



# On the Lyapunov theory for fractional order systems



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## ABSTRACT

We provide the main features of Lyapunov theory when it is formulated for fractional order systems. We give consistent extensions of Lyapunov, LaSalle and Chetaev classical theorems to the case of fractional order systems. We give examples to illustrate the applications of the concepts and propositions introduced.

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## 1. Introduction

Our aim is to extend the theory of Lyapunov, as exposed and generalized by LaSalle [1,2], in order to deal with fractional order systems. Not surprisingly, this purpose is not new in the literature ([3–6]). The Lyapunov strategy to draw conclusions on asymptotic properties of the solution of general equations without explicitly knowing the solution itself, makes use of a scalar (or vector) functional of the solution, which is easier to analyze. In addition, Lyapunov theory formalizes the intuitive notion of stability. The utility of this strategy in many areas of mathematics and engineering is therefore manifest. However, many issues remain controversial in fractional order systems: What is the state of a fractional order system? Can a Lyapunov function be defined for variables which are not the state of the system? Are fractional order systems dynamic ones? Do Lyapunov stability concepts apply? Moreover, the Lyapunov theorems has not a complete fractional generalization (as we will see later in Remark 11, the proofs of the proposed second Lyapunov theorem for fractional systems done in [3–5] are incomplete). Therefore, this purpose has not been yet satisfactorily reached in literature.

In calculation aspects, though fractional calculus in its actual state of development has not handy rules of derivation such as the chain rule or product rule as in integer calculus, Lyapunov-like functions of quadratic type can still be constructed and its fractional derivative (in the Caputo sense) be bounded as shown in [7]. Moreover, qualitative asymptotic statements on solutions of equations are possible drawn which are valid for any order in  $(0, 1]$  while the form of the equation let be unchanged (i.e. only changing the order of derivation), by using the same quadratic Lyapunov function.

Our object of study is a general equation of the type

$$D^\alpha x = f(x, t) \quad (1)$$

where  $f : (\mathcal{D} \subseteq \mathbb{R}^n) \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $D^\alpha$  is some fractional derivative previously defined in the literature. When  $f(x, t) = f(x)$ , it is said that system (1) is **autonomous**.

Note that our specific interest in study system (1) marks a distinction with the efforts made using frequency distributed fractional integrator model as in [6], since we are mainly interested in the variable  $x$  of a possibly non linear equation

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meanwhile in [6] they arrive to an integer order linear system which has not clear connection with an expression like (1) when non zero initial conditions are considered for a previously defined and studied fractional derivative. Nevertheless, such an efforts are consistent for instance if the transfer function approach to linear systems is adopted.

For fixing ideas, we use the following definitions.

**Definition 1** (Fractional Integral. [8], Section 2.1). The fractional integral of order  $\alpha > 0$  of a function  $f : [0, \infty) \rightarrow \mathbb{R}$  is defined as

$$I^\alpha f(t) = I^\alpha[f(\cdot)](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \quad (2)$$

**Definition 2** (Caputo derivative. [8], Definition 3.1). The Caputo derivative of order  $\alpha > 0$  of a  $\mathcal{AC}^n[0, \infty)$  function  $f(t)$  is defined as

$$D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t), \quad (3)$$

where  $n = \lceil \alpha \rceil$ .

The statements will be done as general as possible and all what it is required for the chosen derivative is that it holds a kind of fundamental theorem with the commonly accepted definition of fractional integral. In the Caputo settings, it is given by the following property ([8], Lemma 2.21–2.22)

**Property 1.**

- (a) If  $f$  belongs to  $C^n[a, b]$ , the space of continuous functions that have continuous first  $n$  derivatives, then for all  $t \in [a, b]$

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=1}^n \frac{f^{(k)}(0)}{k!} t^k, \quad (4)$$

where  $n = \lceil \alpha \rceil$ .

- (b) If  $f$  belongs to  $\mathcal{L}^\infty[(a, b)]$ , the Lebesgue space of bounded functions on the interval  $(a, b)$ , then for all  $t \in [a, b]$

$$D^\alpha I^\alpha f(t) = f(t) \quad (5)$$

The rest of the paper is organized as follows. Section 2 provides characterization of fractional order systems which is relevant in the construction of Lyapunov theory. Section 3 presents the main concepts and theorems of the proposed extension. Section 4 shows some applications of main results.

## 2. Status of fractional systems

We start establishing the relevant features of fractional order systems in constructing a Lyapunov theory for them: the state notion, the failure to be an abstract dynamical system and to have a monotonic simple rule associated to the fractional derivative.

### 2.1. State notion in fractional order systems

The intuition behind Lyapunov functional is to have an analogue of the energy for physical systems, namely an scalar function providing enough information to completely characterize them at a given instant. Thus, the concept of state of a system, a variable of the system condensing all system past information at a given instant, must be defined to make possible the analogy.

A system is a mathematical relation built in the space  $X := \{f : [t_0, t_1] \rightarrow \mathbb{R}^n \mid t_0, t_1 \in \mathbb{R}, n \in \mathbb{N}\}$  of elements denoted as  $f_{[t_0, t_1]}$ , i.e. a subset of  $X \times X$ , where the first coordinate is called the input  $u$  and the second output  $y$ . Essentially, the state is a variable  $z \in Z$  for some suitable space  $Z$ , that turns this relation in a function. It is always possible to find that variable which will be not unique since it can be relabeled using any bijection of space  $Z$ . For general differential equations, it would be necessary just the initial and/or boundary conditions. But for physical or dynamical systems, the notion of state requires additionally consistency conditions when an arbitrary initial time is used. We will use the axioms stated in [9], which make meaningful (in its integral version) the following question: For the system  $D^\alpha x = f(x, t)$  with  $x(0) = x_0$ , what is the initial condition  $x_{t_0}$  at time  $t_0$  for any  $t_0 \geq 0$  such that the system solution will be continued from the solution of the new system  ${}_{t_0}D^\alpha x = f(x, t)$ , where  ${}_{t_0}D^\alpha x$  is defined as in Eq. (3) with lower limit of integration  $t_0$  instead of 0?

Although one can consider Eq. (1) just as an integral equation and thereby take the state as the initial condition, we need in principle a system state variable to build a Lyapunov function. Thus, we arrive to the following fact.

**Proposition 1.** The system defined by  $D^\alpha y = f(u, t)$ , where  $0 < \alpha < 1$  and  $(u, y) \in X \times X$  has an infinite dimensional state vector where  $f(u, t)$  is locally integrable and Lipschitz in the second variable.

**Proof.** Consider the integral equation

$$y(t) = \psi(t, 0) + I^\alpha [f(u(\cdot), \cdot)](t), \tag{6}$$

where  $\psi(t, 0)$  depends on initial conditions. If  $\psi(t, 0) = y(0)$ , we recover the original equation  $D^\alpha y = f(u, t)$  for Caputo derivative by taking  $\alpha$  derivative of (6). Other derivatives satisfying a version of the fundamental theorem (Property 1) can also be considered by modifying the function  $\psi$  accordingly (e.g. for Riemann–Liouville derivative, take equation (2.1.39) of [8]).

That  $y(t)$  cannot be the state of system (6) at  $t > 0$ , follows from the second auto consistency condition of [9], which states that if the input-output-state relation (6) is satisfied by an input-output pair  $(u_{(0,t]}, y_{(0,t]})$ , then it is also satisfied by any input-output pair  $(u_{(t_1,t]}, y_{(t_1,t]})$  for  $0 < t_1 \leq t$ . Indeed, if one considers  $t_1 > 0$ , one can write  $y(t) = y(0) + I^\alpha_{[0,t_1]} [f(u(\cdot), \cdot)](t) + I^\alpha_{[t_1,t]} [f(u(\cdot), \cdot)](t)$ , where  $\Gamma(\alpha) I^\alpha_{[a,b]} [f(u(\cdot), \cdot)](t) := \int_a^b (t - \tau)^{\alpha-1} f(\tau) d\tau$ , but  $y(t_1) \neq y(0) + I^\alpha_{[0,t_1]} [f(u(\cdot), \cdot)](t)$  since the right hand is a non constant function in general. Therefore the relationship change in comparison with the initial time. For  $\alpha = 1$ ,  $y(t_1) = y(0) + I^1_{[0,t_1]} [f(u(\cdot), \cdot)](t)$  and one can write  $y(t) = y(t_1) + I^1_{[t_1,t]} [f(u(\cdot), \cdot)](t)$  preserving the form of relation (6).

We claim that  $z(\tau) = (y(0), u_{(0,\tau]})$  is a state variable (in the sense defined by [9]) of system (6) at any time  $\tau \geq 0$ . It follows from Eq. (6) that one gets a function  $y_{[\tau,t]} = F(z(\tau), u_{(\tau,t]})$  in the sense that given  $z(\tau)$ , for any  $u_{[\tau,t]}$  there is only one possible output  $y_{[\tau,t]}$  which is given by  $y(t) = \psi(t, 0) + I^\alpha_{[0,\tau]} [f(u(\cdot), \cdot)](t) + I^\alpha_{[\tau,t]} [f(u(\cdot), \cdot)](t)$ . When  $\alpha = 1$  we note that  $I^\alpha_{[0,\tau]} [f(u(\cdot), \cdot)](t)$  is not a function but a constant that can be subsumed in the value in  $\tau$  of  $y$ , whereby  $z(\tau) = y(\tau)$ . In this way, mutual consistency condition (existence of function  $F$ ) and first consistency condition ( $z(\tau)$  is all the information needed from past instant to  $\tau$ ) of [9] are proved. For the second consistency condition of [9] we just define function  $\psi(t, \tau) := y(0) + I^\alpha_{[0,\tau]} [f(u(\cdot), \cdot)](t)$ . The third consistency condition of [9] holds by the definition of our state as explicit function of  $u$ .

Since the space of functions is infinite dimensional and the state space is unique up to a bijection, we conclude the proof.  $\square$

**Remark 1.** The same argument can be carried out for a system defined by  $D^\alpha y = f(u, t)$  for any  $\alpha > 1 \wedge \alpha \notin \mathbb{N}$ , by using  $\psi(t, t_0 = 0) = \sum_{k=1}^n \frac{y^{(k)}(0)}{k!} t^k$  and  $z(\tau) = (y^{(k)}(0), u_{[0,\tau]})$ , where  $u$  must be in addition of class  $C^n[a, b]$  for  $n = \lceil \alpha \rceil$ , to arrive to the infinite dimensionality claim. The same is true for system  $D^\alpha y = f(y, u, t)$  by taking the state to be  $z(\tau) = (y_{[0,\tau]}, u_{[0,\tau]})$ , where the knowledge of function  $f_{[0,\tau]}$  implies the knowledge of its derivatives. For the linear system  $Ly = Hu$  where  $L = \sum_{i=1}^n a_i D^{\alpha_i}$  and  $H = \sum_{i=1}^m b_i D^{\beta_i}$ , one can use the generic solution (Chapter 5 of [8]) instead of Eq. (6) to arrive to the infinite dimensionality of state variable. The fact that the past values of the input appears as part of the state is reflecting the temporal non locality of the fractional equation.

**Remark 2.** Alternatively, one could consider function  $\psi(t, \tau)$  as the state of the system  $D^\alpha y = f(u)$  at instant  $\tau$ . This function can be seen as a generalization of the initialization function for not null initial condition at the remote past (see [10]).

**Remark 3.** It should be noted that though for all  $t \geq 0$ ,  $z(t) \in X$  (by taking the initial condition as a constant function),  $z(0)$  contains less information than  $z(t)$  and in general,  $z(t_1)$  less than  $z(t_2)$  for  $t_1 < t_2$ , since the former is included in the latter. The non local character is also reflected in this larger transference of information from one instant to another in comparison with integer order systems where it is uniform and only is required a constant vector. As a consequence, the specification of the initial condition varies essentially with the time since the data space varies.

**Remark 4.** There exists an analogy with functional equations. For system  $\dot{x} = F(x_t)$  where  $x_t$  is the function  $x_{[t-r,t]}$  with  $r$  a positive fixed real number,  $x_t$  turns out to be the state of the system ([1], Chapter 3). For  $\alpha < 1$ , the equation  $x^{(\alpha)} = f(x)$  can be  $1 - \alpha$  derived to get  $\dot{x} = D^{1-\alpha} f(x) = g(x_{[0,t]})$ . Following the analogy, one can define a Lyapunov–Krasovskii functional (Energy) of the type  $V(z(t)) = \|x\|^2(t) + I^\beta_{[0,t]} \|x(\cdot)\|^2(t)$ . The notable difference for fractional system is that the last integral is not taken in a finite interval of large  $r$  but in a variable interval of length  $t$ , so this functional will have divergence troubles even for bounded variables. This is circumstantially related to models of a condensator with fractional derivatives having infinite energy as shown in [11], contradicting the modeling itself. Thus, it seems not reasonable to take such Lyapunov functional, if one does not want to deal with infinity quantities.

## 2.2. Dynamical systems

It can be deduced from [1] (the proof is omitted because it is just a simple generalization of Chapters 2 and 3) that a Lyapunov theory –where theorems of Lyapunov type hold– can be set for any dynamical autonomous system which has an additional continuity condition respects to initial conditions, namely on a system described at instant  $t$  by a function  $\pi(t, x)$  for initial condition  $x$  at  $t = 0$  that has the following properties:

- (i)  $\pi(0, x) = x$ ,
- (ii)  $\pi(t_2, \pi(t_1, x)) = \pi(t_1 + t_2, x)$ ,
- (iii)  $\pi(t, \cdot)$  is continuous function for any  $t > 0$ .

For instance, one can demonstrate invariance of the limit point set  $\Omega(x) = \bigcap_{0 < \beta < \omega(x)} \overline{\pi([0, \beta], \omega(x), x)}$ , and under pre-compactness condition on trajectory,  $\pi(t, x) \rightarrow \Omega(x)$  meaning that  $d(\pi(t, x), \Omega) \rightarrow 0$  as  $t \rightarrow \infty$ , where  $d(\cdot, \cdot)$  is a distance function.

Therefore, to make use of this well established theory, one must only verify that the relevant variable describes a dynamical system. Consider the autonomous system,  $D^\alpha x = f(x)$  for  $\alpha \leq 1$ . From our previous analyses, we can use the representation at instant  $t$  by the state function  $x_{[0, t]}$ , whereby if we define  $\pi(t, x) = x_{[0, t]}$ , properties (i) and (ii) hold, but property (iii) has a topological ambiguity since  $x_{[0, t_1]}$  lies in a strictly different space than  $x_{[0, t_2]}$  for  $t_1 \neq t_2$ . Furthermore, as it was remarked, Lyapunov functional on this representation will probably lead to divergences even in stable cases.

On the other hand, we will see that the natural choice of representation of system at instant  $t$ ,  $x(t)$ , does not define a dynamical system (properties (i) and (ii)).

**Proposition 2.** *Let  $D^\alpha$  be Caputo derivative. The system  $D^\alpha x = f(x)$  with  $x(0) \in \mathbb{R}^n$  for  $t \geq 0$  is not a dynamical system for variable  $x$  when  $0 < \alpha < 1$ ,  $f$  is not identically zero and  $x(0)$  is not an equilibrium point. However, it does for  $\alpha = 1$ .*

**Proof.** Let us consider first the linear scalar case. When  $\alpha = 1$ ,  $\pi(t, x_0) := x(t) = x_0 \exp(at)$ , and therefore  $\pi(t, \pi(s, x_0)) = x_0 \exp(as) \exp(at) = x_0 \exp(a(s + t))$ . The key point is that the only continuous function that allows the last equality are of type  $C^t$  for constant  $C$  (a basic mathematical fact). Therefore  $\pi(t, \pi(s, x_0)) = \pi(t + s, x_0)$ . Now for  $\alpha < 1$ , the solution takes the form  $x(t) = x_0 E_\alpha(at^\alpha)$  (see for instance Chapter 5 in [8]) and the function  $E_\alpha(at^\alpha)$  is not of type  $C^t$  (see for instance Section 1.8 [8]), and property (ii) can not be hold when  $x_0 \neq 0$ .

For Caputo derivative, we can write the equivalent equation  $x(t) = x(0) + I_{[0, t]}^\alpha f(x)(t)$ , but  $x(t + s) = x(0) + I_{[0, s]}^\alpha f(x)(t) + I_{[s, t+s]}^\alpha f(x)(t) \neq x(s) + I_{[s, t+s]}^\alpha f(x)(t)$  for any  $t > 0$ , since  $I_{[0, s]}^\alpha f(x)(t)$  is not a constant for  $f(\cdot)$  not identically null (by Property 17 in [12],  $I_{[0, s]}^\alpha f(x)(t)$  decays to zero for fixed  $s$  and hence it is not constant) and  $x(0)$  is not an equilibrium point (as we will see in Proposition 4,  $x(\cdot)$  is not a constant function, and hence  $f$  is not null along  $x(t)$ ). Then  $x$  does not hold property (ii) and therefore does not define a dynamical system. For  $\alpha = 1$ ,  $x(s) = x(0) + I_{[0, s]}^\alpha f(x)(t)$  for any  $t > 0$  since  $I_{[0, s]}^\alpha f(x)(t)$  is a constant, and therefore  $\pi(t_2, \pi(t_1, x(0))) = \pi(t_1 + t_2, x(0))$  with  $\pi(t, x(0)) = x(t)$ .  $\square$

### 2.3. Monotonic characterizations of functions

Lyapunov theory studies the qualitative behavior of system variable  $x$  with a functional  $V(x(t), t)$  built on  $x$  whose behavior can be easier understood. In particular, the basic theory requires for  $V(x(t), t)$  the elemental behavior of monotony with respect to time. A fundamental tool to prove the monotony of a scalar function  $V(x(t), t)$  is by means of a condition on the sign of its (integer) derivative, which can be directly related with the dynamic of the system associated to  $x(\cdot)$  by the equation  $\dot{V} = \nabla_x V \dot{x} + \partial V / \partial t$ . Moreover, if  $V(t) = V(x(t), t)$  is monotone decreasing and bounded from below, it converges. However, the fractional generalization of this condition has not so simple correspondent.

**Proposition 3.** *Consider  $0 < \alpha < 1$  and Caputo derivative. If  $D^\alpha V \leq 0$ , with  $V(t) = V(x(t), t)$  a non negative function of class  $C^1$  then  $V$  not necessarily is monotone. Moreover, there exists a non negative function  $V(t)$  such that  $D^\alpha V \leq 0$ , but  $V$  is not monotone and does not necessarily converge.*

**Proof.** By defining  $W(t) = D^\alpha V(t)$ , we have equivalently  $V(t) = V(0) + I^\alpha W(t)$ . Since  $W \leq 0$ ,  $0 \leq V(t) \leq V(0)$ . Therefore  $I^\alpha W$  is negative and bounded from below. Then  $V \rightarrow L$  iff  $I^\alpha W \rightarrow L$ .

First, we show that in contrast with the case  $\alpha = 1$ ,  $I^\alpha W$  is not in general monotone. We have (up to a positive constant)  $I^\alpha W = \int_0^t (t - \tau)^{\alpha-1} W(\tau) d\tau$ . Forming  $I^\alpha W(t_2) - I^\alpha W(t_1)$ , with  $t_2 > t_1$ , we have

$$\int_0^{t_1} W(\tau) [(t_2 - \tau)^{\alpha-1} - (t_1 - \tau)^{\alpha-1}] d\tau + \int_{t_1}^{t_2} (t_2 - \tau)^{\alpha-1} W(\tau) d\tau.$$

When  $\alpha = 1$  the first term vanishes and the second is negative, whereby we have the monotonic relation  $I^\alpha W(t_2) - I^\alpha W(t_1) \leq 0$  whereby  $V(t_2) \leq V(t_1)$ . For  $\alpha < 1$  the first term is positive and therefore we do not have in general the monotonic relation aforementioned.

For instance, take function  $V$  such that  $D^\alpha V = -\exp(-t)V$  with  $V(0) > 0$ , this function satisfies that  $0 \leq V(t) < V(0)$  for  $t > 0$  but  $V(t) \rightarrow V(0)$  as  $t \rightarrow \infty$  by Proposition 4 of [14] (note that  $I^\alpha [\exp(-t)] \rightarrow 0$ ), therefore  $D^\alpha V \leq 0$  but  $V$  it is necessarily not monotone.

Now we use the function  $p$  defined in Proposition 14 of [12]. It is built from pulses  $p_i$  such that its separation diverges in such a way that the fractional integral of  $p$  remains bounded. For any  $t$ , we can write as  $p(t) = \sum_{i=1}^n p_i(t) + p_{n+1}(t)$ ,  $I^\alpha p(t) = \sum_{i=1}^n I^\alpha p_i(t) + I^\alpha p_{n+1}(t)$ . Note that if  $I^\alpha p(t) \rightarrow L$ , then  $\sum_{i=1}^{n-1} I^\alpha p_i(t) \rightarrow L$ . (if  $f(\cdot)$  converges to  $L$ , then  $f(t - T)$  also converges to  $L$  for any  $T$ ). But, there always exists  $\xi_n$  with  $\xi_n \rightarrow \infty$  such that  $I^\alpha p_{n+1}(\xi_{n+1}) = C$  by construction of such pulses. Therefore  $I^\alpha p(\xi_n) = \sum_{i=1}^n I^\alpha p_i(\xi_n) + C$  which contradicts the convergence of both functions to the same  $L$  when  $n$  is sufficiently large. Thus  $I^\alpha p$  does not converge. By choosing  $W(t) = -p(t)$  and  $p(\cdot)$  such that  $I^\alpha p < V(0)$ , we have the  $V$  is not monotone and does not converge.  $\square$

**Remark 5.** If  $D^\alpha V(\cdot)$  is in addition a decreasing function then  $V(\cdot)$  is a decreasing function, for  $\alpha < 1$ . Indeed, if  $W(\cdot) = D^\alpha V(\cdot)$  is a decreasing negative function then  $I^\alpha W$  is a decreasing function, because, under the change of variable  $s = t - \tau$ ,

we have  $I^\alpha W = \int_0^t s^{\alpha-1} W(t-s) ds$  and  $I^\alpha W(t_2) - I^\alpha W(t_1) = \int_0^{t_1} s^{\alpha-1} [W(t_2-s) - W(t_1-s)] ds + \int_{t_1}^{t_2} s^{\alpha-1} W(t_2-s) ds \leq 0$ . The proposition also proves that although  $\dot{x} \leq 0 \Rightarrow D^\alpha x \leq 0$ , the converse statement is therefore not necessarily true.

**Remark 6.** For Riemann–Liouville derivative,  $D^\alpha V = d(I^{1-\alpha} V)/dt \leq 0$  does not implies that  $V$  is a monotone decreasing function. For instance consider again two positive pulses  $p_1$  and  $p_2$  infinitesimally separate and  $V = -p_1 - p_2$ , then  $I^{1-\alpha} V < 0$  but  $V$  is not monotone.

The fact that for  $\alpha < 1$ ,  $D^\alpha V \leq 0$  does not imply that  $V$  is a decreasing function will be the main difference in the analyses that follow with respect to the standard Lyapunov theory.

### 3. Extensions of Lyapunov theory

By the arguments given above, our main contribution will be to extend the Lyapunov theory to systems which are not necessarily dynamical ones, built on possibly non state variables of the system and where the monotonic character of Lyapunov function can not be established a priori. Therefore one can expect weaker versions of the theorems for similar hypotheses than in the integer order case or equivalently, stronger hypotheses to get similar conclusions. This will be done in reference to fractional order system, but in a generic sense.

#### 3.1. Basic notions

Let us consider a differential or integral equation of integer or fractional order in the variable  $x(\cdot)$ , with initial condition  $x(t_0)$ , such that  $x(t)$  belongs to some topological space  $X$  for every  $t$ . The following concepts are referred to this system and based in Chapter 1 of [1].

We say that the set  $H \subseteq X$  is **stable at**  $t_0$  if for any neighborhood  $N$  of  $H$ , there exists a neighborhood  $U$  of  $H$  such that if  $x(t_0) \in U$  then  $x(t) \in N$  for all  $t \geq t_0$ . Note that this condition is stronger than boundedness and therefore the introduce of this notion is justified. Note also that this definition generalizes the Lyapunov stability in the sense that the latter is set for a normed vector space  $X$ .  $H$  is **attractive at**  $t_0$  if there exists a neighborhood  $N$  of  $H$  such that  $x(t_0) \in N$  implies  $x(t) \rightarrow H$ ;  $H$  is **asymptotically stable at**  $t_0$  if  $H$  is attractive and stable at  $t_0$ . One can define then **stability** of  $H$  if it is stable at  $t$  for all  $t \geq t_0$  and also  $H$  is **uniformly stable** if  $H$  is stable and  $U$  does not depend on  $t_0$ . Note that this requires that initial conditions belong to the same space for all  $t_0 \in \mathbb{R}$ . As it was noted in Remark 3 for fractional order system, the initial condition at  $t_1 > 0$  is not  $x(t_1)$  for a fractional system (1) with derivative with lower integration limit  $t = 0$ ; one can then either encompass both initial conditions in a larger topological space or work just formally as if  $x(t_1)$  were the initial condition. The above concepts are **local** if they hold not for any neighborhood in  $X$  of  $H$  but for some neighborhood  $\mathcal{D}$  of  $H$ .  $H$  is **unstable at**  $t_0$  if it is not (locally) stable at  $t_0$ .

**Example 1.** For system  $D^\alpha x = Ax$ ,  $x = 0$  is asymptotically stable at  $t = 0$  (the initial time of the fractional derivative) when  $A$  is a constant matrix with  $|\arg(\text{spec}((A)))| > \frac{\alpha\pi}{2}$  (Theorem 2 in [19]). Moreover, any  $H \subset \mathbb{R}^n$  with  $0 \in H$ , is asymptotically stable at  $t = 0$ .

**Example 2.** For dynamical systems, stability at  $t = 0$  implies stability (by its definition property (ii) in Section 2.2) and therefore this distinction is useless. For autonomous systems and  $0 < \alpha \leq 1$ , stability at  $t = 0$  of system  $D^\alpha x = f(x)$  implies stability at  $t = t_0$  of system  ${}_{t_0}D^\alpha x = f(x)$ , where  ${}_{t_0}D^\alpha x := {}_{t_0}I^{1-\alpha} \dot{x} := \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-\tau)^{-\alpha} \dot{x}$ .

We say that  $H$  is an **invariant** set if  $x(t_0) \in H$  then  $x(t) \in H$  for all  $t \geq t_0$ .  $H$  is an **equilibrium point** if  $H$  is singleton and invariant.

**Proposition 4.** Let us consider the system  $D^\alpha x = f(x, t)$  with  $f(a, t) = 0$ ,  $f(\cdot, t)$  Lipschitz for all  $t \geq 0$  and  $D^\alpha$  in the Caputo sense. Then  $x = a$  is an equilibrium point of the system.

**Proof.** Note that constant function  $x(t) \equiv a$  is a solution of the system, since Caputo derivative of a constant function is null and it holds the initial condition  $x(0) = a$ . By Lipschitz assumption, the solution is unique (Chapter 5 in [13]) and the claim follows.  $\square$

#### 3.2. Main results

In the following we will assume that  $0 < \alpha \leq 1$  and Caputo derivative (or any fractional derivative which holds Property 1).

**Definition 3** ([1], Chapter 1). For system (1) with  $x(0) = x_0 \in \mathbb{R}^n$  and  $f : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that the solutions are continuous,  $p \in \mathcal{D}$  is a limit point of the solution  $x(t) t \geq 0$ , if there exists a divergent sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $x(t_n) \rightarrow p$  as  $n \rightarrow \infty$ . We denote the set of all limits points of  $x(t)$  as  $\Omega(x_0)$ .

The following proposition gives elementary properties of  $\Omega(x_0)$ , and generalizes those corresponding to the case of  $\alpha = 1$  (see Chapter 2 of [1]), where the fact that  $x$  defines a dynamical autonomous system is exploited. Since it is not the case for  $\alpha < 1$ , we will give a detailed proof to avoid confusion.



**Proposition 5.** *If the solution  $x(t)$ ,  $t \geq 0$ , to (1) with  $x(0) = x_0$  is bounded, then  $\Omega(x_0)$  is nonempty, closed, compact, connected and  $x(t) \rightarrow \Omega(x_0)$  as  $t \rightarrow \infty$ . For autonomous system (1),  $\Omega(x_0)$  does not depend of initial time of fractional derivative.*

**Proof.** Since  $x(\cdot)$  is bounded in  $\mathbb{R}^n$ , the set of points  $x(t)$  for all  $t \geq 0$  is precompact. Thus any sequence  $x(t_n)$  has a convergent subsequence and therefore  $\Omega(x_0)$  is nonempty. Let the sequence  $(p_n)_{n \in \mathbb{N}} \subseteq \Omega(x_0)$  be such that  $p_n \rightarrow p$  as  $n \rightarrow \infty$ , then for every  $n$  there exists sequence  $x(t_{n,m}) \rightarrow p_n$  as  $m \rightarrow \infty$ , whereby  $\|x(t_{n,n}) - p\| \leq \|x(t_{n,n}) - p_n\| + \|p_n - p\| < \epsilon$  for sufficiently great  $n$  and so,  $p \in \Omega(x_0)$  and  $\Omega(x_0)$  is closed. Since  $x(\cdot)$  is bounded (because  $x(\cdot)$  is bounded) and closed in  $\mathbb{R}^n$ , it is compact. Suppose that  $x(t)$  does not converge to  $\Omega(x_0)$ , i.e there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $d(x(t_n), \Omega(x_0)) > \epsilon$  with  $d(\cdot, \cdot)$  distance function in  $\mathbb{R}^n$ . But since  $x(t_n)$  is bounded, there exists a subsequence  $x(t_{n_m})_{m \in \mathbb{N}}$  that converges, so that  $x(t_{n_m}) \in \Omega(x_0)$ , which is a contradiction. Suppose that  $\Omega(x_0)$  is not connected, then it is the union of disjoint closed sets  $\Omega_1, \Omega_2$  and there are disjoint open sets  $U_1, U_2$  such that  $\Omega_1 \subset U_1$  and  $\Omega_2 \subset U_2$ . Since  $x(\cdot)$  is continuous, there is a sequence  $(t_i)_{i \in \mathbb{N}}$  such that  $x(t_i) \notin U_1 \cup U_2$ , and a subsequence  $(t_{i_n})_{n \in \mathbb{N}}$  such that  $x(t_{i_n})$  converges to a point  $p \notin \Omega(x_0) \subset U_1 \cup U_2$ , which is a contradiction with the definition of the set  $\Omega(x_0)$ .

We can express the solution of (1) for initial time  $t_1$  of fractional derivative in the autonomous case as  $x(t) = x_0 + I_t^\alpha f(x)(t)$  where  $I_t^\alpha f(x)(t) = \int_{t_1}^t (t - \tau)^{\alpha-1} f(x(\tau)) d\tau = \int_0^{t-t_1} (t - t_1 - \tau)^{\alpha-1} f(x(\tau + t_1)) d\tau$ . By defining  $y(t) = x(t + t_1)$ , it follows that  $y(t) = x_0 + I_t^\alpha f(y)(t)$ . Therefore if  $x(t_n) \rightarrow p$  as  $n \rightarrow \infty$ , we have  $y(\xi_n) \rightarrow p$  for  $\xi_n := t_n - t_1$ , whereby  $\Omega(x_0)$  does not depend of initial time of fractional derivative.  $\square$

**Remark 7.** The invariant property can not be asserted in general for the autonomous system if it is not a dynamical one. If  $\Omega(x_0)$  consists only of equilibrium points the invariance property can be assured for system (1), but, by connectedness of  $\Omega(x_0)$ , it is singleton or has infinite equilibrium points.

The general goal to extend Lyapunov Theory is to study stability characterizations of  $\Omega(x_0)$  for fractional systems with the aid of a scalar functional  $V$  whose fractional derivative hold some simple condition related to the equation defining the fractional system. For instance, we look for condition  $D^\alpha V \leq -w(x)$  instead of  $V(\tau) \leq V(t)$  for all  $\tau > t$ , because the latter is not easily derived from the original equation, meanwhile the former can in principle be directly done from defining equations of the system (see some examples in [7]).

As it is shown in Theorem 8 of [14], boundedness condition of the solution of (1) can be assured by condition  $D^\alpha V(x(t)) \leq 0$  for radially unbounded function  $V(\cdot)$ . If such a  $V(\cdot)$  is additionally continuous, then  $\Omega(x_0) \subseteq V^{-1}(\Omega(V(x_0)))$ , where  $\Omega(V(x_0)) \subseteq \mathbb{R}$ , whereby the complexity is reduced. However, because of Proposition 3,  $\Omega(x_0) \subset V^{-1}(c)$  for some  $c \in \mathbb{R}$ , is only valid if monotony of  $V$  is imposed or if  $\alpha = 1$ .

We recall that a function  $V(x)$  is **positive definite** is  $V(x) > 0$  for  $x \neq 0$  and  $V(0) = 0$ . A function  $V(x, t)$  is positive definite if  $V(0, t) = 0$  and  $V(x, t) \geq \hat{W}(x)$  for all  $t \geq 0$  and some positive definite function  $\hat{W}(\cdot)$ . A function  $V$  is **negative definite** if  $-V$  is positive definite. We assume that the function  $V(\cdot, t)$  is **radially unbounded** i.e.  $\|x\| \rightarrow \infty \Rightarrow V(x, t) \rightarrow \infty$  for any  $t \geq 0$ . Finally, we recall that a function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}_+$  is of class  $K$  if  $\gamma(0) = 0$  and  $\gamma$  is increasing.

**Proposition 6.** *For system (1), let  $V(x(\cdot), \cdot)$  a continuously differentiable and positive definite function. Let  $w(x)$  be a positive definite function continuous at  $x = 0$  such that in the ball  $B(r) \subseteq \mathcal{D}$  around  $x = 0$  with  $x_0 \in B(r)$  we have*

$$D^\alpha V(x, t) \leq -w(x) \leq 0. \tag{7}$$

*Then  $\liminf_{t \rightarrow \infty} \|x\| = 0$  and  $x = 0$  is stable point at  $t = 0$ . In particular,  $x = 0 \in \bigcap_{x \in B(r)} \Omega(x)$ . For  $\alpha = 1$ ,  $\lim_{t \rightarrow \infty} \|x\| = 0$  ( $x = 0$  is asymptotically stable at  $t = 0$ ).*

**Proof.** By  $\alpha$ -integrating Eq. (7) and using Property 1, we have

$$V(x(t), t) - V(x(t_0), t_0) \leq -I^\alpha w(x(t)) \leq 0.$$

In particular,  $0 \leq V(x(t), t) \leq V(x(t_0), t_0)$ . Since  $V$  is positive definite, the stability at  $t = 0$  of  $x = 0$  follows from similar arguments of Theorem 3 in [7] together with the correspondence between class  $K$  functions and positive definite functions (for instance Lemma 4.1 in [15]). Since  $V$  is bounded, the integral  $I^\alpha w(x(t))$  is bounded. By applying Proposition 15 of [12], we have  $\liminf_{t \rightarrow \infty} w(x(t)) = 0$ . Since  $w$  is positive definite and continuous at 0,  $\liminf_{t \rightarrow \infty} \|x\| = 0$ . That  $x = 0 \in \bigcap_{x \in B(r)} \Omega(x)$  follows from the definition of limit inferior. When  $\alpha = 1$ , we can use the monotone property of  $V$  to conclude that it converges. Since  $V$  is positive definite and  $\liminf_{t \rightarrow \infty} \|x\| = 0$ ,  $V$  must converge to zero. By continuity of  $V$ , we conclude that  $x = 0$  is asymptotically stable at  $t = 0$ .  $\square$

**Remark 8.** Note that if in Eq. (7) instead of function  $w$  we bound with  $I^{1-\alpha} w$ , we can assure convergence of  $x$  to zero for any  $\alpha < 1$ , since in that case we obtain  $V(x(t), t) - V(x(t_0), t_0) \leq -\int w(x(\tau)) d\tau$  and convergence of  $x$  to zero follows from Barbalat Lemma (Lemma 9 in [12]). Note also that the claim would be true even if system (1) was defined for order  $\beta$  and (7) held for  $\alpha < 1$  with  $\beta \neq \alpha$ . This observation will be also true for the rest of the paper implying a theoretical tool to study integer order system with fractional derivative.

An extra condition must be imposed to assure convergence of the solution to zero in the case of  $\alpha < 1$ . We will say that a non negative function  $f$  is  **$\alpha$ -Barbalat** if  $I^\alpha f$  bounded implies that  $f$  converges to zero. From the proof above, it is sufficient for convergence of  $x$  to zero that  $w(x(t))$  be  $\alpha$ -Barbalat. When  $\alpha = 1$  it is sufficient for  $w(\cdot)$  be a continuous function, since as  $x$  is bounded, the function  $w(x(t))$  turns out to be uniformly continuous, and Barbalat Lemma (Lemma 9 in [12]) guarantees

convergence of  $w(x(t))$  to zero and therefore of  $x(t)$  to zero, because  $w(\cdot)$  is positive definite continuous at zero. For  $\alpha < 1$ , the condition  $w(\cdot)$  continuous is not enough as it is shown in Proposition 14 of [12], though by Examples 4 and 5 of [14], the eventual sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $\|x(t)\| > \epsilon$  only can occur with the separation  $|t_{n+1} - t_n| \rightarrow \infty$ . We summarized this reasoning in the following corollary

**Corollary 1.** *Under the same assumptions in Proposition 6 and considering additionally that  $w(\cdot)$  is continuous, we have that for  $\alpha < 1$ , it can not exist a divergent sequence  $(t_n)_{n \in \mathbb{N}}$  with  $|t_{n+1} - t_n| < L$  for all  $L > 0$  such that  $\|x(t_n)\| > \epsilon$  for any  $\epsilon > 0$ . If  $w(x(t))$  is in addition  $\alpha$ -Barbalat we have  $\lim_{t \rightarrow \infty} \|x(t)\| = 0$  and therefore  $x = 0$  is asymptotically stable point at  $t = 0$ .*

**Remark 9.** Note that the condition for a function be  $\alpha$ -Barbalat is just synthetic, it has not yet an analytic form which allows us to decide if a specific system holds this condition.

We give the following extension of the second Lyapunov Theorem,

**Theorem 1.** Consider system (1) with  $0 < \alpha \leq 1$ . Let  $V(x, t)$  be a positive definite function and  $V(t) = V(x(t), t)$  a continuously differentiable function, and  $D^\alpha V = D^\alpha V(x, t)$  negative definite function with  $\alpha \leq 1$ . If there exists function  $\gamma$  of class  $K$  with  $V(x) \leq \gamma(x)$  such that if  $V$  increases then  $\gamma$  increases (e.g.  $V(x) = \gamma(x)$ ), then  $x = 0$  is asymptotically stable point at  $t = 0$ .

**Proof.** By Proposition 6, it is enough to prove that  $V$  cannot increase. Note that initially, indeed,  $V$  is not increasing: if  $\dot{V}(0) > 0$ , by continuous differentiability of  $V = V(t)$ , there would exist time  $t_1 > 0$  such that  $\dot{V}(t) > 0$  for  $t \in (0, t_1)$  and then  $D^\alpha V = I^{1-\alpha} \dot{V} > 0$  for  $t \in (0, t_1)$  which contradicts the negative definiteness of  $D^\alpha V$ .

Let us assume that at instant  $T > 0$ ,  $V$  begins to increase for the first time. Hence  $\gamma$  is also increasing, whereby  $x$  also increases. As  $D^\alpha V$  is negative definite, there exists a function  $\gamma_1$  of class  $K$  such that  $D^\alpha V \leq -\gamma_1(x)$  (Lemma 4.1 in [15]). By fractional comparison principle (Lemma 6.1 in [3]), it is enough study  $D^\alpha V = -\gamma_1(x)$  since if for this equation  $V$  converges to zero, then the solution of  $D^\alpha V \leq -\gamma_1(x)$  also converges to zero because  $V$  is not negative. Therefore  $D^\alpha V(t)$  decrease for  $t > T$ . On the other hand, we can write for  $t > T$

$$D^\alpha V(t) = \frac{1}{\Gamma(1-\alpha)} \left( \int_0^T (t-\tau)^{-\alpha} \dot{V}(\tau) d\tau + \int_T^t (t-\tau)^{-\alpha} \dot{V}(\tau) d\tau \right). \tag{8}$$

Note that as  $t$  increases, for fixed  $\tau$ ,  $(t-\tau)^{-\alpha}$  decreases. Since  $\dot{V}(t) \leq 0$  for all  $0 \leq t \leq T$ , the first term of right hand side in (8) increases. Since  $\dot{V} > 0$  for  $T < t$ , the second term is positive and therefore the total sum of (8),  $D^\alpha V(t)$ , increases for  $t > T$  whit respect to its value in  $T$ , which is a contradiction. Therefore  $V$  can not increase.  $\square$

**Remark 10.** Theorem 6.3 in [3] states a similar claim that Theorem 1 (for Caputo derivative), but its proof is incomplete because case (a) there never happens, case (b) only could prove the  $\liminf$  claim, and monotony is not proved. Proof of Theorem 3.5 in [4] (for Caputo derivative) also is incomplete for the same reason. This seems that they are based in the proof for the integer case (for instance Theorem 4.1 in [15]) where there is an implicit use of the monotony of function  $V$ , whereby it is enough proved the  $\liminf$  claim because  $V$  converges. Theorem 3.7 in [5] assume without any proof that  $V$  is decreasing as a function of time (However, it is formulated for Caputo q-derivative, and it is not clear its relation with other commonly accepted fractional derivatives).

We define for the scalar function  $w(x, t)$ , the set  $E(w)$  of all points  $x$  such that  $w(x, t) \rightarrow 0$  as  $t \rightarrow \infty$ . Note that this set contains the set of all points  $x$  such that  $w(x, t) = 0$  for all  $t \geq 0$ . We also define the set  $\Omega_F(x_0) := \{p \in \mathbb{R}^n : (\exists t_n, C)[(\forall n)|t_n - t_{n-1}| < C], t_n \rightarrow \infty, x(t_n) \rightarrow p\}$  and the set  $E = \bigcap_{w \in \Lambda} E(w)$  where  $\Lambda$  is the set of non negative continuous functions  $w(x, t)$  such that for a positive definite function  $V(\cdot)$  we have  $D^\alpha V(x, t) \leq -w(x, t) \leq 0$ .

**Proposition 7.** *If  $x(\cdot)$  is a uniformly continuous solution of (1) such that there exist function  $V(t) = V(x, t)$  of class  $C^1$  and  $D^\alpha V(x, t) \leq -w(x, t) \leq 0$  with  $w$  continuous function, then  $\Omega_F(x_0) \subseteq E$ .*

**Proof.** Let us assume that  $p \in \Omega_F(x_0)$  but  $p \notin E$ , then there exists  $\epsilon > 0$  and an unbounded sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $D^\alpha V(x, t_n) \leq -2\epsilon$ . Since  $x(\cdot)$  is uniformly continuous, there exist intervals  $\mathcal{J}_n$  and constant  $l > 0$  with  $t_n \in \mathcal{J}_n$  and  $\mu(\mathcal{J}_n) > l$  for all  $n$ , where  $\mu(\cdot)$  is the Lebesgue measure, such that  $D^\alpha V(x, t) \leq -\epsilon$  for every  $t \in \mathcal{J}_n$ . Since  $D^\alpha V(x, t) \leq 0$ ,  $V(x, t_n) - V(x, 0) \leq -\sum_{i=1}^n I^\alpha [\epsilon p_{\mathcal{J}_i}]$  where  $p_{\mathcal{J}_i}(t)$  is the indicator function for the interval  $\mathcal{J}_i$ . Since there exists  $C > 0$  such that  $|t_n - t_{n+1}| < C$ , the intervals  $\mathcal{J}_n$  are finitely separated. By Example 5 in [14], we have  $V(x, t_n) - V(x, 0) \leq -\infty$  which contradicts that  $V(x, t)$  is bounded ( $D^\alpha V(x, t) \leq 0$  implies  $0 \leq V(x, t) \leq V(x(0), 0)$ ). Therefore  $\Omega_F(x_0) \subseteq E(w)$ . Since  $w$  is continuous,  $E(w)$  is closed. The claim follows since  $w$  can be any continuous function such that  $V(x, t) \leq -w(x, t) \leq 0$   $\square$

The next result can be considered the extension for fractional order of LaSalle Theorem ([2]),

**Theorem 2.** *For system (1), let  $V$  be a continuously differentiable (whit respect to time) not negative radially unbounded function and  $w(\cdot, \cdot)$  be a non negative scalar function such that  $w(t) = w(x(t), t)$  is uniformly continuous in  $t$ . Consider that for any  $x \in B(r)$  (the ball around zero) the following holds.*

$$D^\alpha V(x, t) \leq -w(x, t) \leq 0. \tag{9}$$

If the initial condition  $x_0 \in B(r)$  then we have for  $\alpha = 1$ ,  $x \rightarrow E(w)$ . For  $\alpha < 1$  we have  $E(w) \cap \Omega(x_0) \neq \emptyset$  and if  $w(t, x(t))$  is in addition  $\alpha$ -Barbalat then  $x \rightarrow E(w)$ .

**Proof.** Since  $V$  is bounded ( $D^\alpha V(x, t) \leq 0$  implies  $0 \leq V(x, t) \leq V(x(0), 0)$ ) and radially unbounded, necessarily  $x$  is bounded. By Proposition 5,  $x \rightarrow \Omega(x_0)$ .

For  $\alpha = 1$ , we have  $\int_0^t w(\tau) d\tau$  is bounded since  $0 \leq \int_0^t w d\tau \leq V(0, x(0)) - V(t, x(t))$  (by integration of (9)) and  $V$  is a not negative function. Applying Barbalat Lemma (Lemma 9 in [12]), since  $w$  is uniformly continuous as a function of time,  $w(t, x(t)) \rightarrow 0$  as  $t \rightarrow \infty$ , hence by continuity of  $w$ ,  $\Omega(x_0) \subseteq E(w)$ . For  $\alpha < 1$ , if  $E(w) \cap \Omega(x_0) = \emptyset$  then there would exist a time  $T$  such that for any time  $t \geq T$ ,  $w(t) > \epsilon$  for some  $\epsilon > 0$ , whereby  $I^\alpha w(t, x(t))$  would be unbounded contradicting that  $0 \leq I^\alpha w(t, x(t)) \leq V(0, x(0)) - V(t, x(t))$ . The remaining claim follows by a similar reasoning as in the first part of the proof for  $\alpha = 1$  together with the fact that  $\alpha$ -Barbalat implies  $w(t, x(t)) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

**Remark 11.** More generally, if  $\alpha = 1$  then  $\Omega(x_0) \subseteq E$  and therefore  $x \rightarrow E$ . For autonomous systems,  $w = w(x)$  and  $\alpha = 1$ , this result has a considerable advantage: since  $\Omega(x_0)$  is invariant, in order to know the set  $\Omega(x_0)$  one can explore the largest invariant subset  $M$  of  $E(w)$  (usually, the point that makes  $f(x) = 0$  for the autonomous system (1)), whereby  $\Omega(x_0) \subseteq M$  (usually,  $\Omega(x_0) \subseteq E(f) \cap E(w)$ ). This is essentially the LaSalle Invariance theorem (see [2]). For  $\alpha < 1$ , one can write  $x \rightarrow {}_f E$  meaning  $x \rightarrow E$  up to a sequence  $x(t_n) > \epsilon$  with  $(t_n)_{n \in \mathbb{N}}$  a divergently separated sequence, as it was shown Theorem 1. However, since the invariant property is not a priori held, we can not affirm  $x \rightarrow {}_f M$ . The refinement could base on connectedness parts of  $E$ .

**Remark 12.** In the particular case that  $w(x, t) = D^\alpha V(x)$ , convergence to the zeros of  $D^\alpha V(x)$  can not be asserted a priori when  $\alpha < 1$ , as it is for  $\alpha = 1$ . If there exists a point  $p \in \Omega(x_0)$  such that  $D^\alpha V(p) \neq 0$ , then this point is not of equilibrium, since we can integrate  $V(t = 0^+) - V(t = 0) = I^\alpha D^\alpha V(t^+) \neq 0$ , whereby  $x(0^+) \neq p$ . Therefore, if  $\Omega_e(x_0) \subseteq \Omega(x_0)$  is the set of equilibrium points in  $\Omega(x_0)$ , then  $\Omega_e(x_0) \subseteq E$  and if  $\Omega_e(x_0) = \Omega(x_0)$  then  $x \rightarrow E$ .

**Remark 13.** The condition of uniform continuity of  $w(t, x(t))$  can be based, in most of the cases, on the uniform continuity of  $x(t)$  which in turn can be deduce from boundedness of  $D^\alpha x$ . If  $x(t)$  is bounded and  $f(x, t)$  is bounded when  $x(t)$  does it, the uniform continuity of  $x(t)$  follows from Property 11 of [14].

**Remark 14.** In the case of  $\alpha < 1$  and a monotony condition is imposed additionally on function  $V(x, t)$ , then it follows the convergence of  $x$  to the zeros of  $E(w)$  in Theorem 2.

Finally, we give a fractional generalization of Chetaev instability theorem for system (1) with  $0 < \alpha \leq 1$ .

**Theorem 3.** Consider system (1) with  $0 < \alpha \leq 1$ . Let  $V(x(t))$  be a continuous function on  $\mathcal{D} \subseteq \mathbb{R}^n$  and continuously differentiable as time function.

- (i) If  $D^\alpha V(x)$  and  $V(x)$  are positive definite functions then  $x = 0$  is unstable at  $t = 0$ .
- (ii) If  $D^\alpha V(x(t)) = \lambda V + f$  with  $\lambda > 0$  and  $f(x) > 0$ , then  $x = 0$  is unstable at  $t = 0$ .
- (iii) Suppose that  $V(0) = 0$ ,  $D^\alpha V(x)$  continuous function of  $x$  and that there exists a set  $G_r := \{x \in \mathcal{D} \mid V(x) > 0 \wedge D^\alpha V \geq 0 \wedge \|x\| < r\}$  not empty. Then  $x = 0$  is unstable at  $t = 0$ .

**Proof.** (i) Take any bounded neighborhoods  $N \subseteq \mathcal{D}$  and  $U \subseteq N$  of the origin. Take any point  $x_0 \neq 0 \in U$ . Then  $V(x_0) > 0$ . Since  $D^\alpha V \geq 0$  we have  $V(x(t)) \geq V(x_0)$ . By the positive definiteness of  $V$ ,  $\|x(t)\| > l > 0$  for all  $t > 0$  and some  $l > 0$ . Since  $D^\alpha V$  is positive definite,  $D^\alpha V \geq m > 0$  whereby  $V(x(t)) \geq mt^\alpha$ . Therefore, the trajectory  $x(t)$  can not be totally included in the compact set  $\tilde{N}$  since as  $V$  is continuous, it would be bounded. Then there exists  $T > 0$  such that  $x(T) \notin \tilde{N}$ .

(ii) The analytic solution of  $D^\alpha V(x(t)) = \lambda V + f$  takes the form  $V(x(t)) = V(0)E_\alpha(\lambda t^\alpha) + f(x(t)) * E_{\alpha, \alpha}(t)$  (Section 4.1.3 of [8]), therefore  $V(x(t)) \geq V(0)E_\alpha(\lambda t^\alpha)$  since  $E_{\alpha, \alpha}(t)$  is positive. The claim follows from the same argument of part (i) since  $E_\alpha(\lambda t^\alpha)$  diverges as  $t \rightarrow \infty$ .

(iii) Take  $x(0) \in G_r$ . Suppose that  $x(t) \in G_r$  for all  $t > 0$ . Then  $D^\alpha V \geq 0$  whereby  $V(x(t)) \geq V(x(0))$ . Define  $a := \min_{x \in U} D^\alpha V(x)$  where  $U := \{x \in G_r \mid V(x) \geq V(x(0))\}$ . Since  $U$  is closed and bounded,  $U$  is compact. Hence, since  $D^\alpha V(x)$  is continuous function,  $a > 0$ . Therefore  $V(x(t)) = V(x(0)) + I^\alpha D^\alpha V \geq V(x(0)) + at^\alpha$  but this contradicts the fact that  $V$  is bounded at  $G_r$ . Therefore there exists  $T > 0$  such that  $x(T) \notin G_r$ . By noting that the boundary of  $G_r$  is  $\{x = 0\} \cup \{x \mid \|x\| = r\}$  and that  $V(x(t)) \geq V(x(0)) > 0$ , then there exists  $T_1 > 0$  such that  $\|x(T_1)\| = r$  and therefore the origin is unstable.  $\square$

#### 4. Examples

We will start illustrating that the claim of LaSalle extended theorem (Theorem 2) is effective whereby the fractional order case exhibits different behaviors as compared with the integer order case.

**Example 3.** Consider the system

$$D^\alpha x = -f(t)f(t)^T x, \tag{10}$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  is a uniformly continuous, bounded and continuously differentiable function. For  $\alpha = 1$  and  $e := f(t)^T x(t)$  a measure of error, this equation appears in many adaptive control and identification problems of linear systems with  $x$  is



a vector of adjustable parameters. Finding a general set of functions  $f$  for which  $f(t)^T x(t) \rightarrow 0$  is one of the main concern in adaptive theory.

Since  $f$  is a uniformly continuous, bounded and continuously differentiable function,  $x$  is continuously differentiable (Property 12 of [14]). Then  $x^T x$  is also continuously differentiable. By choosing  $V = x^T x$ , it follows by applying inequality of Lemma 1 [16] that  $D^\alpha V(t) \leq -2[f(t)^T x(t)]^2 \leq 0$ . From Property 10 of [14] we conclude that  $x = 0$  is a stable point at  $t = 0$  and, in particular,  $x$  is bounded function. It follows from Property 11 of [14] that  $x(\cdot)$  is uniformly continuous. In Example 4 in [12], it was proved that uniformly continuous functions  $f_p$  exist such that  $[f_p(t)^T x(t)]^2$  does not converge to zero. However, since  $I^\alpha [f_p(\cdot)^T x(\cdot)]$  is bounded and uniformly continuous, necessarily  $\liminf_{t \rightarrow \infty} [f_p(t)^T x(t)]^2 = 0$ . In the special case of  $\alpha = 1$  one can use Barbalat Lemma (Lemma 9 in [12]) to conclude that  $\lim_{t \rightarrow \infty} [f_p(t)^T x(t)]^2 = 0$ . These facts also follow by applying Theorem 2, whereby we have a corroboration of its claim. In the scalar case, a set that assures convergence is the set of uniformly continuous functions  $f$  such that  $I^\alpha f^2$  diverges, whereby  $x$  converges to zero (Theorem 2 in [14]) or  $f$  converges to zero. In the vector case, we have that  $I^\alpha (f(t)^T x(t))^2$  is bounded and  $f(t)^T x(t)$  is uniformly continuous. A set that assures convergence can be built with functions  $f \in SE(n, \alpha)$ , whereby  $x$  converges to zero (see Theorem 1 in [17]) together with functions  $f$  that converge to zero.

In the following example, we corroborate Theorem 1 by extending the Lyapunov's First Method to fractional order systems.

**Example 4.** If in Proposition 6 we restrict  $w$  to be a quadratic form and imposing that  $V(x, t) \geq x^T x$  we have  $D^\alpha V(x, t) \leq -w(x) \leq -\lambda_m V(x, t)$ , where  $\lambda_m$  is the minimum value of matrix associated to the quadratic form. Therefore, using Theorem 7 of [14], we conclude asymptotic stability at  $t = 0$  of the origin of system (1).

Consider now the following system

$$D^\alpha x = -Ax + g(x), \tag{11}$$

where  $A$  is a positive definite constant matrix, and  $g(x)$  is a convergent power series with terms of degree at least 2 in components of  $x$ . By Theorem 6 of [18],  $x$  is continuously differentiable function for  $t > 0$ . Choosing  $V(x) = x^T x$ , we obtain by applying inequality in [7,16] that  $D^\alpha (x^T x) \leq 2x^T Ax + 2x^T g(x)$ . Then we have  $D^\alpha (x^T x) \leq -2\lambda_m x^T x + p(x)$  where  $p(x)$  is a polynomial with terms of degree at least 3 in  $x$ . For a sufficiently small neighborhood around  $x = 0$ , we can get  $p(x) \leq \lambda_m/2 x^T x$  (it always can be done because  $p(\cdot)$  is a polynomial of degree at least 3). Therefore we get  $D^\alpha (x^T x) \leq -\lambda_m x^T x$ , whereby  $x = 0$  is locally asymptotically stable at  $t = 0$ .

From Theorem 1 we can independently conclude that locally asymptotically stable at  $t = 0$ , since for a sufficiently small neighborhood  $\mathfrak{D}$  of the origin,  $g(x)$  is small enough (by components) in compare with  $Ax$ , whereby  $D^\alpha V$  is negative definite.

We can think of  $x^T Ax + p(x)$  as a Taylor expansion of negative definite function  $-w(\cdot)$  differentiable around  $x = 0$  because in that case matrix  $A$  will be indeed negative definite,  $w(0) = 0$  and  $x = 0$  will be a global maximum. Therefore, we can claim that if  $D^\alpha (x^T x) \leq -w(x)$  for  $-w(\cdot)$  positive definite differentiable around  $x = 0$ , the origin  $x = 0$  is locally asymptotically stable at  $t = 0$ . Alternatively, we can think of  $Ax + g(x)$  as a Taylor expansion of a nonlinear function  $f(x)$  differentiable around  $x = 0$  where  $A = \partial f / \partial x(0)$ ,  $f(0) = 0$  and  $g(x)$  is the rest term. When  $\alpha = 1$ , this is essentially the Lyapunov's Linearization or First Method.

For a concrete example, consider the scalar system  $D^\alpha x = -\lambda x + x^3$  with  $\lambda > 0$ . Therefore for  $x^2 \leq \lambda/2$  we have  $D^\alpha x^2 \leq -\lambda x^2$ , whereby  $x = 0$  is locally asymptotically stable at  $t = 0$ .

If in Proposition 6,  $w(x) \geq a\|x\|^r$  with  $r, a$  positive real numbers with  $r \leq 2$  and  $V = x^T x$ , we can conclude asymptotic stability at  $t = 0$  of the origin of system (1). Indeed, since  $D^\alpha V(x, t) \leq -a\|x\|^r \leq 0$ ,  $V$  is a bounded time function by (say)  $V_0$ . By defining  $V_1 = V/V_0$ , we have  $D^\alpha V_1 \leq -bV_1^{r/2}$ , where  $b = aV_0^{r/2-1} > 0$ . Since  $V_1 \leq 1$ , we have  $D^\alpha V_1 \leq -bV_1^{r/2} \leq -bV_1$  whereby  $V_1, V$  and  $x$  converge to zero, and therefore  $x = 0$  is asymptotic stable at  $t = 0$ . The same argument shows that if  $D^\alpha V(t, x) \leq -\sqrt[n]{V}(x, t)$  for  $n \geq 1$  then  $x$  is asymptotic stable at  $t = 0$ .

The next example shows the application of a Lyapunov function built with no state variables.

**Example 5.** Let the functional dynamical system

$$\dot{x}(t) = -x(t) \int_{t-r}^t \|x\|(\tau) d\tau, \tag{12}$$

with  $x_0 = x_{[-r,0]}$ , with  $r > 0$  a fixed real number. The state variable is  $x_t = x_{[t,t-r]}$  and Lyapunov–Krasovskii functional is the natural choice as a Lyapunov function candidate. However, by choosing  $2V(x(t)) = x(t)^T x(t)$ , it follows that  $\dot{V}(t) = -2V(t) \int_{t-r}^t \|x\|(\tau) d\tau$ . By applying Theorem 2,  $x$  converges to the zeros of  $V(t) \int_{t-r}^t \|x\|(\tau) d\tau$ , whereby  $x$  converges to zero, and  $x_t$  converges to zero. Then  $x_t = 0$  is an asymptotically stable point.

Lastly, we connect the classification of equilibrium points with some practical applications of fractional calculus.

**Example 6.** Consider an autonomous system of type  $D^\alpha x = f(x)$  with  $x(t) \in \mathbb{R}^n$ . By assuming Caputo derivative, if  $x_0$  is an equilibrium point and  $y := x - x_0$  then  $D^\alpha y = f(y + x_0)$  can be expanded around  $x_0$  as  $D^\alpha y = J_f(x_0)y + p_2(y)$  with  $p_2(\cdot)$  a polynomial of degree at least 2 and  $J_f(x_0)$  the Jacobian of  $f$  evaluated at  $x_0$ . By Example 2, it rigorously follows that if  $J_f(x_0) < 0$  then  $x = x_0$  is locally asymptotically stable. In a recent paper [20], using Caputo fractional derivative, an autonomous

system has been proposed to model epidemic propagation of dengue. It was showed that there exists  $E_0 \in \mathbb{R}^5$  such that  $f(E_0) = 0$ . From the proof of Theorem 2 in [20], it follows that  $J_f(E_0) < 0$  for  $R_0 < 1$ .

On the other hand, if  $J_f(x_0) > 0$  (or if  $m$  of its eigenvalues are positives and the other negatives), a similar reasoning as in the Example 2 shows that  $D^\alpha V > 0$  for  $2V = x^T x$  (or choosing  $V = \sum_{i=1}^m \lambda_i x_i^2 + \sum_{i=m+1}^n \lambda_i x_i^2$ , where a coordinate transformation was performed such that  $J_f(x_0)$  is diagonal with its first  $m$  diagonal elements  $\lambda_i > 0$ ). Thus,  $x_0$  is an unstable point by Theorem 3. For instance, the points  $E_1, E_2$  are unstable in [21], where an autonomous fractional system using Caputo derivative was proposed to model love triangle system with competition.

However, if  $J_f(x_0)$  has null or complex eigenvalues, other analysis are required. Null eigenvalues are problematic even in the integer case and it is necessary to consider the nonlinearized system. For complex eigenvalues, the complex diagonalization does not work because Lemma [16] is not extensive to the complex case as to claim  $D^\alpha z\bar{z} \leq zD^\alpha \bar{z}$  (as it could be expected since unstable matrix in integer order can be stable in fractional systems).

## 5. Conclusions

We have set the main concepts and properties which allow consistently to develop a Lyapunov Theory for nonlinear fractional order equations.

Weaker version of Lyapunov and Chetaev theorems were derived in our work using similar assumptions as in the integer order case for fractional system with order  $0 < \alpha \leq 1$  and Caputo derivative. It was additionally required continuous differentiability at time of Lyapunov function to get the same conclusion. LaSalle theorem for fractional system additionally requires a stronger condition to get the same conclusion than its integer order case, however it was given in a synthetic way.

We indicate that the failure of the monotony of Lyapunov function in the fractional order case causes that the proof of previous generalizations of Lyapunov theorem in the literature are not quite correct.

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