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Parabolic Kazhdan–Lusztig polynomials for quasi-minuscule quotients

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ABSTRACT

We study the parabolic Kazhdan–Lusztig polynomials for the quasi-minuscule quotients of Weyl groups. We give explicit closed combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q. Our study implies that these are always either zero or a monic power of q, and that they are not combinatorial invariants. We conjecture a combinatorial interpretation for the parabolic Kazhdan–Lusztig polynomials of type -1.

1. Introduction

In 1979, Kazhdan and Lusztig [16] introduced a family of polynomials, indexed by pairs of elements in a Coxeter group W, which plays an important role in various areas

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of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [1] p. 171 and the references cited there). These celebrated polynomials are now known as the Kazhdan–Lusztig polynomials of W (see, e.g., [1] or [14]). In 1987, Deodhar [7] developed an analogous theory for the parabolic setup. Given any parabolic subgroup W_J in a Coxeter system (W, S), Deodhar introduced two Hecke algebra modules (one for each of the two roots q and -1 of the polynomial $x^2-(q-1)x-q$) and two families of polynomials $\{P^{J,q}_{u,v}(q)\}_{u,v\in W^J}$ and $\{P^{J,-1}_{u,v}(q)\}_{u,v\in W^J}$ indexed by pairs of elements of the set of minimal coset representatives W^J. These polynomials are the parabolic analogues of the Kazhdan–Lusztig polynomials: while they are related to their ordinary counterparts in several ways (see, e.g., §2 and [7], Proposition 3.5), they also play a direct role in several areas such as the geometry of partial flag manifolds [15], the theory of Macdonald polynomials [12,13], tilting modules [24,25], generalized Verma modules [5], canonical bases [10,28], the representation theory of the Lie algebra $\mathfrak{gl}_n$ [20], quantized Schur algebras [29], quantum groups [8], and physics (see, e.g., [11], and the references cited there). The computation of these polynomials is a very difficult task. Although a geometric interpretation for the (ordinary and parabolic) polynomials exists (see [17] and [15]) in the case of Weyl groups and an algebraic interpretation exists for the ordinary ones [9] for all Coxeter systems, there are very few explicit formulas for them (see, e.g., [1], p. 172, and the references cited there).

The purpose of this work is to study the parabolic Kazhdan–Lusztig polynomials for the quasi-minuscule quotients of Weyl groups. These quotients possess noteworthy combinatorial and geometric properties (see, e.g., [18] and [27]). The parabolic Kazhdan–Lusztig polynomials for the minuscule quotients have been computed in [19,2–4]. In this work we turn our attention to the quasi-minuscule quotients that are not minuscule (also known as (co-)adjoint quotients). More precisely, we obtain closed combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q of these quotients for the classical Weyl groups. Our results imply that these are always either zero or a monic power of q for all quasi-minuscule quotients, and that they are not combinatorial invariants. For the parabolic Kazhdan–Lusztig polynomials of type -1 we conjecture explicit combinatorial interpretations.

The organization of the paper is as follows. In Section 2 we recall some definitions, notation and results that are used in the sequel. In Section 3 we give combinatorial descriptions of the quasi-minuscule quotients of classical Weyl groups. In Section 4 we give combinatorial formulas for the parabolic Kazhdan–Lusztig polynomials of type q of (co-)adjoint quotients of classical Weyl groups. Our results imply that these polynomials are always either zero or a monic power of q for all quasi-minuscule quotients, and that they are not combinatorial invariants. In Section 5 we derive some consequences of our results for the classical Kazhdan–Lusztig polynomials. Finally, in Section 6 we present our conjectured combinatorial interpretations for the parabolic Kazhdan–Lusztig polynomials of type -1 of the (co-)adjoint quotients of classical Weyl groups, and the evidence that we have in their favor.

2. Preliminaries

In this section we collect some definitions, notation and results that are used in the rest of this work. We let $\mathbb{P} := \{1, 2, 3, \ldots\}$ and $\mathbb{N} := \mathbb{P} \cup \{0\}$. The cardinality of a set A will be denoted by $|A|$. For $n \in \mathbb{P}$ we let $[n] \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ and $[\pm n] \stackrel{\text{def}}{=} \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$.

We follow [26], Chapter 3, for poset notation and terminology. In particular, given a poset $(P, \leq)$ and $u, v \in P$ we let $[u, v] := \{w \in P | u \leq w \leq v\}$ and call this an *interval* of P. We say that v *covers* u, denoted $u \lhd v$ (or, equivalently, that u is *covered* by v) if $|[u, v]| = 2$. The *Hasse diagram* of P is the graph having P as vertex set and $\{\{u, v\} \subseteq P | u \lhd v \text{ or } v \lhd u\}$ as edge set. Usually, when drawing Hasse diagrams, if $u \leq v$ then u is depicted below v, however in this work we find it convenient to rotate our diagrams clockwise by $\frac{\pi}{4}$. We say that $u, v \in P$ are *comparable* if either $u \leq v$ or $v \leq u$. Given two posets P and Q, we write $P \simeq Q$ to mean that they are isomorphic as posets.

We follow [1] and [14] for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $u \in W$ we denote by $\ell(u)$ the length of u in W, with respect to S, and we define $\ell(u, v) \stackrel{\text{def}}{=} \ell(v) - \ell(u)$. If $s_1, \ldots, s_r \in S$ are such that $u = s_1 \cdots s_r$ and $r = \ell(u)$ then we call $s_1 \cdots s_r$ a reduced word for u. We let $D(u) := \{s \in S : \ell(us) < \ell(u)\}$ be the set of (right) descents of u and we denote by e the identity of W. Given $J \subseteq S$ we let W_J be the parabolic subgroup generated by J and

$$W^J \stackrel{\text{def}}{=} \{u \in W : \ell(su) > \ell(u) \text{ for all } s \in J\}.$$

Note that $W^{\emptyset} = W$. If W_J is finite, then we denote by $w_0(J)$ its longest element. We always assume that W^J is partially ordered by *Bruhat order*. Recall (see e.g. [14], §5.9 and 5.10) that this means that $x \leq y$ if and only if for one reduced word of y (equivalently for all) there exists a subword that is a reduced word for x. Given $u, v \in W^J$, $u \leq v$ we let

$$[u, v]^J := \{w \in W^J : u \leq w \leq v\},$$

and $[u, v] \stackrel{\text{def}}{=} [u, v]^{\emptyset}$.

The following two results are due to Deodhar, and we refer the reader to [7, §§2–3] for their proofs.

Theorem 1. *Let (W, S) be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\{R^{J,x}_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbb{Z}[q]$ such that, for all $u, v \in W^J$:*

i) $R^{J,x}_{u,v}(q) = 0$ *if $u \not\leq v$;*
ii) $R^{J,x}_{u,u}(q) = 1$*;*
iii) *if $u < v$ and $s \in D(v)$ then*

$$R^{J,x}_{u,v}(q) = \begin{cases} R^{J,x}_{us,vs}(q), & \text{if } us < u, \\ (q-1)R^{J,x}_{u,vs}(q) + qR^{J,x}_{us,vs}(q), & \text{if } u < us \in W^J, \\ (q-1-x)R^{J,x}_{u,vs}(q), & \text{if } u < us \notin W^J. \end{cases}$$

Theorem 2. *Let (W, S) be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\{P^{J,x}_{u,v}(q)\}_{u,v\in W^J} \subseteq \mathbb{Z}[q]$, such that, for all $u, v \in W^J$:*

i) $P^{J,x}_{u,v}(q) = 0$ *if* $u \not\leq v$;
ii) $P^{J,x}_{u,u}(q) = 1$;
iii) $deg(P^{J,x}_{u,v}(q)) < \frac{1}{2}\ell(u, v)$ *if* $u < v$;
iv)

$$q^{\ell(u,v)}\, P^{J,x}_{u,v}\left(\frac{1}{q}\right) = \sum_{z\in[u,v]^J} R^{J,x}_{u,z}(q)\, P^{J,x}_{z,v}(q)$$

if $u \leq v$.

The polynomials $R^{J,x}_{u,v}(q)$ and $P^{J,x}_{u,v}(q)$, whose existence is guaranteed by the two previous theorems, are called the *parabolic R-polynomials* and *parabolic Kazhdan–Lusztig polynomials* (respectively) of W^J of type x. It follows immediately from Theorems 1 and 2 and from well known facts (see, e.g., [14, §7.5] and [14, §§7.9–11]) that $R^{\emptyset,-1}_{u,v}(q)$ $(= R^{\emptyset,q}_{u,v}(q))$ and $P^{\emptyset,-1}_{u,v}(q)$ $(= P^{\emptyset,q}_{u,v}(q))$ are the (ordinary) *R-polynomials* and *Kazhdan–Lusztig polynomials* of W which we will denote simply by $R_{u,v}(q)$ and $P_{u,v}(q)$, as customary.

The parabolic Kazhdan–Lusztig and R-polynomials are related to their ordinary counterparts in several ways, including the following one.

Proposition 1. *Let (W, S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J$. Then we have that*

$$R^{J,x}_{u,v}(q) = \sum_{w\in W_J} (-x)^{\ell(w)} R_{wu,v}(q),$$

for all $x \in \{-1, q\}$, and

$$P^{J,q}_{u,v}(q) = \sum_{w\in W_J} (-1)^{\ell(w)} P_{wu,v}(q).$$

Furthermore, if W_J is finite then

$$P^{J,-1}_{u,v}(q) = P_{w_0(J)u,w_0(J)v}(q).$$

A proof of this result can be found in [7] (see Propositions 2.12 and 3.4, and Remark 3.8).

Note that it follows easily from Theorem 2, Proposition 1, and well known facts (see, e.g., [1, Proposition 2.4.4]), that for all $J \subseteq S$ and all $u, v \in W^J$ we have that

$$[q^{\frac{1}{2}(\ell(u,v)-1)}](P^{J,q}_{u,v}) = [q^{\frac{1}{2}(\ell(u,v)-1)}](P_{u,v}). \tag{1}$$

We denote this coefficient by $\mu(u, v)$, as customary. The following result is due to Deodhar, and we refer the reader to [7] for its proof.

Proposition 2. *Let (W, S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J$, $u \leq v$. Then for each $s \in D(v)$ we have that*

$$P^{J,q}_{u,v}(q) = \widetilde{P}_{u,v} - \widetilde{M}_{u,v} \tag{2}$$

where

$$\widetilde{P}_{u,v} = \begin{cases} P^{J,q}_{us,vs} + qP^{J,q}_{u,vs} & \text{if } us < u, \\ qP^{J,q}_{us,vs} + P^{J,q}_{u,vs} & \text{if } u < us \in W^J, \\ 0 & \text{if } u < us \notin W^J, \end{cases}$$

and

$$\widetilde{M}_{u,v} = \sum_{\{u \leq w < vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w,v)}{2}} P^{J,q}_{u,w}(q).$$

The following properties of the parabolic Kazhdan–Lusztig polynomials are certainly known, however, for lack of an adequate reference, and for completeness, we include their proof.

Proposition 3. *Let (W, S) a Coxeter system and $J \subseteq S$. Let $u, v \in W^J$ and $s \in D(v)$. Then*

a) *if $us \notin W^J$ then $P^{J,q}_{u,v}(q) = 0$;*
b) *if $us \in W^J$ then $P^{J,q}_{us,v}(q) = P^{J,q}_{u,v}(q)$;*
c) *if $\mu(u, v) \neq 0$ and $\ell(u, v) > 1$ then $D(v) \subseteq D(u)$.*

Proof. If $us \notin W^J$ then by Proposition 2 we have that

$$P^{J,q}_{u,v}(q) = -\sum_{\{u \leq w < vs : ws < w\}} \mu(w, vs) q^{\frac{\ell(w,v)}{2}} P^{J,q}_{u,w}(q).$$

The sum may be empty or we can apply induction on $\ell(u, v)$ and have $P^{J,q}_{u,w}(q) = 0$. In both cases $P^{J,q}_{u,v}(q) = 0$. For *b*) use the same arguments as in the proof of Proposition 5.1.8 of [1]. For *c*) use *a*) and *b*) together and the fact that $P^{J,q}_{u,v}(q)$ has maximal degree. □

The purpose of this work is to study the parabolic Kazhdan–Lusztig polynomials for the quasi-minuscule quotients of Weyl groups. The parabolic Kazhdan–Lusztig polynomials for the minuscule quotients have been computed in [19,2–4], (see also [23] and [21]). In this work we study the quasi-minuscule quotients that are not minuscule. These quotients (also known as (co-)adjoint quotients) have been classified (see, e.g., [6]) and there are three infinite families and four exceptional ones. Using the standard notation for the classification of the finite Coxeter systems, the non-trivial (co-)adjoint quotients are: $(A_n, S \setminus \{s_1, s_n\})$, $(B_n, S \setminus \{s_{n-2}\})$, $(D_n, S \setminus \{s_{n-2}\})$, $(E_6, S \setminus \{s_0\})$, $(E_7, S \setminus \{s_1\})$, $(E_8, S \setminus \{s_7\})$, and $(F_4, S \setminus \{s_4\})$, where we number the generators as in [1] (see Appendix A1 and Exercises 20, 21, 22, 23 in Chapter 8, and also below). The following result is probably known. Its verification follows from the above classification and standard facts. Given a Weyl group W we denote by $\Phi(W)$ its root system and by $\Phi_\ell(W)$ its set of long roots (see, e.g., [14, §2.10]) where, if W is of type B_n, we mean the root system of type B_n.

Proposition 4. *Let (W, S) be a Weyl group and $J \subseteq S$ be such that (W, J) is a (co-)adjoint quotient. Then $|W^J| = |\Phi_\ell(W)|$.*

It is well known (see, e.g., [1, Chap. 1]) that the symmetric group S_n is a Coxeter group with respect to the generating set $S = \{s_1^A, \dots, s_{n-1}^A\}$ where $s_i^A = (i, i+1)$ for all $i \in [n-1]$. The following result is also well known (see, e.g., [1, §1.5]).

Proposition 5. *Let $v \in S_n$. Then $\ell(v) = |\{(i,j) \in [n]^2 : i < j, v(i) > v(j)\}|$ and $D(v) = \{(i, i+1) \in S : v(i) > v(i+1)\}$.*

For $k \in [n]$ and $U, T \subseteq [n]$ such that $|U| = |T| = k$ let $U \preceq T$ if and only if $u_i \leq t_i$ for all $i \in [k]$ where $\{u_1, \dots, u_k\}_< \stackrel{\mathrm{def}}{=} U$ and $\{t_1, \dots, t_k\}_< \stackrel{\mathrm{def}}{=} T$. Note that $U \preceq T$ if and only if

$$|\{j \in U : j \geq r\}| \leq |\{j \in T : j \geq r\}| \tag{3}$$

for all $r \in [n]$. In particular, $U \preceq T$ if and only if $[n] \setminus T \preceq [n] \setminus U$. The following result is well known (see, e.g., [1, Theorem 2.6.3]).

Theorem 3. *Let $u, v \in S_n$. Then the following are equivalent:*

i) $u \leq v$;
ii) $u([j]) \preceq v([j])$ *for all* $j \in [n-1]$;
iii) $u([j]) \preceq v([j])$ *for all* j *such that* $s_j^A \in D(u)$.

We follow [1, Chap. 8] for combinatorial descriptions of the Coxeter systems of type B_n and D_n as permutation groups. In particular, we let S_n^B be the group of all bijections w of $\{-n, \dots, -1, 1, \dots, n\}$ to itself such that $w(-i) = -w(i)$ for all $i \in [n]$, $s_j \stackrel{\mathrm{def}}{=}$

$(j, j+1)(-j, -j-1)$ for $j = 1, \ldots, n-1$, $s_0 \stackrel{\text{def}}{=} (1, -1)$, and $S_B \stackrel{\text{def}}{=} \{s_0, \ldots, s_{n-1}\}$. If $v \in S_n^B$ then we write $v = [a_1, \ldots, a_n]$ to mean that $v(i) = a_i$, for $i = 1, \ldots, n$. It is well known that (S_n^B, S_B) is a Coxeter system of type B_n and that the following holds. Given $v \in S_n^B$ we let

$$inv(v) \stackrel{\text{def}}{=} |\{(i, j) \in [n]^2 : \; i < j, \; v(i) > v(j)\}|,$$

$N_1(v) \stackrel{\text{def}}{=} |\{i \in [n] : \; v(i) < 0\}|$ and

$$N_2(v) \stackrel{\text{def}}{=} |\{(i, j) \in [n]^2 : \; i < j, \; v(i) + v(j) < 0\}|.$$

Proposition 6. *Let* $v \in S_n^B$. *Then* $\ell(v) = inv(v) - \sum_{\{j \in [n] : \; v(j) < 0\}} v(j)$, *and* $D(v) = \{s_i \in S_B : \; v(i) > v(i+1)\}$, *where* $v(0) \stackrel{\text{def}}{=} 0$.

We let S_n^D be the subgroup of S_n^B defined by

$$S_n^D \stackrel{\text{def}}{=} \{w \in S_n^B : \; N_1(w) \equiv 0 \pmod 2\}, \tag{4}$$

$\tilde{s}_0 \stackrel{\text{def}}{=} (1, -2)(2, -1)$, and $S_D \stackrel{\text{def}}{=} \{\tilde{s}_0, s_1, \ldots, s_{n-1}\}$. It is then well known that (S_n^D, S_D) is a Coxeter system of type D_n, and that the following holds (see, e.g., [1, §8.2]).

Proposition 7. *Let* $v \in S_n^D$. *Then* $\ell(v) = inv(v) + N_2(v)$, *and* $D(v) = \{s_i \in S_D : \; v(i) > v(i+1)\}$, *where* $v(0) \stackrel{\text{def}}{=} -v(2)$.

Given $w \in S_n^B$ and $i \in [n]$, define an array $A(w)_i := (A(w)_{i,1}, \ldots, A(w)_{i,n+1-i})$ by letting

$$\{A(w)_{i,1}, \ldots, A(w)_{i,n+1-i}\}_< \stackrel{\text{def}}{=} \{k \in [\pm n] : w(k) \geq i\}_<.$$

For the following criterion see Exercise 6.8 in [1].

Proposition 8. *For* $u, v \in S_n^B$ *the following are equivalent:*

(1) $u \leq v$.
(2) $A(u)_{i,j} \geq A(v)_{i,j}$, *for all* $i \in [n]$, $j \in [n+1-i]$.

Say that two vectors $(a_1, \ldots, a_k), (b_1, \ldots, b_k) \in \mathbb{Z}^k$ are D-compatible if the following condition is satisfied:

$$\text{if } \{|a_i|, \ldots, |a_j|\} = \{|b_i|, \ldots, |b_j|\} = [j-i+1] \text{ for some } 1 \leq i \leq j \leq k, \text{ then}$$
$$N_1(a_i, \ldots, a_j) \equiv N_1(b_i, \ldots, b_j) \mod 2.$$

For the following criterion see Exercise 11.8 in [1].

Proposition 9. *For $u, v \in S_n^D$ the following are equivalent:*

(1) $u \leq v$.
(2) $A(u)_{i,j} \geq A(v)_{i,j}$, *for all* $i \in [n]$, $j \in [n+1-i]$, *and the two vectors* $A(u)_i$ *and* $A(v)_i$ *are* D*-compatible for all* $i \in [n]$.

3. Co-adjoint quotients

In this section we describe combinatorially the (co-)adjoint quotients of types A, B, and D. More precisely, we describe combinatorially their elements, length, descent sets, and Bruhat order.

Let $v \in S_n^{[2,n-2]}$. Then by Proposition 5 we have that

$$S_n^{[2,n-2]} = \{v \in S_n : \ v^{-1}(2) < \cdots < v^{-1}(n-1)\}.$$

Hence the map $v \mapsto (v^{-1}(1), v^{-1}(n))$ is a bijection between $S_n^{[2,n-2]}$ and $\{(i,j) \in [n]^2 : i \neq j\}$. For this reason we will freely identify these two sets and write $v = (i,j)$ if $v \in S_n^{[2,n-2]}$ and $i = v^{-1}(1)$, $j = v^{-1}(n)$.

Proposition 10. *Let* $(a,b), (i,j) \in S_n^{[2,n-2]}$*. Then* $(a,b) \leq (i,j)$ *if and only if* $a \leq i$ *and* $b \geq j$*. Furthermore* $\ell((a,b)) = a - b + n - 1 - \chi(a > b)$.

Proof. It is well known (see, e.g., [1, Cor. 2.2.5]) that $u \leq v$ if and only if $u^{-1} \leq v^{-1}$. Therefore we conclude from (3) and Theorem 3 that $u \leq v$ if and only if $u^{-1}([1]) \preceq v^{-1}([1])$ and $u^{-1}([n-1]) \preceq v^{-1}([n-1])$. The result follows. $\square$

Let $v \in B_n^{(n-2)}$. Then, by Proposition 6, we have that

$$B_n^{(n-2)} = \{v \in B_n : \ 0 < v^{-1}(1) < \cdots < v^{-1}(n-2), v^{-1}(n-1) < v^{-1}(n)\}.$$

Hence the map $v \mapsto (v^{-1}(n-1), v^{-1}(n))$ is a bijection between $B_n^{(n-2)}$ and $\{(i,j) \in [\pm n]^2 : \ i < j, \ i \neq -j\}$. For this reason we will freely identify these two sets and write $v = (i,j)$ if $v \in B_n^{(n-2)}$ and $i = v^{-1}(n-1)$, $j = v^{-1}(n)$.

Lemma 1. *Let* $u \in B_n^{(n-2)}$, $u = (a,b)$, *and* $i \in [0, n-1]$*. Then* $s_i \in D(u)$ *if and only if*

$$i \in \{a, b, -a-1, -b-1\} \setminus \{b-1, -b\}.$$

Proof. Let $s_i \in D(u)$. Suppose $i > 0$. Then, since $u \in B_n^{(n-2)}$, either $u(i) \geq n-1$ or $u(i+1) \leq -(n-1)$. Therefore, either $i \in \{a, b\}$ or $i+1 \in \{-a, -b\}$. Furthermore, $i \neq b-1$ (else $u(i) < n = u(b) = u(i+1)$, which is a contradiction) and $i \neq -b$ (else $u(i) = u(-b) = -n < u(i+1)$).

If $i = 0$ then $u(-1) > 0 > u(1)$ so, since $u \in B_n^{(n-2)}$, either $u(1) = -n$ or $u(1) = -(n-1)$. If $u(1) = -n$ then $-b = 1$ so $0 \in \{a, b, -a-1, -b-1\} \setminus \{b-1, -b\}$. If $u(1) = -(n-1)$ then $a = -1$ and we conclude as above.

Conversely, let $i \in \{a, b, -a-1, -b-1\} \setminus \{b-1, -b\}$. Suppose $i = a$. Then $u(i) = n-1$ and $u(i+1) \neq n$ (else $i+1 = b$), so $s_i \in D(v)$. If $i = b$ then $u(i) = n > u(i+1)$ so $s_i \in D(v)$. If $i = -a-1$ then $u(i+1) = -(n-1)$ but $i \neq -b$ so $u(i) > -n$ and $s_i \in D(v)$. If $i = -b-1$ then $u(i+1) = -n$ so $s_i \in D(v)$.

If $i = 0$ then either $a = -1$ or $b = -1$. In both cases $u(1) < 0$ so $s_0 \in D(u)$. □

Proposition 11. *Let $(a, b), (i, j) \in B_n^{(n-2)}$. Then $(a, b) \leq (i, j)$ if and only if $a \geq i$ and $b \geq j$. Furthermore, $\ell((a, b)) = 2n - 1 - a - b - N_1(a, b, a + b)$.*

Proof. Let $u = (a, b)$ and $v = (i, j)$. It is well known (see, e.g., [1, Cor. 8.1.9]) that $u \leq v$ in S_n^B if and only if $u \leq v$ in $S([\pm n])$. This, in turn, happens if and only if $u^{-1} \leq v^{-1}$ in $S([\pm n])$ (see, e.g., [1]). Hence, by Theorem 3, we conclude that $u \leq v$ in S_n^B if and only if $u^{-1}(\{-n, -(n-1)\}) \preceq v^{-1}(\{-n, -(n-1)\})$ and $u^{-1}(\{n-1, n\}) \succeq v^{-1}(\{n-1, n\})$ and the result follows. The second statement is a routine verification using Proposition 6. □

Let $v \in (D_n)^{(n-2)}$. Then, by Proposition 7, we have that

$$(D_n)^{(n-2)} = \{v \in D_n : \ v^{-1}(-2) < v^{-1}(1) < \ldots < v^{-1}(n-2), \ v^{-1}(n-1) < v^{-1}(n)\}.$$

Hence, if $v \in D_n^{(n-2)}$, then $v^{-1}(-1) < v^{-1}(2)$ and $v^{-1}(-2) < v^{-1}(2)$ so $0 < v^{-1}(2) < v^{-1}(3) < \ldots < v^{-1}(n-2)$ and $v^{-1}(-2) < v^{-1}(1)$, $v^{-1}(-1) < v^{-1}(2)$. Since $N_1(v) \equiv 0 \pmod 2$ for all $v \in S_n^D$ we conclude that the map $v \mapsto (v^{-1}(n-1), v^{-1}(n))$ is a bijection between $(D_n)^{(n-2)}$ and $\{(i, j) \in [\pm n]^2 : \ i < j, \ i \neq -j\}$.

Lemma 2. *Let $u \in D_n^{(n-2)}$, $u = (a, b)$, and $i \in [n-1]$. Then $s_i \in D(u)$ if and only if*

$$i \in \{a, b, -a-1, -b-1\} \setminus \{b-1, -b\}.$$

Furthermore, $\tilde{s}_0 \in D(u)$ if and only if

$$0 \in \{-a-2, -b-2, -a-1, -b-1\} \setminus \{a-1, b-1, a-2, b-2\}.$$

Proof. The first formula follows exactly as in the proof of Lemma 1.

Suppose now that $\tilde{s}_0 \in D(u)$. Then, by Proposition 7, $u(1) + u(2) < 0$. Therefore, since $u \in D_n^{(n-2)}$, either $u(1) \in \{-n, -n+1\}$ or $u(2) \in \{-n, -n+1\}$. So $1 \in \{-b, -a, -b-1, -a-1\}$. On the other hand $1 \notin \{b, a, b-1, a-1\}$. In fact, if $a = 1$ then $u(1) = n-1$ and $b > 1$ so $-b < -1$ hence $u(2) > -n$ so $u(1) + u(2) > 0$, which contradicts our assumption. Similarly, if $1 = a - 1$ then $u(2) = n - 1$ so $b > 2$ and we conclude as above. Furthermore, if $b \in \{1,2\}$ then necessarily $u(1)+u(2) > 0$ which is a contradiction.

Conversely, suppose that $1 \in \{-a-1, -b-1, -a, -b\} \setminus \{a, b, a-1, b-1\}$. If $b \in \{-1,-2\}$ then $-n \in \{u(1), u(2)\}$ so $u(1)+u(2) < 0$ and the result follows from Proposition 7. If $1 = -a$ then $b \neq 2$ so $u(1) = -(n-1)$ and $u(2) < n$ hence $\tilde{s}_0 \in D(u)$ by Proposition 7, as desired. Similarly if $1 = -a-1$. $\square$

Proposition 12. *Let $u, v \in D_n^{(n-2)}$, $u = (a,b)$, $v = (i,j)$. Then $(a,b) \leq (i,j)$ if and only if $a \geq i$, $b \geq j$, $a = 1$ implies $i \neq -1$, $b = 1$ implies $j \neq -1$, $(a,b) = (1,2)$ implies $(i,j) \neq (-2,1)$, and $(a,b) = (-1,2)$ implies $(i,j) \neq (-2,-1)$. Furthermore, $\ell((a,b)) = 2n-1-a-b-2N_1(a,b)-N_1(a+b)$.*

Proof. Let $A(u)_k$ and $A(v)_k$ as in Proposition 9. Then $A(u)_{n-1} = (a,b)$, $A(u)_n = (b)$, $A(v)_{n-1} = (i,j)$ and $A(v)_n = (j)$.

Suppose first that $u \leq v$. This, by Proposition 9, implies that $a \geq i$, $b \geq j$ and the two vectors (a,b) and (i,j) are D-compatible, as are the vectors (b) and (j). But the pairs $\{(1,b), (-1,j)\}$, $\{(a,1), (i,-1)\}$, $\{(1,2), (-2,1)\}$ and $\{(-1,2), (-2,-1)\}$ are not D-compatible, which proves one direction of our statement.

We now prove the other implication. Let $u, v \in D_n^{(n-2)}$, $u = (a,b)$, $v = (i,j)$, be such that $a \geq i$, $b \geq j$, $a = 1$ implies $i \neq -1$, $b = 1$ implies $j \neq -1$, $(a,b) = (1,2)$ implies $(i,j) \neq (-2,1)$, and $(a,b) = (-1,2)$ implies $(i,j) \neq (-2,-1)$. We prove that $u \leq v$ by induction on $\ell(v)$.

If $\ell(v) = 0$ then there is nothing to prove, since then $v = (n-1,n)$. So suppose that $\ell(v) > 0$. In $D(v)$, let s_k be the element with the greatest index k. If $s_k \in D(u)$ then one can check, using Lemma 2, that us_k and vs_k still satisfy our hypotheses. Then by induction $us_k \leq vs_k$ and so $u \leq v$.

We may therefore assume that $s_k \notin D(u)$. We claim that $a = 1$ implies $(vs_k)^{-1}(n-1) \neq -1$, that $b = 1$ implies $(vs_k)^{-1}(n) \neq -1$, that $u = (1,2)$ implies $vs_k \neq (-2,1)$, and that $u = (-1,2)$ implies $vs_k \neq (-2,-1)$.

Note first that if $k > 2$ then $vs_k(1) = v(1)$ and $vs_k(2) = v(2)$ so our claim coincides with our hypotheses.

Suppose first that $k = 0$. Then, by Proposition 7, $v(1)+v(2) < 0$, $0 > v(1) < v(2) < \cdots < v(n)$ and $u(1)+u(2) > 0$. Suppose first that $a = 1$ and, by contradiction, that $(v\tilde{s}_0)^{-1}(n-1) = -1$. Then $i = 2$ which is a contradiction. Similarly, if $b = 1$ and $(v\tilde{s}_0)^{-1}(n) = -1$ then $j = 2$ which is also impossible. If $(a,b) = (1,2)$ and, by contradiction, $v\tilde{s}_0 = (-2,1)$ then $v^{-1}(n) = -2 < 1 = v^{-1}(n-1)$ which contradicts the fact that $v \in D_n^{(n-2)}$. Finally, if $u = (-1,2)$ and, by contradiction, $v\tilde{s}_0 = (-2,-1)$ then $i = 1$ and $-1 = a$ which is again a contradiction.

Suppose now that $k = 1$. Then, by Proposition 7, $v(1) > v(2) < v(3) < \cdots < v(n)$ and $u(1) < u(2)$. Note that this implies that $u^{-1}(n) \neq 1$. If $a = 1$ and, by contradiction $(vs_1)^{-1}(n-1) = -1$ then $i = -2$ and $u(2) = n$ so $u = (1,2)$. Hence $-2 < j \leq 2$ and this, since $v(1) > v(2)$, implies that $j = 1$. So $v = (-2,1)$ which is a contradiction. If

$u = (1, 2)$ and, by contradiction, $vs_1 = (-2, 1)$, then $v = (-1, 2)$ which is a contradiction because $v(1) > v(2)$. Finally, if $u = (-1, 2)$ and $vs_1 = (-2, -1)$ then $v^{-1}(n) = -2 < -1 = v^{-1}(n-1)$ which again contradicts the fact that $v \in D_n^{(n-2)}$.

Finally, assume that $k = 2$. Note that this implies that $u^{-1}(n) \neq 2$, so $u \neq (1, 2)$ and $u \neq (-1, 2)$. Also, if $a = 1$ and $(vs_2)^{-1}(n-1) = -1$ then $i = -1$ which is a contradiction, and similarly if $b = 1$.

This proves our claim. Therefore we conclude by induction that $u \leq vs_k$ and hence that $u \leq v$, as desired.

The second statement is a routine verification using Proposition 7. $\square$

By Propositions 10 and 11 the map $(i, j) \mapsto (-i, j)$ from $S_n^{[2,n-2]}$ to $\{(i, j) \in B_n^{(n-2)} : i < 0 < j\}$ is a poset isomorphism, so $S_n^{[2,n-2]} \cong [(-1, n), (-n, 1)]^{(n-2)}$. In fact, this map also preserves the corresponding parabolic Kazhdan–Lusztig and R-polynomials, as we now show. For $w = (i, j) \in S_n^{[2,n-2]}$ let $\tilde{w} \stackrel{\text{def}}{=} (-i, j) \in B_n^{(n-2)}$.

Proposition 13. *Let $u, v \in S_n^{[2,n-2]}$. Then*

$$R_{u,v}^{[2,n-2],x} = R_{\tilde{u},\tilde{v}}^{(n-2),x}$$

and

$$P_{u,v}^{[2,n-2],x} = P_{\tilde{u},\tilde{v}}^{(n-2),x},$$

for all $x \in \{-1, q\}$.

Proof. The first equation follows immediately from Theorem 1, by induction on $\ell(v) \geq 0$, using the fact that, if $w \in S_n^{[2,n-2]}$ and $k \in [n-1]$, then $ws_k^A < w$ if and only if $\tilde{w}s_k < \tilde{w}$ and $w < ws_k^A \in S_n^{[2,n-2]}$ if and only if $\tilde{w} < \tilde{w}s_k \in B_n^{(n-2)}$, and that, in these cases, $\widetilde{ws_k^A} = \tilde{w}s_k$.

The second statement follows immediately from the first one, by induction on $\ell(u, v)$, using Theorem 2 and the fact that the map $w \mapsto \tilde{w}$ is a poset isomorphism from $S_n^{[2,n-2]}$ to $[(-1, n), (-n, 1)]^{(n-2)}$. $\square$

The result of the previous Proposition for the parabolic Kazhdan–Lusztig polynomials also follows from Corollary 5.15 in [22].

4. Parabolic Kazhdan–Lusztig polynomials

In this section we prove the main result of this work. Namely that the parabolic Kazhdan–Lusztig polynomials of type q are always either zero or a monic power of q for all quasi-minuscule quotients of all Weyl groups.

We begin with the following preliminary result which follows easily from Proposition 2, and whose verification is therefore omitted.

Lemma 3. *Let $u, v \in B_n^{(n-2)}$, $u \le v$, $u = (a, b)$, $v = (i, j)$. Suppose that either $a = i$ or $b = j$. Then*

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \text{if } \ell(u,v) \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $v \in B_n^{(n-2)}$, $v = (i, j)$. Suppose that $j - i > 1$. We let

$$Q[v] \stackrel{\text{def}}{=} \begin{cases} \{(i,j),(i,j+2),(i+1,j+1),(i+1,j+2), & \\ \quad (i+2,j),(i+2,j+1)\}, & \text{if } i+j=-1, \\ \{-1,1\} \times \{j,j+1\}, & \text{if } i=-1, \\ \{i,i+1\} \times \{-1,1\}, & \text{if } j=-1, \\ \{i,i+1\} \times \{j,j+1\}, & \text{otherwise.} \end{cases}$$

Suppose now that $j - i = 1$. We then let

$$C_R[v] \stackrel{\text{def}}{=} \begin{cases} \{(-2,-1),(-2,1)\}, & \text{if } j=-1, \\ \{(i,j),(i,j+1)\}, & \text{otherwise,} \end{cases}$$

and

$$C_L[v] \stackrel{\text{def}}{=} \begin{cases} \{(-1,3),(1,3)\}, & \text{if } j=-1, \\ \{(-2,1),(-1,2)\}, & \text{if } j=-2, \\ \{(i+1,j+2),(i+2,j+2)\}, & \text{otherwise.} \end{cases}$$

Note that $Q[v], C_L[v], C_R[v] \subseteq [-n, n+1] \times [-n+1, n+2]$.

We further define $Q^*(v) \stackrel{\text{def}}{=} \{v^* : v \in Q[v]\}$ and similarly for $C_L^*(v)$, and $C_R^*(v)$ (recall that $u^* = (-b, -a)$ if $u = (a, b)$). Note that, if $u \in Q^*[v]$ (resp. $C_L^*(v)$, $C_R^*(v)$) and $u < v$, then $\ell(v) > 2(n-1)$ (respectively, $2n-1$, $2(n-1)$).

We can now state and prove the first main result of this section.

Theorem 4. *Let $u, v \in B_n^{(n-2)}$, $u \le v$, $v = (i, j)$. Then*

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \text{if } u \in Q[v], \\ q, & \text{if } u \in \{(1,3),(2,3)\}, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

if $v = (-2, 1)$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \text{if } u \in C_R[v], \\ q, & \text{if } u \in C_L[v], \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

if $v = (-2, -1)$,

$$P^{(n-2),q}_{u,v} = \begin{cases} 1, & \text{if } u \in C_R[v], \\ q, & \text{if } u \in C_L[v], \\ q^{d_v - 1}, & \text{if } u \in C^*_R[v], \\ q^{d_v - 2}, & \text{if } u \in C^*_L[v], \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

if $j - i = 1$ *and* $j \neq -1$, *while*

$$P^{(n-2),q}_{u,v} = \begin{cases} 1, & \text{if } u \in Q[v], \\ q^{d_v - 1}, & \text{if } u \in Q^*[v], \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

if $j - i > 1$ *and* $v \neq (-2, 1)$, *where* $d_v \stackrel{\text{def}}{=} \ell(v) - 2(n-1)$.

Proof. Let, for simplicity, $P_{x,y} \stackrel{\text{def}}{=} P^{(n-2),q}_{x,y}$ for all $x, y \in B^{(n-2)}_n$, and, for convenience $\tilde{P}_{u,v}$ be the polynomial defined, for all $u, v \in B^{(n-2)}_n$, by the right hand sides of equations (8) through (5). So we have to prove that $P_{u,v} = \tilde{P}_{u,v}$ for all $u, v \in B^{(n-2)}_n$.

Note first that it is a routine verification to check that for all $u, v \in B^{(n-2)}_n$ and all $s \in D(v)$ we have that

$$\tilde{P}_{u,v} = \begin{cases} 0, & \text{if } us \notin B^{(n-2)}_n, \\ \tilde{P}_{us,v}, & \text{otherwise}. \end{cases} \tag{9}$$

(Indeed, it suffices to check that if $s \in D(v)$ then u is in any one of the sets on the right hand sides of equations (8) through (5) if and only if us is in the same set.)

We proceed by induction on $\ell(v)$. The result being clear if $v = e$. Fix $v \in B^{(n-2)}_n$ such that $\ell(v) > 1$. Let $u \in B^{(n-2)}_n$. We may clearly assume that $u \leq v$. We prove the claim by induction on $\ell(u, v)$, the result being clear if $\ell(u, v) = 0$ (i.e., if $u = v$). So assume that $u < v$.

Assume first that $D(v) \nsubseteq D(u)$. Let $s \in D(v) \setminus D(u)$. If $us \in B^{(n-2)}_n$ then, by Proposition 3, (9), and our induction hypothesis, we have that $P_{u,v} = P_{us,v} = \tilde{P}_{us,v} = \tilde{P}_{u,v}$, as desired. If $us \notin B^{(n-2)}_n$ then by (9) and Proposition 3 we have similarly that $P_{u,v} = 0 = \tilde{P}_{u,v}$, and the result again follows.

We may therefore assume that $D(v) \subseteq D(u)$.

Note that, by Lemma 1, $|D(v)| = 2$ if and only if $j - i > 1$, $j < n$, and $i + j \neq -1$. Let v be such that $|D(v)| = 2$. Then $j - i > 1$, $j < n$, and $i + j \neq -1$. Let $u \in B^{(n-2)}_n$, $u < v$ be such that $D(v) \subseteq D(u)$ (note that then necessarily $D(v) = D(u)$). Then, by Proposition 6 and Lemma 1, $i < -1$ and $i + j \neq -2$ and $u \in \{(-i-1, j)\}$ if $i + j > 0$ while $u \in \{(-j-1, -i-1), (j, -i-1), (i, -j-1)\}$ if $i + j \leq -3$. We distinguish these cases.

1) $u = (-i-1, j)$, $i + j > 0$.

Then, since $i \leq -2$, by Lemma 3 and our definitions, we have that $P_{u,v} = 0 = \tilde{P}_{u,v}$, as desired.

2) $u = (-j-1, -i-1)$, $i + j \leq -3$.

Then $j \neq -1$. Let $s \stackrel{\text{def}}{=} (j, j+1)(-j-1, -j)$. Then $vs = (i, j+1)$ and $us = (-j, -i-1)$. Hence, by what was remarked above, $|D(vs)| = 2$. Therefore if $w \in B_n^{(n-2)}$ is such that $u \leq w < vs$, $\ell(w, vs) > 1$, and $ws < w$ then $\mu(w, vs) = 0$. In fact, if $\mu(w, vs) \neq 0$ then $D(vs) \subseteq D(w)$, but $s \notin D(vs)$, so $|D(w)| = 3$, which is a contradiction. Moreover, if $\ell(w, vs) = 1$ then either $w \in \{(i+1, j+1), (i, j+2)\}$, if $j \neq -2$, or $w \in \{(i+1, -1), (i, 1)\}$, if $j = -2$, so, since $s \in D(w)$, $w = (i, 1)$ and $j = -2$ in which case $P_{u,w} = 0$ by our induction hypotheses (8). By our induction hypotheses (8) we have that $P_{us,vs} = 0$ and $P_{u,vs} = q^{d_{vs}-1}$. Hence by Proposition 2 we have that $P_{u,v} = q^{d_{vs}} = q^{d_v - 1} = \tilde{P}_{u,v}$ as claimed.

3) $u = (j, -i-1)$, $i + j \leq -3$.

Assume first that $j \neq -1$. Let $s = (j, j+1)(-j-1, -j)$. Then $vs = (i, j+1)$ and $us = (j+1, -i-1)$. Hence, by what was remarked above, $|D(vs)| = 2$. Reasoning as in the previous case we conclude that if $w \in B_n^{(n-2)}$ is such that $u \leq w < vs$, $\ell(w, vs) > 1$ and $ws < w$ then $\mu(w, vs) = 0$ and that if $\ell(w, vs) = 1$ then $w = (i, 1)$ and $j = -2$, so $P_{u,w} = 0$ by our induction hypotheses (8). By our induction hypotheses (8) we have that $P_{us,vs} = P_{u,vs} = 0$. Hence by Proposition 2 and our definitions we have that $P_{u,v} = 0 = \tilde{P}_{u,v}$ as claimed.

Assume now that $j = -1$. Then $i \leq -3$. Let $s = s_0$. Then $vs = (i, 1)$ and $us = (1, -i-1)$. Hence, by what was remarked above, $|D(vs)| = 2$, so if $w \in B_n^{(n-2)}$ is such that $u \leq w < vs$ and $ws < w$ then $\ell(w, vs) > 1$ and $\mu(w, vs) = 0$. By our induction hypotheses (8) we have that $P_{us,vs} = 0$ and $P_{u,vs} = q^{d_{vs}-1}$. Hence by Proposition 2 and our definitions we have that $P_{u,v} = q^{d_{vs}} = q^{d_v - 1} = \tilde{P}_{u,v}$ as claimed.

4) $u = (i, -j-1)$, $i + j \leq -3$.

Then, since $u < v$, $j < 0$, so $j \leq -2$. By Lemma 3, Proposition 11, and our definitions, we then have that $P_{u,v} = 0 = \tilde{P}_{u,v}$, as desired.

Suppose now that $|D(v)| = 1$. Then, by what was remarked above, either $j - i = 1$, or $j = n$, or $i + j = -1$. Let $u \in B_n^{(n-2)}$, $u < v$ be such that $D(v) \subseteq D(u)$. Then, by Proposition 6 and Lemma 1, $u \in \{(i+1, b) : b > i+2\}$ if $j - i = 1$ and $i \geq 1$, $u \in \{(i+1, b) : b > i+2\} \cup \{(-i-2, b) : b \geq -i\} \cup \{(a, -i-2) : a \geq i\}$ if $j - i = 1$ and $i < -2$, $u \in \{(-1, b) : b > 1\}$ if $v = (-2, -1)$, $u \in \{(-i-1, n)\}$ if $j = n$ and $i < -1$ (if $j = n$ and $i \geq -1$ then there are no such u), and $u \in \{(-i-1, b) : b \geq -i+1\} \cup \{(i, b) : b \geq -i+1\} \cup \{(a, -i-1) : a \geq i+2\}$ if $i + j = -1$. We distinguish these cases.

1) $u = (i+1, b)$, $b > i+2$, $j - i = 1$, $i \neq -2$.

Then $i < n-2$. Let $s = (i+1, i+2)(-i-2, -i-1)$. Then $vs = (i, i+2)$ and $us = (i+2, b)$. Hence, by what was remarked above, $|D(vs)| = 2$. Therefore if $w \in B_n^{(n-2)}$ is such that $u \leq w < vs$ and $ws < w$ then $\mu(w, vs) = 0$. Moreover there are no w for which $\ell(w, vs) = 1$ and $ws < w$. By our induction hypotheses (8) we have that $P_{us,vs} = 0$, $P_{u,vs} = 1$ if $b = i+3$ while $P_{u,vs} = 0$ if $b > i+3$. Hence by Proposition 2 we have that $P_{u,v} = q$ if $b = i+3$, while $P_{u,v} = 0$ if $b > i+3$ and the result follows from our definition (7) of $\tilde{P}_{u,v}$.

2) $u = (-i-2, b)$, $b \geq -i$, $j - i = 1$, $i < -2$.

Let $s = s_{-i-2}$ and suppose first that $i < -3$. Then $vs = (i, i+2)$ and $us = (-i-1, b)$. $|D(vs)| = 2$ and there are no w such that $\ell(w, vs) = 1$ and $ws < s$ so we conclude from Proposition 2 that $P_{u,v} = P_{us,vs} + q\, P_{u,vs}$. By our induction hypotheses (8) we have that $P_{us,vs} = 0$ for all $b \geq -i$, $P_{u,vs} = 0$ if $b > -i$ and $P_{u,vs} = q^{d_{vs}-1} = q^{d_v - 2}$ if $b = -i$. Hence by Proposition 2 we have that $P_{u,v} = 0$ if $b > -i$ and $P_{u,v} = q^{d_v - 1}$ if $b = -i$, and the result follows from our definition (7) of $\tilde{P}_{u,v}$.

Suppose now that $i = -3$. Then everything follows as above except that $w = (-3, 1)$ is such that $u < ws < w < vs$, $\ell(w, vs) = 1$ but, by our induction hypotheses (8), $P_{u,w} = 0$. So, since by our induction hypotheses (8) $P_{u,vs} = 0$ if $b > 3$ and $P_{u,vs} = q^{d_v - 2}$ if $b = 3$, while $P_{us,vs} = 0$, by Proposition 2 we have that $P_{u,v} = 0$ if $b > 3$ and $P_{u,v} = q^{d_v - 1}$ if $b = 3$ and the result follows as in the previous case.

3) $u = (a, -i-2)$, $a \geq i$, $j - i = 1$, $i < -2$.

Suppose first that $i < -3$. Let $s = s_{-i-2}$. Then $vs = (i, i+2)$ and $us = (a, -i-1)$ if $a \neq i+1$ while $us = (i+2, -i-1)$ if $a = i+1$. Hence, by what was remarked above, $|D(vs)| = 2$, so we conclude again that $\mu(w, vs) = 0$ for all $w \in B_n^{(n-2)}$ such that $u \leq w < vs$ and $ws < w$ except possibly if $w \lhd vs$. But $\{w \in B_n^{(n-2)} : w \lhd vs\} = \{(i, i+3), (i+1, i+2)\}$ so $\{w \in B_n^{(n-2)} : w \lhd vs, ws < w\} = \emptyset$. By our induction hypotheses (8) we have that $P_{u,vs} = 0$, $P_{us,vs} = q^{d_{vs}-1}$ if $a = -i-3$ while $P_{us,vs} = 0$ if $a < -i-3$. Hence by Proposition 2 we have that $P_{u,v} = q^{d_{vs}-1} = q^{d_v-2}$ if $a = -i-3$, while $P_{u,v} = 0$ if $a < -i-3$ and the result follows from our definition (7) of $\tilde{P}_{u,v}$.

Suppose now that $i = -3$. Then $-3 \leq a \leq -2$ and everything works exactly as above except that now $\{w \in B_n^{(n-2)} : w \lhd vs, ws < w\} = \{(-3, 1)\}$ and by our induction hypotheses (8) we have that $P_{u,vs} = 1$, $P_{us,vs} = 0$ if $a = -3$, $P_{us,vs} = q^{d_{vs}-1} = q$ if $a = -2$, and $P_{u,(-3,1)} = 1$. Hence by Proposition 2 we have that $P_{u,v} = q - q = 0$ if $a = -3$ while $P_{u,v} = 2q - q = q$ if $a = -2$ and the result follows from our definition (7) of $\tilde{P}_{u,v}$ (note that if $a = -2$ then $u \in C_L[v] \cap C_L^*[v]$, but $d_v = 3$ so the two definitions are consistent).

4) $u = (-1, b)$, $b > 1$, $j - i = 1$, $i = -2$.

Let $s = s_0$. Then $vs = (-2, 1)$ and $us = (1, b)$; moreover $\{w \in B_n^{(n-2)} : u \leq w < vs, ws < w\} = \{(-1, a) : 1 < a \leq b\}$. Then, by our induction hypotheses (5) we have that $P_{us,vs} = 0$ if $b \neq 3$ and $P_{us,vs} = q$ if $b = 3$, $P_{u,vs} = 0$ if $b > 3$ and $P_{u,vs} = 1$ if $1 < b < 4$. By Proposition 3 we have that $\mu(w, vs) = 0$ if $a > 2$ (since in this case $D(vs) = \{s_1\} \not\subseteq D(w)$) and $\mu(w, vs) = 1$ if $a = 2$; by our induction hypotheses (8) $P_{u,(-1,2)} = 1$ if $2 \leq b \leq 3$ and $P_{u,(-1,2)} = 0$ if $b > 3$. Hence by Proposition 2 we have that $P_{u,v} = 0$ if $b > 3$, $P_{u,v} = q + q - \mu((-1, 3), vs)q^{3/2} - qP_{(-1,3),(-1,2)} = q$ if $b = 3$ and $P_{u,v} = q - q = 0$ if $b = 2$, and the result follows from our definition (6) of $\tilde{P}_{u,v}$.

5) $u = (-i - 1, n)$, $j = n$, $i < -1$.

Then, by Lemma 3, $P_{u,v} = 0$ so the result follows from our definition (8) of $\tilde{P}_{u,v}$.

6) $u = (-i - 1, b)$, $b \geq -i + 1$, $i + j = -1$.

Let $s = s_j$. Then $vs = v^*$, $us = (-i, b)$ and $-n < i < -1$. In this case, since $|D(vs)| = 2$ and there are no $w \in B_n^{(n-2)}$ such that $\ell(w, vs) = 1$ and $s \in D(w)$, we conclude from Proposition 2 that $P_{u,v} = P_{us,vs} + q\, P_{u,vs}$. By our induction hypotheses (8) we have that $P_{us,vs} = 0$ for all $b \geq -i+1$, $P_{u,vs} = 0$ if $-i > 2$, $P_{u,vs} = 1$ if $-i = 2$ and $b = 3$, $P_{u,vs} = 0$ if $-i = 2$ and $b > 3$. Hence we have that $P_{u,v} = q$ if $i = -2$ and $b = 3$, and $P_{u,v} = 0$ otherwise, and the result follows from our definitions (8) and (5) of $\tilde{P}_{u,v}$.

7) $u = (i, b)$, $b > -i$, $i + j = -1$.

Then, by Lemma 3, $P_{u,v} = 1$ if $b = -i + 1$ and $P_{u,v} = 0$ if $b > -i + 1$, and the result follows from our definitions (8) and (5) of $\tilde{P}_{u,v}$.

8) $u = (a, -i - 1)$, $a > i + 1$, $i + j = -1$.

This case is analogous to the previous one. Namely, $P_{u,v} = 1$ if $a = i + 2$ and $P_{u,v} = 0$ if $a > i + 2$, and the result follows from our definition (8) of $\tilde{P}_{u,v}$ (note that $i < -2$ in this case).

This concludes the induction step and hence the proof. $\square$

For lower intervals the preceding theorem takes a particularly simple form.

Corollary 1. *Let $v \in B_n^{(n-2)}$, $n \geq 4$. Then*

$$P_{e,v}^{(n-2),q} = \begin{cases} q, & \text{if } v = (n-3, n-2), \\ 1, & \text{if } v \in \{(n-2, n), (n-1, n)\}, \\ q^{2n-4}, & \text{if } v = (-n, -n+1), \\ 0, & \text{otherwise.} \end{cases}$$

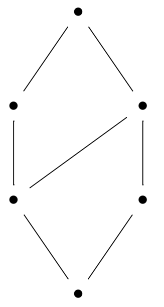

Fig. 1. Hasse diagram of the intervals $[(1,3),(-2,1)]^{(n-2)}$ and $[(2,4),(-1,3)]^{(n-2)}$.

One of the most celebrated conjectures about the Kazhdan–Lusztig polynomials is the so-called "combinatorial invariance conjecture" (see e.g. [1] p. 161 and the references cited there). This conjecture states that the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ depends only on the isomorphism class of $[u,v]$ as a poset. It is natural to wonder about the corresponding statement for the parabolic Kazhdan–Lusztig polynomials. Namely if $P^{J,q}_{u,v}(q)$ (equivalently, $P^{J,-1}_{u,v}(q)$, $R^{J,q}_{u,v}(q)$, $R^{J,-1}_{u,v}(q)$) depends only on the isomorphism class of $[u,v]^J$ as a poset. For the minuscule quotients this is known to be true (see [4, Corollary 4.8]). For the quasi-minuscule quotients, however, this is not the case. For example, by Theorem 4, one has that $P^{(n-2),q}_{(1,3),(-2,1)} = q$, and $P^{(n-2),q}_{(2,4),(-1,3)} = 0$, but from Proposition 11 we have that $[(1,3),(-2,1)]^{(n-2)} \cong [(2,4),(-1,3)]^{(n-2)}$ (this interval is shown in Fig. 1).

Using Proposition 13 we deduce from Theorem 4 the following result which computes explicitly the parabolic Kazhdan–Lusztig polynomials of type q of the co-adjoint quotients of the symmetric groups. Recall that if $v \in S_n^{[2,n-2]}$ we write $v=(i,j)$ to mean that $i = v^{-1}(1)$ and $j = v^{-1}(n)$, and we let $\tilde{v} = (-i,j)$.

Corollary 2. *Let $u, v \in S_n^{[2,n-2]}$, $u \le v$, $v=(i,j)$. Then*

$$P^{[2,n-2],q}_{u,v} = \begin{cases} 1, & \textit{if } \tilde{u} \in Q[\tilde{v}], \\ 0, & \textit{otherwise}, \end{cases}$$

if $v=(2,1)$ while

$$P^{[2,n-2],q}_{u,v} = \begin{cases} 1, & \textit{if } \tilde{u} \in Q[\tilde{v}], \\ q^{\ell(v)-n+1}, & \textit{if } \tilde{u} \in Q^*[\tilde{v}], \\ 0, & \textit{otherwise}, \end{cases}$$

if $v \neq (2,1)$.

Proof. This follows immediately from Proposition 13 and Theorem 4 noting that if $v \in S_n^{[2,n-2]}$ then $\ell(\tilde{v}) = \ell(v) + n - 1$ by Propositions 10 and 11 where the first length is computed in $B_n^{(n-2)}$ while the second one in $S_n^{[2,n-2]}$. $\square$

We now compute the parabolic Kazhdan–Lusztig polynomials of the co-adjoint quotients in type D.

Let $u = (i, j) \in \mathbb{Z}^2$; we define $\hat{v} = (i, -j)$, and $\mathbf{1} = (1, 1)$. For a set $A \subseteq \mathbb{Z}^2$ we define $\tilde{A} = \{\tilde{u} : u \in A\}$ and analogously $\hat{A}$ and A^*. Note that $u \in D_n^{(n-2)}$ if and only if $u^* \in D_n^{(n-2)}$.

For $u = (i, j) \in D_n^{(n-2)}$ we define

$$Q[(i,j)] \stackrel{\text{def}}{=} \begin{cases} \{-1,2\} \times \{j, j+1\}, & \text{if } i = -1, \\ \{i, i+1\} \times \{-1, 2\}, & \text{if } j = -1, \\ \{i, i+1\} \times \{j, j+1\}, & \text{otherwise,} \end{cases}$$

$$E[(i,j)] \stackrel{\text{def}}{=} \{(i,j), (i,j+1), (i+1,j), (i-1,j+1), (i+1,j-1), (i-1,j-1)\},$$

$$C_U[(i,j)] \stackrel{\text{def}}{=} \{i-1\} \times \{j-1, j\},$$

$$C_D[(i,j)] \stackrel{\text{def}}{=} \{i, i+1\} \times \{j+1\}.$$

Note that $Q[u]$, $E[u]$, $C_U[u]$, $C_D[u] \subseteq \mathbb{Z}^2$. We can now state and prove the second main result of this section.

Theorem 5. *Let $u, v \in D_n^{(n-2)}$, $u \leq v$, $v = (i, j)$. Then*

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in [(-1,3), (-1,2)], \\ q, & \textit{if } u \in [(3,4), (2,4)], \\ 0, & \textit{otherwise,} \end{cases} \tag{10}$$

if $v = (-1, 2)$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in [(\mp 1,3), (-2,\pm 1)], \\ q, & \textit{if } u \in [(2,4), (\pm 1,4)], \\ 0, & \textit{otherwise,} \end{cases} \tag{11}$$

if $v = (-2, \pm 1)$,

$$P_{u,v}^{(n-2),q} = \begin{cases} q^{j-1}, & \textit{if } u \in C_D[(j,-i)], \\ q^{j-2}, & \textit{if } u \in C_U[(j,-i)], \\ 1, & \textit{if } u \in E[(-j,-i)], \\ 0, & \textit{otherwise,} \end{cases} \tag{12}$$

if $i + j = -1$ and $j \neq 1$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in C_U[(j,j+1)], \\ q, & \textit{if } u \in C_D[(j,j+1)], \\ q^{|j|-2}, & \textit{if } u \in E^*[(j+1,-j)], \\ 0, & \textit{otherwise,} \end{cases} \tag{13}$$

if $j-i=1$ *and* $j \in [-n+1,-3] \cup [2,n]$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in Q[v], \\ q^{|j|-2}, & \textit{if } u \in \hat{Q}[v], \\ 0, & \textit{otherwise}, \end{cases} \tag{14}$$

if $j-i>1$ *and* $j<-2$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in [(i+1,2),v] \cup [(2,j+1),v] \cup [\hat{v},v] \cup [\tilde{v},v], \\ q\delta_{v,(-3,-2)}, & \textit{if } u \in [\hat{v},v]^*, \\ 0, & \textit{otherwise}, \end{cases} \tag{15}$$

if $i=-2$ *and* $j \in [3,n]$, *or* $j=-2$,

$$P_{u,v}^{(n-2),q} = \begin{cases} 1, & \textit{if } u \in Q[v], \\ q^{|i|-N_1(i+j)|j|-2}, & \textit{if } u \in Q^*[v] \cup \tilde{Q}[v], \\ 0, & \textit{otherwise}, \end{cases} \tag{16}$$

otherwise, where, if x *or* y *don't lie in* $D_n^{(n-2)}$, *then* $[x,y] \stackrel{\text{def}}{=} \emptyset$.

Proof. Let, for simplicity, $P_{x,y} \stackrel{\text{def}}{=} P_{x,y}^{(n-2),q}$ for all $x,y \in D_n^{(n-2)}$, and, for convenience, $\tilde{P}_{u,v}$ be the polynomial defined, for all $u,v \in D_n^{(n-2)}$, by the right hand sides of equations (10) through (16). So we have to prove that $P_{u,v} = \tilde{P}_{u,v}$ for all $u,v \in D_n^{(n-2)}$. We proceed by induction on $\ell(v) \geq 0$.

By reasoning as in the proof of Theorem 4 it is enough to show that $P_{u,v} = \tilde{P}_{u,v}$ for all $u,v \in D_n^{(n-2)}$ such that $D(v) \subseteq D(u)$. Note that, by Lemma 2, $|D(v)| \leq 3$ for all $v \in D_n^{(n-2)}$. Let $v \in D_n^{(n-2)}$, $v=(i,j)$, be such that $\ell(v) \geq 1$ and $u \in D_n^{(n-2)}$, $u<v$, be such that $D(v) \subseteq D(u)$.

Suppose first that $|D(v)|=3$. Note first that, by Lemma 2, $|D(v)|=3$ if and only if $j=-2$ and $i<-3$ or $i=-2$ and $2<j<n$. In particular, $\{\tilde{s}_0, s_1\} \subseteq D(v)$. Since $u<v$ and $D(u) \supseteq D(v)$ we conclude that $|D(u)|=3$ so either $u=(a,-2)$ with $a<-3$ or $u=(-2,b)$ with $2<b<n$. But $u<v$ and $D(u)=D(v)$ so $j=-2$, $i<-3$, and $u=(-2,-i-1)$. Let $s \in D(v) \setminus \{\tilde{s}_0, s_1\}$. Then, by Lemma 2, $|D(vs)|=3$ except if $v=(-4,-2)$ (if $v=(-2,n-1)$ then there are no $u<v$ such that $D(v) \subseteq D(u)$). If $|D(vs)|=3$ then $P_{u,vs} = P_{us,vs} = 0$ by our induction hypotheses (15). Furthermore, if $w \in D_n^{(n-2)}$ is such that $u<w \leq vs$, $ws<w$, and $\mu(w,vs) \neq 0$ then by Proposition 3 we have that either $D(w) \supseteq D(vs)$ or $w \lhd vs$. The first case implies that $D(w) = \{s\} \cup D(vs)$ so $|D(w)|=4$ which is impossible. The second case implies that $w \in \{(i+2,-2),(i+1,-1),(i+1,1)\}$ and hence that $ws>w$, which is again a contradiction. Hence we conclude that all summands on the right hand side of (2) are zero, so $P_{u,v}=0$. If $v=(-4,-2)$ then $u=(-2,3)$ so $us=(-2,4)$ and from Proposition 2 one concludes that $P_{u,v} = q^2-q^2=0$.

On the other hand, it is easy to check from our definition (15) that $\tilde{P}_{u,v} = 0$ so $P_{u,v} = \tilde{P}_{u,v}$, as desired.

Suppose now that $|D(v)| = 2$ and $D(v) \neq \{\tilde{s}_0, s_1\}$. Note first that, if $z \in D_n^{(n-2)}$, $z = (x, y)$, then $|D(z)| = 1$ if and only if either $x+y = -1$ or $y-x = 1$ and $x \neq -3$, or $y = n$ and $x \notin \{-2, n-1\}$ or $z = (-1, 2)$. Therefore, since $|D(v)| = 2$ and $D(v) \neq \{\tilde{s}_0, s_1\}$, $i \neq -2$ and $j \neq -2$. Hence either $i < j < -2$ or $i < -2 < j$ or $-2 < i < j$. Assume first that $i < j < -2$. Then, since $|D(v)| = 2$, $j - i > 1$. Hence $D(v) = \{s_{-i-1}, s_{-j-1}\}$ so $D(u) \supseteq \{s_{-i-1}, s_{-j-1}\}$. From Lemma 2 (or directly) we therefore deduce that either $-i-1 \in \{a, b\}$ or $i \in \{a, b\}$ and that either $-j-1 \in \{a, b\}$ or $j \in \{a, b\}$. So since $j-i > 1$, we conclude that $(a, b) \in \{(-i-1, -j-1), (-i-1, j), (-j-1, -i-1), (j, -i-1), (i, j), (i, -j-1), (-j-1, i)(j, i)\}$. But $a < b$ so, since $i+1 < j < -2$, $(a, b) \in \{(-j-1, -i-1), (j, -i-1), (i, j)(i, -j-1)\}$. Furthermore, $u < v$ so we conclude from Proposition 12 that $u \in \{(-j-1, -i-1), (j, -i-1), (i, -j-1)\}$. We treat only one of these three cases, the others being analogous, and simpler.

Let $u = (j, -i-1)$. Let $s \stackrel{\text{def}}{=} s_{-i-1}$. Then $vs = (i+1, j)$, $us = (j, -i)$ and $\{w \in D_n^{(n-2)} : u \leq w < vs, ws < w\} = \{(a, -i-1) : i+2 \leq a \leq j\}$ (for if $w \in D_n^{(n-2)}$, $w = (x, y)$, $u \leq w < vs$, and $ws < w$ then, by Proposition 12 and Lemma 2 we have that $j \geq x \geq i+1$, $-i-1 \geq y \geq j$, and either $-i-1 \in \{x, y\}$ or $i \in \{x, y\}$). From our induction hypotheses (14) we have that, if $j-i \neq 2$, then $P_{us,vs} = 0$ and $P_{(a,-i-1),vs} = 0$ for all $i+2 \leq a \leq j$, so $P_{u,v} = 0$ and the result follows from our definition (14) of $\tilde{P}_{u,v}$. If instead $j - i = 2$ then from our induction hypothesis (13) we have that $P_{us,vs} = 0$ and $P_{u,vs} = 0$. Therefore $P_{u,v} = 0$ and the result again follows from our definition (14) of $\tilde{P}_{u,v}$.

Assume now that $-2 < i < j$. Then, since $|D(v)| = 2$, $j - i > 1$ and $j < n$. Assume first that $i > 0$. Then $D(v) = \{s_i, s_j\}$ so $D(u) \supseteq \{s_i, s_j\}$ and this, by reasoning as in the case $i < j < -2$, implies that $i \in \{a, b, -a-1, -b-1\}$ and $j \in \{a, b, -a-1, -b-1\}$. Hence $(a, b) \in \{(i, j), (j, i), (-i-1, j), (j, -i-1), (-i-1, -j-1), (-j-1, -i-1), (-j-1, i), (i, -j-1)\}$. But $a < b$ so $(a, b) \in \{(i, j), (-i-1, j), (-j-1, -i-1), (-j-1, i)\}$. Finally, $u < v$, so by Proposition 12 we have that $a \geq i$, $b \geq j$, and $(a, b) \neq (i, j)$ so we conclude that there are no $u < v$ such that $D(u) \supseteq D(v)$. If $i = -1$ then, since $|D(v)| = 2$, $j \geq 3$. Also, $D(v) = \{\tilde{s}_0, s_j\}$ so $\{\tilde{s}_0, s_j\} \subseteq D(u)$ and this, by reasoning as above and using Lemma 2, implies that $j \in \{a, b, -a-1, -b-1\}$ and $0 \in \{-a-2, -b-2, -a-1, -b-1\}$. Hence $(a, b) \in \{(j, -1), (j, -2)(-1, j), (-2, j), (-j-1, -1), (-j-1, -2), (-1, -j-1), (-2, -j-1)\}$. But $a < b$ and $u < v$ so $a \geq -1$, $b \geq j$ and $(a, b) \neq (-1, j)$ so we conclude again that there are no $u < v$ such that $D(u) \supseteq D(v)$.

Finally, assume that $i < -2 < j$. Then, since $|D(v)| = 2$, $j < n$ and $i+j \neq -1$. Assume first that $i + j > -1$. Then $j > 2$, $D(v) = \{s_{-i-1}, s_j\}$, and reasoning exactly as above we conclude that the only $u < v$ such that $D(u) \supseteq D(v)$ is $u = (-i-1, j)$. If $i+j < -1$ and $j \geq 2$ then $D(v) = \{s_{-i-1}, s_j\}$ and we conclude as above that the only $u < v$ such that $D(u) \supseteq D(v)$ are $u = (j, -i-1)$ and $u = (-j-1, -i-1)$. If $i+j < -1$ and $j = 1$ then the conclusion is exactly the same (namely that $u \in \{(1, -i-1), (-2, -i-1)\}$) if $i < -3$ and is that there are no such u if $i = -3$ (because $D((1, 2)) \not\supseteq \{s_1, s_2\}$).

Finally, if $i+j<-1$ and $j=-1$ then $D(v)=\{\tilde{s}_0, s_{-i-1}\}$ so $D(u)\supseteq\{\tilde{s}_0, s_{-i-1}\}$. Hence we conclude that $i\in\{a,b,-a-1,-b-1\}$ and $0\in\{-a-2,-b-2,-a-1,-b-1\}$ so $(a,b)\in\{(-i-1,-1),(-i-1,-2),(-1,-i-1),(-2,-i-1),(i,-1),(i,-2)(-1,i)(-2,i)\}$. But $a<b$ and $u<v$ so from Proposition 12 we deduce that there are no such u if $i=-3$ and that $u\in\{(-1,-i-1),(-2,-i-1)\}$ if $i<-3$.

We treat one of these cases, the others being analogous and simpler.

Let $i+j<-1$ and $u=(j,-i-1)$. Assume first that $i+j<-2$. Let $s=s_{-i-1}$. Then $us=(j,-i)$, $vs=(i+1,j)$, and $\{w\in D_n^{(n-2)}:\ u\le w<vs,\ ws<w\}=\{(x,-i-1):\ i+2\le x\le j\}$ (for if $w=(x,y)$, $u\le w<vs$, and $ws<w$ then $i\in\{x,y,-x-1,-y-1\}$ and $j\ge x\ge i+1$, $-i-1\ge y\ge j$). From our induction hypothesis (16) we have that $P_{u,vs}=P_{us,vs}=0$ and $P_{(a,-i-1),vs}=q^{-i-j-3}$ if $a\in\{-j,-j-1\}$ while $P_{(a,-i-1),vs}=0$ otherwise. But from Proposition 12 we have that $\ell((-j,-i-1),vs)=-2i-2j-3=\ell((-j-1,-i-1),vs)+1$ so $\mu((x,-i-1),vs)=0$ for all $i+2\le x\le j$. Hence we conclude from (2) that $P_{u,v}=0$ and the result follows from our definition (16) of $\tilde{P}_{u,v}$.

If $i+j=-2$ and $j\ge 2$ then the reasoning is exactly the same except that now from our induction hypothesis (12) we have that $P_{(x,-i-1),vs}=1$ if $x=-j+1$, $P_{(x,-i-1),vs}=q^{j-2}$ if $x=j-1$, and $P_{(x,-i-1),vs}=0$ otherwise. From Proposition 12 we have that $\ell((-j+1,j+1),vs)=2$ and $\ell((j-1,j+1),vs)=2j-2$ so $\mu((x,-i-1),vs)=0$ for all $-j\le x\le j$ and we conclude that $P_{u,v}=0=\tilde{P}_{u,v}$ exactly as above.

Finally, if $v=(-3,1)$, then let $s\overset{\text{def}}{=}s_2$. Then $us=(1,3)$, $vs=(-2,1)$, and it follows immediately from Proposition 12 that $u\not\le vs$ so we have from Proposition 2 that $P_{u,v}=P_{us,vs}$. But from our induction hypothesis (11) we have that $P_{us,vs}=0$ and the result again follows from our definition (16) of $\tilde{P}_{u,v}$.

Suppose now that $D(v)=\{\tilde{s}_0,s_1\}$. Then from Lemma 2 we have that $1\in\{i,j,-i-1,-j-1\}$ and $0\in\{-i-1,-j-1,-i-2,-j-2\}$. So we conclude that $v\in\{(-2,x):\ -1\le x\le n\}\cup\{(x,-2):\ -n\le x\le -3\}$. But $|D(v)|<3$, so by what was remarked at the beginning of this proof (or directly) we have that $v\in\{(-2,n),(-3,-2)\}$. Assume first that $v=(-2,n)$. Then $D(u)\supseteq\{\tilde{s}_0,s_1\}$ so by what we have just remarked either $a=-2$ and $b\ge -1$ or $b=-2$ and $a\le -3$. But $u<v$ so, by Proposition 12, $a\ge -2$ and $b\ge n$ which shows that there are no $u<v$ such that $D(u)\supseteq D(v)$. If $v=(-3,-2)$ then reasoning in exactly the same way one sees that if $u<v$ is such that $D(u)\supseteq D(v)$ then $u\in\{(-2,b):\ \ b\ge 3\}$ (for $D((-2,-1))=\{\tilde{s}_0\}$, $D((-2,1))=\{s_1\}$). So let $u=(-2,b)$, $b\ge 3$. Let $s\overset{\text{def}}{=}s_1$. Then $vs=(-3,-1)$, $us=(-1,b)$ and $\{w\in D_n^{(n-2)}:\ u\le w<vs,\ ws<w\}=\{(-2,y):\ \ 3\le y\le b\}$ (for if $w\in D_n^{(n-2)}$, $w=(x,y)$, is such that $u\le w<vs$ and $ws<w$ then $1\in\{x,y,-x-1,-y-1\}$ and $-2\ge x\ge -3$, $b\ge y\ge -1$ so $w\in\{(-2,y):\ b\ge y\ge -1\}\cup\{(-3,1)\}$ but, again by Proposition 12, $(-3,1)\not\le vs$, $(-2,1)\not\le vs$ and, by Lemma 2, or directly, $s\notin D((-2,-1))$). By our induction hypothesis (16) we have that $P_{us,vs}=0$, $P_{u,vs}=1$ if $b=3$ while $P_{u,vs}=0$ otherwise, and $P_{(-2,y),vs}=1$ if $y=3$, $P_{(-2,y),vs}=0$ otherwise. But from Proposition 12 we have that $\ell((-2,3),vs)=2$ so $\mu((-2,y),vs)=0$ for all $3\le y\le b$. Hence from Proposition 2 we have that $P_{u,v}=q$ if $b=3$ and $P_{u,v}=0$ if $b\ge 4$ and the result follows from our definition (15) of $\tilde{P}_{u,v}$.

Finally, suppose that $|D(v)| = 1$. Then either $i + j = -1$ (so $j \in [n-1]$), or $j - i = 1$ and $i \neq -3$, or $j = n$ and $i \notin \{-2, n-1\}$. We distinguish these cases.

Assume first that $i + j = -1$. Then $D(v) = \{s_j\}$ so $D(u) \supseteq \{s_j\}$ and $u < v$. Since $u \in D_n^{(n-2)}$ and $s_j \in D(u)$ (and $1 \leq j \leq n-1$) we have that either $u(j) \geq n-1$ or $u(j+1) \leq -n+1$. So we conclude from Proposition 12 that $u \in \{(j,b) : j+1 < b \leq n\} \cup \{(a,j) : -j-1 \leq a < j\} \cup \{(i,b) : j < b\}$. We do one of these cases, the others being similar, but easier.

Let $u = (j,b)$, $b > j+1$. Assume first that $i = -2$, so $v = (-2,1)$. Let $s \stackrel{\text{def}}{=} s_j$. Then $vs = (-1,2)$, $us = (2,b)$, and, by Proposition 12, $u \not\leq vs$ so $\{w \in D_n^{(n-2)} : u \leq w < vs, ws < w\} = \emptyset$. Hence from (2) and our induction hypotheses (16) we have that $P_{u,v} = P_{us,vs} = q$, if $b = 4$, while $P_{u,v} = P_{us,vs} = 0$ if $b \neq 4$, and the result follows from our definition (11) of $\tilde{P}_{u,v}$.

Assume now that $i = -3$. Then $v = (-3,2)$. Let $s \stackrel{\text{def}}{=} s_2$. Then $vs = (-2,3)$, $us = (3,b)$ and $\{w \in D_n^{(n-2)} :\ u \leq w < vs,\ ws < w\} = \{(2,a) :\ 4 \leq a \leq b\}$.

From our induction hypotheses (15) we have that $P_{u,vs} = 1$ if $b = 4$ while $P_{u,vs} = 0$ otherwise, $P_{us,vs} = 0$, while from (16) we have that $P_{u,(2,a)} = 1$ if $b-1 \leq a \leq b$ while $P_{u,(2,a)} = 0$ otherwise. But from our induction hypotheses (15) we have that $deg(P_{(2,a),vs}) = 0$ so $\mu((2,b),vs) = \mu((2,b-1),vs) = 0$ since $\ell((2,b-1),vs) \geq 3$ because $b \geq 4$. Hence we conclude from Proposition 2 that $P_{u,v} = q$ if $b = 4$ while $P_{u,v} = 0$ otherwise and the result follows from our definition (12) of $\tilde{P}_{u,v}$.

Assume now that $i < -2$. Let $s \stackrel{\text{def}}{=} s_j$. Then $vs = (-j, j+1)$, $us = (j+1, b)$, and $\{w \in D_n^{(n-2)} : u \leq w < vs, ws < w\} = \{(j,a) : j+2 \leq a \leq b\}$. From our induction hypotheses (16) we have that $P_{us,vs} = 0$, $P_{u,vs} = 0$ if $b > j+2$, while $P_{u,vs} = q^{j-2}$ if $b = j+2$, $P_{(j,a),vs} = 0$ if $j+2 < a$ while $P_{(j,a),vs} = q^{j-2}$ if $a = j+2$. But $\ell((j,j+2),vs) = 2j+1$ so $\mu((j,j+2),vs) = 0$, so we have from (2) that $P_{u,v} = 0$, if $b > j+2$ while $P_{u,v} = q^{j-1}$ if $b = j+2$, and the result follows from our definition (12) of $\tilde{P}_{u,v}$.

Assume now that $j - i = 1$ and $i \neq -3$. By reasoning as in the previous case ($i + j = -1$) we then conclude that $u \in \{(-i-2,b) : -i \leq b\} \cup \{(a,-i-2) : i \leq a\} \cup \{(i+1,b) : i+2 < b\}$ if $i < -2$, $u \in \{(-2,b) : 3 \leq b\} \cup \{(-1,b) : 2 < b\}$, if $i = -2$, while $u \in \{(i+1,b) : i+2 < b\}$ if $i > 0$. We treat one of these cases, the others being similar, but easier.

Let $u = (-i-2,b)$, $-i \leq b$, $i < -2$. Then, $i < -3$.

Assume first that $i < -4$. Let $s \stackrel{\text{def}}{=} s_{-i-2}$. Then $vs = (i,i+2)$, $us = (-i-1,b)$, and $\{w \in D_n^{(n-2)} : u \leq w < vs, ws < w\} = \{(-i-2,a) : -i \leq a \leq b\} \cup \{(a,-i-2) : i \leq a\} \cup \{(i+1,a) : i+3 \leq a \leq b\}$ (for if $ws < w$ then $w(-i-2) > w(-i-1)$ which implies that either $w(-i-2) \geq n-1$ or $w(-i-1) \leq -(n-1)$). By Proposition 3 we have that if $\mu(w,vs) \neq 0$ then either $D(w) \supseteq D(vs)$ or $w \lhd vs$. But $D(vs) = \{s_{-i-1}, s_{-i-3}\}$ so if $w \in D_n^{(n-2)}$ is such that $u \leq w \leq vs$, $w \not\lhd vs$, $\mu(w,vs) \neq 0$, and $ws < w$ then necessarily $w(-i) < w(-i-1) < w(-i-2) < w(-i-3)$, which is impossible. Hence $\{w \in D_n^{(n-2)} :\ u \leq w \leq vs,\ ws < w,\ \mu(w,vs) \neq 0\} = \{w \in D_n^{(n-2)} : u \leq$

$w \lhd vs,\ ws < w\}$. But $\{w \in D_n^{(n-2)} :\ w \lhd vs\} = \{(i+1, i+2), (i, i+3)\}$ and none of these have a descent at $s = s_{-i-2}$. From our induction hypotheses (14) we have that $P_{us,vs} = P_{u,vs} = 0$. Therefore we conclude from (2) that $P_{u,v} = 0$, and the result follows from our definition (13) of $\tilde{P}_{u,v}$.

Assume now that $i = -4$. Let $s \stackrel{\text{def}}{=} s_2$. Then $vs = (-4, -2)$, $us = (3, b)$, and everything goes exactly as above except that now $D(vs) = \{\tilde{s}_0, s_1, s_3\}$ so $\{w \in D_n^{(n-2)} :\ u \leq w < vs,$ $\mu(w, vs) \neq 0,\ ws < w\} = \{w \in D_n^{(n-2)} :\ u \leq w \lhd vs,\ ws < w\} = \emptyset$ since $\{w \in D_n^{(n-2)} :$ $w \lhd vs\} = \{(-3, -2), (-4, -1), (-4, 1)\}$ and none of these have s as a descent. So we conclude as above that $P_{u,v} = 0 = \tilde{P}_{u,v}$, as desired.

Assume now that $j = n$ and $i \notin \{-2, n-1\}$. Then reasoning as in the two previous cases we conclude that $u = (-i-1, n)$ and $-n < i < -2$ (if $i > -2$ then there are no $u < v$ such that $D(v) \subseteq D(u)$).

Let $s \stackrel{\text{def}}{=} s_{-i-1}$. Then $vs = (i+1, n)$, $us = (-i, n)$, and $\{w \in D_n^{(n-2)} : u \leq w < vs,$ $ws < w\} = \{u\}$. By our induction hypotheses (16) (and (15) if $i = -3$) we have that $P_{us,vs} = 0$, and $P_{u,vs} = q^{-i-3}$. But, by Proposition 12 we have that $\ell(u, vs) = -2i - 4$ so $\mu(u, vs) = 0$. Hence by (2) we have that $P_{u,v} = q^{-i-2}$ and the result follows from our definition (16) of $\tilde{P}_{u,v}$.

This concludes the induction step and hence the proof. $\square$

For lower intervals the preceding result takes a particularly simple form.

Corollary 3. *Let $v \in D_n^{(n-2)}$, $n \geq 5$. Then*

$$P_{e,v}^{(n-2),q} = \begin{cases} q^{n-3}, & \text{if } v \in \{(-n+1, n), (n+1, n-2)\}, \\ q, & \text{if } v = (n-3, n-2), \\ 1, & \text{if } v \in \{(n-2, n), (n-1, n)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is a straightforward consequence of Theorem 5. $\square$

As a consequence of the last two theorems we obtain the main result of this work.

Corollary 4. *Let (W, S) be a Weyl group and $J \subseteq S$ be such that W^J is a quasi-minuscule quotient. Then $P_{u,v}^{J,q}(q)$ is either zero or a monic power of q for all $u, v \in W^J$, $u \leq v$.*

Proof. If W^J is a minuscule quotient (also known as a Hermitian symmetric pair) the result follows from Corollary 4.4 of [4]. If W^J is a quasi-minuscule quotient that is not minuscule (i.e., a (co-)adjoint quotient) then the result follows from Proposition 13, Theorems 4 and 5, and from computer calculations. $\square$

The parabolic Kazhdan–Lusztig polynomials for the exceptional (co-)adjoint quotients have been computed by implementing in Maple 9 the recursions given by Theorems 1 and 2 and by Proposition 2.

5. Applications

In this section we derive some consequences of our main result for the ordinary Kazhdan–Lusztig polynomials.

Corollary 5. *Let (W, S) be a Weyl group and $J \subseteq S$ be such that W^J is a quasi-minuscule quotient. Then*

$$\sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q)$$

is either zero or a monic power of q, for all $u, v \in W^J$, $u \leq v$.

Proof. This follows immediately from Corollary 4 and Proposition 1. □

Note that the exact power of q in Corollary 5 is explicitly determined in Theorems 4 and 5, in Theorems 4.1, 4.2 and 4.3 of [4] and in Theorem 5.1 of [3].

From (1) and Theorems 4 and 5 we obtain the following explicit expressions for the coefficient of maximum possible degree of the ordinary Kazhdan–Lusztig polynomials indexed by elements of B_n and D_n that lie in the respective quasi-minuscule quotients.

Corollary 6. *Let $u, v \in B_n^{(n-2)}$, $u \leq v$, $v = (i, j)$, $\ell(u, v) > 1$. Then $\mu(u, v)$ equals either 0 or 1. Furthermore $\mu(u, v) = 1$ if and only if either $u = (1, 3)$ and $v = (-2, 1)$, or $u = (-1, 3)$ and $v = (-2, -1)$, or $u = (-2, 1)$ and $v = (-3, -2)$, or $u \in \{(i + 1, j + 2), (-j-2, -i-2)\}$ and $j - i = 1$, $j \neq \{-1, -2\}$, or $u = (-1, -i-1)$ and $j - i > 1$, $j = -1$, or $u = (-j - 1, -i - 1)$ and $j - i > 1$, $i + j < -1$, $i \neq -1$, $j \neq -1$.*

Proof. This is a routine, though somewhat long, check using Proposition 11 and Theorem 4. One distinguishes the cases according to the statement of Theorem 4 (so, since $\ell(u, v) > 1$, we have six cases to consider). We treat one of these cases, the others being similar, and simpler. Assume $j - i = 1$, $i \neq -2$, and $u \in C_R^*[v]$. Then $u \in \{(-i - 1, -i), (-i - 2, -i)\}$. From Proposition 11 we have that $\ell((i, i + 1)) = 2n - 1 - 2i - 1 - N_1(i, i + 1, 2i + 1)$ and similarly that $\ell((-i - 1, -i)) = 2n - 1 + 2i + 1 - N_1(-i, -i - 1, -2i - 1)$ and $\ell((-i - 2, -i)) = 2n - 1 + 2i + 2 - N_1(-i - 2, -i, -2i - 2)$. Therefore we have that $\ell((-i - 1, -i), (i, i + 1)) = -4i - 2 + 3sgn(i)$ and $\ell((-i-2, -i), (i, i+1)) = \ell((-i-1, -i), (i, i+1))-1$. Hence $\mu((-i-2, -i), (i, i+1)) = 0$ and $\mu((-i-1, -i), (i, i+1)) = 0$ if $i \geq 1$. But $d_{(i,i+1)} - 1 = \ell((i, i+1)) - 2n+1 = -2i-4$ so $\mu((-i - 1, -i), (i, i + 1)) = 0$ also if $i \leq -3$. □

Corollary 7. *Let $u, v \in D_n^{(n-2)}$, $u \leq v$, $v = (i, j)$, $\ell(u, v) > 1$. Then $\mu(u, v)$ is either 0 or 1. Furthermore, $\mu(u, v) = 1$ if and only if either $v = (-1, 2)$ and $u = (2, 4)$, or $v = (-2, \pm 1)$ and $u = (\pm 1, 4)$, or $i+j = -1$, $j \neq \{1, -2\}$ and $u = (j - 1, j)$, or $j - i = 1$, $j \notin \{-n, -2, -1\}$ and $u \in \{(j - 1, -j - 1), (j, -j - 2), (j, j + 2)\}$, or $j - i > 1$, $j < -2$*

and $u = (i, -j-1)$, *or* $v = (-3, -2)$ *and* $u = (-2, 3)$, *or* $i + j < -1$, $j - i > 1$, $j > -1$ *and* $u = (-j-1, -i-1)$, *or* $i \leq -3$, $j = -1$ *and* $u = (-2, -i-1)$, *or* $i + j > -1$, $j - i > 1$, $i \leq -1$, *and* $u = (-i-1, j)$.

Proof. The proof is similar to that of Corollary 6 using Theorem 5 and Proposition 12. One again distinguishes the cases according to the statement of Theorem 5 (so, since $\ell(u, v) > 1$, there are 9 cases to consider). We treat one of these cases, the others being similar, and simpler.

Suppose that $j - i = 1$, $j \in [-n+1, -3] \cup [2, n]$ and $u \in E^*[(j+1, -j)]$. Then $u \in \{(j, -j-1), (j-1, -j-1), (j, -j-2), (j-1, -j)(j+1, -j-2), (j+1, -j)\}$ and, since $u < v$, $j \leq -3$ by Proposition 12. By Proposition 12 we have that $\ell((j, -j-1)) = \ell((j-1, -j)) = \ell((j+1, -j-2)) = 2n-3$, $\ell((j-1, -j-1)) = \ell((j, -j-2)) = 2n-2$, $\ell((j+1, -j)) = 2n-4$, and $\ell((j-1, j)) = 2n-2j-5$. Hence we conclude that $\ell((j, -j-1), v) = \ell((j-1, -j), v) = \ell((j+1, -j-2), v) \equiv 0 \pmod 2$, $\ell((j-1, -j-1), v) = \ell(j, -j-2), v) = -2j-3$, and $\ell((j+1, -j), v) = -2j-1$ so we have from Theorem 5 that $\mu(u, v) = 1$ if and only if $u = (j-1, -j-1)$ or $u = (j, -j-2)$. $\square$

6. Open problems

In this section we present some conjectures that have arisen from the present work, together with the evidence that we have for them.

This paper completes the computation of the parabolic Kazhdan–Lusztig polynomials of type q for the quasi-minuscule quotients of Weyl groups. The parabolic Kazhdan–Lusztig polynomials of type -1 have been computed for the minuscule quotients (see [2] and [19]), but not for the (co-)adjoint quotients. We have been unable to compute these polynomials for the co-adjoint quotients of types B and D. However, we have a conjectural combinatorial interpretation for these polynomials, which we now explain. Throughout this section we assume, for simplicity, $n \geq 4$.

Let $v \in B_n^{(n-2)}$, $v = (i, j)$. We define $\tilde{M}(v)$ by

$$\tilde{M}(v) \stackrel{\text{def}}{=} \begin{cases} \{(-2,1), (1,3), (3,4)\}, & \text{if } v = (-2,1), \\ \{(i,1), (j, j+1), (-i+1, -i+2), (1, -i+1)\}, & \text{if } i < -2 = -j-1, \\ \{(i,j), (-i+1, -i+2)\}, & \text{if } j - i = 1, \\ \{(i,j), (j, j+1)\}, & \text{if } 1 < j - i < 2j, \\ \{(i,-1), (-1,2), (-i+1, -i+2), (-j+1, -i+1)\}, & \text{if } i < -2 = j-1, \\ \{(i,j), (j, j+1), (-i+1, -i+2), (-j+1, -i+1)\}, & \text{otherwise,} \end{cases}$$

$M(v) \stackrel{\text{def}}{=} \tilde{M}(v) \cap B_n^{(n-2)}$, and $w_v : M(v) \to \mathbb{N}$ by $w_v(v) \stackrel{\text{def}}{=} 0$,

$$w_v((1,3)) \stackrel{\text{def}}{=} 1, \quad w_v((3,4)) \stackrel{\text{def}}{=} 2,$$

if $v = (-2, 1)$, and

$$w_v((j,j+1)) \stackrel{\text{def}}{=} w_v((-1,2)) \stackrel{\text{def}}{=} 1,$$
$$w_v((-j+1,-i+1)) \stackrel{\text{def}}{=} w_v((1,-i+1)) \stackrel{\text{def}}{=} \ell(v)-2n+3,$$

and

$$w_v((-i+1,-i+2)) \stackrel{\text{def}}{=} \ell(v)-2n+4,$$

otherwise. Thus, for example, $M((2,3)) = \{(2,3)\}$, $M((3,4)) = \{(3,4),(-2,-1)\}$, $M((i,n)) = \{(i,n)\}$ if $i < n-1$, and $M((n-1,n)) = \{(n-1,n),(-n+2,-n+3)\}$. We can now state our first conjecture.

Conjecture 1. *Let $u,v \in B_n^{(n-2)}$, $u \leq v$. Then*

$$P_{u,v}^{(n-2),-1}(q) = \sum_{x \in M(v)\cap[u,v]} q^{w_v(x)}.$$

For example, if $u = [1,2,3,8,4,9,5,6,7]$ and $v = [1,2,9,-8,3,4,5,6,7]$ then $u,v \in B_9^{(7)}$, $u = (4,6)$, $v = (-4,3)$, $\ell(v) = 16$, $\tilde{M}(v) = M(v) = \{(-4,3),(3,4),(5,6),(-2,5)\}$, $M(v) \cap [u,v] = \{(-4,3),(3,4),(-2,5)\}$, $w_v((-4,3)) = 0$, $w_v((3,4)) = 1$, and $w_v((-2,5)) = 16-18+3$ so $P_{u,v}(q) = 1+2q$. This conjecture has been verified for $n \leq 11$.

By Proposition 13 we obtain the following special case of Conjecture 1. Let $u,v \in S_n^{[2,n-2]}$, $u \leq v$. Then, if $u = (a,b)$ and $v = (i,j)$,

$$P_{u,v}^{[2,n-2],-1}(q) = \begin{cases} 1+q^{i-j}, & \text{if } a<j<i<b, \\ 1, & \text{otherwise.} \end{cases} \tag{17}$$

In fact, by the combinatorial description of the quotient $S_n^{[2,n-2]}$ we have that $a,b,i,j > 0$ and, by Proposition 10, $a \leq i$ and $j \leq b$. From Conjecture 1 and Proposition 13 it follows that $P_{u,v}^{[2,n-2],-1}(q) = \sum_{x \in M(\tilde{v})\cap[\tilde{u},\tilde{v}]} q^{w_{\tilde{v}}(x)}$. Since in this case

$$M(\tilde{v}) \cap [\tilde{u},\tilde{v}] = \begin{cases} \{\tilde{v},(-j+1,i+1)\}, & \text{if } a<j<i<b, \\ \{\tilde{v}\}, & \text{otherwise,} \end{cases}$$

and, by Proposition 11, $w_{\tilde{v}}((-j+1,i+1)) = \ell(\tilde{v})-2n+3 = i-j$, we find (17).

Our conjecture for the coadjoint quotients of D_n is similar, but we find it convenient to embed $D_n^{(n-2)}$, as a poset, into a slightly larger poset. More precisely, we let

$$\hat{D}_n^{(n-2)} \stackrel{\text{def}}{=} D_n^{(n-2)} \cup \{(i,i) : 2 \leq |i| \leq n-1\}$$

(recall that we identify $D_n^{(n-2)}$ with $\{(i,j) \in [\pm n]^2 : \; i<j, \; i \neq -j\}$), and we partially order $\hat{D}_n^{(n-2)}$ by taking Proposition 12 as a definition (it is easy to see that this indeed defines a partial order on $\hat{D}_n^{(n-2)}$).

For $v=(k,\ell)\in D_n^{(n-2)}$ we let

$$\tilde{M}(v)\stackrel{\text{def}}{=}\begin{cases}\{(k,\ell),(-k+1,-k+2)\}, & \text{if } |\ell|=1,\ k=-2,\\ \{(k,\ell),(-k+1,-k+2),(\ell,-k+1)\}, & \text{if } |\ell|=1,\ k=-3,\\ \{(k,\ell),(-k+1,-k+2),(\ell,-k+1),(\ell,2)\}, & \text{if } |\ell|=1,\ k<-3,\\ \{(k,\ell),(\ell,\ell+1)\}, & \text{if } \ell>2,\ \ell>k+1\geq 0,\\ \{(k,\ell),(-k+1,\ell),(\ell,\ell),(\ell,\ell+1)\}, & \text{if } k<-1,\ 1-k<\ell,\\ \{(k,\ell),(\ell,\ell+1),(\ell,\ell)\}, & \text{if } k<-1,\ 1-k=\ell,\\ \{(k,\ell),(\ell,\ell+1),(-k+1,-k+2),(-\ell+1,-k+1), & \\ (-k+1,-k+1),(\ell,-k+1)\}, & \text{if } 2<\ell<-k,\\ \{(k,\ell),(\ell,\ell+1),(-k+1,-k+2),(-\ell+1,-k+1), & \\ (-k+1,-k+1),(\ell-1,-k+1)\}, & \text{if } 2=\ell<-k,\\ \{(k,\ell),(k,-\ell+1),(-\ell+1,-\ell+2),(\ell,\ell), & \\ (-\ell+1,-k+1),(-k+1,-k+1), & \\ (-k+1,-k+2),(\ell,-k+1)\}, & \text{if } k+1<\ell<-1,\\ \{(k,\ell),(k,-k+1),(-k+1,-k+1), & \\ (-k+1,-k+2)\}, & \text{if } k+1=\ell<-1,\\ \{(k,\ell)\}, & \text{otherwise.}\end{cases}$$

$M(v)\stackrel{\text{def}}{=}\tilde{M}(v)\cap\hat{D}_n^{(n-2)}$ and $w_v:M(v)\to\mathbb{N}$ be defined by $w_v(v)\stackrel{\text{def}}{=}0$ and

$$w_v((-k+1,-k+2))\stackrel{\text{def}}{=}-k-1,\quad w_v((\ell,-k+1))\stackrel{\text{def}}{=}-k-2,$$
$$w_v((\ell,2))\stackrel{\text{def}}{=}1,$$

if $|\ell|=1$,

$$w_v((\ell,\ell+1))\stackrel{\text{def}}{=}1,$$

if $k\geq -1$,

$$w_v((-k+1,\ell))\stackrel{\text{def}}{=}-k-1,\quad w_v((\ell,\ell))\stackrel{\text{def}}{=}1,$$
$$w_v((\ell,\ell+1))\stackrel{\text{def}}{=}-k,$$

if $k<-1$, $1-k<\ell$,

$$w_v((\ell,\ell+1))\stackrel{\text{def}}{=}1,\quad w_v((\ell,\ell))\stackrel{\text{def}}{=}\ell-2,$$

if $k<-1$ and $1-k=\ell$,

$$w_v((\ell,\ell+1))\stackrel{\text{def}}{=}1,\quad w_v((-k+1,-k+2))\stackrel{\text{def}}{=}-\ell-k+1,$$
$$w_v((-\ell+1,-k+1))\stackrel{\text{def}}{=}-\ell-k,\quad w_v((-k+1,-k+1))\stackrel{\text{def}}{=}-k-1,$$

$$w_v((\ell,-k+1)) \stackrel{\mathrm{def}}{=} w_v((\ell-1,-k+1)) \stackrel{\mathrm{def}}{=} -k-2,$$

if $1 < \ell < -k$

$$w_v((k,-\ell+1)) \stackrel{\mathrm{def}}{=} -\ell-1, \quad w_v((-\ell+1,-\ell+2)) \stackrel{\mathrm{def}}{=} -\ell,$$
$$w_v((-\ell+1,-k+1)) \stackrel{\mathrm{def}}{=} -\ell-k-3, \quad w_v((-k+1,-k+1)) \stackrel{\mathrm{def}}{=} -k-1,$$
$$w_v((-k+1,-k+2)) \stackrel{\mathrm{def}}{=} -\ell-k-2, \quad w_v((\ell,-k+1)) \stackrel{\mathrm{def}}{=} -k-2;$$
$$w_v((\ell,\ell)) \stackrel{\mathrm{def}}{=} 1,$$

if $k+1 < \ell < -1$, and

$$w_v((k,-k+1)) \stackrel{\mathrm{def}}{=} -k-2, \quad w_v((-k+1,-k+1)) \stackrel{\mathrm{def}}{=} -k-1,$$
$$w_v((-k+1,-k+2)) \stackrel{\mathrm{def}}{=} -2k-3,$$

if $k+1 = \ell < -1$.

We can now state the second conjecture of this section.

Conjecture 2. *Let $u, v \in D_n^{(n-2)}$, $u \leq v$. Then*

$$P_{u,v}^{(n-2),-1}(q) = \sum_{x \in M(v) \cap [u,v]} q^{w_v(x)}.$$

Note that the interval $[u,v]$ above is to be taken in $\hat{D}_n^{(n-2)}$, so $[u,v] \stackrel{\mathrm{def}}{=} \{x \in \hat{D}_n^{(n-2)} : u \leq x \leq v\}$. For example, $[(-3,-2),(-4,-3)] = \{(-3,-2),(-4,-2),(-3,-3),(-4,-3)\}$. The preceding conjecture has been verified for $n \leq 11$.

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