the same vertex of $1 \in V'_n$*

Proof. Fix an arbitrary integer $n \in \mathbb{N}_0$. Suppose by contradiction that the desired property does not hold. This means that for every $m > n$, when looking at the paths from level $n$ to level $m$, there is always one minimal path whose starting vertex in $V_n$ is not the same as the vertex visited by $x^-$ in $V_n$. Extend such paths up to $v_0 \in V_0$ following minimal edges, and denote this new paths as $f_j \in E(v_0, j)$ for every $j > n$. Now for each $j > N$ take a point $x^{(j)} \in [f_j]$. As $X$ is compact there exists a sequence $(j_k)_{k \in \mathbb{N}} \nearrow \infty$ such that $x^{(j_k)} \to x$ for some $x \in X$ as $k \to \infty$. It is easy to see that $x$ is a global minimal path and that $x \neq x^-$. For this take an arbitrary integer $N \in \mathbb{N}$. Using the convergence of the sequence $\{x^{(j_k)}\}_{k \in \mathbb{N}}$ we can find $K \in \mathbb{N}$ such that for every $k \geq K$ we have $x_{1:N}^{(j_k)} = x_{1:N}$. In particular, taking $k \geq K$ such that $j_k \geq N$, we see that all edges $x_1, \ldots, x_N$ are minimal. As $N$ is arbitrary we see that $x$ is a global minimal path. Now considering in particular $N = n$ we see that $x_n^{(j_K)} = x_n$, which shows that $x_n$ arrives at a vertex in $V_n$ different from the vertex visited by $x^-$. We then see that $x \neq x^-$. We have so found a global minimal path $x \neq x^-$, which contradicts the existence of a unique global minimal path. □

Using this last proposition together with an inductive argument we get the following result:

**Proposition 1.55** *Let $(B, \geq)$ be a properly ordered Bratteli diagram. Then there exists*

$\mathcal{N} = \{n_k\}_{k\in\mathbb{N}_0}$ *an increasing sequence of non-negative integers with* $n_0 = 0$ *such that the* $\mathcal{N}$*-telescope* $(B', \geq')$ *of* $(B, \geq)$ *satisfies the following property:*

$$\forall n \in \mathbb{N}, \exists\ 1 \in V'_{n-1}\colon \textit{all minimal edges of } E'_n \textit{ start from the same vertex } 1 \in V'_{n-1}$$

Again, although we will not use Propositions 1.54 and 1.55 explicitly it is important to keep them in mind as they help to understand the purpose of some of the assumptions made in many of the results of this thesis.

## 1.4 Classification of Cantor Minimal Systems

We end this chapter with a brief survey of some criteria for classifying Cantor minimal systems according to the properties of their Bratteli-Vershik representations. For making our definitions precise we will need the following result which standardizes the Bratteli-Vershik representations to be considered. This result follows directly from Propositions 1.37 and 1.54 and an appropriate use of the telescoping procedure.

**Proposition 1.56** *Let* $(X, T)$ *be a Cantor minimal system. Then* $(X, T)$ *can be represented by a properly ordered Bratteli diagram* $(B, \geq)$ *with the following properties:*

1. $A^{(1)} = (1, \ldots, 1)^T$.
2. $\forall n \in \mathbb{N}\colon\ A^{(n)} > 0$.
3. $\forall n \in \mathbb{N}\colon$ *all minimal edges in* $E_n$ *arrive at the same vertex in* $V_{n-1}$.

*Such diagram* $(B, \geq)$ *will be referred to as a good Bratteli-Vershik representation of* $(X, T)$.

We begin by introducing the class of substitutive Cantor minimal systems. For some nice results regarding the origin of this class of systems see [14].

**Definition 1.57** *We say that a Cantor minimal system* $(X, T)$ *is substitutive if it admits a good Bratteli-Vershik representation* $(B, \geq)$ *that has the same adjacency matrix at all levels but the first, i.e. there exists a positive matrix* $A > 0$ *so that* $A^{(n)} = A$ *for all* $n \geq 2$.

We now introduce a bigger class of Cantor minimal systems: linearly recurrent Cantor minimal systems. It will be clear from the definition provided that every substitutive Cantor minimal system is also linearly recurrent. For some characterizations and results regarding this class of systems see [11].

**Definition 1.58** *We say that a Cantor minimal system* $(X, T)$ *is linearly recurrent if it admits a good Bratteli-Vershik representation* $(B, \geq)$ *that has finitely many different adjacency matrices, i.e. there exists a finite collection of positive matrices* $A_1, \ldots, A_k > 0$*, with* $k \in \mathbb{N}$*, so that* $A^{(n)} \in \{A_1, \ldots, A_k\}$ *for all* $n \geq 2$.

We now introduce the class of Cantor minimal systems of Toeplitz type. For some nice results regarding the origin of this class of systems see [18].

**Definition 1.59** *We say that a Cantor minimal system $(X, T)$ is of Toeplitz type if it can be represented by a properly ordered Bratteli diagram $(B, \geq)$ such that for all $n \in \mathbb{N}$ the number of edges in $E_n$ finishing at a fixed vertex of $V_n$ is constant independently of the vertex.*

We now introduce the notion of topological rank of Cantor minimal systems and the class of Cantor minimal systems of finite topological rank. For some nice results regarding this class of systems see [8] and [2].

**Definition 1.60** *We say that a Cantor minimal system $(X, T)$ is of finite topological rank if it can be represented by a properly ordered Bratteli diagram $(B, \geq)$ such that the number of vertices per level is uniformly bounded by some integer $\mathrm{d} \in \mathbb{N}$, i.e. $|V_n| \leq \mathrm{d}$ for all $n \in \mathbb{N}_0$. The minimum possible value of $\mathrm{d}$ is called the topological rank of the system. If no such $\mathrm{d} \in \mathbb{N}$ exists we say $(X, T)$ is of infinite topological rank.*

Finally we highlight a particularly interesting result by T. Downarowicz and A. Maass [13] which classifies Cantor minimal systems of finite topological rank in an extremely precise way. We first introduce the definition of expansiveness for invertible dynamical systems:

**Definition 1.61** *Let $(X, \mathrm{d})$ be a compact metric space and $T\colon X \to X$ be a homeomorphism, i.e. $(X, T)$ is an invertible dynamical system. We say $(X, T)$ is expansive if there exists a constant $c > 0$, called the expansivity constant of $(X, T)$ for the metric $\mathrm{d}$, such that for any pair of points $x, y \in X$ with $x \neq y$ there exists an integer $n \in \mathbb{Z}$ such that $\mathrm{d}(T^n x, T^n y) > c$. This property is independent of the metric considered, though the expansivity constant may change.*

Subshifts are easily seen to be expansive. A classical result by Hedlund, see for instance [28], shows that for Cantor minimal systems expansiveness is equivalent to being topologically conjugate to a subshift. In particular, every expansive Cantor minimal system is topologically conjugate to the symbolic factor of some level of any given Bratteli-Vershik representation.

We now introduce a class of Cantor minimal systems called odometers. The definition we provide makes use of the concept of inverse limit of a factoring sequence of dynamical systems, which we introduce later in Definition 2.24.

**Definition 1.62** *An invertible dynamical system $(X, T)$ is said to be an odometer if it is topologically conjugate to the inverse limit of a factoring sequence of invertible periodic dynamical systems, i.e. invertible dynamical systems consisting of exactly one finite orbit.*

It is well known that every odometer is a Cantor minimal system of topological rank $K = 1$. Conversely, every Cantor minimal system of topological rank $K = 1$ is easily seen to be an odometer.

We are finally ready to state the classification result previously mentioned:

**Theorem 1.63** *Let $(X, T)$ be a Cantor minimal system of finite topological rank $K < +\infty$. Then $K = 1$ if only if $(X, T)$ is an odometer and $K \geq 2$ if and only if $(X, T)$ is expansive if and only if $(X, T)$ is topologically conjugate to a subshift.*

# Chapter 2

# Boshernitzan's Condition

In this chapter we explore Boshernitzan's condition in different contexts. We begin by introducing the original definition of this condition for minimal subshifts. We then extend this definition to general dynamical systems via a formulation that involves the metric of the phase space. We prove a variation of M. Boshernitzan's result regarding the unique ergodicity of minimal subshifts that satisfy Boshernitzan's condition, for dynamical systems over compact ultrametric spaces. We use this variation to prove M. Boshernitzan's original result. We then study the proposed general formulation of Borshernitzan's condition in the context of Cantor minimal systems and show how it relates to the symbolic factors of such systems. We prove a unique ergodicity criterium for Cantor minimal systems analogous to Boshernitzan's. We briefly discuss how to extend the subshift property of having sublinear complexity in a subsequence to Cantor minimal systems and show diagram related conditions for such property to hold. This serves as inspiration for the last section of this chapter in which we discuss a series of necessary and/or sufficient conditions on properly ordered Bratteli diagrams so that their associated Bratteli-Vershik systems satisfy Boshernitzan's condition.

## 2.1 Boshernitzan's Condition for Minimal Subshifts

In what follows we will always consider, unless otherwise stated, a finite alphabet $\mathcal{A}$, a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, the left-shift transformation $\sigma\colon X \to X$, and a $\sigma$-invariant probability measure $\mu \in \mathcal{M}(X, \sigma)$. We begin with the definition of the quantities $\varepsilon_n(X, \mu)$ that appear in Boshernitzan's condition:

**Definition 2.1** *Let $(X, \sigma)$ be a subshift and let $\mu \in \mathcal{M}(X, \sigma)$ be an invariant probability measure. For each integer $n \in \mathbb{N}_0$ we define $\varepsilon_n(X, \mu)$ as:*

$$\varepsilon_n(X, \mu) = \min_{\omega \in L_n(X)} \mu({}_0[\omega])$$

We now state Boshernitzan's condition:

**Definition 2.2** *We say a subshift* $(X,\sigma)$ *satisfies Boshernitzan's condition if:*

$$\exists \mu \in \mathcal{M}(X,\sigma)\colon \quad \limsup_{n\to\infty} \left[n \cdot \varepsilon_n(X,\mu)\right] > 0$$

From now on we will abbreviate Boshernitzan's condition as (BC). This abbreviation will be used in any context in which we introduce this condition.

We now state the famous result by M. Boshernitzan [6] regarding the unique ergodicity of minimal subshifts that satisfy (BC):

**Theorem 2.3** *Every minimal subshift* $(X,\sigma)$ *that satisfies (BC) is uniquely ergodic.*

We also state a result by M. Boshernitzan [4] regarding the cardinality of the set of ergodic measures of minimal subshifts:

**Theorem 2.4** *Let* $K \geq 1$ *be an integer. A minimal subshift* $(X,\sigma)$ *such that:*

$$\left\lfloor \liminf_{n\to\infty} \frac{p_X(n)}{n} \right\rfloor \leq K$$

*admits at most* $K$ *ergodic measures.*

Using this result, a first step in the proof of Theorem 2.3 can be provided:

**Proposition 2.5** *Every minimal subshift* $(X,\sigma)$ *that satisfies (BC) is finitely ergodic.*

Proof. Because of (BC) there exists $C > 0$ such that for infinitely many $n \in \mathbb{N}$ we have $n \cdot \varepsilon_n(X,\mu) > C$. From the definition of $\varepsilon_n(X,\mu)$ we see that for all $\omega \in L_n(X)$ we have $\mu({}_0[\omega]) > C/n$. As all these cylinder sets are disjoint and as $\mu(X) = 1$ we have $p_X(n) = |L_n(X)| \leq n/C$. It follows that, $\liminf_{n\to\infty} p_X(n)/n \leq 1/C < \infty$. Then from Theorem 2.4 we see that $(X,\sigma)$ has finitely many ergodic measures. $\square$

Before providing a complete proof of Theorem 2.3 we will formulate Boshernitzan's condition for general dynamical systems. When working in a metric space $(X,\mathrm{d})$ we will denote by $B(x,r)$ the open ball of radius $r > 0$ centered at $x \in X$. The objects we now define will play the same role as the cylinder sets in the new definition of the quantities $\varepsilon_n(X,T,\mu,\delta)$.

**Definition 2.6** *Let* $(X,\mathrm{d})$ *be a compact metric space and let* $T\colon X \to X$ *be a continuous function, i.e.* $(X,T)$ *is a dynamical system. For any point* $x \in X$*, real number* $\delta > 0$*, and integer* $n \in \mathbb{N}_0$ *we define the set:*

$$B_n(x,\delta) = \{y \in X\colon \mathrm{d}(T^k x, T^k y) < \delta,\ \forall k = 0,\dots,n-1\}$$

*We can conveniently write this set as:*

$$B_n(x,\delta) = \bigcap_{k=0}^{n} T^{-k} B(T^k x, \delta)$$

*Notice also that for a fixed $n \in \mathbb{N}_0$ the sets $B_n(x,\delta)$ are open balls when considering the metric $\mathrm{d}_n\colon X \times X \to \mathbb{R}_{\geq 0}$ defined as:*

$$\mathrm{d}_n(x,y) = \max_{k=0,\ldots,n-1} \mathrm{d}(T^k x, T^k y)$$

Let us state the definition of the quantities $\varepsilon_n(X,T,\mu,\delta)$ in this general context:

**Definition 2.7** *Let $(X,\mathrm{d})$ be a compact metric space and let $T\colon X \to X$ be a continuous function, i.e. $(X,T)$ is a dynamical system. Let $\mu \in \mathcal{M}(X,T)$ be an invariant probability measure. For any $n \in \mathbb{N}_0$ and any $\delta > 0$ define:*

$$\varepsilon_n(X,T,\mu,\delta) = \min_{x \in X} \mu(B_n(x,\delta))$$

We now formulate Boshernitzan's condition in this general context:

**Definition 2.8** *Let $(X,\mathrm{d})$ be a compact metric space and let $T\colon X \to X$ be a continuous function. We say the dynamical system $(X,T)$ satisfies Boshernitzan's condition if:*

$$\exists \mu \in \mathcal{M}(X,T),\ \forall \delta > 0\colon\ \limsup_{n\to\infty}\, [n \cdot \varepsilon_n(X,T,\mu,\delta)] > 0$$

It is straightforward to check that the original formulation of (BC) is equivalent to this new formulation when $X$ is a subshift, $\mathrm{d}\colon X \times X \to \mathbb{R}_{\geq 0}$ is the natural metric of the subshift, and $\sigma\colon X \to X$ is the left shift on $X$.

We would like to have a unique ergodicity criteria using this new formulation of (BC) analogous to Theorem 2.3. For this we will restrict ourselves to ultrametric spaces. Let us recall the definition of ultrametric and some basic properties that we will state without proof:

**Definition 2.9** *A metric $\mathrm{d}\colon X \times X \to \mathbb{R}_{\geq 0}$ defined over a set $X$ is said to be an ultrametric if it satisfies the following condition:*

$$\forall x,y,z \in X\colon\ \mathrm{d}(x,z) \leq \max\{\mathrm{d}(x,y), \mathrm{d}(y,z)\}$$

*In this case the metric space $(X,\mathrm{d})$ is said to be an ultrametric space.*

**Proposition 2.10** *Let $\mathrm{d}\colon X \times X \to \mathbb{R}_{\geq 0}$ be an ultrametric defined over a set $X$. Then the following properties hold:*

1. *Every triangle is isosceles or equilateral, i.e.:*

$$\forall x,y,z \in X\colon\ \mathrm{d}(x,y) = \mathrm{d}(y,z) \text{ or } \mathrm{d}(x,z) = \mathrm{d}(y,z) \text{ or } \mathrm{d}(x,y) = \mathrm{d}(z,x)$$

2. *Every point inside an open ball is its center, i.e.:*

$$\forall x,y \in X,\ \forall r > 0\colon\ \mathrm{d}(x,y) < r \Rightarrow B(x,r) = B(y,r)$$

*3. Intersecting open balls are contained in each other, i.e.:*

$$\forall x, y \in X,\ \forall r, s > 0\colon$$
$$B(x, r) \cap B(y, s) \neq \emptyset \Rightarrow B(x, r) \subseteq B(y, s) \text{ or } B(y, s) \subseteq B(x, r)$$

*4. All open balls are both open and closed sets in the induced topology.*

*5. For any fixed $r > 0$, the set of all open balls of radius $r$ partitions $X$.*

*Analogous properties hold for closed balls.*

**Proposition 2.11** *Let $\mathrm{d}\colon X \times X \to \mathbb{R}_{\geq 0}$ be a metric defined over a set $X$. The the following conditions are equivalent:*

1. $\mathrm{d}$ *is an ultrametric*
2. *For all $r > 0$, the set of all open balls of radius $r$ partitions $X$.*
3. *For all $r > 0$, the set of all closed balls of radius $r$ partitions $X$.*

It is straightforward to check that the natural metric of any subshift is an ultrametric. General compact ultrametric spaces do not stray too far away in topological terms from subshifts. The following result, which we state without proof, justifies such claim:

**Theorem 2.12** *Let $(X, \mathrm{d})$ be a compact ultrametric space. Then there exists a family of finite alphabets $\{\mathcal{A}^{(n)}\}_{n\in\mathbb{N}}$, such that $X$ is isomorphic to a subset of the product space $\prod_{n\in\mathbb{N}} \mathcal{A}^{(n)}$, where we consider each alphabet $\mathcal{A}^{(n)}$ with the discrete topology.*

In the context of the theorem above, given a compact ultrametric space $(X, \mathrm{d})$, the sequence $\{|\mathcal{A}^{(n)}|\}_{n\in\mathbb{N}}$ need not be constant, not even bounded. In this sense, the class of ultrametric spaces is bigger than the class of subshifts. Still, the real difference between working with subshifts and dynamical systems over ultrametric spaces will come from the nature of the dynamics considered. Notice also that compact ultrametric spaces are very close to being Cantor spaces; the only missing condition is perfectness. Conversely, every Cantor space admits an ultrametric that induces its topology. One such ultrametric can be obtained via homeomorphism with the fullshift $\{0, 1\}^{\mathbb{Z}}$.

We are interested in finding a good generating family of sets for the borelians of a compact ultrametric space. Let $(X, \mathrm{d})$ be a compact metric space. Then it is first countable, i.e. it admits a countable basis of open sets, which in this particular case are open balls. It is easy then to check that open balls generate the borelians of $X$. When $\mathrm{d}$ is an ultrametric it follows from Property 3 of Proposition 2.10 that open balls form a $\pi$-system. Adding all up, in every ultrametric space $(X, \mathrm{d})$ open balls form a generating $\pi$-system of the borelians. This yields the following result:

**Proposition 2.13** *Let $(X, \mathrm{d})$ be an ultrametric space and let $\mu, \nu$ be borelian probability measures defined on $X$. If the following holds:*

$$\forall x \in X,\ \forall r > 0\colon \mu(B(x, r)) = \nu(B(x, r))$$

*Then both measures coincide, i.e. $\mu = \nu$.*

In a more dynamical context we have the following result:

**Proposition 2.14** *Let $(X, \mathrm{d})$ be a compact metric space and let $T\colon X \to X$ be a continuous function, i.e. $(X, T)$ is a dynamical system. If $\mathrm{d}$ is an ultrametric then $\mathrm{d}_n$ defined as in Definition 2.6 is also an ultrametric.*

Proof. Checking that $\mathrm{d}_n$ is a metric is straightfoward. For verifying that $\mathrm{d}_n$ is an ultrametric we consider arbitrary $x, y, z \in X$ and calculate:

$$\begin{aligned}
\mathrm{d}_n(x, z) &= \max_{k=0,\dots,n-1} \mathrm{d}(T^k x, T^k z) \\
&\leq \max_{k=0,\dots,n-1} \max\{\mathrm{d}(T^k x, T^k y), \mathrm{d}(T^k y, T^k z)\} \\
&= \max\left\{\max_{k=0,\dots,n-1} \mathrm{d}(T^k x, T^k y), \max_{k=0,\dots,n-1} \mathrm{d}(T^k y, T^k z)\right\} \\
&= \max\{\mathrm{d}_n(x, y), \mathrm{d}_n(y, z)\}
\end{aligned}$$

□

We now define Rauzy graphs for dynamical systems over compact ultrametric spaces:

**Definition 2.15** *Let $(X, \mathrm{d})$ be a compact ultrametric space and let $T\colon X \to X$ be a continuous function. For any $r > 0$ and $n \in \mathbb{N}$ we define the Rauzy graph $G_{r,n}(X)$ as the directed graph whose set of vertices is given by all the distinct balls $B_n(x, r)$ for $x \in X$, which by the results above form a partition of $X$, and whose set of edges is defined by the rule:*

$$\forall x, y \in X\colon B_n(x, r) \to B_n(y, r) \Leftrightarrow T[B_n(x, r)] \cap B_n(y, r) \neq \emptyset$$

We can now prove the following version of M. Boshernitzan's unique ergodicity criteria. The proof here provided is based on the arguments exposed in [16].

**Theorem 2.16** *Let $(X, \mathrm{d})$ be an ultrametric compact space and let $T\colon X \to X$ be a continuous minimal function. Suppose that the system $(X, T)$ is finitely ergodic and satisfies (BC). Then $(X, T)$ is uniquely ergodic.*

Proof. We divide the proof in several steps:

1. **MEASURE GAP:**
   Assume by contradiction that $(X, T)$ is not uniquely ergodic. Then, there exists $x_0 \in X$ and $r > 0$ such that the set $E = \{\mu(B(x_0, r)) \mid \mu \in \mathcal{E}(X, T)\}$ has finite cardinality greater than one. We choose two ergodic measures $\mu_1$ and $\mu_2$ which correspond to consecutive elements in $E$, i.e. such that $\mu_1(B(x_0, r)) < \mu_2(B(x_0, r))$ and $(\mu_1(B(x_0, r)), \mu_2(B(x_0, r))) \cap E = \emptyset$. Let $r$ and $s$ be real numbers such that $\mu_1(B(x_0, r)) < r < s < \mu_2(B(x_0, r))$. The interval $(r, s)$ is what we call the measure gap.

2. **FROM MEASURE TO ORBIT INFORMATION:**
Let $\mu \in \mathcal{M}(X,T)$ be an invariant probability measure via which $(X,T)$ satisfies (BC). We want to obtain information about the orbits of the system from what we know about the measures (measure gap). The natural tool to use in this context is Birkhoff's ergodic theorem. For any $x \in X$ we will denote $S_n(x) := \sum_{k=0}^{n-1} \mathbf{1}_{B(x_0,r)}(T^n x)$. For any $\nu \in \mathcal{E}(X,T)$ we have by Birkhoff's ergodic theorem that $\frac{S_n(x)}{n} \to \nu(B(x_0,r)) \notin [r,s]$ as $n \to \infty$ for $\nu$-a.e. point $x \in X$. We want to describe this particular orbit structure in terms of $\mu$. As the measure $\mu$ is not necessarily an ergodic measure, we will need to lose some information (but not much). For this let us consider the sets $F_n = \{x \in X \mid \frac{S_n(x)}{n} \in [r,s]\}$. Due to the convergence previously stated, we see that for any $\nu \in \mathcal{E}(X,T)$ we have $\nu(\bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} X \setminus F_n) = 1$. Using measure continuity we get $\nu(F_N) \leq \nu(\bigcup_{n\geq N} F_n) \to \nu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} F_n) = 0$ as $N \to \infty$. Thus we see that for any $\nu \in \mathcal{E}(X,T)$ we have $\nu(F_n) \to 0$ as $n \to \infty$. We extend this property to any invariant measure by using ergodic decomposition. The quantity $\mu(F_n)$ will act as a bound for a simple expression involving $n \cdot \varepsilon_n(X,T,\mu,r)$ which when taking limits will lead to a contradiction.

3. **GENERIC POINTS AND TRAVERSING THE GAP:**
To get the desired bound we will traverse the measure gap in the Rauzy graphs $G_{r,n}(X)$ while also using a simple combinatorial argument. How to start before the gap and end after it? The answer comes via generic points. As $\mu_1$ and $\mu_2$ are ergodic measures, we can find generic points $y$ of $\mu_1$ and $z$ of $\mu_2$. Indeed one can find this points almost everywhere with respect to the corresponding measures. We have then that $\frac{S_n(y)}{n} \xrightarrow{n\to\infty} \mu_1(B(x_0,r)) < r$ and $\frac{S_n(z)}{n} \xrightarrow{n\to\infty} \mu_2(B(x_0,r)) > s$. Let $N \in \mathbb{N}$ be big enough so that for every $n \geq N$ we have $\frac{S_n(y)}{n} < r$ and $\frac{S_n(z)}{n} > s$. In what follows we will consider a fixed $n \geq N$. As $(X,T)$ is minimal we have that the Rauzy graph $G_{r,n}(X)$ is strongly connected. Then there is a path from $B_n(y,r)$ to $B_n(z,r)$. Let us denote this path by $B_n(y,r) = B_n(v_1,r) \to B_n(v_2,r) \to \cdots \to B_n(v_{p-1},r) \to B_n(v_p,r) = B_n(z,r)$. We choose a shortest such path. In particular it will have no loops. This will provide the necessary disjointness for calculating the bounds in the next step.

4. **COMBINATORIAL ARGUMENT:**
How does the Birkhoff sum change as we move through the path $B_n(v_1,r) \to \cdots \to B_n(v_p,r)$? To answer this question the following fact will be essential: the sum $S_n(x)$ is the same for all points in a fixed ball $B_n(v,r)$. To see this we take any $v \in X$ and any $x \in B_n(v,r)$ and calculate:

$$\begin{aligned} S_n(v) &= \sum_{k=0}^{n-1} \mathbf{1}_{B(x_0,r)}(T^k v) = \sum_{k=0}^{n-1} \mathbf{1}_{B(T^k v,r)}(x_0) \\ &= \sum_{k=0}^{n-1} \mathbf{1}_{B(T^k x,r)}(x_0) = \sum_{k=0}^{n-1} \mathbf{1}_{B(x_0,r)}(T^k x) \\ &= S_n(x) \end{aligned}$$

where in the third equality we have used Property 2 of Proposition 2.10 applied to the ultrametric d. Now notice that for $1 \leq \mathrm{i} \leq p-1$ we have that $S_n(v_{\mathrm{i}+1}) \leq S_n(v_{\mathrm{i}}) + 1$, hence $\frac{S_n(v_{i+1})}{n} \leq \frac{S_n(v_i)}{n} + \frac{1}{n}$. This means that for each step in the path the Birkhoff sum cannot increase by more that $\frac{1}{n}$. Remember that we have $\frac{S_n(v_1)}{n} < r$ and $\frac{S_n(v_p)}{n} > s$, so the Birkhoff sum in the path starts before the gap and ends after the gap. As a consequence we see that the set $I = \{1 \leq i \leq p \mid \frac{S_n(v_i)}{n} \in [r,s]\}$ has cardinality greater than or equal to $\lfloor \frac{(s-r)}{1/n} \rfloor$, which in turn is greater than $n(s-r)-1$. Hence, we see that $F_n$ contains the disjoint union $\bigsqcup_{i \in I} B_n(v_i, r)$. This disjointness comes from the fact that the path considered has no loops. Using the definition of $\varepsilon_n(X,T,\mu,r)$ we see that $\mu(F_n) \geq (n(s-r)-1) \cdot \varepsilon_n(X,T,\mu,r) = (s-r) \cdot n \cdot \varepsilon_n(X,T,\mu,r) - \varepsilon_n(X,T,\mu,r)$. As a consequence we have that $\varepsilon_n(X,T,\mu,r) \to 0$ as $n \to \infty$. Taking limits in the same inequality we see that $n \cdot \varepsilon_n(X,T,\mu,r) \to 0$ as $n \to \infty$, contradicting (BC).

□

The result above is a partial generalization of Theorem 2.3, in the sense that it applies to general minimal dynamical systems over ultrametric spaces, though it requires the additional hypothesis of finite ergodicity. This hypothesis is directly satisfied when considering minimal subshifts that satisfy (BC), as Proposition 2.5 shows.

We want to use Theorem 2.16 on Cantor minimal systems. With this purpose in mind we explore a method for constructing a metric on a given Cantor minimal system starting from a topology generating filtered family of Kakutani-Rholin partitions. The following theorem, which we state without proof, will be the main tool for this purpose:

**Proposition 2.17** *Let $X$ be a Hausdorff topological space. Let $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}_0}$ be a filtered sequence of partitions that generates the topology of $X$. For any $x \in X$ and any $n \in \mathbb{N}_0$ we will denote by $\mathcal{P}^{(n)}_x$ the element of the partition $\mathcal{P}^{(n)}$ to which $x$ belongs. Consider the function $\Delta \colon X \times X \to \mathbb{R}_{\geq 0}$ defined for all $x, y \in X$ as:*

$$\Delta(x,y) = \min\{n \in \mathbb{N}_0 \colon \mathcal{P}^{(n)}_x \neq \mathcal{P}^{(n)}_y\}$$

*Consider the function $\mathrm{d} \colon X \times X \to \mathbb{R}_{\geq 0}$ defined for all $x, y \in X$ as:*

$$\mathrm{d}(x,y) = 2^{-\Delta(x,y)}$$

*Then $\mathrm{d}$ is an ultrametric that generates the topology of $X$.*

In the context of the proposition above, open balls of the metric d are directly related to the sets that conform the partitions $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}_0}$ through the relation:

$$\forall x \in X, \forall m \in \mathbb{N} \colon B(x, 2^{-m}) = \mathcal{P}^{(m)}_x$$

Given any $x \in X$, $n \in \mathbb{N}$, and $m \in \mathbb{N}$ we can also represent the sets of Definition 2.6 as:

$$B_n(x, 2^{-m}) = \bigcap_{k=0}^{n-1} T^{-k}\left[\mathcal{P}^{(m)}_{T^k x}\right]$$

Taking the above facts into consideration we can now appropriately reformulate (BC) for Cantor minimal systems:

**Definition 2.18** *Let $(X,T)$ be a Cantor minimal system and let $\mathcal{P} := \{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$ be a topology generating filtered family of Kakutani-Rholin partitions. For any $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ consider the following set of admisible sequences:*

$$L_{n,m}(\mathcal{P}) = \left\{ (A_k)_{k=0}^{n-1} \in \left[\mathcal{P}^{(m)}\right]^n \;\middle|\; \bigcap_{k=0}^{n-1} T^{-k} A_k \neq \emptyset \right\}$$

*Let $\mu \in \mathcal{M}(X,T)$ be an invariant measure. For any $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ define:*

$$\varepsilon_n(X,T,\mu,m) = \min_{(A_k)_{k=0}^{n-1} \in L_{n,m}(\mathcal{P})} \mu\left(\bigcap_{k=0}^{n-1} T^{-k} A_k\right)$$

*We then say that the system $(X,T)$ satisfies Boshernitzan's condition if:*

$$\exists \mu \in \mathcal{M}(X,T),\ \forall m \in \mathbb{N}\colon\ \limsup_{n\to\infty} n \cdot \varepsilon_n(X,T,\mu,m) > 0$$

This formulation of $(BC)$ still requires further analysis. In particular, we would like to check that it does not depend on the choice of $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}_0}$. This leads us to the main object of interest of the next section: symbolic factors of Cantor minimal systems.

## 2.2 Boshernitzan's Condition for Cantor Minimal Systems

We begin this section by introducing the following terminology:

**Definition 2.19** *Let $(X,\sigma)$ be a subshift defined over a finite alphabet $\mathcal{A}$ and let $\mu \in \mathcal{M}(X,\sigma)$ be an invariant probability measure. We define the Boshernitzan's constant of the pair $(X,\mu)$ as the number:*

$$c(X,\mu) = \limsup_{n\to\infty} \left[n \cdot \varepsilon_n(X,\mu)\right]$$

*where as in Definition 2.1 we consider for all $n \in \mathbb{N}_0$:*

$$\varepsilon_n(X,\mu) = \min_{\omega \in L_n(X)} \mu({}_0[\omega])$$

*In this context we will say (BC) holds for the pair $(X,\mu)$ if its Boshernitzan's constant is positive, i.e. $c(X,\mu) > 0$.*

We now study the behaviour of Boshernitzan's constants under factor maps:

**Proposition 2.20** *Let $(X,\sigma_X)$ and $(Y,\sigma_Y)$ be subshifts defined over finite alphabets $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\pi\colon (X,\sigma_X) \to (Y,\sigma_Y)$ be a factor map between these subshifts. Consider an invariant probability measure $\mu \in \mathcal{M}(X,\sigma_X)$ and let $\pi\mu \in \mathcal{M}(Y,\sigma_Y)$ be the $\pi$-pushforward of the measure $\mu$. Then we have:*

$$c(Y,\pi\mu) \geq c(X,\mu)$$

*In particular, if (BC) holds for the pair $(X,\mu)$ it will also hold for $(Y,\pi\mu)$.*

Proof. By Theorem 1.11 we know that the factor map $\pi\colon (X, \sigma_X) \to (Y, \sigma_Y)$ is a sliding block code. Let us denote its memory by $m \in \mathbb{N}_0$, its anticipation by $a \in \mathbb{N}_0$, and its block code by $\phi\colon L_{a+1+m}(X) \to \mathcal{B}$. When we write $\phi$ acting on a word of length bigger than $a+1+m$ we will be refering to the natural extension of this function to words of such length. Given arbitrary $n \in \mathbb{N}$ and $\omega \in L_{a+n+m}(X)$ let $u = \phi(\omega) \in L_n(Y)$. We then have $\pi({}_0[\omega]) \subseteq {}_a[u]$. Equivalently, ${}_0[\omega] \subseteq \pi^{-1}({}_a[u])$. As $\pi$ is surjective we have that for every $u \in L_n(Y)$ there exists a $\omega_u \in L_{a+n+m}(X)$ such that $\phi(\omega_u) = u$. We then have the following bounds:

$$\begin{aligned}
\varepsilon_n(Y, \pi\mu) &= \min_{u \in L_n(Y)} \pi\mu([u]) = \min_{u \in L_n(Y)} \mu(\pi^{-1}([u])) \\
&\geq \min_{u \in L_n(Y)} \mu([\omega_u]) \geq \min_{\omega \in L_{a+n+m}(X)} \mu([\omega]) \\
&= \varepsilon_{a+n+m}(X, \mu)
\end{aligned}$$

It follows that:

$$\begin{aligned}
c(Y, \pi\mu) &= \limsup_{n\to\infty} n \cdot \varepsilon_n(Y, \pi\mu) \geq \limsup_{n\to\infty} n \cdot \varepsilon_{a+n+m}(X, \mu) \\
&= \limsup_{n\to\infty} \frac{n}{a+n+m} \cdot (a+n+m) \cdot \varepsilon_{a+n+m}(X, \mu) \\
&= \limsup_{n\to\infty} (a+n+m) \cdot \varepsilon_{a+n+m}(X, \mu) \\
&= \limsup_{n\to\infty} n \cdot \varepsilon_n(X, \mu) = c(X, \mu)
\end{aligned}$$

which concludes the proof. □

To better understand the formulation of Boshernitzan's condition for Cantor minimal systems provided in Definition 2.18 it will be extremely helpful to introduce the notion of symbolic factors. Roughly speaking, symbolic factors will provide enough information to understand the original system completely. Let us then introduce the following definition:

**Definition 2.21** *Let $(X, T)$ be an invertible dynamical system. Any subshift $(Y, \sigma)$ over a finite alphabet $\mathcal{A}$ that is a factor of $(X, T)$ is said to be a symbolic factor of $(X, T)$.*

We can provide a more explicit description of the symbolic factors of any given invertible dynamical system through the following definition:

**Definition 2.22** *Let $(X, T)$ be an invertible dynamical system. Let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a finite partition of $X$ by clopen sets. Consider the function $\pi_{\mathcal{P}}\colon X \to \mathcal{P}^{\mathbb{Z}}$ that associates to each $x \in X$ the sequence $\pi_{\mathcal{P}}(x) = (A_{\mathrm{i}_n})_{n\in\mathbb{Z}}$ of elements of $\mathcal{P}$ such that for each $n \in \mathbb{Z}$ one has $T^n x \in A_{\mathrm{i}_n}$. The function $\pi_P$ is a factor map from $(X, T)$ to the subshift $(\pi_{\mathcal{P}}(X), \sigma)$. The system $(\pi_{\mathcal{P}}(X), \sigma)$ will be referred to as the $\mathcal{P}$-symbolic factor of $(X, T)$.*

The symbolic factors described in this last definition are easily seen to account for all possible symbolic factors of the system $(X, T)$ up to change of alphabets. It will also be helpful for our purposes to introduce the notion of symbolic factors of ordered Bratteli diagrams:

**Definition 2.23** *Let $(B, \geq)$ be an ordered Bratteli diagram with unique minimal and maximal infinite paths and let $(X_B, \phi)$ be the associated Bratteli-Vershik system. For any $n \in \mathbb{N}$*

*consider the partition of $X_B$ given by the collection of all level $n$ cylinders $\mathcal{P}^{(n)} = \{[\bar{\mathrm{e}}] \colon \bar{\mathrm{e}} \in E(0,n)\}$. We will refer to the $\mathcal{P}^{(n)}$-symbolic factor of $(X_B, \phi)$ as the symbolic factor of level $n$ of the ordered Bratteli diagram $(B, \geq)$.*

We now introduce the notion of factoring sequence of dynamical systems and the concept of inverse limit of such sequences:

**Definition 2.24** *A sequence of dynamical systems $\{(X_n, T_n)\}_{n\in\mathbb{N}}$ together with a sequence of factor maps $\{\pi_n \colon (X_{n+1}, T_{n+1}) \to (X_n, T_n)\}_{n\in\mathbb{N}}$ among these systems will be refered to as a factoring sequence of dynamical systems. We represent such sequence by the diagram:*

$$(X_1, T_1) \xleftarrow{\pi_1} (X_2, T_2) \xleftarrow{\pi_2} (X_3, T_3) \leftarrow \ldots \leftarrow (X_n, T_n) \xleftarrow{\pi_n} (X_{n+1}, T_{n+1}) \leftarrow \ldots$$

*The inverse limit $(X, T)$ of a factoring sequence of dynamical systems is defined in the following way. The phase space $X$ is given by the subset $X \subseteq \prod_{n\in\mathbb{N}} X_n$ defined as:*

$$X = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \;\middle|\; \forall n \in \mathbb{N} \colon \pi_n(x_{n+1}) = x_n \right\}$$

*This set is a closed subset of the compact space $\prod_{n\in\mathbb{N}} X_n$ when considering the product topology. We provide $X$ with the transformation $T \colon X \to X$ defined for all $(x_n)_{n\in\mathbb{N}} \in X$ as:*

$$T((x_n)_{n\in\mathbb{N}}) = (T_n(x_n))_{n\in\mathbb{N}}$$

*It is straightforward to check that $(X, T)$ is a dynamical system. Moreover each of the systems $(X_n, T_n)$ is a factor of $(X, T)$ through the factor map $\Pi_n \colon (X, T) \to (X_n, T_n)$ defined as the projection to the $n$-th coordinate.*

Using the notation of the definition above, let us briefly discuss some particularities of the topology of inverse limits. As we are assuming that the spaces $\{X_n\}_{n\in\mathbb{N}}$ are all compact metric spaces, we have in particular that they are second countable. It follows that the countable product $\prod_{n\in\mathbb{N}} X_n$ is also second countable. As a consequence, the $\pi$-system of cylinder sets of $\prod_{n\in\mathbb{N}} X_n$ generates the borelians of $\prod_{n\in\mathbb{N}} X_n$. It follows that the $\pi$-system of cylinder sets of $X$, i.e. the restriction of the cylinder sets of $\prod_{n\in\mathbb{N}} X_n$ to $X$, generates the borelians of $X$. It is not hard to see that the cylinders sets of $X$ coincide with the cylinder sets of $X$ that restrict just one coordinate. Indeed one can prove that if $A_{\mathrm{i}} \subseteq X_{\mathrm{i}}$ for $\mathrm{i} = 1, \ldots, n$, are any given subsets, then one has:

$$X \cap \left( \prod_{\mathrm{i}=1}^{n} A_{\mathrm{i}} \times \prod_{\mathrm{i}=n+1}^{\infty} X_{\mathrm{i}} \right) = X \cap \left( \prod_{\mathrm{i}=1}^{n-1} X_{\mathrm{i}} \times A \times \prod_{\mathrm{i}=n+1}^{\infty} X_{\mathrm{i}} \right)$$

where $A \subseteq X_n$ is the set:

$$A = A_n \cap (\pi_{n-1})^{-1}(A_{n-1}) \cap \cdots \cap (\pi_1 \circ \cdots \circ \pi_{n-1})^{-1}(A_1)$$

Notice in particular that if the sets $A_{\mathrm{i}}$ are clopen for all $\mathrm{i} = 1, \ldots, n$, then the set $A$ will also be clopen. From this discussion we can also see that borelian measures on $X$ are uniquely determined by their values on cylinder sets that restrict just one coordinate.

We now return to Cantor minimal systems. The following proposition uses the concept of inverse limit to formalize the idea that symbolic factors of Cantor minimal systems provide enough information to understand the original system completely.

**Theorem 2.25** *Let $(X,T)$ be a Cantor minimal system. Let $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ be a topology generating filtered family of clopen partitions. For each $n \in \mathbb{N}$ let $(X_n, \sigma_n)$ be the $\mathcal{P}^{(n)}$-symbolic factor of $(X,T)$ and let $\Pi_n \colon (X,T) \to (X_n, \sigma_n)$ be the corresponding factor map. For each $n \in \mathbb{N}$ consider also the factor map $\pi_n \colon (X_{n+1}, \sigma_{n+1}) \to (X_n, \sigma_n)$ which given any bi-infinite word $x \in X_{n+1}$ replaces each $\mathcal{P}^{(n+1)}$-symbol with the corresponding $\mathcal{P}^{(n)}$-symbol according to the partition filtering $\mathcal{P}^{(n+1)} \succeq \mathcal{P}^{(n)}$. Let $(\hat{X}, \hat{T})$ be the inverse limit of the factoring sequence of dynamical systems $\{(X_n, \sigma_n)\}_{n\in\mathbb{N}}$ with factor maps $\{\pi_n \colon (X_{n+1}, \sigma_n) \to (X_n, \sigma_n)\}_{n\in\mathbb{N}}$. Then $(X,T)$ is topologically conjugate to $(\hat{X}, \hat{T})$.*

Proof. We directly construct the conjugacy map between the systems $(X,T)$ and $(\hat{X}, \hat{T})$. For this we define the map $\varphi \colon (X,T) \to (\hat{X}, \hat{T})$ that to each $x \in X$ assigns the sequence of images $\varphi(x) = (\Pi_1 x, \Pi_2 x, \dots)$. We will prove that this map is a continous biyective map which conmutes with the dynamics of the systems. For the surjectivity we will use the following conmutativity property: $\pi_{\mathrm{i}} \circ \Pi_{\mathrm{i}+1} = \Pi_{\mathrm{i}},\ \forall \mathrm{i} \in \mathbb{N}$. This fact follows directly from the definitions provided in the statement of this theorem.

1. **CONTINUITY:**
   As the space $\hat{X}$ has the induced product topology, it suffices to check that the map $\varphi \colon X \to \hat{X}$ is continuous component wise. But for all $\mathrm{i} \in \mathbb{N}$ the i-th coordinate of this map is precisely the map $\Pi_{\mathrm{i}} \colon X \to X_{\mathrm{i}}$, which we already know is a factor map.

2. **CONMUTATIVITY:**
   Recalling that the maps $\Pi_{\mathrm{i}} \colon X \to X_{\mathrm{i}}$ are factor maps, we have that for every $x \in X$: $\varphi \circ T(x) = (\Pi_1 \circ T(x), \Pi_2 \circ T(x), \dots) = (\sigma_1 \circ \Pi_1(x), \sigma_2 \circ \Pi_2(x), \dots) = \hat{T} \circ \varphi(x)$.

3. **SURJECTIVITY:**
   Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots) \in \hat{X}$ be any point. As the maps $\Pi_{\mathrm{i}} \colon X \to X_{\mathrm{i}}$ are surjective for all $\mathrm{i} \in \mathbb{N}$, the sets $C_{\mathrm{i}} := \Pi_{\mathrm{i}}^{-1}(\{\hat{x}_{\mathrm{i}}\})$ are nonempty for all $\mathrm{i} \in \mathbb{N}$. These sets are also closed as the maps $\Pi_{\mathrm{i}} \colon X \to X_{\mathrm{i}}$ are continuous and the spaces $X_{\mathrm{i}}$ are Hausdorff for all $\mathrm{i} \in \mathbb{N}$. It is straightforward to check that these sets are nested in the sense that $C_{\mathrm{i}+1} \subseteq C_{\mathrm{i}}$ for all $\mathrm{i} \in \mathbb{N}$. Indeed, given any $\mathrm{i} \in \mathbb{N}$, if $x \in C_{\mathrm{i}+1}$, then we have that $\Pi_{\mathrm{i}+1}(x) = \hat{x}_{\mathrm{i}+1}$. It follows that $\pi_{\mathrm{i}} \circ \Pi_{\mathrm{i}+1}(x) = \pi_{\mathrm{i}}(\hat{x}_{\mathrm{i}+1})$. Using the conmutativity property discussed at the beginning of this proof and the definition of $\hat{X}$ we have that $\Pi_{\mathrm{i}}(x) = \hat{x}_{\mathrm{i}}$. This shows that $C_{\mathrm{i}+1} \subseteq C_{\mathrm{i}}$. We then have a sequence of nested closed sets $C_1 \supseteq C_2 \supseteq \dots$ in the compact metric space $X$, so by Cantor's intersection theorem we have that $\bigcap_{\mathrm{i}\in\mathbb{N}} C_{\mathrm{i}} \neq \emptyset$. Any point $x \in \bigcap_{\mathrm{i}\in\mathbb{N}} C_{\mathrm{i}}$ is such that $\varphi(x) = \hat{x}$.

4. **INYECTIVITY**:

Let $\hat{x} = (\hat{x}^{(1)}, \hat{x}^{(2)}, \dots) \in \hat{X}$ be any point. We then see that:

$$\begin{aligned}\varphi^{-1}(\{\hat{x}\}) &= \{x \in X\colon \varphi(x) = \hat{x}\} = \{x \in X\colon \Pi_{\mathrm{i}}(x) = \hat{x}^{(\mathrm{i})}, \forall \mathrm{i} \in \mathbb{N}\} \\ &\subseteq \{x \in X\colon (\Pi_{\mathrm{i}}(x))_0 = \hat{x}_0^{(\mathrm{i})}\} = \bigcap_{\mathrm{i}\in\mathbb{N}} \hat{x}_0^{(\mathrm{i})}\end{aligned}$$

As the sequence of partition $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ generates the topology of $X$, this last intersection has to be a singleton, proving the desired inyectivity.

□

The last theorem can lead one to think that every Cantor minimal system is topologically conjugate to a subshift over an infinite alphabet, it being an inverse limit of subshifts. This is not the case as the topology of inverse limits does not behave as the topology of subshifts.

Symbolic factors of Cantor minimal systems have a very useful property: not only are they factors of the original system but also of one and consequently all of the subsequent dynamical systems that conform a factoring sequence of dynamical system as the one described in Theorem 2.25. We now provide a proof of this fact:

**Proposition 2.26** *Let $(X, T)$ be a Cantor minimal system. Let $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ be a topology generating filtered family of clopen partitions. For each $n \in \mathbb{N}$ let $(X_n, \sigma_n)$ be the $\mathcal{P}^{(n)}$-symbolic factor of $(X, T)$ and let $\Pi_n\colon (X, T) \to (X_n, \sigma_n)$ be the corresponding factor map. For each $n \in \mathbb{N}$ consider also the factor map $\pi_n\colon (X_{n+1}, \sigma_{n+1}) \to (X_n, \sigma_n)$ which given any bi-infinite word $x \in X_{n+1}$ replaces each $\mathcal{P}^{(n+1)}$-symbol with the corresponding $\mathcal{P}^{(n)}$-symbol according to the partition filtering $\mathcal{P}^{(n+1)} \succeq \mathcal{P}^{(n)}$. We know that $(X, T)$ is topologically conjugate to the inverse limit $(\hat{X}, \hat{T})$ of the factoring sequence of dynamical systems $\{(X_n, \sigma_n)\}$ with factor maps $\{\pi_n\colon (X_{n+1}, \sigma_{n+1}) \to (X_n, \sigma_n)\}_{n\in\mathbb{N}}$. Now let $(Y, \sigma)$ be any symbolic factor of $(X, T)$ through some factor map $\pi\colon (X, T) \to (Y, \sigma)$. Then there exists an integer $n \in \mathbb{N}$ such that $(Y, \sigma)$ is a symbolic factor of the system $(X_n, \sigma_n)$ through a factor map $\pi'\colon (X_n, \sigma_n) \to (Y, \sigma)$ which satisfies the conmutativity relation $\pi = \pi' \circ \Pi_n$.*

Proof. For simplicity of notation we will consider that the maps $\{\Pi_n\colon (X, T) \to (X_n, \sigma_n)\}_{n\in\mathbb{N}}$ and the map $\Pi\colon (X, T) \to (Y, \sigma)$ have as domain the system $(\hat{X}, \hat{T})$. This makes sense as the systems $(X, T)$ and $(\hat{X}, \hat{T})$ are topologically conjugate. We will also consider $(Y, \sigma)$ as a subshift defined over the alphabet $\mathcal{B} = \{b_1, \dots, b_m\}$.

We will choose the integer $n \in \mathbb{N}$ and define the function $\pi'\colon (X_n, \sigma_n) \to (Y, \sigma)$ directly and later verify that they satisfy the desired properties. Consider the clopen sets $C_{\mathrm{i}} = \pi^{-1}({}_0[b_{\mathrm{i}}]) \subseteq \hat{X}$ for $\mathrm{i} = 1, \dots, m$. As these sets are clopen we know that each one of them can be written as $C_{\mathrm{i}} = \hat{X} \cap (X_1 \times \cdots \times X_{k_{\mathrm{i}}-1} \times A_{\mathrm{i}} \times X_{k_{\mathrm{i}}+1} \times \dots)$ where $A_{\mathrm{i}} \subseteq X_{k_{\mathrm{i}}}$ is a clopen set. As the alphabet $\beta$ is finite we can consider without loss of generality that all $k_{\mathrm{i}}$ are the same integer, which we will denote by $n \in \mathbb{N}$. In this way we obtain a partition of $X_n$ given by $\mathcal{P} = \{A_1, \dots, A_m\}$. The desired map $\pi'\colon (X_n, \sigma_n) \to (Y, \sigma)$ is the $\mathcal{P}$-symbolic factor map of $(X_n, \sigma_n)$. Even if we know $\pi'\colon (X_n, \sigma_n) \to (Y, \sigma)$ is continuous and conmutes with the dynamics of the systems, we do not know yet if $\pi'(X_n) \subseteq Y$ and if this map is surjective. These facts follow directly from the desired conmutativity property, which we know prove.

We now show that $\pi = \pi' \circ \Pi_n$. Take any $\hat{x} = (x^{(1)}, x^{(2)}, \dots) \in \hat{X}$. Then for any $k \in \mathbb{Z}$:

$$\begin{aligned}(\pi(\hat{x}))_k = b_{\mathrm{i}} &\Leftrightarrow (\sigma^k(\pi(\hat{x})))_0 = b_{\mathrm{i}} \Leftrightarrow (\pi(T^k(\hat{x})))_0 = b_{\mathrm{i}} \Leftrightarrow T^k(\hat{x}) \in C_{\mathrm{i}} \\ &\Leftrightarrow (\sigma_1^k(x^{(1)}), \sigma_2^k(x^{(2)}), \dots) \in \hat{X} \cap (X_1 \times \dots \times X_{n-1} \times A_{\mathrm{i}} \times X_{n+1} \times \dots) \\ &\Leftrightarrow \sigma_n^k(x^{(n)}) \in A_{\mathrm{i}} \Leftrightarrow (\pi'(x^{(n)}))_k = b_{\mathrm{i}} \Leftrightarrow (\pi'(\Pi_n(\hat{x})))_k = b_{\mathrm{i}}\end{aligned}$$

which proves the desired property. □

With the concept of symbolic factor at hand, we can now give a more comprehensive description of the (BC) formulation for Cantor minimal systems provided in Definition 2.18. According to such formulation a Cantor minimal system $(X, T)$ is said to satisfy (BC) if there exists an invariant probability measure $\mu \in \mathcal{M}(X, T)$ and a topology generating filtered sequence of clopen partitions $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ the pair $(X_n, \Pi_n\mu)$, where $(X_n, \sigma_n)$ is the $\mathcal{P}^{(n)}$-symbolic factor of $(X, T)$ and $\Pi_n\colon (X, T) \to (X_n, \sigma_n)$ is the corresponding factor map, satisfies (BC). Using Propositions 2.20 and 2.26 it is straightforward to prove the following result which provides an equivalent but also clearer formulation of (BC):

**Proposition 2.27** *Let $(X, T)$ be a Cantor minimal system. Then the following properties are equivalent:*

1. *There exists an invariant probability measure $\mu \in \mathcal{M}(X, T)$ and a topology generating filtered sequence of clopen partitions $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ the pair $(X_n, \Pi_n\mu)$, where $(X_n, \sigma_n)$ is the $\mathcal{P}^{(n)}$-symbolic factor of $(X, T)$ and $\Pi_n\colon (X, T) \to (X_n, \sigma_n)$ is the corresponding factor map, satisfies (BC).*
2. *There exists an invariant probability measure $\mu \in \mathcal{M}(X, T)$ such that for every symbolic factor $(Y, \sigma)$ of $(X, T)$ the pair $(Y, \pi\mu)$, where $\pi\colon (X, T) \to (Y, \sigma)$ is the corresponding factor map, satisfies (BC).*

*If any of these conditions hold we will say the system $(X, T)$ satisfies (BC).*

Returning briefly to general dynamical systems, it is not hard to see that the inverse limit of a factoring sequence of uniquely ergodic dynamical systems is itself uniquely ergodic. We now provide a proof of this fact:

**Proposition 2.28** *Let $(X, T)$ be the inverse limit of a factoring sequence of uniquely ergodic dynamical systems $\{(X_n, T_n)\}_{n \in \mathbb{N}}$ with factor maps $\{\pi_n\colon (X_{n+1}, T_{n+1}) \to (X_n, T_n)\}_{n \in \mathbb{N}}$. Then $(X, T)$ is uniquely ergodic.*

Proof. We will use the notation introduced in Definition 2.24. In particular, for all $n \in \mathbb{N}$, we will denote by $\Pi_n\colon (X, T) \to (X_n, T_n)$ the projection of the inverse limit to the $n$-th coordinate. Let $\mu_1, \mu_2 \in \mathcal{M}(X, T)$ be two invariant measures. We will prove that $\mu_1 = \mu_2$. We just need to check that these measures coincide on the cylinder sets of $X$ that fix exactly one coordinate. In other terms we need to check that:

$$\forall n \in \mathbb{N},\ \forall A_n \subseteq X_n \text{ open}:\ \mu_1(\Pi_n^{-1}(A_n)) = \mu_2(\Pi_n^{-1}(A_n))$$

We can write this in the following way:

$$\forall n \in \mathbb{N},\ \forall A_n \subseteq X_n \text{ open}:\ \Pi_n\mu_1(A_n) = \Pi_n\mu_2(A_n)$$

As $\Pi_n\mu_1$ and $\Pi_n\mu_2$ are invariant probability measures of the uniquely ergodic system $(X_n, T_n)$, they must coincide. This proves the desired result. $\square$

This last proposition together with Theorem 2.3 allows us to obtain a new and even clearer equivalent formulation of (BC):

**Proposition 2.29** *Let $(X, T)$ be a Cantor minimal system. Then the following properties are equivalent:*

1. *There exists an invariant probability measure $\mu \in \mathcal{M}(X, T)$ and a topology generating filtered sequence of clopen partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ the pair $(X_n, \Pi_n\mu)$, where $(X_n, \sigma_n)$ is the $\mathcal{P}^{(n)}$-symbolic factor of $(X, T)$ and $\Pi_n\colon (X, T) \to (X_n, \sigma_n)$ is the corresponding factor map, satisfies (BC).*
2. *There exists a topology generating filtered sequence of clopen partitions $\{\mathcal{P}^{(n)}\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ the $\mathcal{P}^{(n)}$-symbolic factor $(X_n, \sigma_n)$ of $(X, T)$ satisfies (BC).*
3. *There exists an invariant probability measure $\mu \in \mathcal{M}(X, T)$ such that for every symbolic factor $(Y, \sigma)$ of $(X, T)$ the pair $(Y, \pi\mu)$, where $\pi\colon (X, T) \to (Y, \sigma)$ is the corresponding factor map, satisfies (BC).*
4. *Every symbolic factor $(Y, \sigma)$ of $(X, T)$ satisfies (BC).*

*If any of these conditions hold we will say the system $(X, T)$ satisfies (BC).*

As a direct consequence of Proposition 2.28 we have the following result, analogous to Boshernitzan's unique ergodicity criteria for minimal subshifts stated in Theorem 2.3:

**Theorem 2.30** *Let $(X, T)$ be a Cantor minimal system that satisfies (BC). Then $(X, T)$ is uniquely ergodic.*

It is important to note at this point that not all Cantor minimal systems are expansive, i.e. topologically conjugate to a subshift. A trivial example is given by odometers. Yet, even though Theorem 1.63 tells us that every Cantor minimal systems of finite topological rank which is not an odometer must be expansive, there exist examples of Cantor minimal systems of infinite topological rank which are neither odometers nor expansive. As a consequence, Theorem 2.30 covers a bigger class of systems than Theorem 2.3.

Given a properly ordered Bratteli diagram $(B, \geq)$ and its associated Bratteli-Vershik system $(X, T)$, we are interested in finding conditions on the structure of $(B, \geq)$ that are necessary and/or sufficient for $(X, T)$ to satisfy (BC). We will draw inspiration from the study of a simpler property for Cantor minimal systems: sublinear complexity in subsequences.

## 2.3 Sublinear Complexity in a Subsequence for Bratteli-Vershik Systems

We begin this section with the definition of the following property for subshifts:

**Definition 2.31** *Let $(X, \sigma)$ be a subshift. We say that $(X, \sigma)$ has sublinear complexity in a subsequence if the following holds:*

$$\liminf_{n \to \infty} \frac{p_X(n)}{n} < +\infty$$

From now on we will abbreviate the property of having sublinear complexity in a subsequences as (SCS). If a subshift $(X, \sigma)$ has such property we will say it satisfies (SCS).

From the proof of Proposition 2.5 the following result, which shows that (BC) is stronger than (SCS), holds:

**Proposition 2.32** *Let $(X, \sigma)$ be a subshift. If $(X, \sigma)$ satisfies (BC) then it satisfies (SCS).*

We now extend (SCS) to Cantor minimal systems in the same way as we did with (BC):

**Definition 2.33** *Let $(X, T)$ be a Cantor minimal system. We say $(X, T)$ satisfies (SCS) if every symbolic factor $(Y, \sigma)$ of $(X, T)$ satisfies (SCS).*

It is straightforward from Proposition 2.32 that (BC) is also stronger than (SCS) in the case of Cantor minimal systems. We now state such result:

**Proposition 2.34** *Let $(X, T)$ be a Cantor minimal system. If $(X, T)$ satisfies (BC) then it satisfies (SCS).*

As in the case of (BC), (SCS) for Cantor minimal systems can be studied by looking at the symbolic factors of any given Bratteli-Vershik representation. Indeed, following the same arguments as in the the case of (BC), one arrives at the following characterization:

**Proposition 2.35** *Let $(X, T)$ be a Cantor minimal system. Then the following properties are equivalent:*

1. *There exists a topology generating filtered family of clopen partitions $\{\mathcal{P}^{(n)}\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ the $\mathcal{P}^{(n)}$-symbolic factor $(X_n, \sigma_n)$ of $(X, T)$ satisfies (SCS).*
2. *Every symbolic factor $(Y, \sigma)$ of $(X, T)$ satisfies (SCS).*

*If any of these conditions hold we will say the system $(X, T)$ satisfies (SCS).*

Given a properly ordered Bratteli diagram $(B, \geq)$ and its associated Bratteli-Vershik system $(X, T)$, we are interested in finding conditions on the structure of $(B, \geq)$ that are necessary and/or sufficient for $(X, T)$ to satisfy (SCS). We begin with the following result:

**Proposition 2.36** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\{(X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the symbolic factors of $(B, \geq)$ and let $\{\Pi_n \colon (X, T) \to (X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the corresponding factor maps. Then the following bound* *holds for all $k \in \mathbb{N}$:*

$$\liminf_{n\to\infty} \frac{p_{X_k}(n)}{n} \leq \liminf_{m\to\infty} \left[ |V_m|^2 \cdot \frac{\max_{\mathrm{i}\in V_m} h_{\mathrm{i}}^{(m)}}{\min_{j\in V_m} h_j^{(m)}} \right]$$

*In particular if the following holds:*

$$\liminf_{m\to\infty} \left[ |V_m|^2 \cdot \frac{\max_{\mathrm{i}\in V_m} h_{\mathrm{i}}^{(m)}}{\min_{j\in V_m} h_j^{(m)}} \right] < +\infty$$

*Then $(X,T)$ satisfies (SCS).*

Proof. Take any $k \in \mathbb{N}$. Consider an arbitrary $m \geq k$. Given any vertex $v \in V_m$ it is straightforward to see that any point $x \in B_v^{(m)}$, where we recall $B_v^{(m)}$ is the base of tower $v$ of level $m$, has the same code of length $h_v^{(m)}$ in $(X_k, \sigma_k)$, i.e. there exists $\omega_v^{(m)} \in L_{h_v^{(m)}}(X_k)$ such that $B_v^{(m)} \subseteq \Pi_k^{-1}({}_0[\omega])$. For the sake of keeping notation simple we denote $H^{(m)} = \min_{j\in V_m} h_j^{(m)}$. Now given any $\omega \in L_{H^{(m)}}(X_k)$ notice that $\omega \sqsubseteq \omega_u^{(m)}\omega_v^{(m)}$ for some $u, v \in V_m$. Using this fact we can bound the complexity function of $(X_k, \sigma_k)$ by $p_{(X_k)}(H^{(m)}) \leq |V_m|^2 \cdot \max_{\mathrm{i}\in V_m} h_{\mathrm{i}}^{(m)}$. It follows that $p_{X_k}(H^{(m)})/H^{(m)} \leq |V_m|^2 \cdot \max_{\mathrm{i}\in V_m} h_{\mathrm{i}}^{(m)} / \min_{j\in V_m} h_j^{(m)}$. As $(B, \geq)$ is properly ordered we have by proposition 1.50 that $H^{(m)} \nearrow +\infty$ when $m \to \infty$. We then obtain the desired bound. The second claim follows directly from (SCS) definition. $\square$

Proposition 2.36 together with some trivial bounds yield the following (SCS) criteria:

**Proposition 2.37** *Let $(X,T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\{(X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$, an integer $\mathrm{d} \in \mathbb{N}$, and a positive constant $\delta > 0$ such that for all $m \in I$ the following conditions hold:*

1. $|V_m| \leq \mathrm{d}$.
2. $\forall \mathrm{i} \in V_m,\ \forall j \in V_m:\ h_{\mathrm{i}}^{(m)} \leq \delta \cdot h_j^{(m)}$.

*Then the following bound holds for all $k \in \mathbb{N}$:*

$$\liminf_{n\to\infty} \frac{p_{X_k}(n)}{n} \leq \liminf_{m\to\infty} \left[ |V_m|^2 \cdot \frac{\max_{\mathrm{i}\in V_m} h_{\mathrm{i}}^{(m)}}{\min_{j\in V_m} h_j^{(m)}} \right] \leq \mathrm{d}^2 \cdot \delta < +\infty$$

*In particular $(X,T)$ satisfies (SCS).*

We now look for alternatives to condition 2 on the set $I$ in Proposition 2.37 that can be described directly in terms of the adjacency matrices of the diagram $(B, \geq)$. With this in mind, the following simple linear algebra lemma will be of great help:

**Lemma 2.38** *Let $A \in M_{n\times m}(\mathbb{R})$ be a $n \times m$ matrix with real coeficients such that $A > 0$, i.e. every coeficient is positive. Consider the real positive number $\alpha \in \mathbb{R}_{>0}$ given by:*

$$\alpha := \frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}}{\max_{js} A_{js}} > 0$$

*Then the image $A\mathbb{R}^m_{\geq 0}$ satisfies:*

$$\forall v \in A\mathbb{R}^m_{\geq 0},\ \forall \mathrm{i}, j \in \{1, \ldots, n\} \colon v_{\mathrm{i}} \geq \alpha \cdot v_j$$

Proof. Given any $u \in \mathbb{R}^m_{\geq 0}$, we see that:

$$Au = \begin{pmatrix} A_{1\cdot}u \\ \vdots \\ A_{n\cdot}u \end{pmatrix}$$

The proof for $u = 0$ is direct. When $u \neq 0$ we just consider the following calculations:

$$\frac{A_{\mathrm{i}\cdot}u}{A_{j\cdot}u} = \frac{\sum_k A_{\mathrm{i}k}u_k}{\sum_l A_{jl}u_l} \geq \frac{\min_r A_{\mathrm{i}r}}{\max_s A_{js}} \cdot \frac{\sum_k u_k}{\sum_l u_l} = \frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}}{\max_{js} A_{js}} = \alpha > 0$$

$\square$

Using Lemma 2.38 it is easy to find a condition on the adjacency matrices of the diagram $(B, \geq)$ to replace condition 2 in Proposition 2.37. Indeed, recalling Proposition 1.24 one has:

$$\forall n \in \mathbb{N}_0 \colon\ h^{(n+1)} = A^{(n+1)}h^{(n)}$$

Then condition 2 can be replaced by the following new condition:

*2'.* $\frac{\min_{\mathrm{i},r} A^{(m)}_{\mathrm{i}r}}{\max_{js} A^{(m)}_{js}} > \delta$.

We then obtain the following (SCS) criteria:

**Proposition 2.39** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\{(X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$, an integer $\mathrm{d} \in \mathbb{N}$, and a postive constant $\delta > 0$ such that for all $m \in I$ the following conditions hold:*

1. $|V_m| \leq \mathrm{d}$.
2. $\frac{\min_{\mathrm{i},r} A^{(m)}_{\mathrm{i}r}}{\max_{js} A^{(m)}_{js}} > \delta$.

*Then the following bound holds for all $k \in \mathbb{N}$:*

$$\liminf_{n\to\infty} \frac{p_{X_k}(n)}{n} \leq \liminf_{m\to\infty} \left[ |V_m|^2 \cdot \frac{\max_{\mathrm{i}\in V_m} h^{(m)}_{\mathrm{i}}}{\min_{j\in V_m} h^{(m)}_j} \right] \frac{\mathrm{d}^2}{\delta} < +\infty$$

*In particular $(X, T)$ satisfies (SCS).*

We can do another final simplification to the (SCS) criteria provided by Proposition 2.39 by replacing the proportionality condition on the adjacency matrices by a stronger repetition conditions. This final formulation is stated in the following corollary:

**Corollary 2.40** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$ and a positive matrix $A > 0$ such that for each $m \in I$ the adjacency matrix of level $m$ of $(B, \geq)$ is given by $A^{(m)} = A$. Then $(X, T)$ satisfies (SCS).*

In the next section we will follow a similar scheme to obtain diagram related conditions for a Bratteli-Vershik system to satisfy (BC).

## 2.4 Boshernitzan's Condition for Bratteli-Vershik Systems

Given a properly ordered Bratteli diagram $(B, \geq)$ and its associated Bratteli-Vershik system $(X, T)$, we are interested in finding conditions on the structure of $(B, \geq)$ that are necessary and/or sufficient for $(X, T)$ to satisfy (BC). This will be the main concern of this section. We will first state a formula for calculating the quantities $\varepsilon_n(X_k, \Pi_k\mu)$ for a given invariant probability measure $\mu \in \mathcal{M}(X, T)$ and integers $n, k \in \mathbb{N}$. We will use this formula to state a necessary and sufficient condition for $(X, T)$ to satisfy (BC). This condition, even if true, will be extremely abstract, hard to interpret, and difficult to check. By doing some trivial bounds on the expressions involved we will be able to obtain a simpler though still abstract sufficient condition. We will then look for simpler not too restrictive conditions that imply the hypotheses of this last result. We will end with a sufficient condition which aside from a standardization hypothesis on the order of $(B, \geq)$ looks exclusively at the adjacency matrices of the diagram. In the next chapter we will provide examples of applications of these results.

Let $(B, \geq)$ be a properly ordered Bratteli diagram and let $(X, T)$ be the associated Bratteli-Vershik system. We now show a formula for computing the values $\varepsilon_n(X_k, \Pi_k\mu)$ for any given invariant probability measure $\mu \in \mathcal{M}(X, T)$ and integers $n, k \in \mathbb{N}$:

**Proposition 2.41** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X, T)$ be an invariant probability measure. Let $\{(X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the symbolic factors of $(B, \geq)$ and let $\{\Pi_n : (X, T) \to (X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the corresponding factor maps. Let $I \subseteq \mathbb{N}$ be the set of integers $n \in \mathbb{N}$ such that all minimal edges in $E_n$ start at the same vertex, say $1 \in V_{n-1}$. Fix integers $n, k \in \mathbb{N}$ and suppose there exists an $m \in I$ such that $m > k$ and $h_1^{(m-1)} \geq n - 1$. Then all points belonging to the same floor of a tower of level $m$ have the same code of length $n$ in the symbolic factor $X_k$, i.e.:*

$$\forall \bar{\mathrm{e}} \in E(0, m), \exists \omega \in L_n(X_k) \colon [\bar{\mathrm{e}}] \subseteq \Pi_k^{-1}({}_0[\omega])$$

*In particular the following formula holds:*

$$\varepsilon_n(X_k, \Pi_k\mu) = \min_{\omega \in L_n(X_k)} \sum_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega)$$

*where for each $\omega \in L_n(X_k)$ and each $\mathrm{i} \in V_m$ the integer $\#_{\mathrm{i}}^{(m)}(\omega)$ is the number of floors of tower $\mathrm{i}$ that have $\omega$ as their code of length $n$ in $X_k$.*

Proof. We begin by proving that all points belonging to the same floor of a tower of level $m$ have the same code of length $n$ in the symbolic factor $X_k$. Let $\bar{\mathrm{e}} \in E(0,m)$ be an arbitrary finite path of length $m$ in $B$ starting at the vertex $v_0$ so that $[\bar{\mathrm{e}}] \subseteq X$ is an arbitrary floor of some tower of level $m$, say $\mathrm{i} \in V_m$. Notice that all points $x \in [\bar{\mathrm{e}}]$ have the same code $\omega_0 \dots \omega_{t-1} \in L_t(X_k)$, where $t \in \mathbb{N}_0$ is the smallest integer such that $T^t([\bar{\mathrm{e}}])$ is the top floor of tower i. After reaching the top floor of tower i all points $x \in [\bar{\mathrm{e}}]$ return to the base of some tower of level $m$. No matter which tower this is, its base must be contained in the base of tower 1 of level $m-1$, because $m \in I$. As $m > k$, this allows us to extend the code $\omega_0 \dots \omega_{t-1}$ to a code $\omega_0 \dots \omega_{t+h_1^{(m-1)}-1} \in L_{t+h_1^{(m-1)}}(X_k)$. As $h_1^{(m-1)} \geq n-1$ this shows that for $\omega = \omega_0, \dots, \omega_{n-1} \in L_n(X_k)$ one has $[\mathrm{e}_1, \dots, \mathrm{e}_m] \subseteq \Pi_k^{-1}({}_0[\omega])$.

We have just proved that any floor of any tower of level $m$ of $(B, \geq)$ must be contained in a set $\Pi_k^{-1}({}_0[\omega])$ for some $\omega \in L_n(X_k)$. Such $\omega \in L_n(X_k)$ is clearly seen to be unique. As the floors of towers of level $m$ of $(B, \geq)$ form a partition of $(X,T)$, the following holds:

$$\Pi_k\mu({}_0[\omega]) = \mu(\Pi_k^{-1}({}_0[\omega])) = \sum_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega)$$

We then conclude that:

$$\varepsilon_n(X_k, \Pi_k\mu) = \min_{\omega \in L_n(X_k)} \Pi_k\mu({}_0[\omega]) = \min_{\omega \in L_n(X_k)} \sum_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega)$$

□

Thus, for calculating $\varepsilon_n(X_k, \Pi_k\mu)$ one must find an $m \in \mathbb{N}$ such that $m > k$ and $h_1^{(m-1)} \geq n-1$, calculate the codes of length $n$ in the symbolic factor $(X_k, \sigma)$ of each of the floors of the towers of level $m$, calculate the number $\#_{\mathrm{i}}^{(m)}(\omega)$ of times a code $\omega \in L_n(X_k)$ appears in each of the towers $\mathrm{i} \in V_m$, use these quantities to calculate for each $\omega \in L_n(X)$ the value of $\mu(\Pi_k^{-1}[\omega]) = \sum_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega)$, and finally find the minimum of these quantities over all $\omega \in L_n(X_k)$. From this formula the following (BC) criteria can be proved directly:

**Theorem 2.42** *Let $(X,T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X,T)$ be an invariant probability measure of $(X,T)$. Let $\{(X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$ and let $\{\Pi_n \colon (X,T) \to (X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the corresponding factor maps. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$ with the property that for each $n \in I$ all minimal edges in $E_n$ start at the same vertex, say $1 \in V_{n-1}$. Recall the definition of Boshernitzan's constants introduced in Definition 2.19. Then the following formula holds for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k\mu) = \limsup_{\substack{n \to \infty \\ m \in I \\ h_1^{(m-1)} \geq n-1}} \left[ n \cdot \min_{\omega \in L_n(X_k)} \sum_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \right]$$

*In particular, the following are equivalent:*

1. *$(X,T)$ satisfies (BC).*

*2.* $\forall k \in \mathbb{N}$: $\limsup_{\substack{n\to\infty \\ m\in I \\ h_1^{(m-1)} \geq n-1}} \left[ n \cdot \min_{\omega \in L_n(X_k)} \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \right] > 0$

Proof. The formula for $c(X_k, \Pi_k\mu)$ follows directly from Proposition 2.41 and Definition 2.19. The second claim follows directly from Proposition 2.29. $\square$

Regarding the hypotheses of Proposition 2.41 and Theorem 2.42, we warn the reader that the existence of the infinite set $I \subseteq \mathbb{N}$, which is a condition on the order of the diagram, is not a restriction but rather a standardization. This claim follows directly from Propositions 1.54 and 1.55, which show that such set $I \subseteq \mathbb{N}$ always exists when considering an appropriate telescoping of the original diagram.

Even though Theorem 2.42 provides a necessary and sufficient condition on the diagram $(B, \geq)$ for the associated Bratteli-Vershik system $(X, T)$ to satisfy (BC), this condition is very hard to interpret. In the spirit of getting simpler (BC) criteria we will progressively simplify Theorem 2.42 to obtain a collection of nice diagram related sufficient conditions for a Bratteli-Vershik system to satisfy (BC). We begin with the following basic simplification:

**Proposition 2.43** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X, T)$ be an invariant probability measure of $(X, T)$. Let $\{(X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$ and let $\{\Pi_n : (X, T) \to (X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the corresponding factor maps. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$ with the property that for each $n \in I$ all minimal edges in $E_n$ start at the same vertex, say $1 \in V_{n-1}$. Recall the definition of Boshernitzan's constants introduced in Definition 2.19. Then the following bound holds for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k\mu) \geq \limsup_{\substack{m\to\infty \\ m\in I}} \left[ h_1^{(m-1)} \cdot \min_{\omega \in L_{h_1^{(m-1)}}(X_k)} \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \right]$$

*In particular, if for all $k \in \mathbb{N}$:*

$$\limsup_{\substack{m\to\infty \\ m\in I}} \left[ h_1^{(m-1)} \cdot \min_{\omega \in L_{h_1^{(m-1)}}(X_k)} \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \right] > 0$$

*Then $(X, T)$ satisfies (BC).*

Proof. The bound for $c(X_k, \Pi_k\mu)$ is obtained by using Theorem 2.42 and looking at the integers $n = h_1^{(m-1)}$ for $m \in I$, which trivially satisfy $h_1^{(m-1)} \geq n-1$ and for which $h_1^{(m-1)} \to \infty$ by Proposition 1.50. The second claim follows directly from Proposition 2.29. $\square$

Even though Proposition 2.43 provides a simplification of Theorem 2.42, the condition stated still remains cryptic. Nevertheless, in the next chapter, we will see an example in which the use of Proposition 2.43 will allow us to verify that certain systems, on which the other simpler criteria we provide do not apply, satisfy (BC). We now make some trivial bounds on the formulas of Proposition 2.43 to obtain the following result:

**Proposition 2.44** *Let $(X,T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B,\geq)$. Let $\mu \in \mathcal{M}(X,T)$ be an invariant probability measure of $(X,T)$. Let $\{(X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B,\geq)$ and let $\{\Pi_n\colon (X,T)\to (X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the corresponding factor maps. Suppose there exists an infinite set of integers $I\subseteq\mathbb{N}$ with the property that for each $n\in I$ all minimal edges in $E_n$ start at the same vertex, say $1\in V_{n-1}$. Recall the definition of Boshernitzan constants introduced in Definition 2.19. Then the following bound holds for all $k\in\mathbb{N}$:*

$$c(X_k,\Pi_k\mu) \geq \limsup_{\substack{m\to\infty\\ m\in I}} \left[h_1^{(m-1)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)}\right]$$

*In particular, if the following holds:*

$$\limsup_{\substack{m\to\infty\\ m\in I}} \left[h_1^{(m-1)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)}\right] > 0$$

*then $(X,T)$ satisfies (BC).*

Proof. Straightforward from Proposition 2.43 by applying the following trivial bound for all $m\in I$ and all $\omega\in L_{h_1^{(m-1)}}(X_k)$:

$$\sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \geq \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \sum_{\mathrm{i}\in V_m} \#_{\mathrm{i}}^{(m)}(\omega) \geq \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot 1$$

□

Using Proposition 2.44 as the main tool, we now provide another sufficient condition for a Bratteli-Vershik system $(X,T)$ represented by a properly ordered Bratteli diagram $(B,\geq)$ to satisfy (BC). This new condition makes strong use of the structure of the measure and height vectors of the diagram $(B,\geq)$ and removes the need for calculating limits. The same order standardization condition used in previous results still needs to be considered:

**Proposition 2.45** *Let $(X,T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B,\geq)$. Let $\mu\in\mathcal{M}(X,T)$ be an invariant probability measure. Let $\{(X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B,\geq)$ and let $\{\Pi_n\colon (X,T)\to (X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the corresponding factor maps. Suppose there exist constants $\delta>0$, $\alpha>0$, and an infinite set of integers $I\subseteq\mathbb{N}$ such that for each $m\in I$ the following conditions hold:*

1. *All minimal edges in $E_m$ start at the same vertex, say $1\in V_{m-1}$.*
2. *$\sum_{j\in V_m} h_{\mathrm{i}}^{(m)} \leq \delta\cdot h_1^{(m-1)}$.*
3. *$\forall \mathrm{i}\in V_m, \forall j\in V_m\colon \mu_{\mathrm{i}}^{(m)} \geq \alpha\cdot\mu_j^{(m)}$.*

*Then the following bound hold for all $k\in\mathbb{N}$:*

$$c(X_k,\Pi_k\mu) \geq \limsup_{\substack{m\to\infty\\ m\in I}} \left[h_1^{(m-1)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)}\right] \geq \frac{\alpha}{\delta} > 0$$

*In particular $(X,T)$ satisfies (BC).*

Proof. The only fact that needs to be proved is:

$$\limsup_{\substack{m\to\infty\\ m\in I}} \left[h_1^{(m-1)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)}\right] \geq \frac{\alpha}{\delta}$$

We start with condition 3, i.e.:

$$\forall m \in I, \forall \mathrm{i} \in V_m, \forall j \in V_m \colon \mu_{\mathrm{i}}^{(m)} \geq \alpha \cdot \mu_j^{(m)}$$

For each $j \in V_m$ we multiply these inequalities by $h_j^{(m)}$ and then add over all $j \in V_m$:

$$\forall m \in I, \forall \mathrm{i} \in V_m \colon \sum_{j\in V_m} h_j^{(m)} \cdot \mu_{\mathrm{i}}^{(m)} \geq \alpha \cdot \sum_{j\in V_m} \mu_j^{(m)} \cdot h_j^{(m)} = \alpha$$

Taking minimum over all $\mathrm{i} \in V_m$ we get:

$$\forall m \in I \colon \sum_{j\in V_m} h_j^{(m)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \geq \alpha$$

Finally using condition 2 we get:

$$\forall m \in I \colon h_1^{(m-1)} \cdot \min_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \geq \frac{\alpha}{\delta}$$

Which in particular implies the desired condition. □

Taking into consideration the importance of finite rank Bratteli diagrams, i.e. Bratteli diagrams for which the cardinality of the vertex set at each level is uniformly bounded, in the theory of Cantor minimal systems, we provide a slight variation of Proposition 2.45 by replacing condition 2 on the set $I$ by the following pair of conditions:

*2.1.* $|V_m| \leq \mathrm{d}$.

*2.2.* $\forall \mathrm{i} \in V_m \colon h_{\mathrm{i}}^{(m)} \leq \delta \cdot h_1^{(m-1)}$.

where $\mathrm{d} \in \mathbb{N}$ is a previously fixed integer. Using these two conditions we see that:

$$\sum_{j\in V_m} h_j^{(m)} \leq \sum_{j\in V_m} \delta \cdot h_1^{(m-1)} \leq \mathrm{d} \cdot \delta \cdot h_1^{(m-1)}$$

Which then gives us the following result:

**Proposition 2.46** *Let $(X,T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B,\geq)$. Let $\mu \in \mathcal{M}(X,T)$ be an invariant probability measure of $(X,T)$. Let $\{(X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B,\geq)$ and let $\{\Pi_n \colon (X,T) \to (X_n,\sigma_n)\}_{n\in\mathbb{N}}$ be the corresponding factor maps. Suppose there exist constants $\delta > 0$, $\alpha > 0$, $\mathrm{d} \in \mathbb{N}$, and an infinite set of integers $I \subseteq \mathbb{N}$ such that for each $m \in I$ the following holds:*

1. *All minimal edges in $E_m$ start at the same vertex, say $1 \in V_{m-1}$.*
2. *$|V_m| \leq \mathrm{d}$.*
3. *$\forall \mathrm{i} \in V_m \colon h_{\mathrm{i}}^{(m)} \leq \delta \cdot h_1^{(m-1)}$.*

*4.* $\forall \mathrm{i} \in V_m, \forall j \in V_m \colon \mu_{\mathrm{i}}^{(m)} \geq \alpha \cdot \mu_j^{(m)}$.

*Then the following bounds hold for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k \mu) \geq \limsup_{\substack{m \to \infty \\ m \in I}} \left[ h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \right] \geq \frac{\alpha}{\mathrm{d} \cdot \delta} > 0$$

*In particular $(X, T)$ satisfies (BC).*

We can provide a second variation of Proposition 2.45 by replacing the hypothesis on the measure of the base of towers by a condition on the measure of whole towers. This new variation is particularly interesting having in mind the concept of exact finite rank Bratteli diagram introduced in [2].

**Proposition 2.47** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X, T)$ be an invariant probability measure of $(X, T)$. Let $\{(X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$ and let $\{\Pi_n \colon (X, T) \to (X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the corresponding factor maps. Suppose there exist constants $\delta > 0$, $\alpha > 0$, and an infinite set of integers $I \subseteq \mathbb{N}$ such that for each $m \in I$ the following conditions hold:*

1. *All minimal edges in $E_m$ start at the same vertex, say $1 \in V_{m-1}$.*
2. $\forall \mathrm{i} \in V_m \colon h_j^{(m)} \leq \delta \cdot h_1^{(m-1)}$.
3. $\forall \mathrm{i} \in V_m \colon \mu\left(\tau_{\mathrm{i}}^{(m)}\right) \geq \alpha$.

*Then the following bounds hold for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k \mu) \geq \limsup_{\substack{m \to \infty \\ m \in I}} \left[ h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \right] \geq \frac{\alpha}{\delta} > 0$$

*In particular $(X, T)$ satisfies (BC).*

Proof. The only fact that needs to be proved is:

$$\limsup_{\substack{m \to \infty \\ m \in I}} \left[ h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \right] \geq \frac{\alpha}{\delta}$$

For this we just need to consider the following bounds for each $m \in I$:

$$h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} = \min_{\mathrm{i} \in V_m} \left[ h_1^{(m-1)} \cdot \mu_{\mathrm{i}}^{(m)} \right] \geq \min_{\mathrm{i} \in V_m} \left[ \frac{1}{\delta} \cdot h_{\mathrm{i}}^{(m)} \cdot \mu_{\mathrm{i}}^{(m)} \right] \geq \frac{1}{\delta} \cdot \min_{\mathrm{i} \in V_m} \mu\left(\tau_{\mathrm{i}}^{(m)}\right) \geq \frac{\alpha}{\delta}$$

Which in particular implies the desired condition. □

We now look for alternatives to conditions 2 and 3 on the set $I$ in Proposition 2.45 that can be described directly in terms of the adjacency matrices of the diagram $(B, \geq)$. As in the case of (SCS) Lemma 2.38 will be of great help. Indeed, using Lemma 2.38 it is easy to find a condition on the adjacency matrices of the diagram $(B, \geq)$ to replace condition 3 in Proposition 2.45. Recalling Proposition 1.39 one has:

$$\forall n \in \mathbb{N}_0 \colon \; \mu^{(n)} = \left[A^{(n+1)}\right]^T \mu^{(n+1)} \in \left[A^{(n+1)}\right]^T \mathbb{R}_{\geq 0}^{V_{n+1}}$$

Then condition 3 can be replaced by the following new condition:

*3'.* $\frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}^{(m+1)}}{\max_{js} A_{js}^{(m+1)}} > \alpha.$

Replacing condition 2 in Proposition 2.45 turns out to be slightly more complicated. Notice that any matrix $A \in M_{n\times m}(\mathbb{R})$ defines a linear bounded operator $A \colon \mathbb{R}^m \to \mathbb{R}^n$ via matrix product, independent of the choice of norms in $\mathbb{R}^n$ and $\mathbb{R}^m$. Endowing $\mathbb{R}^m$ with the $||\cdot||_\infty$ norm and $\mathbb{R}^n$ with the $||\cdot||_1$ norm, we see that for any $u \in \mathbb{R}^m$ it holds that $||Au||_1 \leq ||A||_{\infty,1} \cdot ||u||_\infty$. Recalling Proposition 1.24 we have:

$$\forall n \in \mathbb{N}_0 \colon \; h^{(n+1)} = A^{(n+1)} h^{(n)}$$

Following the previous discussion with the adjacency matrices of the diagram $(B, \geq)$ in place of the matrix $A$ we get the following bounds:

$$\forall n \in \mathbb{N}_0 \colon \sum_{\mathrm{i} \in V_{n+1}} h_{\mathrm{i}}^{(n+1)} \leq ||A^{(n)}||_{\infty,1} \cdot \max_{j \in V_n} h_j^{(n)}$$

From this we see that condition 2 in Proposition 2.45 can be replaced by the following pair of conditions:

*2.1.* $||A^{(m)}||_{\infty,1} < C.$

*2.2.* $\forall \mathrm{i} \in V_{m-1} \colon h_{\mathrm{i}}^{(m-1)} \leq \delta \cdot h_1^{(m-1)}.$

where $C > 0$ is a previously fixed constant. Using again Lemma 2.38 and Proposition 1.24 we can replace condition 2.2. just stated by the new condition:

*2.2'* $\frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}^{(m-1)}}{\max_{js} A_{js}^{(m-1)}} > \delta.$

Adding all up we get the following result:

**Proposition 2.48** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X, T)$ be an invariant probability measure of $(X, T)$. Let $\{(X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$ and let $\{\Pi_n \colon (X, T) \to (X_n, \sigma_n)\}_{n\in\mathbb{N}}$ be the corresponding factor maps. Suppose there exist constants $C > 0$, $\delta > 0$, $\alpha > 0$, and an infinite set of integers $I \subseteq \mathbb{N}$ such that for each $m \in I$ the following holds:*

1. *All minimal edges in $E_m$ start at the same vertex, say $1 \in V_{m-1}$.*
2. $||A^{(m)}||_{\infty,1} < C.$
3. $\frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}^{(m-1)}}{\max_{js} A_{js}^{(m-1)}} > \delta.$
4. $\frac{\min_{\mathrm{i},r} A_{\mathrm{i}r}^{(m+1)}}{\max_{js} A_{js}^{(m+1)}} > \alpha.$

*Then the following bounds hold for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k \mu) \geq \limsup_{\substack{m \to \infty \\ m \in I}} \left[ h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \right] \geq \frac{\alpha}{C \cdot \delta} > 0$$

*In particular $(X, T)$ satisfies (BC).*

We can do another final simplification to the (BC) criteria provided by Proposition 2.48 by replacing the proportionality conditions on the adjacency matrices by stronger repetition conditions. This final formulation is stated in the following corollary:

**Corollary 2.49** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$. Let $\mu \in \mathcal{M}(X, T)$ be an invariant probability measure. Suppose there exist matrices $A, B, C \neq 0$ with $A, C > 0$ and an infinite set of integers $I \subseteq \mathbb{N}$ such that for each $m \in I$ the following conditions hold:*

1. *All minimal edges in $E_m$ start at the same vertex, say $1 \in V_{m-1}$.*
2. *$(A^{(m-1)}, A^{(m)}, A^{(m+1)}) = (A, B, C)$.*

*Then $(X, T)$ satisfies (BC).*

Although this last criteria may appear as too restrictive, it still allows one to prove that linearly recurrent and substitutive Bratteli-Vershik systems satisfy (BC). Its scope remains limited as we will see via very simple examples in the next chapter.

# Chapter 3

# Examples and Applications

In this chapter we show some applications of the (BC) criteria provided in Chapter 2. We begin with the study of Bratteli-Vershik systems of Bi-Toeplitz type and apply Proposition 2.44 to get a simple (BC) criteria for this class of systems. This in turn provides simple conditions for verifying that such systems are uniquely ergodic without the need of explicitly analyzing the measure cones. By means of a particular family of examples of systems of this class we show that Proposition 2.44 is not enough to determine in all cases if a given Bratteli-Vershik system satisfies (BC). We use Proposition 2.43 to verify (BC) for this family of examples. This shows that (BC) is heavily influenced by a variety of combinatorial mechanisms not captured in Proposition 2.44. We then turn our attention to Sturmian subshifts. By means of some particular Bratteli-Vershik representations derived from Rauzy-Veech induction we prove via Proposition 2.44 that every Sturmian subshift whose angle has partial quotients with a special kind of bounded subsequence satisfies (BC). Nevertheless, we are not able to provide a complete proof of the known fact that every Sturmian subshift satisfies (BC). This in turn shows again that (BC) involves a series of combinatorial mechanisms that are not captured by Proposition 2.44.

## 3.1 Bratteli-Vershik Systems of Bi-Toeplitz Type

We first define what it means for a Bratteli diagram to be of Toeplitz type:

**Definition 3.1** *Let $B$ be a Bratteli diagram. We say $B$ is of Toeplitz type if for all $n \in \mathbb{N}$ the number of edges in $E_n$ finishing at a fixed vertex of $V_n$ is constant independently of the vertex. We denote this number by $q_n$. The sequence $\{q_n\}_{n\in\mathbb{N}}$ is referred to as the characteristic sequence of the diagram $B$. We also denote $p_n := q_n \cdots q_1$, for all $n \in \mathbb{N}$.*

Notice that Bratteli diagrams of Toeplitz type can be characterized in terms of their adjacency matrices. In this regard, a Bratteli diagram $B$ is of Toeplitz type if and only if for all $n \in \mathbb{N}$ the sum of the entries of a fixed row of its adjacency matrix $A^{(n)}$ of level $n$ is constant independently of the row. In this case the sum of the entries of a fixed row of $A^{(n)}$

is equal to the integer $q_n$. Notice also that for all $n \in \mathbb{N}$, the sum of the entries of a fixed row of the matrix $A^{(0,n)} = A^{(n)} \cdots A^{(1)}$ is equal to the integer $p_n = q_n \cdots q_1$.

The height vectors of Bratteli diagrams of Toeplitz type are easily calculated by considering the previous discussion together with Proposition 1.24:

**Proposition 3.2** *Let $B$ be a Toeplitz type Bratteli diagram with characteristic sequence $\{q_n\}_{n\in\mathbb{N}}$. Then the height vectors of the diagram $B$ are given by:*

$$\forall n \in \mathbb{N},\ \forall v \in V_n:\ h_v^{(n)} = p_n = q_1 \dots q_n$$

We now define what it means for a Bratteli diagram to be of transpose Toeplitz type:

**Definition 3.3** *Let $B$ be a Bratteli diagram. We say $B$ is of transpose Toeplitz type if for all $n \in \mathbb{N}_0$ the number of edges in $E_n$ starting at a fixed vertex of $V_{n-1}$ is constant independently of the vertex. We denote this number by $r_n$. The sequence $\{r_n\}_{n\in\mathbb{N}}$ is referred to as the characteristic transpose sequence of the diagram $B$. We also denote $s_{n,m} := r_{n+1} \cdots r_m$, for all $n, m \in \mathbb{N}_0$ with $m > n$.*

Notice that Bratteli diagrams of transpose Toeplitz type can be characterized in terms of their adjacency matrices. In this regard, a Bratteli diagram $B$ is of transpose Toeplitz type if and only if for all $n \in \mathbb{N}$ the sum of the entries of a fixed column of its adjacency matrix $A^{(n)}$ of level $n$ is constant independently of the column. In this case the sum of the entries of a fixed column of $A^{(n)}$ is equal to the integer $r_n$. Notice also that for all $n, m \in \mathbb{N}_0$ with $m > n$, the sum of the entries of any given column of the matrix $A^{(n,m)} = A^{(m)} \cdots A^{(n+1)}$ is equal to the integer $s_{n,m} = r_{n+1} \cdots r_m$.

A very nice property holds for Bratteli diagrams of transpose Toeplitz type; the "uniform" probability measure is always an invariant probability measure:

**Proposition 3.4** *Let $(B, \geq)$ be a properly ordered Bratteli diagram such that $B$ is a Bratteli diagram of transpose Toeplitz type and denote the associated Bratteli-Vershik system by $(X_B, \phi)$. For each $n \in \mathbb{N}_0$ denote by $\mathbf{1}^{(n)} := (1, \dots, 1)^T \in \mathbb{R}^{V_n}$. Consider the sequence of uniform vectors $\{p^{(n)}\}_{n\in\mathbb{N}_0}$ defined by:*

$$\forall n \in \mathbb{N}_0,\ \forall v \in V_n:\ p_v^{(n)} = \frac{1}{\langle \mathbf{1}^{(n)}, h^{(n)} \rangle}$$

*Then there exists a unique invariant probability measure $\mu \in \mathcal{M}(X_B, \phi)$ with $\{p^{(n)}\}_{n\in\mathbb{N}_0}$ as its sequence of $\mu$-measure vectors.*

Proof. For every $n \in \mathbb{N}_0$ it is straightforward from the definition of $p^{(n)}$ that $\langle h^{(n)}, p^{(n)} \rangle = 1$. It only remains to show that $p^{(n)} = \left[A^{(n+1)}\right]^T p^{(n+1)}$ for all $n \in \mathbb{N}_0$. Considering an arbitrary integer $n \in \mathbb{N}_0$ and an arbitrary vertex $v \in V_n$ we calculate:

$$\begin{aligned}
\left(\left[A^{(n+1)}\right]^T p^{(n+1)}\right)_v &= \sum_{w \in V_{n+1}} \left(\left[A^{(n+1)}\right]^T\right)_{v,w} \cdot p_w^{(n+1)} = \sum_{w \in V_{n+1}} A_{w,v}^{(n+1)} \cdot \frac{1}{\langle \mathbf{1}^{(n)}, h^{(n+1)} \rangle} \\
&= \frac{r_{n+1}}{\langle \mathbf{1}^{(n)}, A^{(n+1)} h^{(n)} \rangle} = \frac{r_{n+1}}{\left\langle \left[A^{(n+1)}\right]^T \mathbf{1}^{(n)}, h^{(n)} \right\rangle} = \frac{r_{n+1}}{\langle r_{n+1} \mathbf{1}^{(n)}, h^{(n)} \rangle} \\
&= \frac{1}{\langle \mathbf{1}^{(n)}, h^{(n)} \rangle} = p_v^{(n)}
\end{aligned}$$

This shows that $\{p^{(n)}\}_{n \in \mathbb{N}_0}$ satisfies the conditions of Theorem 1.40, so there exists a unique invariant probability measure $\mu \in \mathcal{M}(X_B, \phi)$ with $\{p^{(n)}\}_{n \in \mathbb{N}_0}$ as its sequence of $\mu$-measure vectors. □

We now define what it means for a Bratteli diagram to be of bi-Toeplitz type:

**Definition 3.5** *Let $B$ be a Bratteli diagram. We way $B$ is of bi-Toeplitz type if it is both of Toeplitz type and transpose Toeplitz type. We will denote the characteristic sequence of $B$ by $\{q_n\}_{n \in \mathbb{N}}$ while the characteristic transpose sequence of $B$ will be denoted by $\{r_n\}_{n \in \mathbb{N}}$.*

We now apply Proposition 2.44 to obtain a necessary condition for Bratteli-Vershik systems represented by Bratteli diagrams of bi-Toeplitz type to satisfy (BC):

**Proposition 3.6** *Let $(X, T)$ be a Bratteli-Vershik system represented by a properly ordered Bratteli diagram $(B, \geq)$ such that $B$ is a Bratteli diagram of bi-Toeplitz type. Let $\mu \in \mathcal{M}(X, T)$ be the "uniform" invariant probability measure of $(X, T)$ described in Proposition 3.4. Let $\{(X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the symbolic factors of the diagram $(B, \geq)$ and let $\{\Pi_n \colon (X, T) \to (X_n, \sigma_n)\}_{n \in \mathbb{N}}$ be the corresponding factor maps. Suppose there exists an infinite set of integers $I \subseteq \mathbb{N}$ with the property that for each $n \in I$ all minimal edges in $E_n$ start at the same vertex, say $1 \in V_{n-1}$. Recall the definition of Boshernitzan's constants introduced in Definition 2.19. Then the following bound holds for all $k \in \mathbb{N}$:*

$$c(X_k, \Pi_k \mu) \geq \limsup_{\substack{m \to \infty \\ m \in I}} \left[ \frac{1}{|V_m| \cdot q_m} \right]$$

*In particular, if there exists a constant $C > 0$ and an infinite subset $J \subseteq I$ such that for all $m \in J$ it holds that $|V_m| \cdot q_m \leq C$ then $(X, T)$ satisfies (BC) and in particular is uniquely ergodic. The unique invariant probability measure $\mu \in \mathcal{M}(X, T)$ has $\mu$-measure vectors:*

$$\forall n \in \mathbb{N}_0,\ \forall v \in V_n \colon\ \mu_v^{(n)} = \frac{1}{|V_n| \cdot p_n}$$

The unique ergodicity of Bratteli-Vershik systems represented by Bratteli diagrams as the one in Figure 3.1 was studied in detail by S. Ferenczi, A. M. Fisher, and M. Talet in [15], where a necessary and sufficient condition for such systems to be uniquely ergodic is provided. While this result is clearly stronger than the one obtained by applying Proposition 3.6 to this type of diagrams, it requires specific calculations that strongly rely on the particular

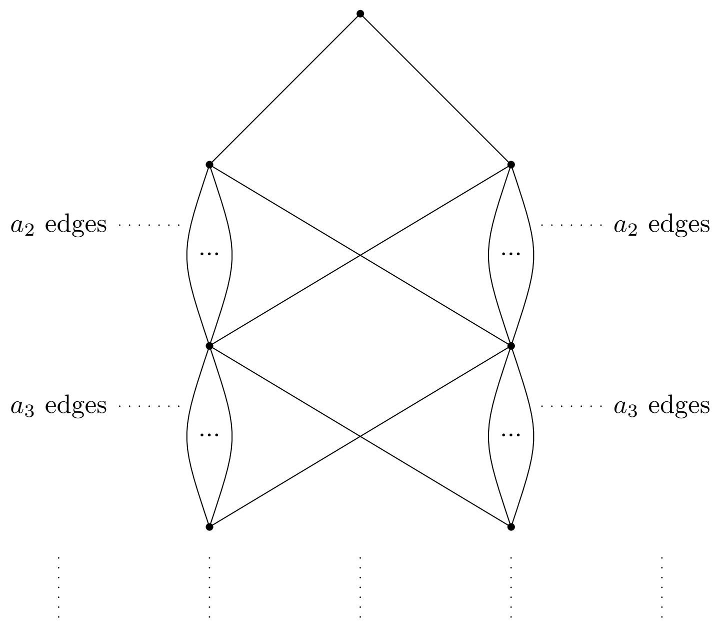


Figure 3.1: Special class of Bi-Toeplitz Bratteli diagrams studied in [15].

structure of the diagrams considered and which do not appear to be easily extendable to other cases.

We now show an example of a Bratteli-Vershik system $(X, T)$ represented by a properly ordered Bratteli diagram $(B, \geq)$, with $B$ a Bratteli diagram of bi-Toeplitz type, which satisfies (BC) but does not satisfy the hypotheses of Proposition 3.6. We will use Proposition 2.43 to show that such example satisfies (BC).

We consider Bratteli diagrams $B$ with $|V_n| = 2$ for all $n \geq 1$ and adjacency matrices $A^{(n)}$ given by $A^{(1)} = (1, 1)^T$ and:

$$\forall n \geq 2: \ A^{(n)} = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$$

where $a_n$ and $b_n$ are positive integers such that $a_n \geq b_n$ for all $n \geq 2$. Such diagrams are of bi-Toeplitz type. For each $n \geq 1$ we denote $V_n = \{A_n, B_n\}$. For each $n \geq 2$ and each $v \in V_n$, the linear order among the edges in $r^{-1}(\{v\})$ is given by:

$$A_n \mapsto (A_{n-1}B_{n-1})^{b_n}(A_{n-1})^{a_n - b_n}$$
$$B_n \mapsto (A_{n-1}B_{n-1})^{b_n}(B_{n-1})^{a_n - b_n}$$

See Figure 3.2. It is straightforward to check that such order defines a properly ordered Bratteli diagram $(B, \geq)$. It is also directly seen from the definition of $\geq$ that for each $n \in \mathbb{N}$ all minimal edges of $E_n$ start from the same vertex $A_{n-1} \in V_{n-1}$. For this class of systems the following (BC) criteria can be obtained by using Proposition 2.43:

**Proposition 3.7** *Let $(B, \geq)$ be a properly ordered Bratteli diagram in the class just described and let $(X, T)$ be the associated Bratteli-Vershik system. Suppose there exists $C > 0$ and an infinite set $J \subseteq \mathbb{N}$ such that $3 \cdot b_n \leq a_n \leq C \cdot b_n$ for all $n \in J$. Then $(X, T)$ satisfies (BC).*

Proof. Let $\mu \in \mathcal{M}(X,T)$ be the "uniform" invariant probability measure of $(X,T)$ described in Proposition 3.4. Recall the notation introduced in Proposition 2.41. Take arbitrary $k \in \mathbb{N}$, $m \in J$ with $m > k$, and $\omega \in L_{h_1^{(m-1)}(X_k)}$. We bound $\Pi_k\mu([\omega]) = \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega)$. Recall that the order $\geq$ in $r^{-1}(\{v\})$ is defined for each $v \in V_m$ by:

$$A_m \mapsto (A_{m-1}B_{m-1})^{b_m}(A_{m-1})^{a_m-b_m} \mid A_{m-1}$$
$$B_m \mapsto (A_{m-1}B_{m-1})^{b_m}(B_{m-1})^{a_m-b_m} \mid A_{m-1}$$

where the symbol that follows $\mid$ represents the tower of level $m-1$ visited right after reaching the top of the respective tower of level $m$. Each tower of level $m-1$, say $u \in V_{m-1}$, has an associated code of length $p_{m-1}$ in $X_k$, say $\omega_u^{(m-1)} \in L_{p_{m-1}}(X_k)$, constructed by coding in $X_k$ the orbit between times $t=0$ and $t = p_{m-1}-1$ of any point in the base of such tower. Notice that $\omega$ must appear as a substring of the concatenation of the codes of two towers $u, v \in V_{m-1}$, i.e. $\omega \sqsubseteq \omega_u^{(m-1)}\omega_v^{(m-1)}$. Due to the repetitions of such concatenations in the tower traversal described by the order $\geq$ we have the following bounds:

$$\omega \sqsubseteq A_{m-1}B_{m-1} \Rightarrow \#_{A_m}^{(m)}(\omega) \geq b_m;\ \#_{B_m}^{(m)}(\omega) \geq b_m$$
$$\omega \sqsubseteq B_{m-1}A_{m-1} \Rightarrow \#_{A_m}^{(m)}(\omega) \geq b_m;\ \#_{B_m}^{(m)}(\omega) \geq b_m$$
$$\omega \sqsubseteq A_{m-1}A_{m-1} \Rightarrow \#_{A_m}^{(m)}(\omega) \geq a_m - b_m;\ \#_{B_m}^{(m)}(\omega) \geq 0$$
$$\omega \sqsubseteq A_{m-1}B_{m-1} \Rightarrow \#_{A_m}^{(m)}(\omega) \geq 0;\ \#_{B_m}^{(m)}(\omega) \geq a_m - b_m$$

Depending on the case we have one of the following two possible bounds:

$$\Pi_k\mu([\omega]) = \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \geq \frac{2b_m}{2p_m}$$
$$\Pi_k\mu([\omega]) = \sum_{\mathrm{i}\in V_m} \mu_{\mathrm{i}}^{(m)} \cdot \#_{\mathrm{i}}^{(m)}(\omega) \geq \frac{a_m - b_m}{2p_m}$$

The condition $a_m \geq 3 \cdot b_m$ assures that the first bound is the smallest of the two. This together with condition $a_m \leq C \cdot b_m$ yields:

$$h_1^{(m-1)} \cdot \varepsilon_{h_1^{(m-1)}}(X_k, \Pi_k\mu) \geq p_{m-1} \cdot \frac{b_m}{p_m} = \frac{b_m}{b_m + a_m} = \frac{1}{1 + \frac{a_m}{b_m}} \geq \frac{1}{1+C} > 0$$

As $J \subseteq \mathbb{N}$ is infinite we get:

$$\limsup_{\substack{m\to\infty \\ m\in I}} \left[h_1^{(m-1)} \cdot \varepsilon_{h_1^{(m-1)}}(X_k, \Pi_k\mu)\right] \geq \frac{1}{1+C} > 0$$

Which shows that $(X_k, \sigma_k)$ satisfies (BC). We conclude that $(X,T)$ satisfies (BC). □

Considering the previous class of systems with sequences $\{a_n\}_{n\geq 2}$ and $\{b_n\}_{n\geq 2}$ which satisfy $3\cdot b_n \leq a_n \leq C\cdot b_n$ for all $n \geq 2$ and such that $a_n, b_n \nearrow +\infty$ we get a family of examples which satisfy (BC) but do not satisfy the hypotheses of Proposition 3.6. We cannot hope then for Proposition 2.44 to yield a sufficient and necessary condition for Bratteli-Vershik systems to satisfy (BC). Clearly there exist a variety of combinatorial mechanisms, as the one used in the proof of Proposition 3.7, which influence the values of the quantities $\varepsilon_n(X_k, \Pi_k\mu)$ in non-trivial ways.

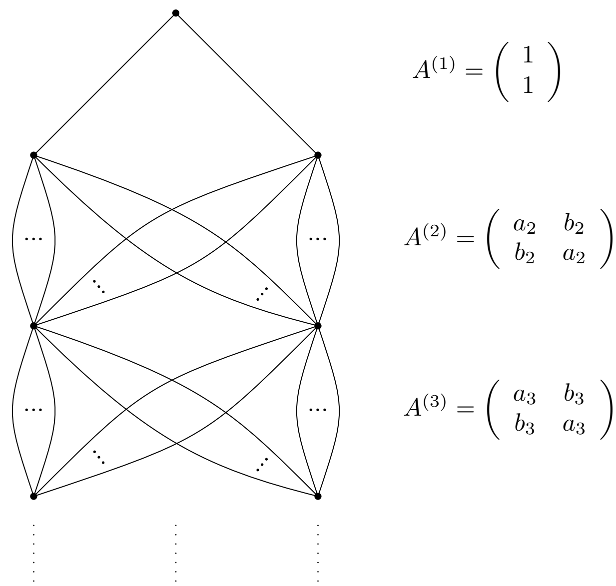


Figure 3.2: Bratteli-Vershik diagram of Bi-Toeplitz type considered in example.

## 3.2 Irrational Rotations and Sturmian Subshifts

We are interested in studying (BC) for Sturmian subshifts via their Bratteli-Vershik representations. It is known that every Sturmian subshift satisfies (BC). See for example [12]. We will prove a weaker result, though still considerably strong, by using the (BC) criteria provided in Chapter 2. Let us begin with the definition of irrational rotations of the circle:

**Definition 3.8** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be a real irrational number. Consider the map $R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $R_\alpha(x) := x + \alpha \pmod{\mathbb{Z}}$ for each $x \in \mathbb{R}/\mathbb{Z}$. Then $(\mathbb{R}/\mathbb{Z}, R_\alpha)$ is a minimal and uniquely ergodic dynamical system. Its unique invariant probability measure is the Lebesgue measure. We will refer to $(\mathbb{R}/\mathbb{Z}, R_\alpha)$ as the irrational rotation of angle $\alpha$ of the circle.*

Let us now introduce the definition of Sturmian subshifts:

**Definition 3.9** *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be a real irrational number and let $(\mathbb{R}/\mathbb{Z}, R_\alpha)$ be the irrational rotation of angle $\alpha$ of the circle. Consider on $\mathbb{R}/\mathbb{Z}$ the partition $\mathcal{P} = \{[0, 1-\alpha), [1-\alpha, 1)\}$. Let $\pi_\mathcal{P} \colon \mathbb{R}/\mathbb{Z} \to \mathcal{P}^\mathbb{Z}$ be the map that associates to each $x \in \mathbb{R}/\mathbb{Z}$ the sequence $\pi_\mathcal{P}(x) = (A_{\mathrm{i}_n})_{n\in\mathbb{Z}}$ of elements of $\mathcal{P}$ such that for each $n \in \mathbb{Z}$ one has $R_\alpha^n x \in A_{\mathrm{i}_n}$. Notice that $X_\alpha := \overline{\pi_P(X)} \subseteq \mathcal{P}^\mathbb{Z}$ is a closed $\sigma$-invariant subset of the fullshift $\mathcal{P}^\mathbb{Z}$. It follows that $(X_\alpha, \sigma)$ is a subshift over the alphabet $\mathcal{P}$. Moreover there exists an almost one to one factor map $\pi \colon (X_\alpha, \sigma) \to (\mathbb{R}/\mathbb{Z}, R_\alpha)$ which is not one to one at precisely one orbit of the system $(\mathbb{R}/\mathbb{Z}, R_\alpha)$. The system $(X_\alpha, \sigma)$ will be referred to as the Stumian subshift of angle $\alpha$.*

We provide Bratteli-Vershik representations of Sturmian subshifts in order to apply the (BC) criteria provided in Chapter 2. For this purpose we recall some basic facts about simple continued fractions. For more details see for instance [20]. To each real irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we associate via Euclid's division algorithm a unique sequence of non-negative integers $(a_n)_{n\in\mathbb{N}_0}$ with $a_n \geq 1$ for all $n \geq 1$ called the sequence of partial quotients of $\alpha$. We denote this by $\alpha = [a_0; a_1, a_2, \ldots]$. As we will work exclusively with real irrational numbers $\alpha \in [0,1)$ we will always have $a_0 = 0$. We consider the increasing sequences of integers $(p_n)_{n\geq -2}$ and $(q_n)_{n\geq -2}$ defined recursively by:

$$\begin{array}{lll} p_{-2} = 0, & p_{-1} = 1, & p_n = a_n p_{n-1} + p_{n-2}, \quad \forall n \in \mathbb{N}_0 \\ q_{-2} = 1, & q_{-1} = 0, & q_n = a_n q_{n-1} + q_{n-2}, \quad \forall n \in \mathbb{N}_0 \end{array}$$

Then the following relation holds for all $n \in \mathbb{N}_0$:

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_n}}}$$

This defines a sequence $(p_n/q_n)_{n\in\mathbb{N}_0}$ of rational numbers converging to $\alpha$ with convergence rate given by:

$$\frac{1}{q_n(q_{n+1}+q_n)} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

This convergence is also denoted by:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

The subsequences $(p_{2n}/q_{2n})_{n\in\mathbb{N}_0}$ and $(p_{2n+1}/q_{2n+1})_{n\in\mathbb{N}_0}$ are monotonically increasing and decreasing respectively. We will also consider the sequence of real numbers $(\theta_n)_{n\geq -2}$ given by $\theta_n = q_n\alpha - p_n$ and the sequence of absolute values $(|\theta_n|)_{n\geq 2}$. Notice that $\operatorname{sgn}(\theta_n) = (-1)^n$. In particular $|\theta_n| = (-1)^n\theta_n$. It is straightforward to check the following recurrence:

$$|\theta_{-2}| = \alpha, \qquad |\theta_{-1}| = 1, \qquad |\theta_{n+1}| = |\theta_{n-1}| - a_n|\theta_n|, \quad \forall n \in \mathbb{N}_0$$

It is also easy to check the following bounds:

$$\frac{1}{q_{n+1}q_n} < |\theta_n| < \frac{1}{q_n}$$

Let $\alpha \in [0,1)$ be a real irrational number with partial quotients given by $\alpha = [0; a_1, a_2, \ldots]$ and let $(X_\alpha, \sigma)$ be the Sturmian subshift of angle $\alpha$. Using Rauzy-Veech induction, see for instance [27], we can provide a Bratteli-Vershik representation $(B, \geq)$ of $(X_\alpha, \sigma)$ by the following procedure. Level 1 of the diagram is given by the adjacency matrix $A^{(1)} = (1,1)^T$. For each $n \geq 2$ there are two possibilities for level $n$ of the diagram: transitions of type 0 and transitions of type 1. This possibilities are represented in Figure 3.5. Level 1 of the diagram is followed by $a_1 - 1$ transitions of type 1, $a_2$ transitions of type 0, $a_3$ transitions of type 1, $a_4$ transitions of type 0, and so on. This yields the properly ordered diagram in Figure 3.4. It is not hard to see that the symbolic factor of level 1 of this diagram is topologically conjugate to the Sturmian subshift of angle $\alpha$.

Using Proposition 1.24 we can calculate the height vectors of this diagram:

**Proposition 3.10** *Let $(B, \geq)$ be the properly ordered Bratteli diagram in Figure 3.4. Then for any given* $\mathrm{i} \in \mathbb{N}_0$ *and* $0 \leq k \leq a_{\mathrm{i}+1}$, *with the exception of* $\mathrm{i} = 0$ *for which* $0 < k \leq a_1$ *is*

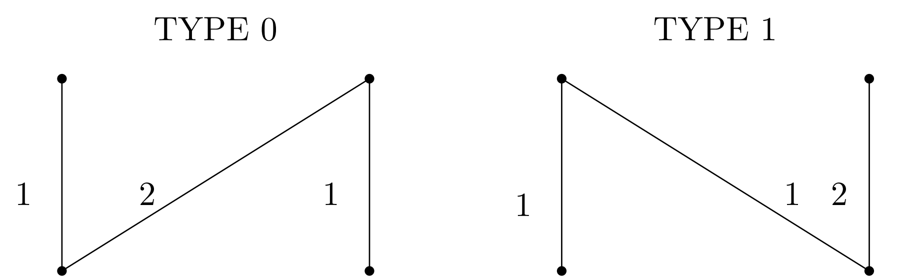


Figure 3.3: Transition types for Bratteli-Vershik representation of Sturmian subshift.

*considered instead, the height vector of level* $a_1 + \cdots + a_\mathrm{i} + k$ *is given by:*

$$h^{(a_1+\cdots+a_\mathrm{i}+k)} = \begin{cases} \begin{pmatrix} q_\mathrm{i} \\ q_{\mathrm{i}-1} + kq_\mathrm{i} \end{pmatrix} & \text{if } \mathrm{i} \text{ even} \\ \\ \begin{pmatrix} q_{\mathrm{i}-1} + kq_\mathrm{i} \\ q_\mathrm{i} \end{pmatrix} & \text{if } \mathrm{i} \text{ odd} \end{cases}$$

Using Theorem 1.40 we can find a natural invariant measure for this diagram:

**Proposition 3.11** *Let* $(B, \geq)$ *be the properly ordered Bratteli diagram in Figure 3.4 and let* $(X_B, \phi)$ *be the associated Bratteli-Vershik system. Let* $p^{(0)} = (1)$*. For each* $\mathrm{i} \in \mathbb{N}_0$ *and* $0 \leq k \leq a_{\mathrm{i}+1}$*, with the exception of* $\mathrm{i} = 0$ *for which* $0 < k \leq a_1$ *is considered instead, let* $p^{(a_1+\cdots+a_\mathrm{i}+k)}$ *be the vector given by:*

$$p^{(a_1+\cdots+a_\mathrm{i}+k)} = \begin{cases} \begin{pmatrix} |\theta_\mathrm{i}| \\ |\theta_{\mathrm{i}-1}| - k|\theta_\mathrm{i}| \end{pmatrix} & \text{if } \mathrm{i} \text{ even} \\ \\ \begin{pmatrix} |\theta_{\mathrm{i}-1}| - k|\theta_\mathrm{i}| \\ |\theta_\mathrm{i}| \end{pmatrix} & \text{if } \mathrm{i} \text{ odd} \end{cases}$$

*Then there exists a unique invariant probability measure* $\mu \in \mathcal{M}(X_B, \phi)$ *with* $\{p^{(n)}\}_{n\in\mathbb{N}_0}$ *as its sequence of* $\mu$*-measure vectors.*

Using the (BC) criteria provided in Proposition 2.44 we get the following result:

**Proposition 3.12** *Let* $(X, T)$ *be the Sturmian subshift of irrational angle* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*. Suppose the partial quotients of* $\alpha$ *are given by* $\alpha = [0; a_1, a_2, \ldots]$*. Suppose there exists an infinite set of even integers* $I \subseteq 2\mathbb{N}$ *and a constant* $C \geq 1$ *such that for all* $\mathrm{i} \in I$ *one of the following conditions holds:*

1. $1 < a_{\mathrm{i}+1} \leq C$.
2. $a_{\mathrm{i}+1} = 1$ *and* $1 \leq a_{\mathrm{i}+2} \leq C$.

*Then* $(X, T)$ *satisfies* $(BC)$*.*

Proof. Let $\mathrm{i} \in I$ be arbitrary even integer in the set $I$. We look at the level $m_\mathrm{i} = a_1 + \cdots + a_\mathrm{i} + 1$ of the diagram $(B, \geq)$. Notice that all minimal edges in $E_{m_\mathrm{i}}$ start at the same

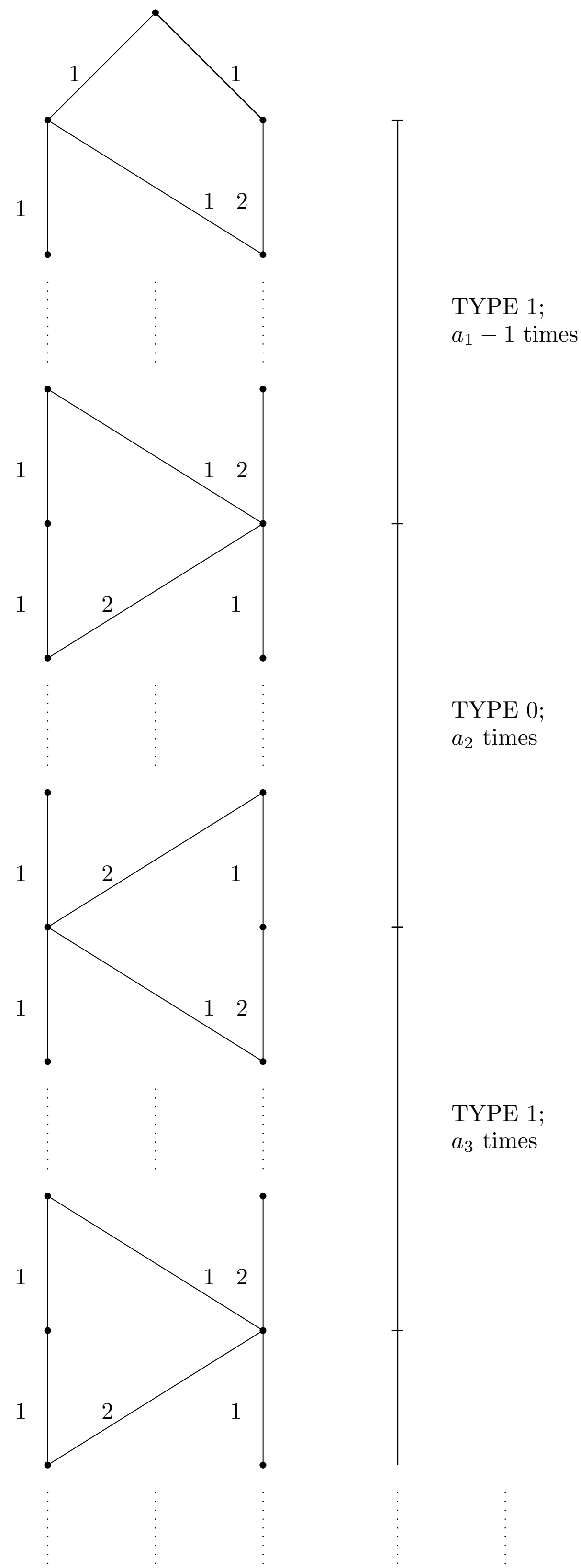


Figure 3.4: Bratteli-Vershik representation of Sturmian subshift.

vertex $1 \in V_{m_{\mathrm{i}}-1}$. From Proposition 3.10 we see that $h_1^{(m_{\mathrm{i}}-1)} = q_{\mathrm{i}}$. Let $\mu \in \mathcal{M}(X_B, \phi)$ be the invariant probability measure of $(X_B, \phi)$ described in Proposition 3.11. We deal with the two possible conditions for i separately.

Suppose first that $1 < a_{\mathrm{i}+1} \leq C$. Then $\min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} = |\theta_{\mathrm{i}+1}|$. It follows that:

$$h_1^{(m_{\mathrm{i}}-1)} \cdot \min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} = q_{\mathrm{i}} |\theta_{\mathrm{i}}| \geq \frac{q_{\mathrm{i}}}{q_{\mathrm{i}+1} + q_{\mathrm{i}}} = \frac{1}{\frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} + 1}$$

To bound the denominator of this fraction we consider the recursive definition $q_{\mathrm{i}+1} = a_{\mathrm{i}+1} q_{\mathrm{i}} + q_{\mathrm{i}-1}$. From this we see that $\frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} = a_{\mathrm{i}+1} + \frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} \leq a_{\mathrm{i}+1} + 1 \leq C + 1$. It follows then that:

$$h_1^{(m_{\mathrm{i}}-1)} \cdot \min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} \geq \frac{1}{C+2}$$

Suppose that $a_{\mathrm{i}+1} = 1$ and $1 \leq a_{\mathrm{i}+2} \leq C$. Then $\min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} = |\theta_{\mathrm{i}+1}|$. It follows that:

$$h_1^{(m_{\mathrm{i}}-1)} \cdot \min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} = q_{\mathrm{i}} |\theta_{\mathrm{i}+1}| \geq \frac{q_{\mathrm{i}}}{q_{\mathrm{i}+2} + q_{\mathrm{i}+1}} = \frac{1}{\frac{q_{\mathrm{i}+2}}{q_{\mathrm{i}}} + \frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}}}$$

To bound the denominator of this fraction we consider first the recursive definition $q_{\mathrm{i}+1} = a_{\mathrm{i}+1} q_{\mathrm{i}} + q_{\mathrm{i}-1} = q_{\mathrm{i}} + q_{\mathrm{i}-1}$. From this we see that $\frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} = a_{\mathrm{i}+1} + \frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} \leq 1 + 1 = 2$. We also consider the recursive definition $q_{\mathrm{i}+2} = a_{\mathrm{i}+2} q_{\mathrm{i}+1} + q_{\mathrm{i}}$. From this we see that $\frac{q_{\mathrm{i}+2}}{q_{\mathrm{i}}} = a_{\mathrm{i}+2} \frac{q_{\mathrm{i}+1}}{q_{\mathrm{i}}} + 1 \leq 1 + 1 = 2a_{\mathrm{i}+2} + 1 \leq 2C + 1$. It follows that:

$$h_1^{(m_{\mathrm{i}}-1)} \cdot \min_{j \in V_{m_{\mathrm{i}}}} \mu_j^{(m_{\mathrm{i}})} \geq \frac{1}{2C+3}$$

Notice now that as $I$ is infinite we have $(m_{\mathrm{i}})_{\mathrm{i} \in I} \nearrow \infty$. It follows that:

$$\limsup_{\substack{m \to \infty \\ m \in I}} \left[ h_1^{(m-1)} \cdot \min_{\mathrm{i} \in V_m} \mu_{\mathrm{i}}^{(m)} \right] \geq \min \left\{ \frac{1}{C+2}, \frac{1}{2C+3} \right\} = \frac{1}{2C+3} > 0$$

By Proposition 2.44 we conclude that $(X_B, \phi)$ satisfies (BC) and so does $(X, T)$, it being topologically conjugate to the symbolic factor of level 1 of the diagram $(B, \geq)$. □

Using a slight variation of Rauzy-Veech induction we can generate a different Bratteli-Vershik representation $(B', \geq')$ of $(X, T)$. We use the same procedure as before but with transition types given by Figure 3.5. This again yields a properly ordered Bratteli diagram whose symbolic factor of level 1 is topologically conjugate to the Sturmian subshift of angle $\alpha$. As the Bratteli diagrams $B$ and $B'$ are the same (only the order is changed) the calculations provided in Propositions 3.10 and 3.11 still hold. Using the same proof as in Proposition 3.12 but over the diagram $(B', \geq')$ we get the following result:

**Proposition 3.13** *Let $(X, T)$ be the Sturmian subshift of irrational angle $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Suppose the partial quotients of $\alpha$ are given by $\alpha = [0; a_1, a_2, \ldots]$. Suppose there exists an infinite set of odd integers $I \subseteq 1 + 2\mathbb{N}$ and a constant $C \geq 1$ such that for all $\mathrm{i} \in I$ one of the following conditions holds:*

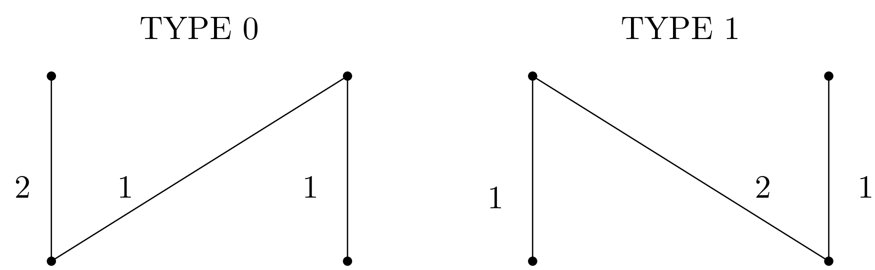


Figure 3.5: Alt. transition types for Bratteli-Vershik representation of Sturmian subshift.

*1.* $1 < a_{\mathrm{i}+1} \leq C$.

*2.* $a_{\mathrm{i}+1} = 1$ *and* $1 \leq a_{\mathrm{i}+2} \leq C$.

*Then* $(X, T)$ *satisfies* $(BC)$.

Adding all up we get the following result:

**Corollary 3.14** *Let* $(X, T)$ *be the Sturmian subshift of angle* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. *Suppose the partial quotients of* $\alpha$ *are given by* $\alpha = [0; a_1, a_2, \dots]$. *Suppose there exists an infinite set of integers* $I \subseteq \mathbb{N}$ *and a constant* $C \geq 1$ *such that for all* $\mathrm{i} \in I$ *one of the following conditions holds:*

*1.* $1 < a_{\mathrm{i}+1} \leq C$.

*2.* $a_{\mathrm{i}+1} = 1$ *and* $1 \leq a_{\mathrm{i}+2} \leq C$.

*Then* $(X, T)$ *satisfies* $(BC)$.

Although this last result does not provide a complete proof to the known fact that every Stumian subshift satisfies (BC) it still covers an apparently big family of cases. The fact that we are not able to prove the complete result via Proposition 2.44 tells us that there must be some combinatiorial mechanisms which significantly influences the values $\varepsilon_n(X_k, \Pi_k\mu)$ for the Bratteli-Vershik representations considered.

# Conclusion

Through this thesis we have brought to light the very interesting technique of extending subshift related properties to the more general context of Cantor minimal systems via the concept of symbolic factors, as we did with (BC) and (SCS). What gives real significance to this technique is Proposition 2.26, which allows one to study these extended properties on a given Cantor minimal systems by looking at any particular Bratteli-Vershik representation. It then makes sense to look for diagram related necessary and/or sufficient conditions for these extended properties to hold.

Regarding the study of (BC) for Cantor minimal systems, we were able to provide simple, easily applicable diagram related criteria for Bratteli-Vershik systems to satisfy this condition. In this sense, Proposition 2.44 appears as the most balanced in terms of broadness and applicability. Not only were we able to use this criterium to show that many different Bratteli-Vershik systems satisfy (BC) but it also allowed us to arrive at a very simple unique ergodicity sufficient condition for Bratteli-Vershik systems of Bi-Toeplitz type that does not require an explicit analysis of the measure cones of the system. Nonetheless, this criterium has significant limitations as was seen via several examples on which it was not applicable but which still satisfied (BC). This shows that (BC) is heavily influenced by a variety of combinatorial mechanisms not captured by this criterium.

Although Theorem 2.42 provides necessary and sufficient diagram related conditions for Bratteli-Vershik systems to satisfy (BC), it is hard to check these conditions and even harder to interpret them. In this context an interesting problem arises: To find necessary and sufficient diagram related conditions for Bratteli-Vershik systems to satisfy (BC) that are more understandable and easier to check than those provided by Theorem 2.42. A simpler but still interesting problem is to find simple, easily applicable diagram related criteria for Bratteli-Vershik systems to not satisfy (BC). Checking that a given Bratteli-Vershik system does not satisfy (BC) requires one to bound the quantities $\varepsilon_n(X_k, \Pi_k\mu)$, where we are considering the notation of Proposition 2.41, for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$, which makes this problem apparently hard to solve.

The (BC) criteria provided by Corollary 2.49 also draws attention as, aside from the standardization hypothesis on the order of the diagram, it only requires the repetition of a block of three positive matrices at inifinitely many levels of the diagram. In this regard the following questions come to mind: Does there exist a Bratteli-Vershik system which does not satisfy (BC) but which satisfies the hypothesis of Corollary 2.49 with a block of two positive matrices instead of three? Does there exist a Bratteli-Vershik system which does not satisfy

(BC) but which satisfies the hypothesis of Corollary 2.49 with a block of one positive matrix instead of three? One can also think of variations of these questions, changing the top matrix of the matrix block by a height proportionality condition, or changing the bottom matrix of the matrix block by a measure proportionality condition. The main difficulty of these questions comes from the need to check that a given Bratteli-Vershik system does not satisfy (BC), which, as discussed previously, is apparently hard.

Another aspect that would be nice to see further developed in the future is the understanding of the combinatorial mechanisms involved in (BC). A good starting point for this would be to study the application of the (BC) criteria provided in this thesis to other classes of Cantor minimal systems, particularly to codings of interval exchange transformations. These systems, known to have very rich dynamics, appear simple enough as to be able to understand, at least partially, some of the combinatorial mechanisms that influence (BC). Understanding such mechanisms could ultimately lead to broader, more transparent diagram related (BC) criteria.

It is hard to tell how precise and easily applicable can one expect future (BC) criteria to be. Due to the variety of combinatorial behaviours among the huge class of all Cantor minimal systems it is hard to expect that a characterization of Cantor minimal systems which satisfy (BC) can be more transparent than what Theorem 2.42 provides. Yet an appropriate classification of Cantor minimal systems according to the combinatorial behaviour of their Bratteli-Vershik representations would be very helpful in the search of progressively more understandable (BC) characterizations.

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